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SPECIAL SECTION: AN HOMAGE TO MANFREDO P. DO CARMO

Bott–Thom isomorphism, Hopf bundles and Morse theory

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Abstract

Based on Morse theory for the energy functional on path spaces we develop a deformation theory for mapping spaces of spheres into orthogonal groups. This is used to show that these mapping spaces are weakly homotopy equivalent, in a stable range, to mapping spaces associated to orthogonal Clifford representations. Given an oriented Euclidean bundle $V \to X$ of rank divisible by four over a finite complex X we derive a stable decomposition result for vector bundles over the sphere bundle $\mathbb{S}(\mathbb{R} \oplus V)$ in terms of vector bundles and Clifford module bundles over X. After passing to topological K-theory these results imply classical Bott–Thom isomorphism theorems.

Keywords Vector bundles · Path space · Morse theory · Centrioles · Hopf bundles · Atiyah–Bott–Shapiro map · Thom isomorphism

Mathematics Subject Classification Primary: 53C35 · 15A66 · 55R10; Secondary: 55R50 · 58E10 · 58D15

Dedicado à memória de Manfredo do Carmo

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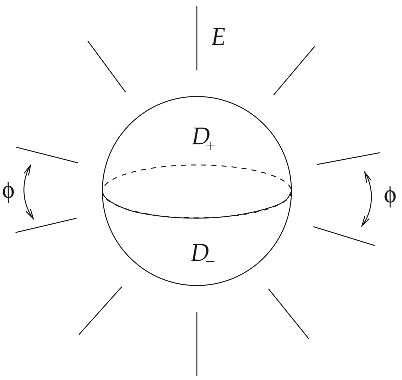
1 Introduction

In their seminal paper on Clifford modules Atiyah et al. [2] describe a far-reaching interrelation between the representation theory of Clifford algebras and topological K-theory. This point of view inspired Milnor's exposition [11] of Bott's proof of the periodicity theorem for the homotopy groups of the orthogonal group. The unique flavor of Milnor's approach is that a very peculiar geometric structure (centrioles in symmetric spaces) which is related to algebra (Clifford representations) leads to basic results in topology, via Morse theory on path spaces.

In the paper at hand we rethink Milnor's approach and investigate how far his methods can be extended. In fact, they allow dependence on arbitrary many extrinsic local parameters. Thus we may replace the spheres in Milnor's computation of homotopy groups by sphere bundles over any finite CW-complex. Among others this leads to a geometric perspective of Thom isomorphism theorems in topological K-theory.

This interplay of algebra, geometry and topology is characteristic for the mathematical thinking of Manfredo do Carmo. We therefore believe that our work may be a worthwhile contribution to his memory.

Recall that a Euclidean vector bundle E of rank p over a sphere $\mathbb{S}^n$ can be described by its *clutching map* $\phi : \mathbb{S}^{n-1} \to \mathrm{SO}_p$. In fact, over the upper and lower hemisphere E is the trivial bundle $\underline{\mathbb{R}}^p$, and ϕ identifies the two fibers $\mathbb{R}^p$ along the common boundary $\mathbb{S}^{n-1}$ as in the following picture.



Milnor in his book on Morse theory [11] describes a deformation procedure that can be used to simplify these clutching maps ϕ. The main idea in [11] is viewing the sphere as an iterated suspension and the map ϕ as an iterated path family in SO_p with prescribed end points, and then Morse theory for the energy functional on each path space is applied. However in a strict sense, Morse theory is not applied but *avoided*: it is shown that the non-minimal critical points (geodesics) have high index, so they do not obstruct the deformation of the path space onto the set of minima (shortest geodesics) via the negative gradient flow of the energy. Thus the full path space is deformed onto the set of shortest geodesics whose midpoint set can be nicely described in terms of certain totally geodesic submanifolds P_j ("centrioles"). In fact there is a chain of iterated centrioles $\mathrm{SO}_p \supset P_1 \supset P_2 \supset \cdots$ such that the natural inclusion of P_j into the path space of P_{j-1} is d-connected for some large d and for all j (that is, it induces an isomorphism in homotopy groups π_k for $k < d$ and a surjection on π_d). This is sufficient for Milnor's purpose to understand the topology of the path spaces in order to compute the stable homotopy groups of SO_p (Bott periodicity).

In [8] we went one step further and deformed the whole map ϕ into a special form: the restriction of a certain linear map $\phi_o : \mathbb{R}^n \to \mathbb{R}^{p \times p}$. The latter defines a module structure on $\mathbb{R}^p$ for the Clifford algebra Cl_{n-1}, which turns the given bundle E into the Hopf bundle for this Clifford module. As shown in [8] this leads to a conceptual proof of [2, Theorem (11.5)], expressing the coefficients of topological K-theory in terms of Clifford representations, and thus gives a positive response to the remark in [2, page 4]: "It is to be hoped that Theorem (11.5) can be given a more natural and less computational proof".

In the present paper we will put this deformation process into a family context, aiming at a description of vector bundles over sphere bundles in terms of Clifford representations. More specifically, let $V \to X$ be a Euclidean vector bundle over a finite CW-complex X and let $\hat{V} = \mathbb{S}(\mathbb{R} \oplus V) \to X$ be the sphere bundle of the direct sum bundle $\mathbb{R} \oplus V$. It is a sphere bundle with two distinguished antipodal sections $(\pm 1, 0)$. Similar as before a vector bundle $\mathscr{E} \to \hat{V}$ can be constructed by a fiberwise clutching function along the "equator spheres" $\mathbb{S}(V)$ in each fibre, and one may try to bring this clutching function into a favorable shape by a fiberwise deformation process similar as the one employed in [8].

We will realize this program if V is oriented and of rank divisible by four in order to derive bundle theoretic versions of classical Bott–Thom isomorphism theorems in topological K-theory. For example let $\mathrm{rk}\, V = 8m$ and assume that $V \to X$ is equipped with a spin structure. Let $\mathscr{S} \to \hat{V}$ be the spinor Hopf bundle associated to the chosen spin structure on V and the unique (ungraded) irreducible Cl_{8m}-representation, compare Definition 8.8. Then each vector bundle $\mathscr{E} \to \hat{V}$ is – after addition of trivial line bundles and copies of $\mathscr{S}$ – isomorphic to a bundle of the form $E_0 \oplus (E_1 \otimes \mathscr{S})$, where E_0, E_1 are vector bundles over X. Moreover the stable isomorphism types of E_0 and E_1 are determined by the stable isomorphism type of $\mathscr{E}$, see Remark 9.3. In K-theoretic language this amounts to the classical Thom isomorphism theorem in orthogonal K-theory, compare part (a) of Theorem 10.7.

Atiyah in his book on K-theory [1, p. 64] proved an analogous statement for complex vector bundles $\mathscr{E} \to \hat{L}$ where $L \to X$ is a Hermitian line bundle and $\hat{L} = \mathbb{P}(\mathbb{C} \oplus L) = \mathbb{S}(\mathbb{R} \oplus L) \to X$ is the complex projective bundle with fibre $\mathbb{CP}^1 = \mathbb{S}^2$. Now the clutching map of $\mathscr{E}$ is defined on the circle bundle $\mathbb{S}(L)$ and can be described fiberwise by Fourier polynomials with values in $\mathrm{Gl}_p(\mathbb{C})$. Then the higher Fourier modes are removed by some deformation on $\mathrm{Gl}_p(\mathbb{C})$ after enlarging $p = \mathrm{rk}(\mathscr{E})$; only the first (linear) Fourier mode remains. In our case of higher dimensional $\mathbb{S}^n$-bundles, Fourier analysis is no longer available. However, using Milnor's ideas, we still can linearize the clutching map along every fibre. But linear maps from the sphere to SO_p are nothing else than Clifford representations (cf. Proposition 2.2).

Our paper is organized as follows. In Sect. 2 we recall some notions from the theory of Clifford modules. Section 3 relates the theory of Clifford modules to iterated centrioles in symmetric spaces. This setup, which implicitly underlies the argument in [11], provides a convenient and conceptual frame for our later arguments.

A reminder of the Morse theory of the energy functional on path spaces in symmetric spaces is provided in Sect. 4 along the lines in [11]. This is accompanied by some explicit index estimates for non-minimal geodesics in Sect. 5. Different from [11] we avoid curvature computations using totally geodesic spheres instead.

After these preparations Sect. 6 develops a deformation theory for pointed mapping spaces $\mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)$, based on an iterative use of Morse theory on path spaces in symmetric spaces. When $\mathbb{R}^p$ is equipped with a Cl_k-representation, $\mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)$ contains the subspace of affine Hopf maps associated to Clifford sub-representations on $\mathbb{R}^p$ (compare Definitions 2.1, 7.9). Our Theorem 7.10 gives conditions under which this inclusion is highly connected.

Section 8 recalls the construction of vector bundles over sphere bundles by clutching data and provides some examples. The central part of our work is Sect. 9, where we show that if $V \to X$ is an oriented Euclidean vector bundle of rank divisible by four, then vector bundles $\mathscr{E} \to \hat{V}$ are, after stabilization, sums of bundles which arise from $\mathrm{Cl}(V)$-module bundles over X by the clutching construction and bundles pulled back from X. We remark that up to this point our argument is not using topological K-theory.

The final Sect. 10 translates the results of Sect. 9 into a K-theoretic setting and derives the $\mathrm{Cl}(V)$-linear Thom isomorphism theorem 10.3 in this language. This recovers Karoubi's Clifford–Thom isomorphism theorem [9, Theorem IV.5.11] in the special case of oriented vector bundles $V \to X$ of rank divisible by four. In this respect we provide a geometric approach to this important result, which is proven in [9] within the theory of Banach categories; see Discussion 10.10 at the end of our paper for more details. Together with the representation theory of Clifford algebras it also implies the classical Thom isomorphism theorem for orthogonal K-theory. Finally, for completeness of the exposition we mention the analogous periodicity theorems for unitary and symplectic K-theory, which are in part difficult to find in the literature.

2 Recollections on Clifford modules

Let $(V, \langle\ ,\ \rangle)$ be a Euclidean vector space. Recall that the Clifford algebra $\mathrm{Cl}(V)$ is the $\mathbb{R}$-algebra generated by all elements of V with the relations $vw + wv = -2\langle v, w\rangle \cdot 1$ for all $v, w \in V$, or equivalently, for any orthonormal basis $(e_1, \ldots, e_n)$ of V,

$$e_i e_j + e_j e_i = -2\delta_{ij}\,. \tag{2.1}$$

For $V = \mathbb{R}^n$ with the standard Euclidean structure we write $\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{R}^n)$.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. An (ungraded) $\mathrm{Cl}(V)$-*representation* is a $\mathbb{K}$-module L together with a homomorphism of $\mathbb{R}$-algebras

$$\rho : \mathrm{Cl}(V) \to \mathrm{End}_{\mathbb{K}}(L)\,.$$

In other words, L is a $\mathrm{Cl}(V) \otimes \mathbb{K}$-module. We also speak of *real, complex*, respectively *quaternionic* $\mathrm{Cl}(V)$-representations.

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V and put $J_i = \rho(e_i)$. Due to (2.1) these are anticommuting $\mathbb{K}$-linear complex structures on L, that is $J_i^2 = -I$ and

$J_i J_k = -J_k J_i$ for $i \neq k$. We also speak of a *Clifford family* $(J_1, \ldots, J_n)$. This implies that all J_i are orientation preserving (for $\mathbb{K} = \mathbb{C}, \mathbb{H}$ this already follows from $\mathbb{K}$-linearity).

In the following we restrict to real $\mathrm{Cl}(V)$-modules; for complex or quaternionic $\mathrm{Cl}(V)$-modules similar remarks apply. We may choose an inner product on L such that $J_i \in \mathrm{SO}(L)$; equivalently all J_i are skew adjoint. In this case we also speak of an *orthogonal* $\mathrm{Cl}(V)$-representation. With the inner product

$$\langle A, B\rangle := \frac{1}{\dim L} \operatorname{tr}(A^T \circ B)$$

on $\mathrm{End}(L)$ this implies $J_i \perp J_k$ and $J_i \perp \mathrm{id}_L$ for $1 \leq i \neq k \leq n$. In particular we obtain an isometric linear map $\mathbb{R} \oplus V \to \mathrm{End}(L)$,

$$(t, v) \mapsto t \cdot \mathrm{id}_L + \rho(v)\,.$$

By the previous remarks it sends the unit sphere $\mathbb{S}(\mathbb{R} \oplus V) \subset \mathbb{R} \oplus V$ into the special orthogonal group $\mathrm{SO}(L) \subset \mathrm{End}(L)$.

Definition 2.1 We call the restriction

$$\mu : \mathbb{S}(\mathbb{R} \oplus V) \to \mathrm{SO}(L)$$

the *Hopf map* associated to the orthogonal Clifford representation ρ.

Isometric linear maps $\mathbb{R} \oplus V \to \mathrm{End}(L)$ are in one-to-one correspondence with isometric embeddings $\mathbb{S}(\mathbb{R} \oplus V) \to \mathbb{S}(\mathrm{End}(L))$ onto great spheres. This leads to the following geometric characterization of Clifford representations.

Proposition 2.2 *Let L be a Euclidean vector space and let*

$$\mu : \mathbb{S}(\mathbb{R} \oplus V) \to \mathbb{S}(\mathrm{End}(L))$$

be an isometric embedding as a great sphere, which satisfies

$$\mu(\mathbb{S}(\mathbb{R} \oplus V)) \subset \mathrm{SO}(L)\,, \quad \mu(1, 0) = \mathrm{id}_L\,.$$

Then μ is the Hopf map of an orthogonal Clifford representation $\mathrm{Cl}(V) \to \mathrm{End}(L)$.

Proof By assumption μ is the restriction of a linear map $\mathbb{R} \oplus V \to \mathrm{End}(L)$. If $A, B \in \mathrm{image}(\mu) \subset \mathrm{SO}(L)$, then $A + B \in \mathbb{R} \cdot \mathrm{SO}(L)$ by assumption, hence

$$(A + B)^T \cdot (A + B) = t \cdot \mathrm{id}$$

for some $t \in \mathbb{R}$. Furthermore

$$(A + B)^T \cdot (A + B) = A^T A + B^T B + A^T B + B^T A = 2 \cdot I + A^T B + B^T A\,,$$

and hence

$$A^T B + B^T A = s \cdot \mathrm{id}$$

with $s = t - 2$.

Taking the trace on both sides we have $s = 2\langle A, B\rangle$. If $A = I$ and $B \perp I$, this implies $B + B^T = 0$. For $A, B \perp \mathrm{id}$ we hence get the Clifford relation $AB + BA = -2\langle A, B\rangle \cdot \mathrm{id}$. □

The structure of the real representations of Cl_n is well known (cf. [10, p. 28]). They are direct sums of irreducible representations ρ_n. These are unique and faithful when $n \not\equiv 3 \mod 4$. Otherwise there are two such ρ_n, which are both not faithful and differ by an automorphism of Cl_n. The corresponding modules S_n and algebras $C_n := \rho_n(\mathrm{Cl}_n)$ are as follows.

Theorem 2.3 (Periodicity theorem for Clifford modules)

$$\begin{array}{cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 8+k \\ S_n & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H} & \mathbb{H}^2 & \mathbb{C}^4 & \mathbb{O} & \mathbb{O} & \mathbb{O}^2 & \mathbb{O}^2 \otimes S_k \\ s_n & 1 & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 16 & 16 s_k \\ C_n & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H} & \mathbb{H}(2) & \mathbb{C}(4) & \mathbb{R}(8) & \mathbb{R}(8) & \mathbb{R}(16) & \mathbb{R}(16) \otimes C_k \end{array} \tag{2.2}$$

where $s_n = \dim S_n$. Here $\mathbb{K}(p)$ denotes the algebra of $(p \times p)$-matrices over $\mathbb{K}$. For $n = 3$ and $n = 7$, the two different module structures on $S_n = \mathbb{K}$ for $\mathbb{K} = \mathbb{H}, \mathbb{O}$ are generated by the left and the right multiplications, respectively, with elements of the "imaginary" subspace $\mathbb{R}^n = \mathrm{Im}\ \mathbb{K} = \mathbb{R}^\perp \subset \mathbb{K}$.

The action of $(x, \xi) \in \mathbb{R}^{k+8} = \mathbb{R}^k \oplus \mathbb{O}$ on $S_{k+8} = \mathbb{O}^2 \otimes S_k = (\mathbb{O} \otimes S_k)^2$ is given by

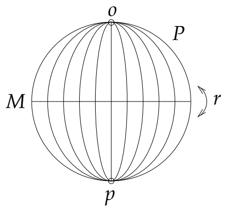
$$\rho_{k+8}(x, \xi) = \begin{pmatrix} x & -L(\bar{\xi})^T \\ L(\xi) & -x \end{pmatrix}$$

where $L(\xi) = L(\xi) \otimes \mathrm{id}_{S_k}$ denotes the left translation on $\mathbb{O}$, and where $x \in \mathbb{R}^k \subset \mathrm{Cl}_k$ acts on $S_k = 1 \otimes S_k$ by ρ_k.

3 Poles and centrioles

Clifford modules bear a close relation to the geometry of symmetric spaces. Let P be a Riemannian symmetric space: for any $p \in P$ there is an isometry s_p of P which is an involution having p as an isolated fixed point. Two points $o, p \in P$ will be called *poles* if $s_p = s_o$. The notion was coined for the north and south pole of a round sphere, but there are many other spaces with poles; e.g. $P = \mathrm{SO}_{2n}$ with $o = I$ and $p = -I$, or the Grassmannian $P = \mathbb{G}_n(\mathbb{R}^{2n})$ with $o = \mathbb{R}^n$ and $p = (\mathbb{R}^n)^\perp$. Of course, pairs of poles are mapped onto pairs of poles by isometries of P.

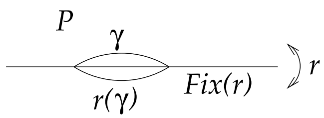
A geodesic γ connecting poles $o = \gamma(0)$ and $p = \gamma(1)$ is reflected into itself at o and p and hence it is closed with period 2.



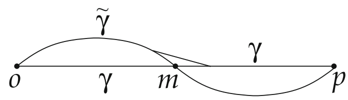
Now we consider the *midpoint set* M between poles o and p,

$$M = \{m = \gamma\left(\tfrac{1}{2}\right) : \gamma \text{ shortest geodesic in } P \text{ with} \gamma(0) = o,\ \gamma(1) = p\}.$$

For the sphere $P = \mathbb{S}^n$ with north pole o, this set would be the equator. In general, M need not be connected, but it is still the fixed point set of a reflection (order-two isometry) r on P.[1] Hence the connected components of M, called *(minimal) centrioles* [6], are totally geodesic subspaces of P – otherwise short geodesic segments γ in the ambient space P with end points in a component of $\mathrm{Fix}(r)$ would not be unique:



Each such midpoint $m = \gamma(\frac{1}{2})$ determines its geodesic γ uniquely, and thus the set of minimal geodesics can be replaced with M: if there is another geodesic $\tilde{\gamma}$ from o to p through m, it can be made shorter by cutting the corner at m, thus it is not minimal:



There are chains of minimal centrioles (centrioles in centrioles):

$$P \supset P_1 \supset P_2 \supset \cdots \tag{3.1}$$

Peter Quast [14,15] classified all such chains starting from a compact simple Lie group $P = G$ with at least 3 steps. The result is (3.2) below. The chains 1,2,3 are introduced in Milnor's book [11].

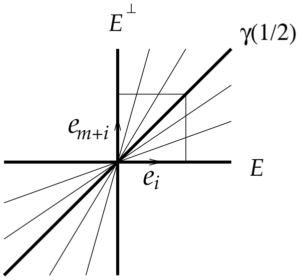
No.	G	P_1	P_2	P_3	P_4	restr.
1	$(\mathrm{S})\mathrm{O}_{4n}$	$\mathrm{SO}_{4n}/\mathrm{U}_{2n}$	$\mathrm{U}_{2n}/\mathrm{Sp}_n$	$\mathbb{G}_m(\mathbb{H}^n)$	Sp_m	$m = \frac{n}{2}$
2	$(\mathrm{S})\mathrm{U}_{2n}$	$\mathbb{G}_n(\mathbb{C}^{2n})$	U_n	$\mathbb{G}_m(\mathbb{C}^n)$	U_m	$m = \frac{n}{2}$
3	Sp_n	$\mathrm{Sp}_n/\mathrm{U}_n$	$\mathrm{U}_n/\mathbb{O}_n$	$\mathbb{G}_m(\mathbb{R}^n)$	SO_m	$m = \frac{n}{2}$
4	Spin_{n+2}	Q_n	$(\mathbb{S}^1 \times \mathbb{S}^{n-1})/\pm$	$\mathbb{S}^{n-2}$	$\mathbb{S}^{n-3}$	$n \geq 3$
5	E_7	$\mathrm{E}_7/(\mathbb{S}^1\mathrm{E}_6)$	$\mathbb{S}^1\mathrm{E}_6/\mathrm{F}_4$	$\mathbb{OP}^2$	–	

(3.2)

[1] There is a covering $\pi : P \to \check{P}$ with $\pi(p) = \pi(o)$ where $\check{P} = \{s_p : p \in P\} \subset Iso(P)$, and $r = \gamma \circ s_o$ where γ is the deck transformation of π with $\gamma(o) = p$.

