# THÈSE

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## Ruine et investissement en environnement markovien

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**JURY**

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## Mais aussi...

*When I hear of an "equity" in a case like this, I am reminded of a blind man in a dark room - looking for a black hat - which isn't there, ...*

Charles Synge Christopher Bowen

Je pense toutefois que cette section serait incomplète sans mentionner le sentiment dominant de mon parcours, à savoir la solitude de l'adversité, qui m'a contraint sans relâche à l'excellence. Il me faut donc aussi remercier mes nombreux rivaux et obstacles qui, à force d'oppositions, m'ont malgré eux mené jusqu'ici.

*...and finds it anyways!*

Lee Dinetan

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# Chapitre 1

# Introduction

L'objet de cette thèse est de modéliser et optimiser les stratégies d'investissement d'un agent soumis à un environnement markovien, et à un risque de liquidité se déclarant quand il ne peut plus faire face à une sortie d'argent faute d'actifs liquides.

## 1.1 Présentation

Afin d'échapper à la faillite, il dispose pour cela d'opportunités d'investissement, lui donnant l'espoir d'accroître ses gains futurs en échange d'une dépense immédiate. L'objet du travail est de déterminer sa meilleure stratégie d'investissement dans cet univers markovien.

### 1.1.1 Contexte

Avant d'introduire son contenu, il convient d'expliquer la nécéssité de cette thèse du point de vue de l'état de l'art, afin de justifier des hypothèses sous lesquelles on se placera. Nous savons qu'en l'absence de "frictions du marché" au sens de Modigliani et Miller ([35]), la valeur des actifs de l'acteur ne dépend pas du ratio entre son capital liquide et ses investissements ; en conséquence, la minimisation des risques d'insolvabilité à long terme se ferait par investissement de l'ensemble du capital disponible, et le recours à la liquidation (voire à l'emprunt) en cas de besoin. La question de la stratégie d'investissement se résoudrait donc par le simple calcul de la rentabilité du projet, puisque tout projet rentable trouverait un financement dans cet univers idéal.

Pour cette raison, nous nous intéressons au contraire à un modèle dans lequel les frictions dûes à l'illiquidité de l'investissement sont totales. Nous n'y tolérons aucune possibilité d'emprunt, et les moyens de liquider (même à perte) un investissement passé ne sont disponibles qu'à titre aléatoire ; par conséquent, l'acteur solvable à long terme (résultat net moyen positif) mais en défaut de paiement immédiat (au sens du fonds de roulement liquidité) doit être considéré en état de faillite et cesser d'exercer. Il convient dès lors de conserver une partie de la liquidité disponible en vue de faire face à ces risques, sacrifiant donc des opportunités d'investissement rentables quand les réserves de liquidité après investimment sont jugées insuffisantes ; ceci signifie que l'on juge que les risques de faillite à court terme (illiquidité) sont trop élevés par rapport à ceux sur le long terme (manque à gagner, dû au refus d'investir). Nous sommes donc intéressés par les risques de faillite de l'acteur, à court ou long terme, d'une manière similaire à [1, 19, 10] : les revenus futurs sont aléatoires et l'investissement est irréversible.

Il est possible de calculer la réponse à la plupart de ces questions grâce aux travaux sur les modèles de Lévy de la littérature ([24, 20] dans les cas browniens, [28, 33] plus généralement). [13] fournit par ailleurs une réponse au problème de l'investissement aléatoire lorsque les revenus suivent un processus de Lévy. Cependant, nous pensons que les processus de Lévy ne sont pas une représentation fidèle de la réalité. Ainsi, l'indépendance des variations successives est sujette à caution, à cause du comportement endogène des investisseurs envers l'inertie ([9, 21]), qui pourrait aussi être une conséquence exogène de la volatilité des prix ([27]) créant des cycles "croissance-crise". Pour cette raison, nous créerons un modèle markovien du capital de l'acteur, et nous nous interrogerons sur l'effet du retrait de l'hypothèse d'indépendance entre les temps successifs sur les décisions d'investissement. Nous noterons en outre qu'il englobe la classe des processus de Lévy et donc que les résultats trouvés restent valides quand on se restreint aux modèles classiques. Nous observerons en réalité que la dépendance temporelle entre les fluctuations aléatoires des prix des actifs détruit de manière significative la qualité prédictive des modèles de Lévy quant à l'évaluation des risques de faillite. Nous avons donc décidé de revoir les questions sur la stratégie d'investissement dans notre modèle markovien.