By $\mathbb{G}_m(\mathbb{K}^n)$ we denote the Grassmannian of m-dimensional subspaces in $\mathbb{K}^n$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Further, Q_n denotes the complex quadric in $\mathbb{CP}^{n+1}$, which is isomorphic to the real Grassmannian $\mathbb{G}_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes in $\mathbb{R}^{n+2}$, and $\mathbb{OP}^2$ is the octonionic projective plane $\mathrm{F}_4/\,\mathrm{Spin}_9$.

A chain is extendible beyond P_k if and only if P_k contains poles again. E.g. among the Grassmannians $P_3 = \mathbb{G}_m(\mathbb{K}^n)$ only those of half dimensional subspaces ($m = \frac{n}{2}$) enjoy this property: Then $(E, E^\perp)$ is a pair of poles for any $E \in \mathbb{G}_{n/2}(\mathbb{K}^n)$, and the corresponding midpoint set is the group $\mathrm{O}_{n/2}, \mathrm{U}_{n/2}, \mathrm{Sp}_{n/2}$ since its elements are the graphs of orthogonal $\mathbb{K}$-linear maps $E \to E^\perp$, see figure below.



For compact connected matrix groups $P = G$ containing $-I$, there is a linear algebra interpretation for the iterated minimal centrioles P_j. We only consider classical groups.

Theorem 3.1 *Let $L = \mathbb{K}^p$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $G \in \{\mathrm{SO}_p, \mathrm{U}_p, \mathrm{Sp}_p\}$ with p even in the real case. Then a chain of minimal centrioles*

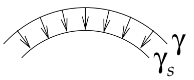
$$G \supset P_1 \supset \cdots \supset P_k$$

corresponds to a Cl_k-representation $J_1, \ldots, J_k$ on L with $J_j \in G$, and each P_j is the connected component through J_j of the set

$$\hat{P}_j = \{J \in G : J^2 = -I,\ JJ_i = -J_i J \text{ for } i < j\}. \tag{3.3}$$

Proof A geodesic γ in G with $\gamma(0) = I$ is a *one-parameter subgroup*, a Lie group homomorphism[2] $\gamma : \mathbb{R} \to G$. When $\gamma(1) = -I$, then $\gamma(\frac{1}{2}) = J$ is a *complex structure*, $J^2 = -I$. Thus the midpoint set $\hat{P}_1$ is the set of complex structures in G.

[2] Let $\gamma : \mathbb{R} \to G$ be a smooth group homomorphism. Its curvature vector field $\eta = \nabla_{\gamma'}\gamma'$ is γ-invariant, $\eta(t) = \gamma(t)_*\eta(0)$. When $\eta \neq 0$, the neighbor curve $\gamma_s(t) = \exp_{\gamma(t)}(s\eta(t))$ is another γ-orbit, $\gamma_s(t) = \gamma(t)g_s$. Deforming in the curvature vector direction shortens a curve, thus γ_s is shorter than γ on any finite interval.



But on the other hand $\gamma_s = R(g_s)\gamma$ has the same length as γ since $R(g_s)$ is an isometry of G. Thus $\eta = 0$.

By induction hypothesis, we have anticommuting complex structures $J_i \in P_i$ for $i \le j$, and P_j is the connected component through J_j of the set $\hat{P}_j$ as in (3.3). Suppose that also

$$-J_j \in P_j. \tag{3.4}$$

Consider a shortest geodesic γ from J_j to $-J_j$ in P_j. Put $J = \gamma(\frac{1}{2}) \in P_j$. Thus J anticommutes with J_i for all $i < j$. It remains to show that J anticommutes with J_j, too. Since P_j is totally geodesic, γ is a geodesic in G, hence $\gamma = \gamma_o J_j$ where γ_o is a one-parameter group with $\gamma_o(1) = -I$ which again implies that $J_o := \gamma_o(\frac{1}{2})$ is a complex structure. But also $J \in P_j$ is a complex structure, $J^{-1} = -J$, and since $J = \gamma(\frac{1}{2}) = J_o J_j$, this means $J_j J_o = -J_o J_j$. Thus both J_o and $J = J_o J_j$ anticommute with J_j, hence $J \in \hat{P}_{j+1}$ as defined in (3.3).

Vice versa, let $J \in \hat{P}_{j+1}$, that is $J \in G$ is a complex structure anticommuting with $J_1, \dots, J_j$. Then $J_o := J J_j^{-1} \in G$ is a complex structure which anticommutes with J_j and commutes with J_i, $i < j$. Further, from $J_o^{-1} = -J_o$ and $J_o^T = J_o^{-1}$ we obtain $J_o^T = -J_o$, thus $J_o \in G \cap \mathfrak{g} \subset \mathrm{End}_{\mathbb{K}}(L)$ where $\mathfrak{g}$ denotes the Lie algebra of G (here we use $G = \mathrm{SO}(L) \cap \mathrm{End}_{\mathbb{K}}(L)$). Putting $\gamma_o(t) = \exp(t\pi J_o)$ we define a geodesic γ_o in G from $\gamma_o(0) = I$ to $\gamma_o(1) = -I$ via $\gamma(\frac{1}{2}) = J_o$. In fact this is shortest in G, being a great circle in the plane spanned by I and J_o. Further, the geodesic $\gamma = \gamma_o J_j$ from J_j to $-J_j$ is contained in P_j, due to the subsequent Lemma 3.2 (applied to $A = \pi J_o$), and it is shortest in P_j (even in the ambient space G). Thus J is contained in the midpoint set of (P_j, J_j). □

Lemma 3.2 *Let* $A \in \mathfrak{g}$. *Then* $\exp(tA)J_j \in P_j$ *for all* $t \in \mathbb{R}$ *if and only if*

$$A \text{ anticommutes with } J_j \text{ and commutes with } J_i \text{ for } i < j. \tag{3.5}$$

Proof [11, p. 137] For generic $t \in \mathbb{R}$ we have: A anticommutes with $J_j \iff J_j^{-1}\exp(tA)J_j = \exp(-tA) \iff \exp(tA)J_j = J_j\exp(-tA) = -(\exp(tA)J_j)^{-1}$ $\iff$ (A): $\exp(tA)J_j$ is a complex structure.
Further, A commutes with $J_i \iff J_i$ commutes with $\exp(tA)$
$\iff$ (B): $\exp(tA)J_j$ anticommutes with J_i (with $i < j$).
(A) and (B) $\iff \exp(tA)J_j \in P_j$. □

Remark 3.3 The proof of Theorem 3.1 shows that the induction step can be carried through as long (3.4) holds. This condition limits the length k of the chain of minimal centrioles.

4 Deformations of path spaces

Minimal centrioles in P are also important from a topological point of view: Since they represent the set of shortest geodesics from o to p, they form tiny models of the path space of P. This is shown using Morse theory of the energy function on the path space [11].

Definition 4.1 An inclusion $A \subset B$ of topological spaces is called *d-connected* for some $d \in \mathbb{N}$ if for any $k \le d$ and any continuous map $\phi : (\mathbb{D}^k, \partial\mathbb{D}^k) \to (B, A)$ there is a continuous deformation $\phi_t : \mathbb{D}^k \to B$, $0 \le t \le 1$, with

$$\phi_0 = \phi\,, \quad \phi_1(\mathbb{D}^k) \subset A\,, \quad \phi_t|_{\partial\mathbb{D}^k} \text{ constant in } t.$$

This implies the same property for $(\mathbb{D}^k, \partial\mathbb{D}^k)$ replaced by a finite CW-pair (X, Y) with $\dim Y < \dim X \le d$ by induction over the dimension of the cells in X not contained in Y and homotopy extension (see Corollary 1.4 in [5, Ch. VII]).

Let (P, o) be a pointed symmetric space and p a pole of o. Let $\Omega := \Omega(P; o, p)$ be the space of all continuous paths $\omega : [0, 1] \to P$ with $\omega(0) = o$ and $\omega(1) = p$ (for short we say $\omega : o \rightsquigarrow p$ in P), equipped with the compact-open topology (uniform convergence). Furthermore let $\Omega^0(P; o, p) \subset \Omega(P; o, p)$ be the subspace of shortest geodesics $\gamma : o \rightsquigarrow p$. The following theorem is essentially due to Milnor [11].

Theorem 4.2 *Let*

$$\Omega_* \subset \Omega(P; o, p)$$

be a connected component of $\Omega(P; o, p)$ *and*

$$\Omega^0_* \subset \Omega_*$$

be the subspace of minimal geodesics $\gamma : o \rightsquigarrow p$ *in* Ω_*. *Let* $d + 1$ *be the smallest index of all non-minimal geodesics* $o \rightsquigarrow p$ *in* Ω_*.

Then the inclusion $\Omega^0_* \subset \Omega_*$ *is d-connected.*

We recall the main steps of the proof. The basic idea is using the *energy function*

$$E(\omega) = \int_0^1 |\omega'(t)|^2 dt. \tag{4.1}$$

for each path $\omega \in \Omega_*$ which is H^1 (almost everywhere differentiable with square-integable derivative). Applying the gradient flow of $-E$ we may shorten all H^1-paths simultaneously to minimal geodesics.[3]

Since the energy is not defined on all of Ω_*, we will apply this flow only on the subspace of *geodesic polygons* $\Omega_n \subset \Omega_*$ for large $n \in \mathbb{N}$ (to be chosen later), where each such polygon $\omega \in \Omega_n$ has its vertices at $\omega(k/n)$, $k = 0, \dots, n$, and the connecting curves are shortest geodesics. For any $r \in \mathbb{N}$ with $n|r$ we have $\Omega_n \subset \Omega_r$. Furthermore $\Omega^0_* \subset \Omega_n$ for all n.

Lemma 4.3 *For all* $k \ge 0$ *there exists an* n *such that the inclusion* $\Omega_n \subset \Omega_*$ *is* k*-connected.*

[3] When ω is reparametrized proportional to arc length (which does not increase energy), we have $E(\omega) = |\omega'|^2 = L(\omega)^2$. Thus L is minimized when E is minimized.

Proof Let $\phi : \mathbb{D}^k \to \Omega^*$ with $\phi(\partial\mathbb{D}^k) \subset \Omega_n$. We consider ϕ as a continuous map $\mathbb{D}^k \times [0, 1] \to P$. This is equicontinuous by compactness of $\mathbb{D}^k \times [0, 1]$.

Let R be the convexity radius on P, which means that for any $q \in P$ and any $q', q'' \in B_R(q)$ the shortest geodesic between q' and q'' is unique and contained in $B_R(q)$. By equicontinuity, when n is large enough and $x \in \mathbb{D}^k$ arbitrary, $\phi(x)$ maps every interval $[\frac{k-1}{n}, \frac{k+1}{n}]$ into $B_R(\phi(x, \frac{k}{n})) \subset P$, for $k = 1, \ldots, n-1$.

Let $\phi_1 : \mathbb{D}^k \to \Omega_n$ such that $\phi_1(x)$ is the geodesic polygon with vertices at $\phi(x)(\frac{k}{n})$ for $k = 0, \ldots, n$. Using the unique shortest geodesic between $\phi(x)(s)$ and $\phi_1(x)(s)$ for each $s \in [0, 1]$, we define a homotopy ϕ_t between ϕ and ϕ_1 with $\phi_t(x) = \phi(x)$ when $x \in \partial\mathbb{D}^k$. □

For $\omega \in \Omega_n$ let $\omega_k = \omega|[\frac{k-1}{n}, \frac{k}{n}]$ for $k = 1, \ldots, n$. Its length is $L(\omega_k) = \frac{1}{n}|\omega_k'|$, its energy $E(\omega_k) = \frac{1}{n}|\omega_k'|^2 = n \cdot L(\omega_k)^2$. The distance between the vertices $\omega(\frac{k-1}{n}), \omega(\frac{k}{n})$ is the length of ω_k which is $\leq \sqrt{E(\omega)/n}$ since $E(\omega) \geq E(\omega_k) = n \cdot L(\omega_k)^2$.

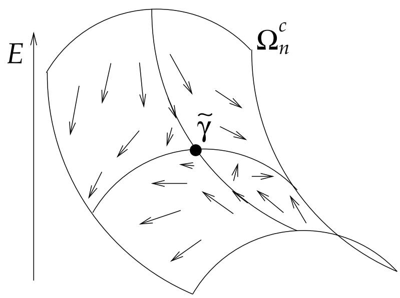
For $c > 0$ let $\Omega_n^c = \{\omega \in \Omega_n : E(\omega) \leq c\}$. We have $\Omega_*^0 = \Omega_n^{c_o}$ where c_o is the energy of a shortest geodesic. By continuity, $E \circ \phi_1$ has bounded image, hence $\phi_1(X) \subset \Omega_n^c$ for some $c > 0$. When n is large enough, more precisely $n > c/R^2$, any two neighboring vertices of every $\omega \in \Omega_n^c$ lie in a common convex ball, hence the joining shortest geodesic segments are unique and depend smoothly on the vertices.

Thus we may consider Ω_n^c as the closure of an open subset of $P \times \cdots \times P$ (($n-1$)-times) with its induced topology.

Lemma 4.4 *The inclusion $\Omega_*^0 \subset \Omega_n^c$ is d-connected for d as in Theorem 4.2.*

Proof Let $\phi : \mathbb{D}^k \to \Omega_n^c$, $0 \leq k \leq d$, with $\phi(\partial\mathbb{D}^k) \subset \Omega_*^0$. The space Ω_n^c is finite dimensional and contains all geodesics of length $\leq \sqrt{c}$ from o to p. It is a closed subset of $P \times \cdots \times P$ (($n-1$)-times), and its boundary points are the polygons in Ω_n whose energy takes its maximal value c. The gradient of $-E$ on Ω_n^c is a smooth vector field and its flow is smooth, too.

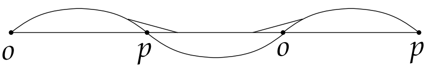
The index of any geodesic is the same in Ω_* and Ω_n, see [11, Lemma 15.4]. Hence the index of a non-minimal geodesic $\tilde{\gamma}$ in Ω_n (a "saddle" for E) is at least $d+1$, and $\phi_n(\mathbb{D}^k)$ can avoid the domains of attraction for all these geodesics. The energy decreases along the gradient lines starting on $\phi_n(\mathbb{D}^k)$, thus these curves avoid the boundary of Ω_n^c and they end up on the minimum set of E, the set of minimal geodesics. Hence we can use this flow to deform ϕ into some $\tilde{\phi} : \mathbb{D}^k \to \Omega_*^0$ without changing $\phi|_{\partial\mathbb{D}^k}$. □



Proof of Theorem 4.2 Let $\phi : \mathbb{D}^k \to \Omega_*$, $k \le d$, with $\phi(\partial\mathbb{D}^k) \subset \Omega_*^0$. By Lemma 4.3 we may (after a deformation which is constant on $\partial\mathbb{D}^k$) assume that $\phi(\mathbb{D}^k) \subset \Omega_n^c$ for some large n. Now the claim follows from Lemma 4.4. □

5 A lower bound for the index

How large is d in Theorem 4.2? This has been computed in [11, §23,24] and [8].[4] We slightly simplify Milnor's arguments replacing curvature computations by totally geodesic spheres. An easy example is the sphere itself, $P = \mathbb{S}^n$. A non-minimal geodesic γ between poles o and p covers a great circle at least one and a half times and can be shortened within any 2-sphere in which it lies (see figure below). There are $n-1$ such 2-spheres perpendicular to each other since the tangent vector $\gamma'(0) = e_1$ (say) is contained in $n-1$ perpendicular planes in the tangent space, Span (e_1, e_i) with $i \ge 2$. Thus the index is $\ge n-1$, in fact $\ge 2(n-1)$ since any such geodesic contains at least 2 conjugate points where it can be shortened by cutting the corner, see figure.



For the groups $P = G = \mathrm{SO}_p, \mathrm{U}_p, \mathrm{Sp}_p$ (p even) and their iterated centrioles we have similar results. The Riemannian metric on G is induced from the inclusion $G \subset \mathrm{End}(\mathbb{K}^p)$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively, where the inner product on $\mathbb{K}^{p\times p} = \mathrm{End}(\mathbb{K}^p)$ is

$$\langle A, B\rangle = \frac{1}{p}\,\mathrm{Re\ trace}\,(A^*B) \tag{5.1}$$

for any $A, B \in \mathbb{K}^{p\times p}$ where $A^* := \bar{A}^T$. In particular, $\langle I, I\rangle = 1$.

Proposition 5.1 *The index of any non-minimal geodesic from I to $-I$ in SO_p is at least $p-2$.*

Proof A shortest geodesic from I to $-I$ in SO_p is a product of $p/2$ half turns, planar rotations by the angle π in $p/2$ perpendicular 2-planes in $\mathbb{R}^p$. A non-minimal geodesic must make an additional full turn and thus a 3π-rotation in at least one of these planes, say in the x_1x_2-plane. We project γ onto a geodesic γ_1 in a subgroup $\mathrm{SO}_4 \subset \mathrm{SO}_p$ sitting in the coordinates x_1, x_2, x_{k-1}, x_k, for any even $k \in \{4, \dots, p\}$. Then γ_1 consists of 3 half turns in the x_1x_2-plane together with (at least) one half turn in the $x_{k-1}x_k$-plane.

In the torus Lie algebra $\mathfrak{t}$ of $\mathfrak{so}_4$, which is $\mathbb{R}^2 \cong \mathfrak{t} = \left\{\begin{pmatrix} aJ & \\ & bJ\end{pmatrix} : a, b \in \mathbb{R}\right\}$ for $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, we have $\gamma_1(t) = \exp \pi t v$ with $v = \begin{pmatrix} 3J & \\ & J\end{pmatrix} = \begin{pmatrix} 3 \\ 1\end{pmatrix}$ for short. Now we use $\mathrm{SO}_4 = L(\mathbb{S}^3)R(\mathbb{S}^3) \cong (\mathbb{S}^3 \times \mathbb{S}^3)/\pm$, where the two $\mathbb{S}^3$-factors correspond to the vectors $\begin{pmatrix} 1 \\ 1\end{pmatrix}, \begin{pmatrix} 1 \\ -1\end{pmatrix} \in \mathfrak{t}$ (the left and right multiplications by $i \in \mathbb{H}$ on $\mathbb{H} = \mathbb{R}^4$). Decomposing v with respect to this basis we obtain $v = 2\begin{pmatrix} 1 \\ 1\end{pmatrix} + \begin{pmatrix} 1 \\ -1\end{pmatrix}$. Thus the lift of

[4] A different argument using root systems was given by Bott [4, 6.7] and in more detail by Mitchell [12,13].

γ_1 in $\mathbb{S}^3 \times \mathbb{S}^3$ has two components: one is striding across a full great circle, the other across a half great circle. The first component passes the south pole of $\mathbb{S}^3$ and takes up index 2. Since there are $(p-2)/2$ such coordinates x_k the index of a non-minimal geodesic in SO_p is at least $p-2$ (compare [11, Lemma 24.2]). □

For U_p we have a different situation. To any path $\omega : I \rightsquigarrow -I$ we assign the closed curve $\det \omega : [0,1] \to \mathbb{S}^1 \subset \mathbb{C}$ (assuming p to be even). Its winding number (mapping degree) $w(\omega) \in \mathbb{Z}$ is obviously constant on each connected components of $\Omega(\mathrm{U}_p; I, -I)$.

Definition 5.2 We will call $w(\omega)$ the *winding number of* ω.