### 1.1.2 Contenu du travail

Cette thèse se constitue de deux articles, précédés par une introduction générale expliquant leur motivation. La première étude introduit le modèle markovien utilisé, et explique pourquoi et comment corriger la possible sous-estimation des risques de faillite d'un actif y étant soumis lors de l'usage d'un processus de Lévy. Après avoir construit le modèle, nous nous intéressons à la résolution de problèmes d'investissement dans un tel milieu markovien ; en particulier, nous verrons comment la particularité de chaque état du marché affecte la décision d'investir pour combattre les risques inhérents à l'illiquidité totale des investissements. Nous nous pencherons finalement sur les notions d'investissement coopératif. Ainsi, nous démontrerons que dans notre univers, deux individus peuvent tirer parti du caractère markovien du marché afin de diminuer tous deux leurs risques de faillite sans apport exogène d'actifs. Nous retrouverons par ailleurs que la fusion entre les deux fonds, qui est optimale dans un modèle de Lévy, reste optimale dans notre cadre plus général. Une section de conclusion présentera des perspectives d'évolution du travail.

### Modèle markovien

On commence par présenter une structure de processus permettant d'introduire la dépendance de l'agent envers le marché. Pour cela, on introduit un modèle doté d'une chaîne de Markov cachée (qui représente le marché) et d'un processus stochastique y étant sujet (qui représente le flux de capitaux de l'agent). Cette structure, du nom de C-processus, permet en outre de modéliser les comportements cycliques du marché, ainsi qu'une partie des phénomènes de persistance des variations des cours des actifs.

Une fois le modèle construit, on cherche à contrôler son temps de ruine via sa transformée de Laplace. Pour cela, on crée une martingale fondée sur l'exposant de Lundberg d'un processus de Lévy ; celle-ci doit tenir compte de l'état du marché et de la valeur des actifs détenus par l'investisseur. Par suite, les propriétés de cette martingale sont exploitées afin d'évaluer les risques de faillite : l'expression trouvée fait apparaître les contributions de chaque "état du marché" à ces risques, mais surtout un paramètre exponentiel indiquant l'effet marginal des actifs liquides sur les risques de faillite. On verra en outre que les phénomènes cycliques créent de l'instabilité et accroissent les risques de faillite.

### Choix à l'investissement

Supposons que l'agent se trouve en présence d'une opportunité d'investissement, et doive choisir de l'accepter ou non en fonction de son objectif qui reste d'éviter la banqueroute : on modélise la dynamique de ses actifs par un C-processus différent suivant son choix. En connaissance des risques de faillite d'un tel processus comme calculés précédemment, l'agent peut quantifier l'amélioration de ses revenus promise par l'investissement, et ainsi juger de la décision à prendre ; on verra donc en quoi le caractère markovien du modèle modifie les méthodes courantes du calcul du meilleur comportement. On s'intéressera en particulier à une révision des problèmes habituels comme le temps d'achat optimal dans ce cadre markovien soumis aux contraintes d'illiquidité.

Ultérieurement, on utilisera ce modèle afin de discuter des notions de coopération entre deux agents tentant tous deux d'échapper à la faillite, dont on considère qu'ils sont tous deux dotés d'un C-processus soumis au même marché, mais à des rendements différents et indépendants (conditionnellement au marché). On montrera qu'il leur est possible de se soutenir mutuellement sans aide exogène, de manière à diminuer tous deux leur risque de faillite. Ce concept de “contrat de support” permet à la victime d'un mauvais état du marché d'être aidée par les bénéficiaires de celui-ci, au prix de la promesse de rendre la faveur quand le marché se retournera. En outre, on pourra en déduire comment investir dans un projet d'amélioration des flux de capitaux de manière coopérative.

# Introduction

## 1.2 History of this work

We originally aimed at dealing with the problem of investment in a liquidity-constrained economy. This generic expression specifically refers to purchasing assets (stocks, bonds, etc.) earning a long-term income at the expense of an immediate price, with the risk of short-term illiquidity if the investment costs are not amortized before the buyer runs out of cash and goes bust. Main features of our model are thus common to old studies ([1, 19, 10]), namely uncertainty of future payoffs (investment incomes are random) and irreversibility of investment. We chose to keep the latter, since full reversibility (liquidity) often means through the Modigliani-Miller theorem ([35]) that one should always prefer the investment producing the highest revenues. However, while [13] tackled the issue of investment under uncertainty with Lévy-shaped payoffs, we believe that the latter are not a realistic representation of reality. Independence of successive variations is questionable, because of endogenous behaviour of investors towards momentum ([9, 21]) ; it may even be an exogenous consequence of price volatility ([27]) generating boom-bust cycles. For this reason, we create a Markovian model for the agent's wealth, and wonder on how common investment decisions change when removing the hypothesis of independence between successive time periods.