Any geodesic in U_p from I to $-I$ is conjugate to $\gamma(t) = \exp(t\pi i D)$ where $D = \mathrm{diag}(k_1, \ldots, k_p)$ with odd integers k_i, and $w(\gamma) = \frac{1}{2}\sum_j k_j$. Let k_j, k_h be a pair of entries with $k_j > k_h$ and consider the projection γ_{jh} of γ onto U_2 acting on $\mathbb{C}e_j + \mathbb{C}e_h$. Let $a = \frac{1}{2}(k_j + k_h)$ and $b = \frac{1}{2}(k_j - k_h)$. Then

$$e^{-\pi i a t}\gamma_{ij}(t) = \mathrm{diag}(e^{\pi i b t}, e^{-\pi i b t})\,.$$

This is a geodesic in $\mathrm{SU}_2 \cong \mathbb{S}^3$ which takes the value $\pm I$ when t is a multiple of $1/b$. For $0 < t < 1$ there are $b-1$ such t-values, and at any of these points γ_{ij} takes up index 2. Thus

$$\mathrm{ind}(\gamma) \geq 2 \sum_{k_j > k_h} \left(\frac{1}{2}(k_j - k_h) - 1\right). \tag{5.2}$$

Proposition 5.3 *Let $\gamma : I \rightsquigarrow -I$ be a geodesic in U_p with $p - 2|w(\gamma)| \geq 2d$ for some $d > 0$. Then $\mathrm{ind}(\gamma) > d$ unless γ is minimal.*

Proof Let k_+ be the sum of the positive k_j and $-k_-$ the sum of the negative k_j. Then $k_+ - k_- = \sum_j k_j = 2w$ and $k_+ + k_- \geq p$. Thus $2k_+ \geq p + 2w \geq p - 2|w|$ and $2k_- \geq p - 2w \geq p - 2|w|$, and the assumption $p - 2|w| \geq 2d$ implies $k_\pm \geq d$. If $k_j \geq 3$ or $k_h \leq -3$ for some j, h, then by (5.2)

$$\mathrm{ind}(\gamma) \geq \sum_{k_h < 0} (3 - k_h - 2) = 1 + k_- > d$$

or

$$\mathrm{ind}(\gamma) \geq \sum_{k_j > 0} (k_j + 3 - 2) = 1 + k_+ > d\,.$$

At least one of these inequalities must hold, unless all $k_i \in \{1, -1\}$. In the latter case γ is minimal: it has length π which is the distance between I and $-I$ in U_p, being the minimal norm (with respect to the inner product (5.1)) for elements of $(\exp|_{\mathfrak{t}})^{-1}(-I) = \{\pi i\, \mathrm{diag}(k_1, \ldots, k_p) : k_1, \ldots, k_p \text{ odd}\} \subset \mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of the maximal torus of diagonal matrices in U_p. □

Next we will show that similar estimates hold for arbitrary iterated centrioles P_ℓ of SO_p. A geodesic $\gamma : J_\ell \rightsquigarrow -J_\ell$ in P_ℓ has the form

$$\gamma(t) = e^{\pi t A} J_\ell\,, \quad t \in [0,1]\,, \tag{5.3}$$

for some skew symmetric matrix A, which commutes with $J_1, \dots, J_{\ell-1}$ and anticommutes with J_ℓ, cf. (3.5). We split $\mathbb{R}^p$ as a direct sum of subspaces,

$$\mathbb{R}^p = M_1 \oplus \cdots \oplus M_r \tag{5.4}$$

such that the subspaces M_j are invariant under $J_1, \dots, J_\ell$ and A and minimal with this property. Then A has only one pair of eigenvalues $\pm i k_j$ on every M_j; otherwise we could split M_j further. Since $\exp(\pi A) = -I$, the k_j are odd integers which can be chosen positive.

When all $k_j = 1$, then γ has minimal energy: $E(\gamma) = |\gamma'|^2 = |\pi A\gamma|^2 \overset{(5.1)}{=} \pi^2$. In general, $J' := A/k_j$ is a complex structure on M_j, and $J_{\ell+1} := J_\ell J'$ is another complex structure on M_j which anticommutes with $J_1, \dots, J_\ell$. Hence each M_j is an irreducible $\mathrm{Cl}_{\ell+1}$-module.

The two cases (a) $\ell \neq 4m-2$ and (b) $\ell = 4m-2$ have to be distinguished (cf. [11, p. 144-148]); they are similar to the previous cases of SO_p and U_p, respectively. By (2.2), all M_j are isomorphic as Cl_ℓ-modules when $\ell = 4m-2$ (Case (b)) and as $\mathrm{Cl}_{\ell+1}$-modules when $\ell \neq 4m-2$ (Case (a)).

Proposition 5.4 *Suppose $\ell \neq 4m-2$. Let $s_{\ell+1}$ be the dimension of the irreducible $\mathrm{Cl}_{\ell+1}$-module $S_{\ell+1}$, see (2.2). Then $s_{\ell+1} | p$, and all non-minimal geodesics $\gamma : J_\ell \rightsquigarrow -J_\ell$ in P_ℓ have index $> r-1$ with $r = p/s_{\ell+1}$.*

Proof Choose any pair of submodules M_j, M_h with $j \neq h$ in (5.4). We may isometrically identify M_j and M_h as $\mathrm{Cl}_{\ell+1}$-modules, but first we modify the module structure on M_h by changing the sign of $J_{\ell+1}$. In this way we view $M_j + M_h = M_j \oplus M_j$. Thus

$$A|_{M_j+M_h} = \begin{pmatrix} k_j J' & \\ & -k_h J' \end{pmatrix} = \begin{pmatrix} aJ' & \\ & aJ' \end{pmatrix} + \begin{pmatrix} bJ' & \\ & -bJ' \end{pmatrix}$$

for $a = \frac{1}{2}(k_j - k_h)$, $b = \frac{1}{2}(k_j + k_h)$. Let $\gamma(t)_{jh} := \gamma(t)|_{M_j+M_h}$. Then

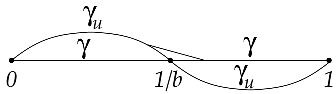
$$\gamma(t)_{jh} = (e^{t\pi A} J_\ell)_{jh} = \begin{pmatrix} e^{t\pi aJ'} & \\ & e^{t\pi aJ'} \end{pmatrix} \begin{pmatrix} e^{t\pi bJ'} & \\ & e^{-t\pi bJ'} \end{pmatrix} \begin{pmatrix} J_\ell & \\ & J_\ell \end{pmatrix} \tag{5.5}$$

Put $B := \begin{pmatrix} & -I \\ I & \end{pmatrix}$ on $M_j + M_h = M_j \oplus M_j$ and $B = 0$ on the other modules M_k for $k \neq j, h$. Then B and e^{uB} commute with $J_1, \dots, J_\ell$ for all $u \in \mathbb{R}$. Put $A_u = e^{uB} A e^{-uB}$. Then all geodesics

$$\gamma_u(t) = e^{uB}\gamma(t)e^{-uB} = e^{t\pi A_u} J_\ell$$

are contained in P_ℓ by (3.5).

The point $\gamma(t)$ is fixed under conjugation with the rotation matrix $e^{uB} = \begin{pmatrix} cI & -sI \\ sI & cI \end{pmatrix}$ on $M_j \oplus M_j$ with $c = \cos u$, $s = \sin u$ if and only if $e^{t\pi bJ'} = e^{-t\pi bJ'}$, see (5.5). This happens precisely when t is an integer multiple of $1/b$. If $k_h > 1$, say $k_h \geq 3$, then $b = \frac{1}{2}(k_j + k_h) \geq 2$ and $1/b \in (0, 1)$. All γ_u are geodesics in P_ℓ connecting J_ℓ to $-J_\ell$. Using "cutting the corner" it follows that γ can no longer be locally shortest beyond $t = 1/b$, see figure below. If there is at least one eigenvalue $k_h > 1$, there are at least $r - 1$ index pairs (j, h) with $\frac{1}{2}(k_j + k_h) \geq 2$, hence by (5.2), the index of non-minimal geodesics is at least $r - 1$.[5] □



Now we consider the case $\underline{\ell = 4m - 2}$. Then $J_o := J_1 J_2 \ldots J_{\ell-1}$ is a complex structure[6] which commutes with A and $J_1, \ldots, J_{\ell-1}$ and anticommutes with J_ℓ (since $\ell - 1$ is odd). Thus A can be viewed as a complex matrix, using J_o as the multiplication by i, and the eigenvalues of A have the form ik for odd integers k. As before, we split $\mathbb{R}^p$ into minimal subspaces M_j which are invariant under $J_1, \ldots, J_\ell$ and A. Let $E_{k_j} \subset M_j$ be a complex eigenspace of $A|_{M_j}$ corresponding to an eigenvalue ik_j. Then E_{k_j} is invariant under $J_1, \ldots, J_{\ell-1}$ (which commute with A and i), and also under J_ℓ (which anticommutes with both A and $i = J_o$). By minimality we have $M_j = E_{k_j}$, hence $A = k_j J_o$ on M_j.

Again we consider two such modules M_j, M_h. As Cl_ℓ-modules they can be identified, $M_j + M_h = M_j \oplus M_j$, see (2.2). This time,

$$A|_{M_j + M_h} = \mathrm{diag}(k_j i, k_h i) = aiI + \mathrm{diag}(bi, -bi)$$

with $a = \frac{1}{2}(k_j + k_h)$ and $b = \frac{1}{2}(k_j - k_h)$. Thus

$$\gamma(t) = e^{\pi t A} J_\ell = e^{\pi t a i}\, \mathrm{diag}(e^{\pi t b i}, e^{-\pi t b i}) J_\ell. \tag{5.6}$$

Consider the linear map B on $\mathbb{R}^p$ which is $\begin{pmatrix} & -I \\ I & \end{pmatrix}$ on $M_j + M_h = M_j \oplus M_j$ and $B = 0$ elsewhere, and the family of geodesics

$$\gamma_u(t) = e^{uB} \gamma(t) e^{-uB} = e^{\pi t A_u} J_\ell, \quad A_u = e^{uB} A e^{-uB}.$$

Since e^{uB} commutes with $J_1, \ldots, J_\ell$, the first equality implies γ_u in P_ℓ, see (3.5). Now $\gamma(t)$ is fixed under conjugation with the rotation matrix $e^{uB} = \begin{pmatrix} cI & -sI \\ sI & cI \end{pmatrix}$ with

5 Another way of saying: J_ℓ, J' and BJ' span a Clifford 2-sphere in $\mathrm{SO}(M_j + M_h)$ anticommuting with $J_1, \ldots, J_{\ell-1}$ and containing γ_{jh} which covers at least three half great circles.

6 Let $J_1, \ldots, J_n$ be a Clifford family and put $w_n = J_1 \cdots J_n$. Then $w_n^2 = J_1 \cdots J_n J_1 \cdots J_n = (-1)^{n-1} w_{n-1}^2 J_n^2 = (-1)^n w_{n-1}^2$. Thus $w_n^2 = (-1)^s I$ with $s = n + (n-1) + \cdots + 1 = \frac{1}{2} n(n+1)$. When $n - 1$ or $n - 2$ are multiples of 4 then s is odd, hence $w_n^2 = -I$. Further, when n is even (odd), w_n anticommutes (commutes) with $J_1, \ldots, J_n$. If the Clifford family extends to $J_1, \ldots, J_{n+1}$, then w_n commutes with J_{n+1} when n is even and anticommutes when n is odd.

$c = \cos u$, $s = \sin u$ if and only if $e^{\pi tbi} = e^{-\pi tbi}$ which happens precisely when t is an integer multiple of $1/b$. When $b > 1$, we obtain an energy-decreasing deformation by cutting $b-1$ corners, see figure above. Thus the index of a geodesic γ as in (5.3) is similar to (5.2):

$$\operatorname{ind}(\gamma) \geq \sum_{k_j > k_h} \left(\frac{1}{2}(k_j - k_h) - 1 \right). \tag{5.7}$$

As before we need a lower bound for this number when γ is non-minimal.

Any $J \in P_\ell$ defines a $\mathbb{C}$-linear map $JJ_\ell^{-1} = -JJ_\ell$ since JJ_ℓ commutes with all J_i and hence with J_o. This gives an embedding[7]

$$P_\ell \hookrightarrow \mathrm{U}_{p/2} : J \mapsto JJ_\ell^{-1} \tag{5.8}$$

Note that p is divisible by four due to the representation theory of Clifford algebras (2.2), since for $\ell = 4m - 2$ the space $\mathbb{R}^p$ admits at least two anticommuting almost complex structures.

Definition 5.5 For any $\omega \in \Omega := \Omega(P_\ell; J_\ell, -J_\ell)$ we define its *winding number* $w(\omega)$ as the winding number of ωJ_ℓ^{-1}, considered as a path in $\mathrm{U}_{p/2}$ from I to $-I$, see Def. 5.2. In particular for a geodesic $\omega = \gamma$ with $\gamma(t) = e^{\pi t A}$ as in (5.6) we have $w = \frac{1}{2} \dim_{\mathbb{C}}(S_\ell) \sum_j k_j$.

Proposition 5.6 *Let $\ell = 4m - 2$. Let $\gamma : J_\ell \rightsquigarrow -J_\ell$ be a geodesic in P_ℓ with winding number w such that*

$$p - 4|w| \geq 4d \cdot s_\ell \tag{5.9}$$

for some $d > 0$ where $s_\ell = \dim_{\mathbb{R}} S_\ell$, see (2.2). Then $\operatorname{ind}(\gamma) > d$ *unless γ is minimal.*

Proof Let r be the number of irreducible Cl_ℓ-representations in $\mathbb{R}^p$, that is $r = p/s_\ell$. Let k_+ be the sum of the positive k_j and $-k_-$ the sum of negative k_j. Then $k_+ + k_- \geq r = p/s_\ell$ and $k_+ - k_- = \sum_j k_j = 2w/\dim_{\mathbb{C}} S_\ell = 4w/s_\ell$. Thus $2k_+ \geq (p + 4w)/s_\ell$ and $2k_- \geq (p - 4w)/s_\ell$, and our assumption $(p \pm 4w)/s_\ell \geq 4d$ implies $k_\pm \geq 2d$. If $k_j \geq 3$ or $k_h \leq -3$ for some j, h, then by (5.7)

$$\operatorname{ind}(\gamma) \geq \sum_{k_h < 0} \frac{1}{2}(3 - k_h - 2) = \frac{1}{2}(1 + k_-) > d$$

or

$$\operatorname{ind}(\gamma) \geq \sum_{k_j > 0} \frac{1}{2}(k_j + 3 - 2) = \frac{1}{2}(1 + k_+) > d\,.$$

[7] This embedding $J \mapsto JJ_\ell^{-1}$ is equivariant: Let $J = gJ_\ell g^{-1}$ for some $g \in \mathrm{SO}_p$ preserving P_ℓ, then $JJ_\ell^{-1} = g\tau(g)^{-1}$ where τ is the involution $g \mapsto J_\ell g J_\ell^{-1}$. It is totally geodesic, a fixed set component of the isometry $\iota \circ \tau$ with $\iota(g) = g^{-1}$ since $\iota(\tau(g\tau(g)^{-1})) = (\tau(g)g^{-1})^{-1} = g\tau(g)^{-1}$. In fact it is the Cartan embedding of P_ℓ into the group $G_\ell = \{g \in \mathrm{SO}_p : gP_\ell g^{-1} = P_\ell\}$.

One of these inequalities holds unless all $k_i \in \{1, -1\}$ which means that γ is minimal, see the subsequent remark. $\square$

Remark 5.7 Let still $\ell = 4m-2$. The geodesic $\gamma_o(t) = e^{\pi t A}$ from I to $-I$ is minimal in $U_{p/2}$ (with length π) if and only if A has only eigenvalues $\pm i$. In this case, when $\gamma = \gamma_o J_\ell$ lies inside P_ℓ, the numbers k_+ of positive signs and k_- of negative signs are determined by the above conditions $k_+ + k_- = p/s_\ell$ and $k_+ - k_- = 4w/s_\ell$. This corresponds to a $(J_1, \dots, J_\ell)$-invariant orthogonal splitting (which in particular is complex linear with respect to $i = J_1 \cdots J_{\ell-1}$),

$$\mathbb{R}^p = L_- \oplus L_+\,, \quad \dim L_\pm = s_\ell k_\pm\,, \tag{5.10}$$

with $A = \pm i$ on $L_\pm$. Then $A^2 = -I$, and we obtain another complex structure $J_{\ell+1} = AJ_\ell = \gamma(\frac{1}{2})$ anticommuting with $J_1, \dots, J_\ell$ and defining a minimal centriole $P_{\ell+1} \subset P_\ell$. Note that

$$J_{\ell+1} = \pm J_1 \cdots J_\ell \text{ on } L_\pm\,.$$

The property of $J_{\ell+1}$ being determined (up to sign) by $J_1, \dots, J_\ell$ only appears for $\ell = 4m-2$.

For later use we formulate a necessary and sufficient condition when this $\mathrm{Cl}_{\ell+1}$-representation can be extended to a $\mathrm{Cl}_{\ell+2}$-representation.

Proposition 5.8 *Let $\gamma(t) = e^{\pi t A} J_\ell$ be a minimal geodesic in $\Omega(P_\ell; J_\ell, -J_\ell)$ with mid-point $J_{\ell+1}$ for $\ell = 4m-2$. Then the following assertions are equivalent.*

(i) There exists a complex structure $J_{\ell+2}$ anticommuting with $J_1, \dots, J_{\ell+1}$,
(ii) $w(\gamma) = 0$.

Proof With the notation from Remark 5.7 recall that $J_{\ell+1} J_\ell^{-1} = e^{(\pi/2)A} = A$ is a complex matrix with eigenvalue i on L_+ and $-i$ on L_-.

When $J_{\ell+2}$ exists, then $J_{\ell+2} J_\ell^{-1}$ is another complex matrix which anticommutes with $J_{\ell+1} J_\ell^{-1}$ and thus interchanges the eigenspaces L_+ and L_-. Therefore $k_+ = k_-$, that is $w = 0$.

Vice versa, when $k_+ = k_- =: k$, then L_+ and L_- may be identified as Cl_ℓ-modules, and by putting $J_{\ell+2} := \begin{pmatrix} & -I_k \\ I_k & \end{pmatrix} \cdot J_\ell$ we define a further complex structure anticommuting with $J_1, \dots, J_{\ell+1}$. $\square$

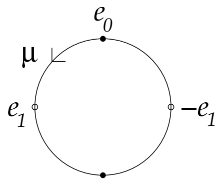
Remark 5.9 We do not need to consider the group $G = \mathrm{Sp}_p$ and its iterated centrioles since $\mathrm{Sp}_p = P_4(\mathrm{SO}_{8p})$, cf. (3.2). For $G = \mathrm{U}_p$, the iterated centrioles are U_q and $\mathbb{G}_{q/2}(\mathbb{C}^q)$ (the complex Grassmannian) for all $q = p/2^m$. U_q has been considered in Proposition 5.3. In $\mathbb{G}_{q/2}(\mathbb{C}^q)$ the index of non-minimal geodesics is high when q is large enough[8] as can be seen from the *real Grassmannian* $\mathbb{G}_{q/2}(\mathbb{R}^q) = P_3(\mathrm{Sp}_q)$. In fact, real and complex Grassmannians have a common maximal torus, therefore any

[8] This example was omitted in [11].

geodesic in the complex Grassmannian can be conjugated into the real Grassmannian. Moreover, the distance between poles (i.e. the length of a minimal geodesic) agrees for the complex Grassmannian and the real one. Hence the index of a non-minimal geodesic in the complex Grassmannian is at least as big as in the real Grassmannian, but the latter one is among the spaces considered in Proposition 5.4.

6 Deformations of mapping spaces

We consider a round sphere $\mathbb{S} = \mathbb{S}^k$, $k \geq 1$, with "north pole" $N = e_0 \in \mathbb{S}$, and its equator $\mathbb{S}' = \mathbb{S} \cap N^\perp$ with "north pole" $N' = e_1 \in \mathbb{S}'$ (or rather "west pole", according to the orientation (e_0, e_1)), where the standard basis of $\mathbb{R}^{k+1}$ is denoted $e_0, \ldots, e_k$. Let $\mu : [0, 1] \to \mathbb{S}$ be the meridian from N to $-N$ through N' and $m = \mu([0, 1]) \subset \mathbb{S}$.



Further let P be the group $G \in \{\mathrm{SO}_p, \mathrm{U}_p, \mathrm{Sp}_p\}$ or one of its iterated centrioles (see Sect. 3). We fix some $J \in P$ and a shortest geodesic

$$\gamma : J \rightsquigarrow -J$$

in P, which we consider as a map $\gamma : m \to P$.

Definition 6.1 Let $\mathrm{Map}_*(\mathbb{S}, P)$ be the space of C^0-maps $\phi : \mathbb{S} \to P$ with $\phi(N) = J$, equipped with the compact-open topology. We denote by $\mathrm{Map}_\gamma(\mathbb{S}, P)$ the subset of maps ϕ with $\phi|_m = \gamma$, equipped with the subspace topology.

Building on the results from Sect. 4 will will develop a deformation theory for these mapping spaces $\mathrm{Map}_*(\mathbb{S}, P)$. The main result, Theorem 7.10, says that stably it can be approximated by the subspace of maps which are block sums of constant maps and Hopf maps associated to Clifford representations (see Definition 2.1).

Lemma 6.2 *The inclusion* $\mathrm{Map}_\gamma(\mathbb{S}, P) \subset \mathrm{Map}_*(\mathbb{S}, P)$ *is a weak homotopy equivalence.*

Proof This can be seen directly by constructing a deformation of arbitrary maps $\phi \in \mathrm{Map}_*(\mathbb{S}, P)$ into maps $\tilde{\phi} \in \mathrm{Map}_\gamma(\mathbb{S}, P)$. However, the proof becomes shorter if we use some elementary homotopy theory, see [5, Ch. VII]. We consider the restriction map

$$r : \mathrm{Map}_*(\mathbb{S}, P) \to \mathrm{Map}_*(m, P), \quad \phi \mapsto \phi|_m,$$

with $\mathrm{Map}_*(m, P) := \{f : m \to P : f(N) = J\}$. Clearly, $r^{-1}(\gamma) = \mathrm{Map}_\gamma(\mathbb{S}, P)$. This map is a fibration, according to Theorem 6.13 together with Corollary 6.16 in [5, Ch. VII], applied to $Y = P$ and $(X, A) = (\mathbb{S}, m)$; the requirement that $A \subset X$ is a cofibration follows from Corollary 1.4 in [5, Ch. VII]. Since the base space is clearly contractible, the inclusion of the fibre

$$\mathrm{Map}_{\gamma}(\mathbb{S}, P) \subset \mathrm{Map}_{*}(\mathbb{S}, P) \tag{6.1}$$

is a weak homotopy equivalence, due to the homotopy sequence of this fibration. □

Let $\mathbb{S}' = \mathbb{S}^{k-1} \subset \mathbb{S}$ be the equator sphere. We may parametrize $\mathbb{S}$ by $\mathbb{S}' \times I$ with $(v, t) \mapsto \mu_v(t)$ where $\mu_v : [0, 1] \to \mathbb{S}$ is the meridian from N to $-N$ through $v \in \mathbb{S}'$. Using the assignment $\phi \mapsto (v \mapsto \phi \circ \mu_v)$ for any $\phi \in \mathrm{Map}_{\gamma}(\mathbb{S}, P)$ and $v \in \mathbb{S}'$ we obtain a canonical homeomorphism

$$\mathrm{Map}_{\gamma}(\mathbb{S}, P) \approx \mathrm{Map}_{*}(\mathbb{S}', \Omega) \tag{6.2}$$

where

$$\Omega = \Omega(P; J, -J)$$

is the space of continuous paths $\omega : [0, 1] \to P$ with $\omega(0) = J$ and $\omega(1) = -J$, and $\mathrm{Map}_{*}(\mathbb{S}', \Omega)$ the space of mappings $\phi : \mathbb{S}' \to \Omega$ with $\phi(N') = \gamma$.

Let $\Omega^0 = \Omega^0(P; J, -J) \subset \Omega$ be the subspace of minimal geodesics from J to $-J$ in P. Since every such minimal geodesic is determined by its midpoint, which belongs to the midpoint set $\hat{P}' \subset P$ whose components are the minimal centrioles $P' \subset P$ (see Sect. 3), we furthermore have a canonical homeomorphism

$$\mathrm{Map}_{*}(\mathbb{S}', \Omega^0) \approx \mathrm{Map}_{*}(\mathbb{S}', \hat{P}')\,. \tag{6.3}$$

The composition

$$\begin{aligned} \Sigma_J \; : \; \mathrm{Map}_{*}(\mathbb{S}', \hat{P}') &\approx \mathrm{Map}_{*}(\mathbb{S}', \Omega^0) \subset \mathrm{Map}_{*}(\mathbb{S}', \Omega) \\ &\approx \mathrm{Map}_{\gamma}(\mathbb{S}, P) \; \subset \mathrm{Map}_{*}(\mathbb{S}, P) \end{aligned} \tag{6.4}$$

sends $\phi' \in \mathrm{Map}_{*}(\mathbb{S}', \hat{P}')$ to $\phi \in \mathrm{Map}_{*}(\mathbb{S}, P)$ defined by

$$\phi(v, t) = \gamma_{\phi'(v)}(t) \quad \text{for all } (v, t) \in \mathbb{S}' \times [0, 1] \tag{6.5}$$

where $\gamma_{\phi'(v)} : J \rightsquigarrow -J$ is the shortest geodesic in P through $\phi'(v)$.

Definition 6.3 The map Σ_J is called the *geodesic suspension* along J.

We will show that Σ_J is highly connected in many cases. At first we will deal with the case $k = \dim \mathbb{S} \geq 2$.

Proposition 6.4 *Let $k \geq 2$ and let $d + \dim(\mathbb{S}')$ be smaller than the index of any non-minimal geodesic in the connected component $\Omega_* \subset \Omega$ containing γ. Then Σ_J is d-connected.*

Proof By Lemma 6.2, the inclusion $\mathrm{Map}_\gamma(\mathbb{S}, P) \to \mathrm{Map}_*(\mathbb{S}, P)$ is a weak homotopy equivalence. Therefore we only have to deal with the inclusion

$$\mathrm{Map}_*(\mathbb{S}', \Omega^0) \subset \mathrm{Map}_*(\mathbb{S}', \Omega)\,.$$

Let $\psi : \mathbb{D}^j \to \mathrm{Map}_*(\mathbb{S}', \Omega)$, $j \leq d$, with $\psi(\partial\mathbb{D}^j) \subset \mathrm{Map}_*(\mathbb{S}', \Omega^0)$.

We consider ψ as a map $\hat{\psi} : \mathbb{D}^j \times \mathbb{S}' \to \Omega$ and observe, using $k \geq 2$, that the image of this map lies in the connected component $\Omega_* \subset \Omega$ determined by γ.

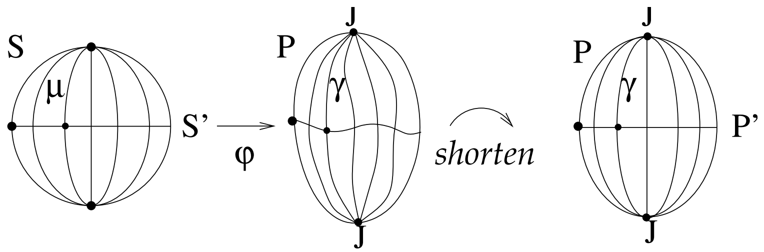
Theorem 4.2 may now be applied to the $(j+\dim\mathbb{S}')$-dimensional CW-pair (compare Definition 4.1)

$$(X, Y) = (\mathbb{D}^j \times \mathbb{S}', (\partial\mathbb{D}^j \times \mathbb{S}') \cup (\mathbb{D}^j \times \{N'\}))$$

where $j+\dim\mathbb{S}' \leq d+\dim\mathbb{S}'$. This results in a deformation of $\hat{\psi}$ to a map with image contained in Ω_*^0 by a deformation which is constant on $(\partial\mathbb{D}^j \times \mathbb{S}') \cup (\mathbb{D}^j \times \{N'\})$.

Hence ψ may be deformed to a map with image contained in $\mathrm{Map}_*(\mathbb{S}', \Omega^0)$ by a deformation which is constant on $\partial\mathbb{D}^j$. □

The following picture illustrates the deformation process employed in this proof where $\varphi \in \mathrm{Map}_\gamma(\mathbb{S}, P)$ lies in the range of ψ.



7 Iterated suspensions and Clifford representations

This process can be iterated. For the sake of exposition we will concentrate on the case $G = \mathrm{SO}_p$ in the following argument. Let $k \geq 1$ and consider an orthogonal Cl_k-representation

$$\rho : \mathrm{Cl}_k \to \mathbb{R}^{p\times p}$$

determined by anticommuting complex structures $J_1, \ldots, J_k \in \mathrm{SO}_p$. Let

$$\mathrm{SO}_p = P_0 \supset P_1 \supset \cdots \supset P_k \tag{7.1}$$

be the associated chain of minimal centrioles (compare Theorem 3.1).

Let $e_0, \ldots, e_k$ denote the standard basis of $\mathbb{R}^{k+1}$ and $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ be the unit sphere. Furthermore, for $\ell \in \{0, \ldots, k\}$ let[9]

$$\mathbb{S}^{k-\ell} := \mathbb{S}^k \cap \operatorname{Span}\{e_\ell, \ldots, e_k\}$$

be the great sphere in the subspace spanned by $e_\ell, \ldots, e_k$ with "north pole" (base point) e_ℓ and equator sphere $\mathbb{S}^{k-\ell-1}$.

Let $\operatorname{Map}_*(\mathbb{S}^{k-\ell}, P_\ell)$, $\ell = 0, \ldots, k$, be the space of maps $\mathbb{S}^{k-\ell} \to P_\ell$ sending e_ℓ to J_ℓ for $\ell \geq 1$ and sending e_0 to $I \in \mathrm{SO}_p$ for $\ell = 0$. Putting $\Sigma_\ell := \Sigma_{J_\ell}$ (the suspension along J_ℓ) with $J_0 := I$ we consider the composition

$$\theta : \operatorname{Map}_*(\mathbb{S}^1, P_{k-1}) \xrightarrow{\Sigma_{k-2}} \operatorname{Map}_*(\mathbb{S}^2, P_{k-2}) \xrightarrow{\Sigma_{k-3}} \cdots \xrightarrow{\Sigma_0} \operatorname{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)\,. \tag{7.2}$$

Proposition 7.1 *For each d there is a $p_0 = p_0(d)$ such that θ is d-connected whenever $p \geq p_0$.*

Proof Since the composition of d-connected maps is d-connected it suffices to show that for all $\ell = 0, \ldots, k-2$ the suspension

$$\Sigma_{J_\ell} : \operatorname{Map}_*(\mathbb{S}^{k-\ell-1}, P_{\ell+1}) \to \operatorname{Map}_*(\mathbb{S}^{k-\ell}, P_\ell)$$

in the composition θ is d-connected once $p \geq p_0$.

This claim follows from Proposition 6.4, where the assumption on indices of non-minimal geodesics holds for the following reasons:

(1) If $\ell + 2$ is not divisible by four it holds by Proposition 5.4.
(2) If $\ell + 2$ is divisible by four it holds by Proposition 5.6. Indeed, let $\gamma : J_\ell \rightsquigarrow -J_\ell$ be the minimal geodesic in P_ℓ running through $J_{\ell+1}$. Then $w(\gamma) = 0$ by Proposition 5.8 since there is a complex structure $J_{\ell+2}$ anticommuting with $J_1, \ldots, J_{\ell+1}$ (recall $\ell+2 \leq k$). Hence $w(\tilde\gamma) = 0$ for each non-minimal geodesic $\tilde\gamma$ in the connected component $\Omega_* \subset \Omega = \Omega(P_\ell; J_\ell, -J_\ell)$ containing γ. □

It remains to investigate the space $\operatorname{Map}_*(\mathbb{S}^1, P_{k-1})$. We restrict to the case $k = 4m-1$. Setting $\ell = k-1$ this means $\ell = 4m-2$, and $P = P_\ell$. From equation (5.8) we recall the canonical embedding

$$P_\ell \hookrightarrow \mathrm{U}_{p/2}\,, \quad J \mapsto JJ_\ell^{-1}\,.$$

Assumption In the remainder of this section we will assume that the given Cl_k-representation ρ satisfies

$$J_k = +J_1 \cdots J_\ell\,.$$

[9] The reason for this unusual choice is that (7.1) is descendent and we would like to map e_1 onto $J_1 \in P_1$ and e_j onto $J_j \in P_j$ as in Theorem 3.1.

Orthogonal Cl_k-representations of this kind and their Hopf maps will be called *positive*.[10]

Furthermore we denote by

$$\gamma : J_\ell \rightsquigarrow -J_\ell$$

the shortest geodesic in P_ℓ through $J_{\ell+1} = J_k$. With the complex structure $i = J_o = J_1 J_2 \cdots J_{\ell-1}$ on $\mathbb{R}^p$ we then have

$$\gamma(t) = e^{\pi i t} J_\ell$$

In particular $w(\gamma) = p/4$.

Definition 7.2 For each $\omega \in \mathrm{Map}_*(\mathbb{S}^1, P_\ell)$ with $\ell = k - 1$ we define the *winding number* $\eta(\omega) \in \mathbb{Z}$ as the winding number of the composition

$$\mathbb{S}^1 = \mathbb{S}^{k-\ell} \xrightarrow{\omega} P_\ell \hookrightarrow \mathrm{U}_{p/2} \xrightarrow{\det} \mathbb{S}^1 .$$

Here we identify $(\alpha, \beta) \in \mathbb{S}^1 \subset \mathbb{R}^2$ with $\alpha \cdot e_{k-1} + \beta \cdot e_k \in \mathbb{S}^{k-\ell} \subset \mathrm{Span}\{e_{k-1}, e_k\}$. Note that η is constant on path components of $\mathrm{Map}_*(\mathbb{S}^1, P_\ell)$.

For $\eta \in \mathbb{Z}$ let

$$\mathrm{Map}_*(\mathbb{S}^1, P_\ell)_\eta \subset \mathrm{Map}_*(\mathbb{S}^1, P_\ell)$$

denote the subspace of loops with winding number equal to η. This is a union of path components of $\mathrm{Map}_*(\mathbb{S}^1, P_\ell)$.

By Proposition 7.1, for sufficiently large p, the map θ in (7.2) induces a bijection of path components. Hence, at least after taking a block sum with a constant map $\mathbb{S}^k \to \mathrm{SO}_{p'}$ for some large p', the previously defined winding number

$$\eta : \mathrm{Map}_*(\mathbb{S}^1, P_\ell) \to \mathbb{Z}$$

induces a map

$$\eta^s : \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p) \to \mathbb{Z},$$

such that $\eta^s \circ \theta = \eta$, which we call the *stable winding number*.

[10] For $k = 3$ and 7 this is the representation of Cl_k on $\mathbb{K} \in \{\mathbb{H}, \mathbb{O}\}$ by left translations. In fact, on $\mathbb{H}$ we have $ijk = -1$ (Hamilton's equation). On $\mathbb{O}$ there are further basis elements ℓ, $p = i\ell$, $q = j\ell$, $r = k\ell$, and the corresponding left translations on $\mathbb{O}$ will be denoted by the same symbols. Then $ijk = -\mathrm{id}$ on $\mathbb{H}$. Further, using anti-associativity of Cayley triples like $(j\ell)r = -j(\ell r)$ we have $\ell pqr = \ell(p((j\ell)r)) = -\ell(p(j(\ell r))) = -\ell(p(j(\ell k\ell))) = -\ell(p(jk)) = -\ell((i\ell)i) = -\ell^2 = 1$ and thus $\ell pqr = \mathrm{id}$ on $\mathbb{H}$ since ℓpqr commutes with i, j, k. Moreover, both ijk and ℓpqr anticommute with ℓ, hence on $\ell\mathbb{H}$ we have $ijk = \mathrm{id}$ and $\ell pqr = -\mathrm{id}$. Together we see $ijk\ell pqr = -\mathrm{id}$ on $\mathbb{O}$.

This is independent of the particular choice of p', and hence well defined on $\mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)$ for the original p. By definition it is constant on path components and therefore may be regarded as a map

$$\eta^s : \pi_k(\mathrm{SO}_p; I) \to \mathbb{Z}\,.$$

Since all positive Cl_k-representations on $\mathbb{R}^p$ are isomorphic, this map is independent from the chosen positive Cl_k-representation ρ.

For $\eta \in \mathbb{Z}$ let

$$\mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)_\eta \subset \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)$$

denote the subspace with stable winding number equal to η.

Example 7.3 Let $\mu : \mathbb{S}^k \to \mathrm{SO}_p$ be the Hopf map (see Definition 2.1) associated to the given Clifford representation ρ. Then $\eta^s(\mu) = p/2$. Indeed we have $\mu = \theta(\omega)$, where $\omega : \mathbb{S}^1 \to P_\ell$ is given by $\alpha \cdot e_{k-1} + \beta \cdot e_k \mapsto \alpha \cdot J_{k-1} + \beta \cdot J_k$, such that with $i = J_1 J_2 \cdots J_{\ell-1}$ the composition $\mathbb{S}^1 \to P_\ell \to \mathrm{U}_{p/2}$ is given by $e^{\pi i t} \mapsto e^{\pi i t} J_\ell$ as $J_k = +J_1 \cdots J_\ell = i J_\ell$.

Remark 7.4 The stable winding number is additive with respect to block sums of Cl_k-representations. More precisely, let

$$\rho_i : \mathrm{Cl}_k \to \mathrm{SO}_{p_i}$$

be positive Cl_k-representations, $i = 1, 2$, with associated chains of minimal centrioles

$$\mathrm{SO}_{p_i} \supset P_1^{(i)} \supset \cdots \supset P_k^{(i)}\,.$$

The chain of minimal centrioles associated to the block sum action

$$\rho = \rho_1 \oplus \rho_2 : \mathrm{Cl}_k \to \mathrm{SO}_{p_1} \times \mathrm{SO}_{p_2} \subset \mathrm{SO}_{p_1+p_2}$$

then takes the form

$$\mathrm{SO}_{p_1} \times \mathrm{SO}_{p_2} \supset P_1^{(1)} \times P_1^{(2)} \supset \cdots \supset P_k^{(1)} \times P_k^{(2)}\,,$$

and each suspension in (7.2) is a product of corresponding suspensions for ρ_1 and ρ_2. Hence for $\phi = \phi_1 \oplus \phi_2 \in \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_{p_1+p_2})$ of block sum form we obtain for the stable winding numbers

$$\eta^s(\phi) = \eta^s(\phi_1) + \eta^s(\phi_2)\,.$$

Let $\Omega = \Omega(P_\ell; J_\ell, -J_\ell)$ be the space of paths in P_ℓ from J_ℓ to $-J_\ell$ and let

$$\hat{P}_k = \Omega^0 \subset \Omega$$

be the subspace of shortest geodesics, cf. (3.3). Consider the geodesic suspension

$$\Sigma_\ell : \mathrm{Map}_*(\mathbb{S}^0, \hat{P}_k) \to \mathrm{Map}_*(\mathbb{S}^1, P_\ell)\,,$$

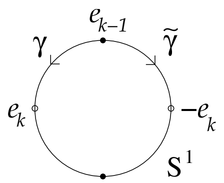
where $\mathrm{Map}_*(\mathbb{S}^0, \Omega^0)$ is the space of maps $\mathbb{S}^0 = \{\pm e_k\} \to \Omega^0$ sending e_k to $\gamma \in \Omega^0$.

Definition 7.5 For $\phi \in \mathrm{Map}_*(\mathbb{S}^0, \Omega)$ we denote by $w(\phi) \in \mathbb{Z}$ the winding number of $\tilde{\gamma} = \phi(-e_k) \in \Omega$ in the sense of Definition 5.5. Note that possibly $w(\tilde{\gamma}) \neq w(\gamma)$, since $\mathbb{S}^0$ is disconnected and hence $\tilde{\gamma} = \phi(-e_k)$ and $\gamma = \phi(e_k)$ may lie in different path components of Ω.

For $w \in \mathbb{Z}$ let

$$\mathrm{Map}_*(\mathbb{S}^0, \Omega)_w \subset \mathrm{Map}_*(\mathbb{S}^0, \Omega) \text{ and } \mathrm{Map}_*(\mathbb{S}^0, \Omega^0)_w \subset \mathrm{Map}_*(\mathbb{S}^0, \Omega^0)$$

be the subspaces of maps of winding number w.



Note that the concatenation map $\tilde{\gamma} \mapsto \gamma * \tilde{\gamma}^{-1}$ (first γ, then $\tilde{\gamma}^{-1}$ where $\tilde{\gamma}^{-1}(t) := \tilde{\gamma}(1-t)$, which together form a loop starting and ending at J_{k-1}) gives a canonical identification

$$\mathrm{Map}_*(\mathbb{S}^0, \Omega)_w = \mathrm{Map}_*(\mathbb{S}^1, P_\ell)_{\eta=(p/4)-w}$$

since $w(\gamma) = p/4$.

After these preparations we obtain the following analogue of Proposition 6.4.

Proposition 7.6 *Let $d \geq 0$ and let $w \in \mathbb{Z}$ satisfy $p - 4|w| \geq 4s_\ell \cdot d$. Then the suspension Σ_ℓ restricts to a d-connected map*

$$\Sigma_\ell : \mathrm{Map}_*(\mathbb{S}^0, \Omega^0)_w = \mathrm{Map}_*(\mathbb{S}^1, P_\ell)_{\eta=(p/4)-w}\,.$$

Proof Similar as in the proof of Proposition 6.4 we only have to deal with the inclusion

$$\mathrm{Map}_*(\mathbb{S}', \hat{P}_k)_w \subset \mathrm{Map}_*(\mathbb{S}', \Omega)_w$$

where we now have $\mathbb{S}' = \mathbb{S}^0$, cf. (6.4). This is equivalent to the inclusion

$$\Omega^0(P_\ell; J_\ell, -J_\ell)_w \subset \Omega(P_\ell; J_\ell, -J_\ell)_w\,.$$

By Theorem 4.2 and Proposition 5.6 with our assumption on $|w|$ this inclusion is d-connected. □

Each $\phi \in \mathrm{Map}_*(\mathbb{S}^0, \Omega^0)$ is determined by its value on $-e_k \in \mathbb{S}^0$, a minimal geodesics $\tilde{\gamma}(t) = e^{\pi i A t} J_\ell$ where A is a self adjoint complex $(\frac{p}{2} \times \frac{p}{2})$-matrix with eigenvalues ± 1 which commutes with $J_1, \dots, J_\ell$. Furthermore $w(\tilde{\gamma}) = \mathrm{trace}_{\mathbb{C}} A$.

As in Remark 5.7 the eigenspaces of A induce an orthogonal splitting

$$\mathbb{R}^p = L_0 \oplus L_1$$

into subspaces invariant under $J_1, \dots, J_\ell$ (hence under $i = J_1 \cdots J_{\ell-1}$) such that

$$\tilde{\gamma}(t) = \gamma(t) = e^{\pi i t} J_\ell \text{ on } L_0\,, \quad \tilde{\gamma}(t) = e^{-\pi i t} J_\ell \text{ on } L_1\,.$$

Hence the geodesic $\tilde{\gamma}$ is of block diagonal form

$$\tilde{\gamma} = \gamma_0 \oplus \gamma_1 : [0, 1] \to \mathrm{SO}(L_0) \times \mathrm{SO}(L_1)$$

with midpoint $(+J_k, -J_k)$ and $w(\tilde{\gamma}) = \frac{1}{2}(\dim_{\mathbb{C}} L_0 - \dim_{\mathbb{C}} L_1)$. The suspension $\omega := \Sigma_{J_\ell}(\phi) \in \mathrm{Map}_*(\mathbb{S}^1, P_\ell)$ is equal to the concatenation

$$\omega = \gamma * \tilde{\gamma}^{-1} : [0, 1] \to P_\ell\,.$$

Therefore ω is also in block diagonal form

$$\omega = \omega_0 \oplus \omega_1 : [0, 1] \to \mathrm{SO}(L_0) \times \mathrm{SO}(L_1)\,,$$

where

$$\omega_0 = \gamma_0 * \gamma_0^{-1}\,, \quad \omega_1(t) = e^{2\pi i t} J_\ell\,.$$

The loop $\gamma_0 * \gamma_0^{-1}$ is homotopic relative to $\{0, 1\}$ to the constant loop with value J_ℓ. This motivates the following definition.