Throughout this work, we aim at minimizing the default risks of investors rather than looking for maximizing one's wealth, or a dividend policy inducing consumption. Therefore, we are actually looking for a "secure" strategy to avoid economic catastrophes ; e.g. we want to avoid an insurance company to collapse during a crisis and leave policy holders exposed to huge risks.

### 1.2.1 The game of Monopoly

We investigate on an economy where agents are subject to random incomes and aim at not going bankrupt ; for this purpose, they are able to invest on properties that produce a permanent rent. The question of the best investment strategy recalls us the Monopoly game. We notice that avoiding bankruptcy is the goal sought by the rules of Monopoly ; assuming that real-life investment strategies are correctly represented in this game, we were interested in how to model it, and most notably, by its possible equilibria : there is not necessarily a single winner, as [41] hints. Therefore, we first translated the rules of Monopoly into mathematical terms.

### Creation of the model

Before discussing on how players may buy properties, we began by creating an adequate model to represent the random fluctuations of agents' assets. The position of a Monopoly game is fully described by the determination of : players' placements on the board, the player to move, the number of double rolls or jail turns in a row for each player, the order of "Chance" and "Community Chest" cards, and the places of "Get out of jail free" cards in the decks or with players. So, we first remarked that the presence of a game board gives a Markovian behaviour of the game, extending the similar work of [41]. Therefore, we create a finite state space $(A_i)_{i \leq A}$ where each possible configuration of the position is represented by a state $A_i$ for some $i \leq A$. At each turn, a dice roll randomly moves the position to another configuration. As dice rolls are naturally deemed independent, and the rules of Monopoly do not change during the game, dice rolls create a time-homogenous Markovian process $M$ on $(A_i)_{i \leq A}$, whose transition probabilities may be explicitly computed thanks to the rules of the game. Considering Monopoly as a toy model to real-life economy, $M$ acts as an exogenous market (commonly called "the conjoncture") and each state $A_i$ translates as the state of such a market, while its transition matrix indicates the likely short-term evolution of the conjoncture.

Now we move on to modelling players' assets as random processes hereby called $C_p$ : e.g. at the beginning of the game, $J_p$ is granted with starting cash reserves $C_p(0)$. The position of $M$ has an incidence on the assets vs. liabilities balance of each player $J_p$, defined through the rules of Monopoly : upon any player landing on each square, drawing a card, etc., there are cash moves between players and the bank depending on $M$. Therefore, for each couple of $M$'s states $(A_i, A_j)$, the rules define payoffs called $D^{(p)}_{i \to j}$ earned by $J_p$ when $M$ crosses the transition $(A_i \to A_j)$. These transition payoffs $D^{(p)}_{i \to j}$ behave as successive increments for the processes $C_p$. They may be randomly drawn (in Monopoly, it may happen that a player is sent to a utility by a card and must roll to determine the bill), but they are independent from the past of the game, thus the determination of $M$ and every $C_p$ at present time has a Markovian behaviour. This encourages us to build a new class of processes in order to model $C_p$'s natural variations. They are called C-processes, and will be central to our whole work.

**Model 1.1** *C-process*

*A C-process is the determination of*

— *A Markovian time-homogeneous process* $(M(t))_{t\in\boldsymbol{N}}$ *with a finite state space* $(A_i)_{i\leq A}$ *with* $A \in \boldsymbol{N}^*$*, whose transition probabilities are defined through*

$$\forall i, j \leq A, P_{i\to j} = \boldsymbol{P}(M(t+1) = A_j | M(t) = A_i)$$

*for any* $t \in \boldsymbol{N}$*. Moreover,* $M(0)$ *is deemed determinisitc ; the* $A_i$ *in* $M$*'s state space such that* $M(0) = A_i$ *almost surely is called* $M$*'s (or* $C$*'s) starting state.*

— *Random variables called transition payoffs, such that*
  — *For every* $i, j \leq A$*, there is a probability distribution over* $\boldsymbol{R} \cup \{+\infty\}$ *defining a random variable* $D_{i\to j}$ *with respect to this distribution ;*
  — *For every* $t \in \boldsymbol{N}^*$*, we define the family* $(D_{i\to j}(t))_{i,j\leq A}$ *to be an independent and identically distributed copy of the family* $(D_{i\to j})_{i,j\leq A}$*, with respect to the time variable* $t \in \boldsymbol{N}^*$*.*

*For every* $i, j \leq A$ *and* $t \in \boldsymbol{N}^*$*,* $D_{i\to j}(t)$ *is called the transition payoff between states* $A_i$ *and* $A_j$ *at time* $t$*.*

— *A process* $C$*, whose initial value* $C(0) = C_0 \in \boldsymbol{R}^+$ *is deterministic and called* $C$*'s starting point, and whose increments hold*

$$\forall t \in \boldsymbol{N}, C(t+1) = C(t) + D_{M(t)\to M(t+1)}(t+1)$$

*We say that* $C$ *is a C-process whose underlying Markovian process is* $M$*, and transition payoffs are (the distributions of) the random variables* $D_{i\to j}$*, for each state numbers* $i, j \leq A$*.*

Notice that the processes $C_p$ of $J_p$'s assets are C-processes only in Monopoly's main variant, since transitions are additive and do not dependent from $C_p$'s present value. Counter-examples are created when playing with some house rules, like the so-called "free parking jackpot", as the value of said jackpot depends on cumulated past taxes before it is won, or the 10% income tax, as it depends on $C_p$ and thus the past incomes.

**Investment**

The rules allow a player $J_p$ to invest when $M$ hits a pre-determined state. When this opportunity is accepted, $J_p$ pays an investment cost, and changes transition payoffs in a favorable way to him by means of a permanent rent. Therefore, we consider that the game changes of C-processes each time investment happens, forming a "tree" of C-processes defining possible timelines for investment. The rules and gameplay of Monopoly may thus be completely expressed thanks to a graph of C-processes. Although investment opportunities may be declined for an arbitrarily long time before acceptation, we shall assume that the time period when players may invest is bounded in order to simplify the model, so that the tree has a finite height.

**Definition 1.2.1** *Concept of a C-game*
*A C-game with $J \in \boldsymbol{N}^*$ players consists in a tree-shaped finite oriented graph $G$, seen as a process with $G(t)$ describing the node hit on the graph at time $t$. Every state in $G$ consists in a C-process, and indicates investment states $A_i$ in its underlying Markovian process' state space, whereupon a pre-determined player $J_p$ chooses the next node $G(t+1)$ incurring investment costs.*

Notice that this definition also contains "dis-investment" or liquidation, if an asset is sold in return for liquidity. Other aspects of investment include how players may trade assets (with a zero-sum income) or even borrow money from other players, although the latter is illegal in Monopoly.

### 1.2.2 Method

The main work consists in two studies. The first one creates the model of a C-process : necessary tools to build them are defined, and then we introduce the notions of martingale parameter and dominant eigenvectors leading to the fundamental approximation. Once done, we get on with the other study, using the fundamental approximation to compute the default risks in C-games, and to investigate on buying decisions and cooperative management between players.

#### Default expectancies

Under the standard rules of Monopoly, players aim at avoiding bankruptcy, as the last standing player is declared the winner ; therefore, we should be interested in minimizing the default probability of each player. However, remembering that we took Monopoly as a model of real-life investment strategies, it seems unnatural to look for the default risks at arbitrarily remote time periods, so we shall investigate the more general problem of the Laplace transforms of players' default times $T_p$, taken at points $a_p \in \mathbf{R}^+$ expressing $J_p$'s discount factor. Moreover, we will not be concerned by the question of subsequent dynamics of the processes $C_p$ once one player defaulted, as we shall commonly assume that $J = 2$ and neglect $J_1$'s risks of default once $J_2$ defaulted, a simplification to be discussed during the work. For the previous reasons, we will look at the Laplace transforms

$$\mathbf{E}\left(e^{-a_p T_p}\right)$$

for each player $J_p$ at some point $a_p \in \mathbf{R}$, called $J_p$'s default expectancy ; we notice that when $a_p = 0$, this becomes the default probability $\mathbf{P}\left(T_p < \infty\right)$.

**Use of the tree**

Assuming that the tree of investment opportunities has a finite height, we aim at expressing these default expectancies on its leaves, then to solve buying decisions by backward induction on nodes. Actually, the induction step works as follows :

1. Taking an investment opportunity whose children have a known default expectancy, so set that the default expectancy at this point is given by the minimal default expectancy, as chosen by the player ;
2. Do this for every investment opportunities in the same node, and compute default expectancies on other states in this node : starting from any state, we compute the expected time and payoff until $C$ hits an investment opportunity, and use its Markovian behaviour to get the result ;
3. Hence, get the default expectancies for this node, in order to restart for parent nodes.

The node indicating $J_p$'s minimal default expectancy is $J_p$'s best choice. Therefore, we may compute the best strategy by backward induction.

Unfortunately, generic results using this scheme are accurate only when numeric computations are done, because of the behaviour of exponential default expectancies. Therefore, we shall mainly focus on the issue of a single investment opportunity in several models, discussing about how the structure of Monopoly as a C-process affects the default expectancies.