Definition 7.7 We define the subspace

$$\mathrm{Map}^0_*(\mathbb{S}^1, P_\ell) \subset \mathrm{Map}_*(\mathbb{S}^1, P_\ell)$$

consisting of loops $\omega \in \mathrm{Map}_*(\mathbb{S}^1, P_\ell)$ such that there is an orthogonal decomposition

$$\mathbb{R}^p = L_0 \oplus L_1$$

into subspaces which are invariant under $J_1, \dots, J_\ell$ and with respect to which ω is in block diagonal form

$$\omega = \omega_0 \oplus \omega_1 : \mathbb{S}^1 \to \mathrm{SO}(L_0) \times \mathrm{SO}(L_1)$$

such that ω_0 is constant with value J_ℓ and $\omega_1 : \mathbb{S}^1 \to \mathrm{SO}(L_1)$ is given by

$$e^{2\pi i t} \mapsto e^{2\pi i t} J_\ell\,,$$

where $i = J_1 \cdots J_{\ell-1}$. For $0 \leq \eta \leq p/2$ we denote by

$$\mathrm{Map}^0_*(\mathbb{S}^1, P_\ell)_\eta \subset \mathrm{Map}^0_*(\mathbb{S}^1, P_\ell)$$

the subspace of loops with winding number η (equal to $\dim_{\mathbb{C}} L_1 \leq p/2$).

We have a canonical homeomorphism

$$h : \mathrm{Map}^0_*(\mathbb{S}^1, P_\ell) \approx \mathrm{Map}_*(\mathbb{S}^0, \Omega^0) \tag{7.3}$$

which replaces the constant map on L_0 by the concatenation $\omega = \gamma * \gamma^{-1}$, restricted to L_0. Since this is homotopic relative to $\{0, 1\}$ to the constant loop with value $I \in \mathrm{SO}(L_0)$ by use of the explicit homotopy

$$\omega_s = \gamma_s * \gamma_s^{-1}\,, \quad \gamma_s(t) := \gamma(st)\,, \quad 0 \leq s, t \leq 1\,,$$

we conclude that the composition

$$\mathrm{Map}^0_*(\mathbb{S}^1, P_\ell) \overset{h}{\approx} \mathrm{Map}_*(\mathbb{S}^0, \hat{P}_k) \xrightarrow{\Sigma_\ell} \mathrm{Map}_*(\mathbb{S}^1, P_\ell)$$

is homotopic to the canonical inclusion

$$j : \mathrm{Map}^0_*(\mathbb{S}^1, P_\ell) \to \mathrm{Map}_*(\mathbb{S}^1, P_\ell)\,.$$

Since homotopic maps induce the same maps on homotopy groups and a map $A \to B$ is d-connected if and only if it induces a bijection on π_j for $0 \leq j \leq d-1$ and a surjection on π_d, we hence obtain the following version of Proposition 7.1 together with Proposition 7.6 where $\Sigma_\ell \circ h$ (cf. (7.3)) is replaced by j.

Proposition 7.8 *Let $d \geq 1$ and $p \geq p_0(d)$ as in Proposition 7.1. Assume $s_\ell \cdot d \leq \eta \leq p/4$. Then the composition*

$$\theta^0 : \mathrm{Map}^0_*(\mathbb{S}^1, P_\ell)_\eta \xrightarrow{j} \mathrm{Map}_*(\mathbb{S}^1, P_\ell)_\eta \xrightarrow{\theta} \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)_\eta \tag{7.4}$$

is d-connected.

Proof Notice that for $\eta \leq p/4$ the winding number $w = (p/4) - \eta$ is non-negative. Hence $p - 4|w| = p - 4w = 4\eta \geq 4s_\ell \cdot d$. The assertion now follows from Propositions 7.1 and 7.6. □

By construction, elements $\phi \in \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)_\eta$ in the image of θ^0 in (7.4) are described as follows. There is an orthogonal splitting

$$\mathbb{R}^p = L_0 \oplus L_1$$

into Cl_k-invariant subspaces with $\dim_{\mathbb{C}} L_1 = \eta$ such that

$$\phi = (\phi_0, \phi_1) : \mathbb{S}^k \to \mathrm{SO}(L_0) \times \mathrm{SO}(L_1) \subset \mathrm{SO}_p$$

where $\phi_0 \equiv I$ and $\phi_1 : \mathbb{S}^k = \mathbb{S}(\mathbb{R} \oplus \mathbb{R}^k) \to \mathbb{S}(\mathrm{End}(L_1))$ is an isometric embedding as a great sphere with image contained in $\mathrm{SO}(L_1)$ which sends e_0 to I and e_i to $J_i|_{L_1}$ for $1 \leq i \leq k$.

Proposition 2.2 then implies that the map ϕ_1 is in fact equal to the Hopf map associated to the positive Cl_k-representation ρ restricted to L_1.

Definition 7.9 We call maps ϕ of this kind (that is $\phi = (\phi_0, \phi_1)$ on $L_0 \oplus L_1$ with $\phi_0 \equiv I$ and ϕ_1 a positive Hopf map on L_1) *affine Hopf maps* of stable winding number $\eta = \dim_{\mathbb{C}} L_1$ associated to ρ (a positive Clifford representation). Let

$$\mathrm{Hopf}_\rho(\mathbb{S}^k, \mathrm{SO}_p)_\eta \subset \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)_\eta$$

be the subspace of affine Hopf maps of stable winding number η.

Summarizing we obtain the following result on deformations of mapping spaces (recall $s_k = s_\ell$ from (2.2)).

Theorem 7.10 *Let $d \geq 1$ and $k = 4m-1$. Then for all $p \geq p_0(d)$ and $s_k \cdot d \leq \eta \leq p/4$ and any positive Clifford representation $\rho : \mathrm{Cl}_k \to \mathbb{R}^{p \times p}$ the canonical inclusion*

$$\mathrm{Hopf}_\rho(\mathbb{S}^k, \mathrm{SO}_p)_\eta \subset \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)_\eta \tag{7.5}$$

is d-connected.

Proof The theorem follows from Proposition 7.8 since $\mathrm{Hopf}_\rho(\mathbb{S}^k, \mathrm{SO}_p)_\eta$ is the image of θ^0 in (7.4). □

8 Vector bundles over sphere bundles: clutching construction

The deformation theory for maps $\phi : \mathbb{S}^k \to \mathrm{SO}_p$ can be applied to the classification of oriented Euclidean vector bundles over n-spheres where from now on

$$n = k + 1 = 4m\,.$$

Choose $N = e_0$ and $S = -e_0$ ("north and south poles") in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ where $e_0, \ldots, e_n$ is the standard basis of $\mathbb{R}^{n+1}$. Let

$$\mathbb{D}^n_\pm = \{v \in \mathbb{S}^n : \pm\langle v, e_0\rangle \geq 0\},$$
$$\mathbb{S}^{n-1} = \{v \in \mathbb{S}^n : \langle v, e_0\rangle = 0\} = \mathbb{D}^n_+ \cap \mathbb{D}^n_-$$

("hemispheres and equator"). Let $\mathscr{E} \to \mathbb{S}^n$ be an oriented Euclidean vector bundle with fibre $\mathbb{R}^p$. Since $\mathbb{D}^n_\pm$ is contractible, $\mathscr{E}|_{\mathbb{D}^n_\pm}$ is a trivial bundle, isomorphic to $\mathbb{D}^n_\pm \times E_\pm$ where

$$E_\pm = \mathscr{E}_{\pm e_n} \cong \mathbb{R}^p ,$$

pulled back to $\mathbb{D}^n_\pm$ along the radial projections onto the midpoint. The bundle E is obtained from these trivial bundles by identifying $\{v\} \times E_+$ to $\{v\} \times E_-$ via an oriented orthogonal map, thus by a mapping $\phi : \mathbb{S}^{n-1} \to \mathrm{SO}(E_+, E_-) \cong \mathrm{SO}_p$ called *clutching function* which is well defined by $\mathscr{E}$ up to homotopy. In this situation we write

$$\mathscr{E} = (E_+, \phi, E_-) .$$

Example 8.1 Let $\rho : \mathrm{Cl}_n \to \mathrm{End}(L)$ be an orthogonal Cl_n-representation. Since $n = 4m$, the longest Clifford product $\omega := e_1 \cdots e_n$ (the "volume element") commutes with all elements of Cl_n^+ (the subalgebra spanned by the products of even length) and has order two, $\omega^2 = 1$.[11] The (± 1)-eigenspaces[12] $L^\pm$ of $\rho(\omega)$ are invariant under Cl_n^+, and $\rho(v)$ interchanges the eigenspaces for any $v \in \mathbb{S}^{n-1}$. Using $\mu := \rho|_{\mathbb{S}^{n-1}}$ we obtain a bundle $\mathscr{L} \to \mathbb{S}^n$,

$$\mathscr{L} = (L^+, \mu, L^-)$$

This is called the *Hopf bundle* over $\mathbb{S}^n$ associated to the Clifford representation ρ.

Remark 8.2 This bundle is isomorphic to (L^+, μ_+, L^+), where

$$\mu_+ := \mu(e_1)^{-1} \cdot \mu : \mathbb{S}^{n-1} \to \mathrm{SO}(L^+) \cong \mathrm{SO}_p$$

is the Hopf map $\mu_+ = \rho_+|_{\mathbb{S}^k}$ $(k = n - 1)$ associated to the representation $\rho_+ : \mathrm{Cl}_k \cong \mathrm{Cl}_n^+ \to \mathrm{SO}(L_+)$,

$$\rho_+ : e_i \mapsto \rho(-e_1 e_{i+1}), \quad i = 1, \ldots, k. \tag{8.1}$$

Note that ρ_+ is positive,[13] that means its Clifford family $J_i = \rho(e_i)$, $i = 1, \ldots, k$, on L^+ satisfies

$$J_k = J_1 \cdots J_{k-1} .$$

[11] 12341234 = −11234234 = 234234 = 223434 = −3434 = 1, 12345678 = 56781234.

[12] Superscripts $\pm$ will always refer to a splitting induced by the volume element.

[13] E.g. $k = 7$: Putting $\mu(e_i) =: i$ we have

$$\mu_+(e_1) \cdots \mu_+(e_7) = (-1)^7 12\underline{1}3\underline{1}4\underline{1}5\underline{1}6\underline{1}7\underline{1}8 = -12\underline{3}4\underline{5}6\underline{7}8 = -\mathrm{id}_{L^+} \text{ on } L^+.$$

In particular the stable winding number of μ_+ (see Definition 7.2) is given by

$$\eta^s(\mu_+) = \frac{1}{2}\dim L^+ = \frac{1}{4}\dim L > 0\,.$$

Compare Example 7.3. The positivity of this winding number will become important later in the proof of Theorem 9.2.

We now replace the sphere $\mathbb{S}^n$ by a sphere bundle with two antipodal sections over a finite CW-complex X. More specifically, starting from an n-dimensional Euclidean vector bundle $V \to X$ we glue two copies $\mathbb{D}_\pm V$ of its disk bundle $\mathbb{D}V \to X$ along the common boundary, the unit sphere bundle $\mathbb{S}V$, by the identity map. Thus we obtain an $\mathbb{S}^n$-bundle $\hat{V} = \mathbb{S}(\mathbb{R} \oplus V) \to X$,

$$\hat{V} = \mathbb{D}_+V \cup_{\mathbb{S}V} \mathbb{D}_-V\,. \tag{8.2}$$

We obtain two sections $s_\pm : X \to \hat{V}$ of the bundle $\hat{V} \to X$, which we regard as north and south poles, defined as the zero sections of $\mathbb{D}^\pm V$.

Let

$$\mathscr{E} \to \hat{V}$$

be a Euclidean vector bundle over the total space of $\hat{V} \to X$. Let

$$E_\pm = s_\pm^*\mathscr{E} \to X$$

which we sometimes tacitly pull back to bundles $E_\pm \to \mathbb{D}_\pm V$ along the canonical fiberwise projection maps $\mathbb{D}_\pm V \to s_\pm(X)$.

Since there are – up to homotopy unique – bundle isometries $\mathscr{E}|_{\mathbb{D}_\pm V} \cong E_\pm$ restricting to the identity over $\{s_\pm\}$, the bundle $\mathscr{E}$ is obtained from $E_\pm$ by a clutching map σ, that is a section of the bundle

$$\mathrm{Map}(\mathbb{S}V, \mathrm{O}(E_+, E_-)) \to X\,,$$

which is uniquely determined up to fiberwise homotopy.

Hence, for any $x \in X$ we have a map $\sigma_x : \mathbb{S}V_x \to \mathrm{O}((E_+)_x, (E_-)_x)$. Note that we may equivalently consider the clutching map as a bundle isometry $\sigma : E_+|_{\mathbb{S}V} \to E_-|_{\mathbb{S}V}$. In this situation we write

$$\mathscr{E} = (E_+, \sigma, E_-)\,.$$

Vice versa, if $E_\pm \to X$ are oriented Euclidean bundles and

$$\sigma \in \Gamma(\mathrm{Map}(\mathbb{S}V, \mathrm{O}(E_+, E_-)))$$

a clutching map, then we obtain a vector bundle $\mathscr{E} = (E_+, \sigma, E_-) \to \hat{V}$ by gluing the pull back bundles of $E_\pm \to \mathbb{D}_\pm V$ along $\mathbb{S}V$ by σ.

Example 8.3 A particular case is $E_+ = E_- =: F$ and $\sigma(v) = \mathrm{id}_{F_v}$ for all $v \in \mathbb{S}V$. We write (F, id, F) for this triple. Since F_v can be identified with $F_{s_\pm}$ for all $v \in \mathbb{D}^\pm V$, this bundle is trivial over every fibre $\hat{V}_x$, $x \in X$, hence it is isomorphic to a bundle over X, pulled back to $\hat{V}$ by the projection $\pi : \hat{V} \to X$.

Vice versa, for any vector bundle $F \to X$, the pull-back bundle $\pi^* F$ is given by the triple (F, id, F).

For a generalization of Example 8.1 to sphere bundles we recall the following definition.

Definition 8.4 Let $\mathrm{Cl}(V) \to X$ be the Clifford algebra bundle associated to the Euclidean bundle V with fibre $\mathrm{Cl}(V)_x = \mathrm{Cl}(V_x)$ over $x \in X$.

A real $\mathrm{Cl}(V)$-*Clifford module bundle* is a vector bundle $\Lambda \to X$ such that each fibre Λ_x, $x \in X$, is a real $\mathrm{Cl}(V_x)$ module. More precisely, we are given a bundle homomorphism $\mu : \mathrm{Cl}(V) \to \mathrm{End}(\Lambda)$ which restricts to an algebra homomorphism in each fibre.

We may and will assume that Λ is equipped with a Euclidean structure such that Clifford multiplication with elements in $\mathbb{S}V$ is orthogonal.

From now on we will in addition assume that $V \to X$ is oriented. Any oriented orthonormal frame[14] of $V|_U$ over $U \subset X$ induces an orientation preserving orthogonal trivialization

$$b : U \times \mathbb{R}^n \cong V|_U \,.$$

Thus $L := \Lambda_{x_o}$ with some fixed $x_o \in U$ becomes a Cl_n-module. Consider the bundle

$$\mathrm{Iso}_{\mathrm{Cl}_n}(\underline{L}, \Lambda|_U) \to U$$

whose fibre over $x \in U$ is the space of Cl_n-linear isomorphisms $\phi_x : L \to \Lambda_x$. If U is contractible it has a section ϕ which intertwines the Cl_n-module multiplications (denoted by $\cdot$) on L and Λ_x, $x \in U$: for all $\xi \in L$ and $v \in \mathbb{R}^n$ and $x \in U$ we have

[14] A frame (basis) $b = (b_1, \ldots, b_n)$ of V_x will be considered as a linear isomorphism $b : \mathbb{R}^n \to V_x$, $v \mapsto bv = \sum_j v_j b_j$.

$$\phi_x(v \cdot \xi) = b_x(v) \cdot \phi_x(\xi). \tag{8.3}$$

In other words: We obtain Euclidean trivializations

$$V|_U \cong U \times \mathbb{R}^n, \qquad \mathrm{Cl}(V)|_U \cong U \times \mathrm{Cl}_n, \quad \Lambda|_U \cong U \times L \tag{8.4}$$

with a Cl_n-module L such that the $\mathrm{Cl}(V)$-module structure on Λ corresponds to the Cl_n-multiplication on L. Notice that in particular the Cl_n-isomorphism type of L, the *typical fibre* of $\Lambda \to X$ is uniquely determined over each path component of X.

Let $n = 4m$. Since $V \to X$ is oriented, the volume element $\omega \in \mathrm{Cl}_n$ defines a section ("volume section") of $\mathrm{Cl}(V) \to X$, and the ± 1-eigenspaces define the positive and negative Clifford algebra bundles $\mathrm{Cl}^{\pm}(V) \to X$ and subbundles $\Lambda^{\pm} \to X$ of $\Lambda \to X$, which are invariant under $\mathrm{Cl}^{+}(V)$ and such that for every $v \in V_x$ the endomorphism $\mu(v)$ interchanges Λ^{+}_x and Λ^{-}_x. Correspondingly the typical fibre L decomposes as $L = L^{+} \oplus L^{-}$ such that the local trivializations (8.4) preserve positive and negative summands.

Definition 8.5 For $n = 4m$ the bundle $\mathscr{L} \to \hat{V}$ defined by the triple

$$\mathscr{L} = (\Lambda^{+}, \mu, \Lambda^{-}) \tag{8.5}$$

is called the *Hopf bundle* associated to the $\mathrm{Cl}(V)$-Clifford module bundle $\Lambda \to X$. Note that each $\mathscr{L}|_{\hat{V}_x} \to \hat{V}_x$, $x \in X$, is a Hopf bundle in the sense of Example 8.1 after passing to local trivializations (8.4).

Example 8.6 As a particular example we consider the $\mathrm{Cl}(V)$-module bundle

$$\Lambda = \mathrm{Cl}(V) \to X$$

with $\mathrm{Cl}(V)$-module structure μ given by (left) Clifford multiplication on $\mathrm{Cl}(V)$. Note that $\Lambda^{\pm} = \mathrm{Cl}^{\pm}(V) \to X$ where $\mathrm{Cl}^{+}(V)$ ($\mathrm{Cl}^{-}(V)$) is generated as a vector bundle by the Clifford products of even (odd) length.

We define the *Clifford-Hopf bundle* $\mathscr{C} \to \hat{V}$ as the triple

$$\mathscr{C} = (\mathrm{Cl}^{+}(V), \mu, \mathrm{Cl}^{-}(V)) \tag{8.6}$$

where $\mu : \mathbb{S}V \to \mathrm{SO}(\mathrm{Cl}^{-}(V), \mathrm{Cl}^{+}(V))$ is the (left) Clifford multiplication.

Example 8.7 Recall that a *spin structure* on $V \to X$ is given by a two fold cover

$$P_{\mathrm{Spin}}(V) \to P_{\mathrm{SO}}(V)$$

of the $\mathrm{SO}(n)$-principal bundle of oriented orthonormal frames in V, which is equivariant with respect to the double cover $\mathrm{Spin}_n \to \mathrm{SO}_n$.

Consulting Theorem 2.3 we see that for $n = 4m$ there is exactly one irreducible Cl_n-module $S = S_n$, and we obtain the *spinor bundle*

$$\Sigma = P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} S.$$

Since the Clifford algebra bundle can be written in the form

$$\mathrm{Cl}(V) = P_{\mathrm{Spin}}(V) \times_{\mathrm{Spin}_n} \mathrm{Cl}_n$$

where $\mathrm{Spin}_n \subset \mathrm{Cl}_n$ acts on Cl_n by conjugation, Σ becomes a $\mathrm{Cl}(V)$-module bundle by setting

$$[p, \alpha] \cdot [p, \sigma] = [p, \alpha\sigma]$$

for $p \in P_{\mathrm{Spin}}(V)$, $\alpha \in \mathrm{Cl}_n$ and $\sigma \in S$.

Definition 8.8 The *Spinor Hopf bundle over* $\hat{V}$ is the Euclidean vector bundle $\mathscr{S} \to \hat{V}$ defined by the triple

$$\mathscr{S} = (\Sigma^+, \mu, \Sigma^-) \tag{8.7}$$

where $\mu : \mathbb{S}V \to \mathrm{SO}(\Sigma^+, \Sigma^-)$ is the restriction of the Clifford module multiplication.