## 1.3 Properties of C-processes

Our first work will be to evaluate correctly $J_p$'s default expectancy without any investment decisions. The task of computing the default expectancy of a stochastic process has already been investigated for simple cases, most notably for Lévy processes ([12, 25]) using the trick of Lundberg's parameter to get Cramér-Lundberg's approximation for the default probability. However, these computations require $C_p$ to be a Lévy process : we briefly recall that a Lévy process is determined by a single distribution for its independent increments $D(t+1)$ and a deterministic starting point $C(0)$ through $C(t+1) = C(t) + D(t+1)$. In general, $C_p$ will not be a Lévy process ; we will explain how to determine when a C-process is actually a Lévy process, although every Lévy process is a C-process with a trivial state space and $A = 1$. The matter is that the structure of a C-process allows to create short-term momentum that hugely distorts the default risks ; on a side note, common wild behaviour of the stock market may be explained by these time dependency effects. It follows that Lévy processes fail at accurately representing $C$'s default expectancy, because the successive increments, namely

$$C(t+1) - C(t) = D_{M(t) \to M(t+1)}$$

are correlated through $M$ ; as we will see, errors made when neglecting dependency may be arbitrarily large, so we could not use research on Lévy processes to find the sought default expectancy.

### 1.3.1 Martingale parameter

However, we managed to extend the notion of Lundberg's parameter to such C-processes. Working with Laplace transforms of transition payoffs (as suggested by [41]), we build the notion of a Laplace transform for a C-process $C$, expressing it as a matrix function $L_C$ instead of a real function. We remark that this Laplace matrix function extends the usual multiplicative properties of Laplace transforms for Lévy processes, and use this property to create the equivalent of Lundberg's parameter for C-processes : this yields its martingale parameter, but also dominant eigenvectors to this matrix that will govern the default expectancy. Actually, for $C$ a non-pathological C-process and $a \in \mathbf{R}^+$ be a Laplace parameter, we have this statement.

**Proposition 1.3.1** *Martingale parameter*
*Except perhaps $\alpha = 0$, there is a single $\alpha \in \boldsymbol{R}^+$ such that $e^a$ is the dominant eigenvalue of $L_C(\alpha)$, called C's martingale parameter at point a. The eigenspace associated with the dominant eigenvalue $e^a$ of $L_C(\alpha)$ has dimension one, and may be directed by $w^{(a)}$ a scaled positive vector, called C's dominant eigenvector.*

Most importantly, this $\alpha$ allows to ensure that the process defined by

$$X_C^{(a)} = \begin{pmatrix} \mathbf{N} & \to & \mathbf{R}^+ \\ t & \to & w_{[M(t)]}^{(a)} e^{-\alpha(a)C(t)} e^{-at} \end{pmatrix}$$

is a martingale. Likewise, the single $-\omega \in \mathbf{R}^-$ holding the same properties is called the negative martingale parameter, and may be used for symmetrical purposes.

### 1.3.2 Default expectancy

We use this martingale property to find the default expectancy, using the default time $T$ as a stopping time. Our scheme is to prove that the distribution of the multiplicative term

$$w_{[M(T)]}^{(a)} e^{-\alpha(a)C(T)}$$

in front of the desired default time is “roughly independent” of $T$, in the sense that $T$ has approximately no incidence on the way $C$ defaults. This issue has already been investigated in [42], but in either the classical Cramér-Lundberg model ([7]) or even the (closer to our study) Markov-modulated risk model ([2]), continuous time and purely negative jumps are required. Our study presents additional advantages of requiring neither of these, and also investigating cases of slow or quick convergence to Cramér-Lundberg's approximation. In our model of C-processes, we call it the fundamental approximation, that holds for non-pathological C-processes.

**Theorem 1.1** *Fundamental approximation*

*For every* $a \in \boldsymbol{R}^+$*, let* $\alpha(a)$ *be* $C$*'s martingale parameter at point* $a$ *and* $w^{(a)}$ *be the associated dominant eigenvector. There is a continuous function* $K : \left(\boldsymbol{R}^+ \to \boldsymbol{R}\right)$ *such that for every* $i \leq A$*, the log-Laplace transform of* $T$ *giving* $M(0) = A_i$ *and* $C(0) = C_0$ *holds*

$$-\Lambda_T(a) \in \left[\left(\alpha(a)C_0 - \ln\left(w^{(a)}_{[i]}\right) + K(a)\right) \pm e(C_0, a)\right]$$

*where* $e$ *is a non-negative error function, uniformly convergent to* 0 *over any compact subset of* $\boldsymbol{R}^+$ *in* $a$*, i.e.*

$$\forall a_0 \in \boldsymbol{R}^+, \forall \epsilon > 0, \exists C_{a_0} \in \boldsymbol{R}^+; \forall C_0 > C_{a_0}, \forall a \leq a_0, e(C_0, a) < \epsilon$$

This is the main result of our first study ; in other words, it states that $C$'s default expectancy holds

$$\mathbf{E}\left(e^{-aT}\right) \approx Z(a) w^{(a)}_{[M(0)]} e^{-\alpha(a)C(0)}$$

While the martingale parameter $\alpha(a)$ is similar to Lundberg's exponent, and $Z(a)$ still expresses some kind of “default severity” as in [18], the dominant eigenvector $w^{(a)}$ expresses a new idea, describing how states of $M$ are helpful or harmful to the player. In some extent, this is a sharper approximation than suggested in [41].