Definition 8.9 Let Λ be a $\mathrm{Cl}(V)$-module bundle with Clifford multiplication μ and a $\mathrm{Cl}(V)$-invariant orthogonal decomposition

$$\Lambda = \Lambda_0 \oplus \Lambda_1$$

into $\mathrm{Cl}(V)$-submodule bundles. This induces a triple

$$\left(\Lambda^+, \omega, \Lambda_0^+ \oplus \Lambda_1^-\right), \quad \omega = \mathrm{id}_{\Lambda_0^+} \oplus \mu|_{\Lambda_1^+} \tag{8.8}$$

equal to the direct sum bundle

$$(\Lambda_0^+, \mathrm{id}, \Lambda_0^+) \oplus (\Lambda_1^+, \mu, \Lambda_1^-) \to X\,.$$

We call $(\Lambda^+, \omega, \Lambda_0^+ \oplus \Lambda_1^-)$ the *affine Hopf bundle* associated to the decomposition $\Lambda = \Lambda_0 \oplus \Lambda_1 \to X$.

In the next section we will prove that *every* vector bundle over $\hat{V}$ is—after suitable stabilization—isomorphic to an affine Hopf bundle.

9 Vector bundles over sphere bundles and affine Hopf bundles

Let $V \to X$ be an oriented Euclidean vector bundle of rank $n = 4m$ with associated sphere bundle $\hat{V} = \mathbb{S}(\mathbb{R} \oplus V) \to X$.

Definition 9.1 (1) Two Euclidean vector bundles $E, \tilde{E}$ over X or $\hat{V}$ are called *stably isomorphic*, written

$$E \cong_s \tilde{E}\,,$$

if $E \oplus \underline{\mathbb{R}}^q \cong \tilde{E} \oplus \underline{\mathbb{R}}^q$ for some trivial vector bundle $\underline{\mathbb{R}}^q \to X$.

(2) Two $\mathrm{Cl}(V)$-module bundles $\Lambda, \tilde{\Lambda} \to X$ are called *stably isomorphic*, written $\Lambda \cong_s \tilde{\Lambda}$, if there is some $\mathrm{Cl}(V)$-linear isometric isomorphism

$$\Lambda \oplus \mathrm{Cl}(V)^q \cong \tilde{\Lambda} \oplus \mathrm{Cl}(V)^q$$

for some q.

The following result is central in our paper.

Theorem 9.2 *Let*

$$\mathscr{E} \to \hat{V}$$

be a Euclidean vector bundle. Then possibly after adding copies of $\mathscr{C}$ (the Clifford-Hopf bundle, see Example 8.6) and trivial vector bundles $\underline{\mathbb{R}}$ to $\mathscr{E}$ we have

$$\mathscr{E} \cong E \oplus \mathscr{L} \tag{9.1}$$

for a Euclidean vector bundle $E \to X$ (pulled back to $\hat{V}$) and a Hopf bundle $\mathscr{L} \to \hat{V}$ associated to some $\mathrm{Cl}(V)$-module bundle $\Lambda \to X$.

Let $E, \tilde{E} \to X$ be Euclidean vector bundles and $\mathscr{L}, \tilde{\mathscr{L}} \to \hat{V}$ Hopf bundles corresponding to $\mathrm{Cl}(V)$-module bundles $\Lambda, \tilde{\Lambda} \to X$. If

$$E \oplus \mathscr{L} \cong_s \tilde{E} \oplus \tilde{\mathscr{L}} \tag{9.2}$$

then $E \cong_s \tilde{E}$ and $\Lambda \cong_s \tilde{\Lambda}$.

Remark 9.3 When V carries a spin structure, we will see in Proposition 10.4 (which is self consistent) that $\Lambda \cong E \otimes \Sigma$ when m is even, and $\Lambda \cong E \otimes_{\mathbb{H}} \Sigma$ when m is odd, where E is some Euclidean vector bundle over X and Σ denotes the Spinor bundle associated to $P_{\mathrm{Spin}}(V)$, cf. Example 8.7. Therefore

$$\mathscr{L} \cong \begin{cases} E \otimes \mathscr{S} & \text{when } m \text{ is even} \\ E \otimes_{\mathbb{H}} \mathscr{S} & \text{when } m \text{ is odd} \end{cases}$$

where $\mathscr{S}$ denotes the spinor-Hopf bundle (cf. Definition 8.8) and E a vector bundle over X, pulled back to $\hat{V}$. Further, the stable isomorphism type of Λ determines the stable isomorphism type of E, see Lemma 10.5.

The proof of Theorem 9.2 will cover the remainder of this section. We represent $\mathscr{E}$ by a triple (E_+, σ, E_-) for Euclidean vector bundles $E_\pm \to X$, which we consider as bundles over $\mathbb{D}_\pm V$ in the usual way.

Remark 9.4 Let $\mathscr{E} = (E_+, \sigma, E_-)$ and $\tilde{\mathscr{E}} = (\tilde{E}_+, \tilde{\sigma}, \tilde{E}_-)$ be triples. Then an orthogonal vector bundle homomorphism

$$\phi : \mathscr{E} \to \tilde{\mathscr{E}}$$

consists of a pair (ϕ_+, ϕ_-) of orthogonal bundle homomorphisms $\phi_\pm : E_\pm \to \tilde{E}_\pm$ over $\mathbb{D}V$ such that the following diagram over $\mathbb{S}V$ commutes:

$$\begin{array}{ccc} E_+|_{\mathbb{S}V} & \xrightarrow{\phi_+} & \tilde{E}_+|_{\mathbb{S}V} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tilde{\sigma}} \\ E_-|_{\mathbb{S}V} & \xrightarrow{\phi_-} & \tilde{E}_-|_{\mathbb{S}V} \end{array} \tag{9.3}$$

If $\phi_+^t : E_+ \to \tilde{E}_+, t \geq 0$, is a homotopy of bundle isomorphisms over $\mathbb{D}V$ with $\phi_+ = \phi_+^0$, we may replace ϕ by $\phi^t = (\phi_+^t, \phi_-)$, replacing the clutching map $\tilde{\sigma} = \phi_- \sigma \phi_+^{-1}$ over $\mathbb{S}V$ by the homotopic clutching map $\tilde{\sigma}^t = \tilde{\sigma}\phi_+(\phi_+^t)^{-1}$ which does not change $\tilde{\mathscr{E}}$.

In the proof of Theorem 9.2 we can assume without loss of generality that X is path connected. Let

$$V|_U \cong U \times \mathbb{R}^n\,, \quad E_+|_U \cong U \times \mathbb{R}^p$$

be orthogonal trivializations over a connected open subset $U \subset X$, where the first trivialization is assumed to be orientation preserving. (We are not assuming that the the bundle $E_+ \to X$ is orientable). We obtain an induced trivialization

$$E_-|_U \cong U \times \mathbb{R}^p$$

using the isomorphisms $\sigma_x(e_1) : (E_+)_x \to (E_-)_x$ for $x \in U$ where $e_1 \in \mathbb{R}^n$ is the first standard basis vector. With respect to theses trivializations $\sigma|_U$ is given by a map

$$\sigma|_U : U \to \mathrm{Map}_*(\mathbb{S}^{n-1}, \mathrm{SO}_p)\,. \tag{9.4}$$

Here we recall that $\mathbb{S}^{n-1}$ is connected since $n \geq 4$ by assumption so that the local clutching function has indeed values in SO_p rather than O_p. The stable winding number $\eta^s(\sigma_x) \in \mathbb{Z}$ (see Definition 7.2) is independent of the chosen trivializations and constant over U. It is hence an invariant of the triple $\mathscr{E} = (E_+, \sigma, E_-)$.

Set $d = \dim X$. After adding trivial bundles and Clifford-Hopf bundles to $\mathscr{E}$ we can assume that the stable mapping degree $\eta = \eta^s(\sigma)$ satisfies the conditions

$$s_k \cdot d \leq \eta \leq p/4 \tag{9.5}$$

from Theorem 7.10 (with $k := n - 1$, recall $s_k = s_\ell$ for $\ell = k - 1$), compare Remarks 7.4 and 8.2. Furthermore, we can assume that p is divisible by s_k. In particular, by Theorem 7.10, with these choices of p and η the canonical inclusion

$$\mathrm{Hopf}_\rho(\mathbb{S}^k, \mathrm{SO}_p)_\eta \subset \mathrm{Map}_*(\mathbb{S}^k, \mathrm{SO}_p)_\eta$$

is d-connected for any positive representation $\rho : \mathrm{Cl}_k \to \mathrm{End}(\mathbb{R}^p)$ (which is equivalent to the (p/s_k)-fold direct sum of the positive S_k). We will work under these assumptions from now on.

Next we note that we may add on both sides of (9.2) a bundle $F \to X$, pulled back via $\pi : \hat{V} \to X$ (which further increases p, but not the winding number η, so that the assumptions of Theorem 7.10 are preserved), that is we add $\pi^* F = (F, \mathrm{id}, F)$. Then the original statement is obtained by embedding F into a trivial bundle over X and adding a complement $F^\perp$ on both sides of (9.2).

We use this freedom to put E_+ into a special form. Since E_+ embeds into a trivial bundle $\underline{\mathbb{R}}^N \to X$ and each bundle over X of rank larger than $\dim(X)$ splits off a trivial line bundle (its Euler class vanishes), E_+ embeds into any sufficiently large vector bundle over X, in particular into Λ_+ for some $\mathrm{Cl}(V)$-module bundle $\Lambda \to X$ (e.g. $\Lambda = \mathbb{R}^q \otimes \mathrm{Cl}(V) = \mathrm{Cl}(V)^q$). Then E_+ is a direct summand of Λ^+.

After adding (F, id, F) to (E_+, σ, E_-) with $F = (E_+)^\perp \subset \Lambda^+$ we may hence assume that

$$E_+ = \Lambda^+ .$$

Note that this bundle is oriented. After these preparations Theorem 9.2 is proven by induction over a cell decomposition of X. The decomposition (9.2) results from the following fact. xxx

Proposition 9.5 *There is an orthogonal decomposition of* Λ *into* $\mathrm{Cl}(V)$*-invariant subbundles*

$$\Lambda = \Lambda_0 \oplus \Lambda \tag{9.6}$$

with the following property: the triple $\mathscr{E} = (E_+, \sigma, E_-)$ *is isomorphic to the triple*

$$\left(\Lambda_0^+ \oplus \Lambda_1^+, \mathrm{id} \oplus \mu, \Lambda_0^+ \oplus \Lambda_1^-\right) = (\Lambda_0^+, \mathrm{id}, \Lambda_0^+) \oplus (\Lambda_1^+, \mu, \Lambda_1^-) \tag{9.7}$$

where $\mu : \mathbb{S}V \times \Lambda_1^+ \to \Lambda_1^-$ *is the* $\mathrm{Cl}(V)$*-module multiplication on* Λ_1.

Furthermore this isomorphism can be chosen as the identity on the first bundle $E_+ = \Lambda^+$. *Hence, by (9.3), it is given by an isomorphism of vector bundles over the total space* $\mathbb{D}V$,

$$f = f_- : E_- \xrightarrow{\cong} \Lambda_0^+ \oplus \Lambda_1^-$$

such that over $\mathbb{S}V$ *we have*

$$f \circ \sigma = \omega := \mathrm{id} \oplus \mu .$$

Proof In the induction step let $X = X' \cup D$ be obtained by attaching a cell D to X' and assume that the assertion holds for the restriction to X' of the bundle

$$\mathscr{E} = (E_+, \sigma, E_-)$$

We denote by $V' \to X'$ the restriction of V. Hence we have the decomposition (9.6) over X' and an isomorphism

$$f : E_-|_{\mathbb{D}V'} \xrightarrow{\cong} (\Lambda_0^+ \oplus \Lambda_1^-)|_{\mathbb{D}V'} \tag{9.8}$$

such that over $\mathbb{S}V'$,

$$f \circ \sigma = \omega = \mathrm{id} \oplus \mu \in \mathrm{SO}(\Lambda_0^+ \oplus \Lambda_1^+, \Lambda_0^+ \oplus \Lambda_1^-) \tag{9.9}$$

where $\mu(v) \in \mathrm{SO}(\Lambda_1^+, \Lambda_1^-)$ is the $\mathrm{Cl}(V)$-module multiplication ($v \in \mathbb{S}V'$).

We need to extend f to a similar isomorphism over $X = X' \cup D$. In particular we need to extend the bundle decomposition $\Lambda = \Lambda_0 \oplus \Lambda_1$ from X' to X. This will ultimately be achieved by applying Theorem 7.10 to the restriction of $\mathscr{E} \to \hat{V}$ to $\hat{V}|_D \cong D \times \mathbb{S}^n$. In fact, trivializing $E_\pm$ over $D \subset X$, the clutching map σ of $\mathscr{E}$ will become a map $\hat{\sigma} : D \times \mathbb{S}^{n-1} \to \mathrm{SO}(L^+)$ where $L^+ = \mathbb{R}^p$ is the standard fibre of E_+. However, in order to apply Theorem 7.10, this map $\hat{\sigma} : D \to \mathrm{Map}(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))$ needs

(i) taking values in $\mathrm{Map}_*(\mathbb{S}^{n-1}, \mathrm{SO}(L^+)) = \{\phi : \mathbb{S}^{n-1} \to \mathrm{SO}(L^+) : \phi(e_1) = I\}$ (see Def. 6.1),

(ii) $\hat{\sigma}(\partial D) \subset \mathrm{Hopf}_{\rho_+}(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))_\eta$ with $\rho_+ : \mathrm{Cl}_k \to \mathrm{End}(L_+)$ as in (8.1).

Requirement (ii) will be met (using the clutching maps σ, ω as trivializations) by transforming f to $\hat{f}$, see (9.13), which will change (9.9) to (9.15). From (9.15) we will see using Remark 8.2 that the clutching map $\hat{f}\hat{\sigma} : \partial D \to \mathrm{Map}_*(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))$ takes values in $\mathrm{Hopf}_{\rho_+}(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))$. In order to meet the requirement (i) we need an extension $\hat{F}$ of $\hat{f}$ from ∂D to all of D such that $\hat{F}(x, e_1) = \mathrm{id}_{L_+}$ for all $x \in D$, see the "Assertion" below. Then we can apply our deformation theorem 7.10 to the clutching map $\tau = \hat{F}\hat{\sigma}$, thus obtaining a new clutching map τ_1 with values in $\mathrm{Hopf}_\rho(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))_\eta$. It fits together with the given clutching map along X' since there was no change along $\partial D = X' \cap D$.

Now we explain these steps in detail. Choose a trivialization of $V|_D$, that is an oriented orthonormal frame over D. It induces trivializations of $\mathbb{D}V$ and $\mathrm{Cl}(V)$. Also we choose a compatible trivialization of Λ, compare Example 8.4:

$$\mathbb{D}V|_D \cong D \times \mathbb{D}^n, \qquad \mathrm{Cl}(V)|_D \cong (\underline{\mathrm{Cl}}_n)_D, \quad \Lambda|_D \cong (\underline{L})_D \tag{9.10}$$

for some Cl_n-module $L = L^+ \oplus L^-$ (recall $n = 4m$) such that for any $x \in D$ the $\mathrm{Cl}(V_x)$-module multiplication $V_x \times \Lambda_x \to \Lambda_x$ is transferred to the Cl_n-module multiplication $\mathbb{R}^n \times L \to L$.

Identifying $\mathbb{S}V|_D$ with $D \times \mathbb{S}^{n-1}$, we choose $e_1 \in \mathbb{S}^n$ as a base point and put

$$\sigma_1(x) := \sigma(x, e_1), \quad \omega_1(x) := \omega(x, e_1)\,. \tag{9.11}$$

We use these bundle isomorphisms as trivializations ϵ and ϑ of the bundles $E_-|_D$ and $(\Lambda_0^+ \oplus \Lambda_1^-)|_{\partial D}$ (which below will be pulled back to $\mathbb{D}V|_D$ and $\mathbb{D}V|_{\partial D}$, respectively):

$$\begin{aligned} \epsilon \ &: \quad E_- \quad \xrightarrow{\sigma_1^{-1}} \quad E_+ \quad &= \Lambda^+ \cong \underline{L}^+ \ \text{ over } D, \\ \vartheta \ &: \Lambda_0^+ \oplus \Lambda_1^- \xrightarrow{\omega_1^{-1}} \Lambda_0^+ \oplus \Lambda_1^+ &= \Lambda^+ \cong \underline{L}^+ \text{ over } \partial D. \end{aligned} \tag{9.12}$$

Thus we can express the isomorphism f in (9.8), restricted to $\mathbb{D}V|_{\partial D}$, as a map

$$\hat{f} = \vartheta \circ f \circ \epsilon^{-1} : \partial D \times \mathbb{D}^n \to \mathrm{SO}(L^+). \tag{9.13}$$

At the base point $e_1 \in \mathbb{S}^{n-1}$ we obtain for all $x \in \partial D$:

$$\begin{aligned} \hat{f}(x, e_1) &= \vartheta(f(\sigma_1(x))) \\ &\overset{(9.9)}{=} \vartheta(\omega_1(x)) \\ &\overset{(9.12)}{=} \omega_1(x)^{-1} \circ \omega_1(x) \\ &= \mathrm{id}_{L^+}. \end{aligned} \tag{9.14}$$

Further, (9.9) is transformed over $\mathbb{S}V|_{\partial D} = \partial D \times \mathbb{S}^{n-1}$ into

$$\hat{f}\hat{\sigma} = \hat{\omega} : \partial D \times \mathbb{S}^{n-1} \to \mathrm{SO}(L^+) \tag{9.15}$$

where $\hat{\sigma} = \epsilon \circ \sigma : D \times \mathbb{S}^{n-1} \to \mathrm{SO}(L^+)$ and $\hat{\omega} = \vartheta \circ \omega : \partial D \times \mathbb{S}^{n-1} \to SO(L^+)$ are evaluated as follows:

$$\begin{aligned} \hat{\sigma}(x, v) &= \sigma(x, e_1)^{-1}\sigma(x, v) \ \text{ for all } \ x \in D, \\ \hat{\omega}(x, v) &= \omega(x, e_1)^{-1}\omega(x, v) \ \text{ for all } x \in \partial D. \end{aligned} \tag{9.16}$$

Now we will extend $\hat{f}$ from $\partial D \times \mathbb{D}^n$ to $D \times \mathbb{D}^n$. The extended map

$$\hat{F} : D \times \mathbb{D}^n \to \mathrm{SO}(L^+)$$

restricted to $D \times \mathbb{S}^{n-1}$, will be used to define another clutching map

$$\tau := \hat{F}\hat{\sigma} : D \times \mathbb{S}^{n-1} \to \mathrm{SO}(L^+) \ \text{ with } \tau = \hat{f}\hat{\sigma} = \hat{\omega} \text{ on } \partial D \times \mathbb{S}^{n-1}. \tag{9.17}$$

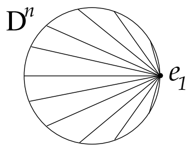
□

Assertion The map $\hat{f} : \partial D \times \mathbb{D}^n \to \mathrm{SO}(L^+)$ in (9.13) extends to a map

$$\hat{F} : D \times \mathbb{D}^n \to \mathrm{SO}(L^+)$$

such that $\hat{F}(x, e_1) = \mathrm{id}_{L^+}$ for all $x \in D$.

Proof of assertion By (9.14), $\hat{f}$ is equal to the identity on the fibre over $\partial D \times \{e_1\}$, where as usual $e_1 \in \mathbb{S}^{n-1} \subset \mathbb{D}^n$. Notice that $\{e_1\} \subset \mathbb{D}^n$ is a strong deformation retract. Let $r_t : \mathbb{D}^n \to \mathbb{D}^n$, $t \in [0,1]$, be a deformation retraction with $r_1 = \mathrm{id}_{\mathbb{D}^n}$ and $r_0 \equiv e_1$:



This gives a homotopy $\hat{f}_t : \mathrm{id}_{L^+} \simeq \hat{f}$,

$$\hat{f}_t = \hat{f} \circ (\mathrm{id}_{\partial D}, r_t) : \partial D \times \mathbb{D}^n \to \mathrm{SO}(L^+)$$

with $\hat{f}_t(\tilde{x}, e_1) = \mathrm{id}_{L^+}$ for all $\tilde{x} \in \partial D$ and $t \in [0,1]$. Considering D as the cone over ∂D, this can be viewed as a map

$$\begin{aligned} \hat{F} : D \times \mathbb{D}^n &\to \mathrm{SO}(L^+)\,, \\ \hat{F}(x, v) &= \hat{f}_{|x|}(\tilde{x}, v) = \hat{f}(\tilde{x}, r_{|x|}(v)) \end{aligned}$$

with $\tilde{x} = x/|x|$ when $x \neq 0$. Moreover, $F(0, v) = \mathrm{id}_{L^+}$ for all $v \in \mathbb{D}^n$ since $r_0 \equiv e_1$ and $\hat{f}(\cdot, e_1) \equiv \mathrm{id}_{L^+}$. This map restricts to $\hat{f}$ over $\partial D \times \mathbb{D}^n$ and it is constant $= \mathrm{id}_{L^+}$ along the fibers over $D \times [0,1]e_1 \subset D \times \mathbb{D}^n$, thus finishing the proof of the assertion. □

The new clutching map $\tau = \hat{F}\hat{\sigma} : D \to \mathrm{Map}(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))$ satisfies

$$\tau(\partial D) \subset \mathrm{Hopf}_{\rho_+}(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))_\eta$$

by (9.17), (9.16), (9.9) (cf. definitions 7.9 and 8.1), where η is the stable winding number of the original clutching map σ in the sense of (9.4).

Since $p \geq p_0(d)$ where $d = \dim X$ and $s_\ell \cdot d \leq \eta \leq p/4$, we can apply Theorem 7.10 to obtain a deformation $(\tau_t)_{t\in[0,1]}$ of $\tau = \tau_0$ such that $\tau_t = \tau$ on ∂D and

$$\tau_1(D) \in \mathrm{Hopf}_{\rho_+}(\mathbb{S}^{n-1}, \mathrm{SO}(L^+))_\eta.$$

Thus the trivial bundle $\underline{L}^+$ decomposes over D into orthogonal Cl_{n-1}-invariant subbundles,

$$(\underline{L}^+)_D = L_0^+ \oplus L_1^+ \;\text{ and }\; \tau_1 = \mathrm{id} \oplus \hat{\mu}, , \tag{9.18}$$

where $\hat{\mu} \overset{(9.16)}{=} \mu(e_1)^{-1}\mu = \rho_+|_{\mathbb{S}^{n-1}}$ is the Hopf map induced by the Cl_{n-1}-module structure on L_1^+ (compare Remark 8.2).

We have to connect these clutching data over D to the given ones over X'. In (9.10) we have chosen a trivialization $\Lambda^+|_D \xrightarrow{\cong} (\underline{L}^+)_D$ which over ∂D transforms the decomposition $\Lambda^+ = \Lambda_0^+ \oplus \Lambda_1^+$ into a bundle decomposition $(\underline{L}^+)_{\partial D} = (L_0^+ \oplus$

$L_1^+)|_{\partial D}$. Now we extend the decomposition (9.6) of $\Lambda^+|_{\partial D}$ to all of D by applying the inverse of this trivialization to (9.18) obtaining a decomposition $\Lambda^+ = \Lambda_0^+ \oplus \Lambda_1^+$ on all of X. Similarly, we obtain subbundles $\Lambda_0^- = \mu(V)\Lambda_0^+$ and $\Lambda_1^- = \mu(V)\Lambda_1^+$ decomposing Λ^-.

Now we can define a bundle isomorphism

$$F : E_- \to \Lambda_0^+ \oplus \Lambda_1^-$$

over $\mathbb{S}V$, with $F = f$ over $\mathbb{S}V|_{X'}$ and

$$F : E_- \xrightarrow{\sigma^{-1}} E_+ = \Lambda^+ \xrightarrow{\mathrm{id}\oplus\mu} \Lambda_0^+ \oplus \Lambda_1^- \quad \text{over } \mathbb{S}V|_D \tag{9.19}$$

This is compatible over $\partial D = X' \cap D$ because over $\mathbb{S}V|_{\partial D}$ we have $f = \omega \circ \sigma^{-1}$ and $\mathrm{id} \oplus \mu = \omega$. It defines an isomorphism

$$\phi = (\mathrm{id}_{E_+}, F) : (E_+, \sigma, E_-) \to (\Lambda^+, \omega, \Lambda_0^+ \oplus \Lambda_1^-)\,.$$

In fact, over X' this is true by assumption, and over D we use (9.19) for the commutativity of the diagram

$$\begin{tikzcd} E_+ \arrow[r, "="] \arrow[d, "\sigma"'] & \Lambda^+ \arrow[d, "\mathrm{id}\oplus\mu"] \\ E_- \arrow[r, "F"] & \Lambda_0^+ \oplus \Lambda_1^- \end{tikzcd}$$

Hence the isomorphism f from Equation (9.8) extends to an isomorphism ϕ over X between the given bundle $\mathcal{E}$ and a bundle of the form (9.7). This finishes the proof of Proposition 9.5 and the existence part of Theorem 9.2.

For the uniqueness statement of Theorem 9.2 we need the following lemma.

Lemma 9.6 *Let $\mathscr{E} = (E_+, \sigma, E_-)$ and $\tilde{\mathscr{E}} = (\tilde{E}_+, \tilde{\sigma}, \tilde{E}_-)$ be triples with*

$$E_+ = \tilde{E}_+ =: F$$

and

$$\phi = (\phi_+, \phi_-) : \mathscr{E} \to \tilde{\mathscr{E}}$$

an orthogonal bundle isomorphism (see Remark 9.4). Let $\mathscr{F} = (F, \mathrm{id}, F) = \pi^*(F)$. Then there exists an orthogonal isomorphism

$$\tilde{\phi} : \mathscr{E} \oplus \mathscr{F} \to \tilde{\mathscr{E}} \oplus \mathscr{F}$$

such that $\tilde{\phi}_+ = \mathrm{id}$.

Proof We consider the isomorphism $\hat{\phi}' = (A, B) : \mathscr{E} \oplus \mathscr{F} \to \mathscr{F} \oplus \tilde{\mathscr{E}}$ given by the following commutative diagram:

$$\begin{tikzcd} E_+ \oplus F \arrow[r, "A"] \arrow[d, "\sigma\oplus\mathrm{id}"'] & F \oplus \tilde{E}_+ \arrow[d, "\mathrm{id}\oplus\tilde{\sigma}"] \\ E_- \oplus F \arrow[r, "B"] & F \oplus \tilde{E}_- \end{tikzcd}$$

with $A = \begin{pmatrix} & -\phi_+^{-1} \\ \phi_+ & \end{pmatrix}$ and $B = \begin{pmatrix} & -\phi_+^{-1} \\ \phi_- & \end{pmatrix}$. Note that A is a complex structure on $F \oplus F$, that is $A^2 = -\mathrm{id}$. This can be deformed into id using the homotopy $A_t = (\cos t)\,\mathrm{id} + (\sin t)A$ with $A_0 = \mathrm{id}$ and $A_{\pi/2} = A$. By Remark 9.4, $\hat{\phi}'$ can be deformed onto an isomorphism $\tilde{\phi}' : \mathscr{E} \oplus \mathscr{F} \to \mathscr{F} \oplus \tilde{\mathscr{E}}$ with $\tilde{\phi}'_+ = \mathrm{id}$.

However, we have yet to interchange the two summands of $\mathscr{F} \oplus \tilde{\mathscr{E}}$. Hence we apply the same construction once more, replacing $\phi : \mathscr{E} \to \tilde{\mathscr{E}}$ by $\phi'' = \mathrm{id} : \tilde{\mathscr{E}} \to \tilde{\mathscr{E}}$. This gives us a bundle isomorphism $\hat{\phi}'' = (A'', B'') : \tilde{\mathscr{E}} \oplus \mathscr{F} \to \mathscr{F} \oplus \tilde{\mathscr{E}}$ such that A'' can be deformed into id, and by Remark 9.4 again $\hat{\phi}''$ can be deformed into an isomorphism $\tilde{\phi}''$ of these bundles with $(\tilde{\phi}'')_+ = \mathrm{id}$. Then the composition $\tilde{\phi} := (\tilde{\phi}'')^{-1}\tilde{\phi}' : \mathscr{E} \oplus \mathscr{F} \to \tilde{\mathscr{E}} \oplus \mathscr{F}$ satisfies $\tilde{\phi}_+ = \mathrm{id}$. □

Now, for the uniqueness part of Theorem 9.2, assume that $E, \tilde{E} \to X$ are Euclidean vector bundles and $\mathscr{L}, \tilde{\mathscr{L}} \to \hat{V}$ the Hopf bundles corresponding to $\mathrm{Cl}(V)$-module bundles $\Lambda, \tilde{\Lambda} \to X$. Furthermore, for $\mathscr{E} := E \oplus \mathscr{L}$ and $\tilde{\mathscr{E}} := \tilde{E} \oplus \tilde{\mathscr{L}}$ (after pulling back E and $\tilde{E}$ to $\hat{V}$) we assume

$$\mathscr{E} \cong \tilde{\mathscr{E}}\,. \tag{9.20}$$

Put $E_\pm = E0 \oplus \Lambda 1^\pm$ and $\tilde{E}_\pm = \tilde{E} \oplus \tilde{\Lambda}1^\pm$. We claim that after adding certain bundles $F, \tilde{F} \to X$ to $E_+, \tilde{E}_+$ with $F \cong_s \tilde{F}$ we may in addition assume that

$$E_+ = \tilde{\Lambda}^+ = \tilde{E}_+ \tag{9.21}$$

for some $\mathrm{Cl}(V)$-module bundle $\hat{\Lambda} \to X$. For proving this claim choose a $\mathrm{Cl}(V)$-module bundle $\Lambda_0 \to X$ such that $E, \tilde{E}$ embed into Λ_0^+ and put $\Lambda = \Lambda \oplus \tilde{\Lambda} \oplus \Lambda$. Put $F = \tilde{\Lambda}^+ \oplus (\Lambda_0^+ \ominus E)$ and $\tilde{F} = \Lambda^+ \oplus (\Lambda_0^+ \ominus \tilde{E})$. Then

$$F \oplus \Lambda^+ \oplus E = \hat{\Lambda}^+ = \tilde{F} \oplus \tilde{\Lambda}^+ \oplus \tilde{E}, \tag{9.22}$$

but $\Lambda 1^+ \oplus E = E_+$ and $\tilde{\Lambda}^+ \oplus \tilde{E} = \tilde{E}_+$ are isomorphic by (9.20) and thus $F \cong_s \tilde{F}$ by (9.22), such that (9.21) can be assumed.

After this preparation we may hence assume that

$$\begin{aligned} \mathscr{E} &= (\Lambda^+ = E \oplus \Lambda_1^+, \omega, E \oplus \Lambda_1^-)\,, \\ \tilde{\mathscr{E}} &= (\Lambda^+ = \tilde{E} \oplus \tilde{\Lambda}_1^+, \tilde{\omega}, \tilde{E} \oplus \tilde{\Lambda}_1^-)\,, \end{aligned}$$

and

$$\phi = (\phi_+, \phi_-) : \mathscr{E} \xrightarrow{\cong} \tilde{\mathscr{E}}$$

is a vector bundle isomorphism, which we may assume to be orthogonal.

Due to (9.21) Lemma 9.6 implies that we can furthermore assume $\phi_+ = \mathrm{id}$, at least after adding some trivial bundle $\underline{\mathbb{R}}^q$ to $\mathscr{E}$ and $\tilde{\mathscr{E}}$ (chosen such that the bundle F in Lemma 9.6 embeds as a subbundle of $\underline{\mathbb{R}}^q$). In particular this implies

$$\tilde{\omega} = \phi_- \circ \omega : \mathbb{S}V \to \mathrm{SO}(\hat{\Lambda}^+, \tilde{E} \oplus \tilde{\Lambda}^-) .$$

We have to show that

$$E \cong_s \tilde{E}\,, \quad \Lambda \cong_s \tilde{\Lambda}\,.$$

But the given data induce a triple $\hat{\mathscr{E}} = (\hat{\Lambda}^+ \times [0,1], \hat{\sigma}, \hat{E}_-)$ over $\hat{V} \times [0,1]$ (the total space of $\hat{V} \times [0,1] \to X \times [0,1]$) by pulling back the triples $\mathscr{E}$ and $\tilde{\mathscr{E}}$ over $\hat{V} \times [0, 1/2]$, respectively $\hat{V} \times [1/2, 1]$ and gluing them along $\hat{V} \times \{1/2\}$ by means of the isomorphism (id, ϕ_-). More precisely, this is done as follows. Recall

$$E_- = E \oplus \Lambda^-\,, \quad \tilde{E}_- = \tilde{E} \oplus \tilde{\Lambda}^-\,.$$

We have the isomorphism $\phi_- : E_- \to \tilde{E}_-$ over $\mathbb{D}_-V$ with $\phi \circ \omega = \tilde{\omega}$ over $\mathbb{S}V$. Let s_- be the 0-section of $\mathbb{D}_-V$ and $\phi_o = \phi_-|_{s_-}$, which is an isomorphism between E_- and $\tilde{E}_-$ over X. Note that ϕ_o and ϕ_- are homotopic.

We define a bundle $\hat{E}_- \to X \times [0,1]$ as follows. We put

$$\hat{E}_- = \big(E_- \times [0, 1/2]\big) \cup_{\phi_o} \big(\tilde{E}_- \times [1/2, 1]\big)$$

using the isomorphism ϕ_o to identify $E_- \times \{\frac{1}{2}\}$ with $\tilde{E}_- \times \{\frac{1}{2}\}$. This defines a vector bundle $\hat{E}_- \to X \times [0,1]$. Further, we define a clutching map

$$\hat{\sigma} : \tilde{\Lambda}^+ \times [0,1] \to \hat{E}_-$$

over $\mathbb{S}V \times [0,1]$ as follows.

$$\hat{\sigma}(v,t) = \begin{cases} \mu(v) & \text{for } \ 0 \le t \le 1/2, \\ \phi_t \circ \mu(v) & \text{for } 1/2 \le t \le 3/4, \\ \tilde{\mu}(v) & \text{for } \ 3/4 \le t \le 1, \end{cases}$$

where $\phi_t : E_- \to \tilde{E}_-$ is a homotopy of bundle isomorphisms over $\mathbb{D}_-V$ for $1/2 \le t \le 3/4$ with $\phi_{1/2} = \phi_o$ and $\phi_{3/4} = \phi_-$.

Then $\hat{\sigma}$ is well defined at $t = 1/2$ since for any $\xi \in \hat{\Lambda}_x$ and $v \in \mathbb{S}V_x$, the element $\mu(v)\xi \in (E_-)_x$ is identified with $\phi_o\mu(v)\xi \in (\tilde{E}_-)_x$. It is also well defined at $t = 3/4$ because $\phi \circ \mu = \tilde{\mu}$, see (9.3) with $\phi_+ = \mathrm{id}$.

Thus we obtain a triple

$$\hat{\tilde{\mathscr{E}}} = (\hat{\Lambda}^+ \times [0,1], \hat{\sigma}, \hat{E}_-)$$

over $\hat{V} \times [0,1]$ which restricts to $\mathscr{E}$ on $\hat{V} \times \{0\}$ and to $\tilde{\mathscr{E}}$ on $\hat{V} \times \{1\}$, as required.

We now apply the previous deformation process in the proof of Equation (9.7) in Proposition 9.5 on $X \times [0,1]$, but relative to $X \times \{0,1\}$, i.e. we start the induction at $X' = X \times \{0,1\}$. Now we have to assume $d = \dim X + 1$ in Equation (9.5). This implies that (stably) the vector bundle $E \sqcup \tilde{E} \to X \times \{0,1\}$ (called Λ_+^0 in Prop. 9.5) extends over $X \times [0,1]$, and the analogous holds for the $\mathrm{Cl}(V)$-module bundle $\Lambda \sqcup \tilde{\Lambda} \to X \times \{0,1\}$ (called Λ_1 in Prop. 9.5).

Hence we obtain stable isomorphisms $E \cong_s \tilde{E}$ and $\Lambda \cong_s \tilde{\Lambda}$, finishing the proof of the uniqueness statement in Theorem 9.2.

10 Thom isomorphism theorems

Theorem 9.2 can be reformulated concisely in the language of topological K-theory. Let X be a finite CW-complex and $V \to X$ an oriented Euclidean vector bundle of rank $n = 4m \geq 0$ with associated Clifford algebra bundle $\mathrm{Cl}(V) \to X$.

Definition 10.1 We denote by $\mathrm{K}^{\mathrm{Cl}(V)}(X)$ the topological $\mathrm{Cl}(V)$-linear K-theory of X. More precisely, elements in $\mathrm{K}^{\mathrm{Cl}(V)}(X)$ are represented by formal differences of isomorphism classes of $\mathrm{Cl}(V)$-module bundles (cf. Example 8.4), with addition induced by the direct sum and neutral element the trivial $\mathrm{Cl}(V)$-module bundle with fibre 0.

Each $\mathrm{Cl}(V)$-module bundle $\Lambda \to X$ is isomorphic to a $\mathrm{Cl}(V)$-submodule bundle of $\mathrm{Cl}(V)^q \to X$ for some q. This holds in the special case $X = \{\text{point}\}$, since each Cl_n-module is a direct sum of irreducible Cl_n-modules, each of which also occurs as a summand in the Cl_n-module Cl_n. [15]

For more general X we use a partition of unity subordinate to a finite cover of X by open trivializing subsets for the given $\mathrm{Cl}(V)$-module bundle, similar as for ordinary vector bundles.

In particular, two $\mathrm{Cl}(V)$-module bundles $\Lambda, \tilde{\Lambda}$ represent the same element in $\mathrm{K}^{\mathrm{Cl}(V)}(X)$, if and only if they are stably isomorphic as $\mathrm{Cl}(V)$-module bundles in the sense of Definition 9.1(2).

Let $\mathrm{K}^{\mathrm{O}}(X)$ denote the orthogonal topological K-theory of X. If X is equipped with a base point x_0 recall the definition of the reduced orthogonal K-theory

$$\tilde{\mathrm{K}}^{\mathrm{O}}(X) = \ker(\mathrm{K}^{\mathrm{O}}(X) \xrightarrow{\text{restr.}} \mathrm{K}^{\mathrm{O}}(x_0)) \subset \mathrm{K}^{\mathrm{O}}(X)\,.$$

Let $X^V = \hat{V}/s_+$ be the Thom space associated to $V \to X$ with base point $[s_+]$ (recall that $s_+ \subset \hat{V}$ denotes the zero section of $\mathbb{D}_+ V$). The fibrewise projection $\pi : \hat{V} \to s_+$ induces a group homomorphism

[15] Recall that by Theorem 2.2 there is, up to isomorphism, just one irreducible Cl_n-module for $n = 4m$.

$$\chi : \mathrm{K}^{\mathrm{O}}(\hat{V}) \to \tilde{\mathrm{K}}^{\mathrm{O}}(X^V)\,, \quad [\mathscr{E}] \mapsto [\mathscr{E}] - \pi^*[\mathscr{E}|_{s_+}]\,. \tag{10.1}$$

Definition 10.2 The group homomorphism

$$\Psi_V : \mathrm{K}^{\mathrm{Cl}(V)}(X) \to \tilde{\mathrm{K}}^{\mathrm{O}}(X^V)\,, \quad [\Lambda] \mapsto \chi([\mathscr{L}])\,,$$

where $\mathscr{L} = (\Lambda^+, \mu, \Lambda^-) \to \hat{V}$ is the Hopf bundle associated to Λ (see Example 8.4), is called the *Clifford–Thom homomorphism*.

In this language Theorem 9.2 translates to the following result.

Theorem 10.3 *The Clifford–Thom homomorphism Ψ_V is an isomorphism.*

We will now point out that Theorem 10.3 implies classical Thom isomorphism theorems for orthogonal, unitary and symplectic K-theory (the orthogonal and unitary cases are also treated in [9, Theorem IV.5.14]).

Assume that $V \to X$ is equipped with a spin structure and let $\Sigma \to X$ be the spinor bundle associated to $V \to X$, compare Example 8.7. In this case $\mathrm{Cl}(V)$-module bundles are of a particular form.

Recall the notion of the *tensor product over* $\mathbb{H}$ (e.g. see [7, p. 18]): Let P a $\mathbb{H}$-right vector space and Q a $\mathbb{H}$-left vector space. Then

$$\begin{aligned} P \otimes_{\mathbb{H}} Q &:= (P \otimes Q)/R\,, \text{ where} \\ R &:= \mathrm{Span}\,\{p\lambda \otimes q - p \otimes \lambda q : p \in P, q \in Q, \lambda \in \mathbb{H}\}. \end{aligned}$$

This is a vector space over $\mathbb{R}$, not over $\mathbb{H}$.

If m is odd then the right $\mathbb{H}$-multiplication on S commutes with the left Cl_n-action (see Table (2.2)), and hence Σ is a bundle of right $\mathbb{H}$-modules in a canonical way. We equivalently regard S and Σ as left $\mathbb{H}$-modules by setting $\lambda \cdot s := s\lambda^{-1}$.

Proposition 10.4 *Let Λ be a $\mathrm{Cl}(V)$-module bundle. Then Λ is a twisted spinor bundle, that is there exists a Euclidean vector bundle $E \to X$ over $\mathbb{R}$ or $\mathbb{H}$ when m is even or odd, respectively, such that*

$$\Lambda \cong \begin{cases} E \otimes \Sigma & \textit{when } m \textit{ is even} \\ E \otimes_{\mathbb{H}} \Sigma & \textit{when } m \textit{ is odd} \end{cases} \tag{10.2}$$

If two such bundles $\Lambda = E \otimes_{(\mathbb{H})} \Sigma$ and $\tilde{\Lambda} = \tilde{E} \otimes_{(\mathbb{H})} \Sigma$ are isomorphic as $\mathrm{Cl}(V)$-module bundles, then $E \cong \tilde{E}$.