## 1.4 Investment

Using the fundamental approximation, players are able to compute their default expectancies in C-games by means of backward induction, and thus solve investment decisions.

### 1.4.1 Incentive and handicap

Most often, we find out that investing for a price $I \in \mathbf{R}^+$ is optimal whenever the player has high enough cash reseves, the threshold amounting to $B = I/\gamma(a) + H$ where

— $\gamma(a)$ is called the incentive to buy, that depends only on the martingale parameters of the C-processes obtained with and without buying. In the case of a single "take it or leave it" investment opportunity, one gets

$$\gamma(a) = 1 - \frac{\alpha_1(a)}{\alpha_2(a)}$$

where $\alpha_i(a)$ denotes the martingale parameter, $i = 1$ corresponding to rejection and $i = 2$ to acceptation of investment.

— $H(a)$ is called the handicap, whose form depends on the model, and most notably on the specificities of each state reached after the investment decision (through the dominant eigenvectors $w$ above). In the same model,

$$H(a) = \frac{\ln\left(\frac{Z_2(a)w_2^{(a)}}{Z_1(a)w_1^{(a)}}\right)}{\alpha_2(a) - \alpha_1(a)}$$

in the terme of the fundamental approximation.

For example, when an investor is presented with a renewable investment opportunity, the concept of "value of waiting" investigated in [33] still appears and is quantified.

**Proposition 1.4.1** *Decision when allowed to wait*

*Let us deem that the C-game at present time $t$ involves*

— *A C-process $C^{[G(t)]}$ before buying, characterized by a default expectancy*

$$\boldsymbol{E}\left(e^{-aT}\right) \approx Z_1(a)e^{-\alpha_1(a)C(0)}$$

*and a negative martingale parameter $\omega(a)$ ;*

— *A default expectancy after buying (on the node $C^{[G(t),2]}$), characterized by an exponential parameter $\alpha_2(a)$ associated with a multiplicative parameter $Z_2(a)$ ;*

— *A threshold $b \in \boldsymbol{R}^+$ such that the player's strategy on node $G(t)$ is to buy whenever an investment opportunity is hit at some time $t$ when $C(t) \geq b$, yielding a stopping time $\tau$.*

*There are computable constants $Y_0(a)$, $Y_1(a)$ and $Y_2(a)$ such that the approximative best threshold $b$ is expressed by*

$$b = \frac{I}{\gamma^*(a)} + H^*(a)$$

*where the new incentive is still $\gamma^*(a) = \gamma(a)$, and the new handicap is $H^*(a) = H(a) + H^+(a) - H^-(a)$ with the additional handicaps are given by*

$$H^+(a) = \frac{\ln\left(\frac{(\alpha_2(a)+\omega(a))}{(\alpha_1(a)+\omega(a))}\right)}{\alpha_2(a) - \alpha_1(a)}$$

*and*

$$H^-(a) = \frac{\ln\left(\frac{Y_1(a)}{Y_2(a)}\right)}{\alpha_2(a) - \alpha_1(a)}$$

*In particular,* $H^+(a)$ *and* $H^-(a)$ *are always positive.*

### 1.4.2 Support contracts

Let us now tackle the issue of cooperative management in a Markovian environment through a real-life simple example. We start by the assumption that $J_1$ and $J_2$ are players whose incomes depend on the same market indicator in contrary motions. For instance, $J_1$ owns an oil company (his earnings increase when oil prices increase) and $J_2$ manufactures cars (when oil prices increase, demand drops and her incomes decrease). In this situation, both players are threatened by an adverse variation of oil prices, and suffer from a high default probability. This market may be modelled by a C-process with two states $A_1$ and $A_2$ coding respectively for high and low oil prices, describing $M$ the oil market, and a tendency to momentum : $M$'s transition probabilities hold $P_{i\to i} > P_{i\to 3-i}$. As we see in the first study, this momentum effect worsens both player's default probabilties, so we look for an agreement between $J_1$ and $J_2$ in order to escape from bankruptcy.

The notions of state specificities in a C-process may indicate that some states of the market generate a discrepancy between individuals : in our example, $A_1$ is profitable to $J_1$ and detrimental to $J_2$, and $A_2$ reverses these effects. This gave us the intuition that an agreement "cancelling" these discrepancies may be profitable to both. Like with the market of call and put options, let us adjudicate some values $v_1 < 0$ to the state $A_1$ and $v_2 > 0$ to $A_2$ so that $J_1$ recieves $v_i$ from $J_2$ when $M$ hits $A_i$ :

— If $M(t+1) = A_1$, $M$ turns profitable to $J_1$, so $J_1$'s incomes are higher than expected while $J_2$'s ones are lower. This is compensated by a payment of $-v_1$ to $J_2$, levelling out the discrepancy of returns induced by $M$.

— Likewise, if $M(t+1) = A_2$, $M$ turns profitable to $J_2$, and this time $J_2$ pays $v_2$ to $J_1$ to offset the discrepancy.

Players thus eliminate the risks due to players' discrepancies with respect to $M$.

**Definition 1.4.1** *Support contract*

*Let* $J_1, J_2$ *be two players subject to C-processes* $C_1$ *and* $C_2$ *in a given node of* $G$*, that are deemed independent conditionally to the underlying Markovian process* $M$*. A support contract is the determination of a C-process* $S$ *over* $M$*'s state space, such that the processes giving* $J_1$*'s and* $J_2$*'s assets under* $S$ *are given by*

$$C_1'(t) = C_1(t) + S(t) \wedge C_2'(t) = C_2(t) - S(t)$$

*Moreover, the transition payoffs of $S$ are deemed to depend only on $M$'s transition at present time, each transition payoff given $M(t) = A_i$ and $M(t+1) = A_j$ is a deterministic value $s_{i\to j} \in \boldsymbol{R}$.*

A support contract may thus be thought of as the averaging of rents in Monopoly : as state specifities are removed, the players may agree on the value of each one's investment, and pay the average permanent mean rent instead of the rent indicated by the rules of the game. We eventually aim at finding the best choices for $s_{i\to j}$ so that both players decrease their default expectancies.

**Proposition 1.4.2** *Best support contract*

*At any equilibrium support contract $S^*$, the players have the same Laplace matrix functions at points $a_p$ up to the discount factors, i.e*

$$L_{C_1+S^*}\left(\alpha_1(a_1)\right)e^{-a_1} = L_{C_2-S^*}\left(\alpha_2(a_2)\right)e^{-a_2}$$

*where $\alpha_p$ refer to the martingale parameters of the modified C-processes $C_p \pm S^*$.*

As a noticeable consequence, the dominant eigenvectors of $C_p \pm S^*$ are identical, which means that the discrepancies between players with respect to $M$ are totally erased.

### 1.4.3 Mutualization of risk

A natural continuation to this concept of support contract, compensating for one's "bad luck" with $M$, is the "pooling" of assets : securitization has the well-known effect of reducing risks ([31, 44]) for Lévy-based models, and we eventually prove that it is still optimal with C-processes. The outline behind optimality lies with the martingale parameters : pooling $C_1$ and $C_2$ into a single $C$ yields a better martingale parameter, thus should reduce default risks.

**Proposition 1.4.3** *Sub-additivity for $\alpha^{-1}$*

*Let $C_1$ and $C_2$ be C-processes with martingale parameters $\alpha_1$ and $\alpha_2$. We deem that $C = C_1 + C_2$ has $\alpha$ as a martingale parameter.*

1. *For every $a \in \boldsymbol{R}^+$,*

$$\frac{1}{\alpha(a)} \leq \frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}$$

2. *This inequality is an equality iff there are a constant $u \in \boldsymbol{R}_+^*$ and a globally constant C-process $C^=$ such that almost surely,*

$$\forall t \in \boldsymbol{N}, C_2(t) = uC_1(t) + C^=(t)$$

In particular, when $J_1$ and $J_2$ have the same Laplace parameter $a$, if they merge their incomes and share them with a ratio $\alpha_2(a)/(\alpha_1(a)+\alpha_2(a))$ to $J_1$, setting

$$x(a) = \alpha(a)\left(\frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}\right) \geq 1$$

then $J_1$'s new martingale parameter is $\alpha_1(a)x(a)$ and $J_2$'s one is $\alpha_2(a)x(a)$ ; both have increased, so both default expectancies should be reduced. We also find equilibrium support contracts for $a_1 \neq a_2$ ; in the main line, the total payoff $C(t+1) - C(t)$ should be split such that

$$C_1(t) = rC(t) + \frac{(a_2 - a_1)r(1-r)}{\alpha\left(ra_1 + (1-r)a_2\right)}t \wedge C_2(t) = (1-r)C(t) - \frac{(a_2 - a_1)r(1-r)}{\alpha\left(ra_1 + (1-r)a_2\right)}t$$

for some $r \in [0,1]$, i.e. in an "affine function" fashion whose constant term favors the longer-sighted player.