Proof Following [3, p. 115], we put

$$E = \mathrm{Hom}_{\mathrm{Cl}(V)}(\Sigma, \Lambda).$$

The isomorphism in (10.2) is given by

$$j : E \otimes \Sigma \to \Lambda, \phi \otimes \xi \mapsto \phi(\xi). \tag{10.3}$$

This is a $\mathrm{Cl}(V)$-linear bundle homomorphism: For all $\alpha \in \mathrm{Cl}(V)$,

$$j(\alpha \cdot (\phi \otimes \xi)) = j(\phi \otimes (\alpha \cdot \xi)) = \phi(\alpha \cdot \xi) = \alpha \cdot \phi(\xi) = \alpha \cdot j(\phi \otimes \xi).$$

To check bijectivity we look at the fibers L of Λ and S of Σ. Set $C := \mathrm{Cl}_n$. The fibre of $E = \mathrm{Hom}_{\mathrm{Cl}(V)}(\Sigma, \Lambda)$ is $\mathrm{Hom}_C(S, L)$ and the homomorphism j is fiberwise the linear map

$$j_o : \mathrm{Hom}_C(S, L) \otimes S \to L,\ (\phi, s) \mapsto \phi(s).$$

Since S is the unique irreducible C-representation we have $L \cong S^p = \mathbb{R}^p \otimes S$ for some p, and $\mathrm{Hom}_C(S, L) = \mathrm{Hom}_C(S, S)^p = \mathrm{End}_C(S)^p$. Applying j_o to $\phi = (0, \dots, \mathrm{id}_S, \dots, 0)$ with the identity at the k-th slot for $k = 1, \dots, p$, we see that this map is onto. When m is even we have $C = \mathrm{End}_{\mathbb{R}}(S)$, and when m is odd, $C = \mathrm{End}_{\mathbb{H}}(S)$ (cf. Table (2.2)). Thus $\mathrm{End}_C(S) = \mathbb{R} \cdot \mathrm{id}_S$ when m is even and $\mathrm{End}_C(S) = \mathbb{H} \cdot \mathrm{id}_S$ when m is odd. Note that in the second case an element $\lambda \in \mathbb{H}$ corresponds to $\hat{\lambda} := R_{\lambda^{-1}} \in \mathrm{End}_C(S)$ (right multiplication with λ^{-1} on S, that is left multiplication with λ with respect to the left $\mathbb{H}$-module structure defined before) in order to make the identification a ring map.[16]

In the first case, $\mathrm{Hom}_C(S, L) = \mathrm{End}_C(S)^p \otimes S = \mathbb{R}^p \otimes S = S^p$. Thus j_o is an isomorphism whence j is an isomorphism.

In the second case we consider $\mathrm{Hom}_C(S, L)$ as a $\mathbb{H}$-right vector space using precomposition $\phi \mapsto \phi \circ \hat{\lambda}$, where $\hat{\lambda} = R_{\lambda^{-1}} \in \mathrm{End}_C(S) = \mathbb{H}$. We hence compute

$$j_o(\phi\lambda, \xi) = (\phi \circ \hat{\lambda})(\xi) = \phi(\xi\lambda^{-1}) = \phi(\lambda \cdot \xi) = j_o(\phi, \lambda \cdot \xi)\,.$$

Thus j_o descends to $\bar{j}_o : \mathrm{Hom}_C(L, S) \otimes_{\mathbb{H}} S \to L$. Since $\mathrm{Hom}_C(L, S) \cong \mathrm{Hom}_C(S^p, S) = \mathrm{End}_C(S)^p = \mathbb{H}^p$ and $\mathbb{H} \otimes_{\mathbb{H}} S \cong S$ via the map $\lambda \otimes_{\mathbb{H}} s = 1 \otimes_{\mathbb{H}} \lambda s \mapsto \lambda s$, we obtain that $\bar{j}_o$ maps $\mathrm{Hom}_C(S, L) \otimes_{\mathbb{H}} S \cong \mathbb{H}^p \otimes_{\mathbb{H}} S = (\mathbb{H} \otimes_{\mathbb{H}} S)^p \cong S^p$ isomorphically onto $L \cong S^p$.

A $\mathrm{Cl}(V)$-linear isomorphism ϕ between two such bundles $E \otimes_{(\mathbb{H})} \Sigma$ and $\tilde{E} \otimes_{(\mathbb{H})} \Sigma$ is fiberwise a C-linear isomorphism $\phi_o : S^p \to S^p$. This is an invertible $(p \times p)$-matrix $A = (a_{in})$ with coefficients in $\mathrm{End}_C(S) = \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ when m is even and $\mathbb{K} = \mathbb{H}$ when m is odd. In the first case we have $S^p = \mathbb{R}^p \otimes S$ and $j_o = A \otimes \mathrm{id}_S$. In the second case we let $S^p = \mathbb{H}^p \otimes_{\mathbb{H}} S$ using the isomorphism $se_i \mapsto e_i \otimes_{\mathbb{H}} s$ for all $s \in S$ and $i = 1, \dots, p$ where $e_1 = (1, 0, \dots, 0), \dots, e_p = (0, \dots, 0, 1)$. Then

$$\phi_o(se_i) = \sum_j a_{ij} s e_j \mapsto \sum_j e_j \otimes_{\mathbb{H}} a_{ij} s = \sum_j e_j a_{ij} \otimes_{\mathbb{H}} s = Ae_i \otimes_{\mathbb{H}} s$$

[16] $\mathrm{End}_C(S)$ is an (associative) division algebra, thus isomorphic to $\mathbb{H}$ or $\mathbb{C}$ or $\mathbb{R}$. In fact, kernel and image of any $A \in \mathrm{End}_C(S)$ are C-invariant subspaces of the irreducible C-module S, hence A is invertible or 0. When m is odd, $C = \mathrm{End}_{\mathbb{H}}(S)$ and therefore $\mathrm{End}_C(S) \supset \mathbb{H}$, hence $\mathrm{End}_C(S) = \mathbb{H}$. When m is even, $C = \mathrm{End}_{\mathbb{R}}(S)$. Any $A \in \mathrm{End}_C(S)$ has a real or complex eigenvalue and hence an invariant line or plane in S. Since it commutes with all endomorphisms on S, *every* line or plane is A-invariant since $\mathrm{GL}(S) \subset \mathrm{End}(S)$ acts transitively on the Grassmannians. Thus A is a real multiple of the identity. (See also Wedderburn's theorem).

and hence $\phi_o = A \otimes_{\mathbb{H}} \mathrm{id}_S$.

Therefore $\phi = f \otimes \mathrm{id}_\Sigma$ for some $\mathbb{R}$-linear isomorphism $f : E \to \tilde{E}$ in the first case, and $\phi = f \otimes_{\mathbb{H}} \mathrm{id}_\Sigma$ for some $\mathbb{H}$-linear isomorphism $f : E \to \tilde{E}$ in the second case. $\square$

Lemma 10.5 *Let $\Lambda, \tilde{\Lambda} \to X$ be $\mathrm{Cl}(V)$-module bundles. Then the following assertions are equivalent.*

(i) *Λ and $\tilde{\Lambda}$ are stably isomorphic as $\mathrm{Cl}(V)$-module bundles.*
(ii) *There is some $q \geq 0$, such that $\Lambda \oplus \Sigma^q \cong \tilde{\Lambda} \oplus \Sigma^q$ are isomorphic $\mathrm{Cl}(V)$-module bundles*

Proof This follows since $\Sigma \to X$ is a $\mathrm{Cl}(V)$-submodule bundle of $\mathrm{Cl}(V)^q \to X$ for some $q \geq 0$, and vice versa $\mathrm{Cl}(V) \to X$ is a $\mathrm{Cl}(V)$-submodule bundle of $\Sigma^q \to X$ for some q. Again this is obvious if X is equal to a point and follows for general X by a partition of unity argument. $\square$

Together with Proposition 10.4 this implies

Proposition 10.6 *Let $V \to X$ be of rank $4m$ and equipped with a spin structure, and let $\Sigma \to X$ denote the associated spinor bundle.*

(a) *For even m the map*

$$\mathrm{K}^{\mathrm{O}}(X) \to \mathrm{K}^{\mathrm{Cl}(V)}(X)\,, \quad [E] \mapsto [E \otimes \Sigma]\,,$$

is an isomorphism.

(b) *For odd m the map*

$$\mathrm{K}^{\mathrm{Sp}}(X) \to \mathrm{K}^{\mathrm{Cl}(V)}(X)\,, \quad [E] \mapsto [E \otimes_{\mathbb{H}} \Sigma]\,,$$

is an isomorphism, where K^{Sp} denotes symplectic (quaternionic) K-theory based on $\mathbb{H}$-right vector bundles.

Proof The maps in (a) and (b) are well defined and one-to-one: Let $E, \tilde{E} \to X$ be vector bundles. Then $\Lambda = E \otimes \Sigma$ is stably isomorphic to $\tilde{\Lambda} = \tilde{E} \otimes \Sigma \overset{10.5}{\iff} \Lambda \oplus \Sigma^q = (E \oplus \underline{\mathbb{R}^q}) \otimes \Sigma$ (for some $q \in \mathbb{N}$ given by 10.5) is isomorphic to $\tilde{\Lambda} \oplus \Sigma^q = (\tilde{E} \oplus \underline{\mathbb{R}^q}) \otimes \Sigma \overset{10.4}{\iff} E \oplus \underline{\mathbb{R}^q} \cong \tilde{E} \oplus \underline{\mathbb{R}^q} \iff E \cong_s \tilde{E}$. Further, these maps are onto by Proposition 10.4. $\square$

Hence we obtain the classical Thom isomorphism by composing the isomorphisms of 10.6 and 10.3 , using that the spinor Hopf bundle $\mathscr{S}$ (cf. 8.8) is the Hopf bundle (cf. 8.1) associated to the spinor bundle Σ:

Theorem 10.7 *Let $V \to X$ be equipped with a spin structure and let $\mathscr{S} \to \hat{V}$ denote the spinor Hopf bundle associated to $V \to X$ (see Definition 8.8).*

(a) *For even m the map*

$$\mathrm{K}^{\mathrm{O}}(X) \to \tilde{\mathrm{K}}^{\mathrm{O}}(X^V), \quad [E] \mapsto \chi([E \otimes \mathscr{S}]),$$

is an isomorphism.

(b) *For odd m the map*

$$\mathrm{K}^{\mathrm{Sp}}(X) \to \tilde{\mathrm{K}}^{\mathrm{O}}(X^V), \quad [E] \mapsto \chi([E \otimes_{\mathbb{H}} \mathscr{S}]),$$

is an isomorphism.

Hence $\chi([\mathscr{S}]) \in \tilde{\mathrm{K}}^{\mathrm{O}}(X^V)$ serves as the "K^{O}-theoretic Thom class" of V.

There are analogues of the theorems 9.2 and 10.3 for complex and quaternionic vector bundles over $\hat{V}$. Detailed proofs of the following statements are left to the reader.

Theorem 10.8 *Let X be a finite CW-complex and $V \to X$ an oriented Euclidean vector bundle with associated Clifford algebra bundle $\mathrm{Cl}(V) \to X$.*

(a) *Let $\operatorname{rk} V = 4m$. Then there is a Clifford–Thom isomorphism*

$$\mathrm{K}^{\mathrm{Cl}(V)\otimes\mathbb{H}}(X) \cong \tilde{\mathrm{K}}^{\mathrm{Sp}}(X^V).$$

(b) *Let $\operatorname{rk} V = 2m$. Then there is a Clifford–Thom isomorphism*

$$\mathrm{K}^{\mathrm{Cl}(V)\otimes\mathbb{C}}(X) \cong \tilde{\mathrm{K}}^{\mathrm{U}}(X^V).$$

The analogues of Proposition 10.4 for complex and quaternionic $\mathrm{Cl}(V)$-module bundles are as follows.

Assume that $V \to X$ is equipped with a spin structure. Recall from (2.2) that the spinor bundle Σ associated to $P_{\mathrm{Spin}}(V)$ is quaternionic when $\operatorname{rk} V = n = 4m$ with m odd and it is real when $n = 4m$ with m even. If Λ is a $(\mathrm{Cl}(V) \otimes \mathbb{H})$-module bundle (with left $\mathrm{Cl}(V)$-multiplication and right $\mathbb{H}$-multiplication), we put $E = \mathrm{Hom}_{\mathrm{Cl}(V)\otimes\mathbb{H}}(\Sigma, \Lambda)$ for m odd and $E = \mathrm{Hom}_{\mathrm{Cl}(V)}(\Sigma, \Lambda)$ if m is even; in both cases precisely one of the bundles E and Σ is (right) quaternionic, and we obtain an isomorphism of $(\mathrm{Cl}(V) \otimes \mathbb{H})$-module bundles analogue to (10.3),

$$j : E \otimes_{\mathbb{R}} \Sigma \to \Lambda, \quad \phi \otimes \xi \mapsto \phi(\xi).$$

For $(\mathrm{Cl}(V) \otimes \mathbb{C})$-module bundles Λ with $\dim V = n = 2m$ we can apply [3, Prop. 3.34]: Let Σ^c be the complex spinor bundle associated to $P_{\mathrm{Spin}^c}(V)$ by means of a Spin^c-structure on V (cf. [10, Appendix D]). Putting $E = \mathrm{Hom}_{\mathrm{Cl}(V)\otimes\mathbb{C}}(\Sigma^c, \Lambda)$ we obtain that

$$j : E \otimes_{\mathbb{C}} \Sigma^c \to \Lambda, \quad \phi \otimes_{\mathbb{C}} \xi \mapsto \phi(\xi)$$

is an isomorphism of $(\mathrm{Cl}(V) \otimes \mathbb{C})$-module bundles.

Together with Theorem 10.8 we hence arrive at the following classical Thom isomorphism theorems, which complement Theorem 10.7.

Theorem 10.9 (a) *Let $V \to X$ be a spin bundle of rank $n = 4m$. Then multiplication with the spinor Hopf bundle $\mathscr{S} \to \hat{V}$ induces Thom isomorphisms*

$$\mathrm{K}^{\mathrm{Sp}}(X) \to \tilde{\mathrm{K}}^{\mathrm{Sp}}(X^V) \textit{ for even } m\,,$$
$$\mathrm{K}^{\mathrm{O}}(X) \to \tilde{\mathrm{K}}^{\mathrm{Sp}}(X^V) \textit{ for odd } m\,.$$

(b) *Let $V \to X$ be a $Spin^c$-bundle of rank $n = 2m$. Then multiplication with the complex spinor Hopf bundle $\mathscr{S}^c \to \hat{V}$ induces a Thom isomorphism*

$$\mathrm{K}^{\mathrm{U}}(X) \to \tilde{\mathrm{K}}^{\mathrm{U}}(X^V)\,.$$

Discussion 10.10 We will point out some connections of the argument at the beginning of this section to the classical monograph [9]. Let Q denote the given (positive definite) quadratic form on the Euclidean bundle $V \to X$. Then the Clifford algebra bundle $\mathrm{Cl}(V) \to X$ in the sense of Definition 8.4 is equal to the Clifford algebra bundle $C(V, -Q) \to X$ in the sense of [9, IV.4.11] [17] and $\mathrm{K}^{\mathrm{Cl}(V)}(X)$ from Definition 10.1 is equal to $\mathrm{K}(\mathscr{E}^{(V,-Q)}(X))$, the Grothendieck group of the Banach category $\mathscr{E}^{(V,-Q)}(X)$ of $C(V, -Q)$-module bundles over X, compare [9, II.1.7].

Since V is oriented and of rank divisible by four we have $C(V, Q) \cong C(V, -Q)$ as algebra bundles, induced by the map $V \to C(V, Q)$, $v \mapsto \omega \cdot v$, where $\omega \in \mathrm{Cl}(V)$ denotes the volume section of $C(V, -Q)$ (note that $v^2 = -\|v\|^2$ and $4 \mid \mathrm{rk}\, V$ imply $(\omega \cdot v)^2 = +\|v\|^2$). Hence

$$\mathrm{K}(\mathscr{E}^{(V,Q)}(X)) \cong \mathrm{K}(\mathscr{E}^{(V,-Q)}(X)) = \mathrm{K}^{\mathrm{Cl}(V)}(X) \tag{10.4}$$

for the given $V \to X$.

In the following we denote by $C(V) \to X$ the algebra bundle $C(V, Q) \to X$ and $\mathscr{C} = \mathscr{E}^V(X)$ the Banach category of module bundles over $C(V) \to X$.

As in [9, Section IV.5.1] let $\mathrm{K}^V(X) = \mathrm{K}(\phi^V)$ be the Grothendieck group of the forgetful functor $\phi^V : \mathscr{E}^{V\oplus\mathbb{R}}(X) \to \mathscr{E}^V(X)$. Using $C(\mathbb{R}, Q) = \mathbb{R}\oplus\mathbb{R}$ for the standard quadratic form Q on $\mathbb{R}$, compare [9, III.3.4], we have $C(V \oplus \mathbb{R}) \cong C(V) \otimes (\mathbb{R} \oplus \mathbb{R}) = C(V) \oplus C(V)$, see [9, Prop. III.3.16.(i)] (note that $C(V, Q) > 0$, which means $\omega^2 = +1$, holds since $4 \mid \mathrm{rk}\, V$).

Hence $\mathscr{E}^{V\oplus\mathbb{R}}(X) \cong \mathscr{C} \times \mathscr{C}$ and the forgetful functor ϕ^V can be identified with the functor $\psi : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, sending a pair of $C(V)$-module bundles (E, F) to their direct sum $E \oplus F$, compare [9, III.4.9] for the cases $q = 0$ and $q = 4$. Since this functor has a right inverse $E \mapsto (E, 0)$ the first and last maps in the exact sequence

$$\mathrm{K}^{-1}(\mathscr{C} \times \mathscr{C}) \xrightarrow{\psi_*} \mathrm{K}^{-1}(\mathscr{C}) \to \mathrm{K}(\phi^V) \to \mathrm{K}(\mathscr{C} \times \mathscr{C}) \xrightarrow{\psi_*} \mathrm{K}(\mathscr{C})$$

from [9, II.3.22] are surjective. We hence obtain isomorphisms

[17] Notice the sign convention for the construction of the Clifford algebras in [9, III.3.1], which is different from ours.

$$\mathrm{K}(\phi^V) \cong \ker\left(\psi_* : \mathrm{K}(\mathscr{C}) \oplus \mathrm{K}(\mathscr{C}) \overset{(a,b)\mapsto a+b}{\longrightarrow} \mathrm{K}(\mathscr{C})\right) \cong \{(a, -a) \mid a \in \mathrm{K}(\mathscr{C})\} \cong \mathrm{K}(\mathscr{C})$$

showing that

$$\mathrm{K}^V(X) \cong \mathrm{K}(\phi^V) \cong \mathrm{K}(\mathscr{C}) \overset{(10.4)}{\cong} \mathrm{K}^{\mathrm{Cl}(V)}(X)\,.$$

Put differently: For the given $V \to X$ Karoubi's theory $\mathrm{K}^V(X)$ is the topological K-theory based on $\mathrm{Cl}(V)$-module bundles over X. We remark that this is not true in general for bundles $V \to X$ of rank not divisible by four.

One can show that with respect to this identification the Clifford–Thom homomorphism in Definition 10.2 is identified with the map $t : \mathrm{K}^V(X) \to \mathrm{K}(B(V), S(V))$ from [9, IV.5.10.]. Hence Theorem 10.3 recovers [9, Theorem IV.5.11] (whose full proof is given in [9, IV.6.21]), for oriented $V \to X$ of rank divisible by four.

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References

1. Atiyah, M.: *K*-Theory, Benjamin Inc. (1967)
2. Atiyah, M., Bott, R., Shapiro, A.: Clifford Modules. Topology **3**(Suppl. 1), 3–38 (1964)
3. Berline, N., Getzler, E., Vergne, M.: Heat Kernels and Dirac Operators. Springer, (1996)
4. Bott, R.: The stable homotopy of the classical groups. Ann. Math. **70**, 313–337 (1959)
5. Bredon, G.E.: Topology and Geometry. Springer, (1991)
6. Chen, B.-Y., Nagano, T.: Totally geodesic submanifolds of symmetric spaces II. Duke Math. J. **45**, 405–425 (1978)
7. Draxl, P.K.: Skew Fields. Lond. Math. Soc. Lect. Notes **81**, (1983)
8. Eschenburg, J.-H., Hanke, B.: Bott Periodicity, Submanifolds, and Vector Bundles, In: Suh, Y.J., Ohnita, Y., Zhou, J., Kim, B.H., Lee, H. (eds.) Hermitian Grassmann Submanifolds. Springer Proceedings in Mathematics & Statistics, vol. 203, pp. 295–309 (2017)
9. Karoubi, M.: K-Theory, an introduction. Springer, (1978)
10. Lawson, H.B., Michelsohn, M.-L.: Spin Geometry. Princeton, (1989)
11. Milnor, J.: Morse Theory. Princeton, (1969)
12. Mitchell, S.A.: The Bott filtration of a loop group. Springer Lect. Notes Math. **1298**, 215–226 (1987)
13. Mitchell, S.A.: Quillen's theorem on buildings and the loops on a symmetric space. Enseign. Math. (II) **34**, 123–166 (1988)
14. Quast, P.: Complex structures and chains of symmetric spaces, Habilitationsschrift Augsburg (2010), urn:nbn:de:bvb:384-opus4-848278, available under https://opus.bibliothek.uni-augsburg.de/opus4/84827
15. Quast, P.: Centrioles in Symmetric Spaces. Nagoya Math. J. **211**, 51–77 (2013)

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