## 1.5 Outlines

We may finally relate the studies with the game of Monopoly, particularly in perspective with [17], to understand the best investment strategies. This paragraph enlights interpretations of the ideas presented during the study in Monopoly terms. For the sake of explanations, we will work with the standard rules and terminology of Monopoly in examples, numbering the board squares in ascending order starting from 0 "Go".

### 1.5.1 Structure of a C-process

Let us assume that $J_1$ owns a whole color group of properties and may elect to develop them. However, he knows that the cards "street repairs" is about to come in the cycle of chance cards. A sound strategy for $J_1$ is to wait for these cards to appear, and only then to develop, even if the expected rent money overcompensates the penalty incurred by having to repair one's properties (when the martingale parameter $\alpha^{[2]}(a_1)$ when developping exceeds $\alpha^{[1]}(a_1)$ when not doing so). The reason behind postponing investment lies in the dominant eigenvector $w^{[1]}$. By postponing investment, $J_1$ escapes from a specifically "bad" state for $C^{[2]}$ (the one with incoming street repairs) and thus avoids an adverse term in the dominant eigenvector : for an identical incentive, $J_1$ has a worse handicap $H(a_1)$ when postponing investment. We also notice that with "take it or leave it" investment (e.g. developping when there is a risk of housing shortage), we do not encounter this phenomenon unless liquidity risks are consequent. Indeed, comparing the approximated default expectancies $F^{[2]}$ with investment and $F^{[1]}$ without investment yields

$$\frac{F^{[2]}}{F^{[1]}} = \frac{Z^{[2]} w_{[M(0)]}^{(a_1),[2]}}{Z^{[1]} w_{[M(0)]}^{(a_1),[1]}} e^{\alpha^{[2]}(a_1) I} e^{-\left(\alpha^{[2]}(a_1) - \alpha^{[1]}(a_1)\right) C(0)}$$

The equations of incentive and handicap lead to the threshold of investment ; the higher the handicap (corresponding to street repairs penalty), the higher $w_{[M(0)]}^{(a_1),[2]}$, and so the threshold increases, even if investment is still beneficial for high enough cash reserves.

Momentum effects are inherent to the structure of a C-process $C$, describing how $C$ is apart from being a Lévy process, and we may quantify them using $C$'s spread term. We chose instead to look at [41], doing the study with the Laplace matrix function and using an approximative Brownian motion (estimating a drift and a variance ; we know after our second study that their drift estimator converges to our mean expectancy $E(C)$, and we also expect their variance estimator to converge to our first study's $V(C)$). They find out that the ruin probability comes close to the one found in their Markovian model, the same way we remarked that $2E(C)/V(C) \approx \alpha(0)$. For this reason, we will henceforth mainly characterize $C$ by $E(C)$ and $V(C)$ during this discussion.

### 1.5.2 Discussion about the investment strategies

It is popular Monopoly wisdom that the orange color group represents good investment, and the green color group a bad one ([17]). The reason behind this is the return on investment rate for these properties : according to the standard rules, the cost of fully developping the orange group is \$2060 for an expected drift of \$85 a turn, with a return on investment time of 24.4 turns, while the green color group costs \$3920 for an expected \$104 a turn, with a return on investment time of 37.7 turns (this computation has been done taking $M$'s invariant measure and computing mean expectancies of developping properties ; we do not count the negligible cost of street repairs, amounting at approximately to \$1 per turn per hotel). We agree with it on a specific point : when liquidity is limited, and investment is limited by players' assets, the orange properties are more advisable, as is recovered by [17]'s simulations.

However, we believe that translating [17] to a two-player model suffers from an oversight concerning "blocked" endgames, when assets are balanced between players and the game proceeds eternally because players' means expectancies are positive (the "Go" salary owerweights all payments to bank). [17] halts the study after 500 game turns, and we believe that a horizon effect appears way later during the game for the following reasons :

— With two-player balanced endgames without any possible development (all color groups are split), green properties actually become the *best* because of higher rents. Assuming e.g. that $J_1$ owns properties 19, 31 and 32 while $J_2$ owns 16, 18 and 34 yields \$1.9 a turn from $J_2$ to $J_1$ but only \$1.5 a turn from $J_2$ to $J_1$. Over 500 turns, the expected flow is \$200 to $J_1$, hardly enough to be observable and rank players by wealth ; however, infinite games should always favor $J_1$ thanks to the law of large numbers.

— When development has been done and the game lasts for more than 500 turns, it is highly likely that rent money is balanced and mean expectancies are positive. Assuming that $J_1$ owns 3 houses per green property against $J_2$'s hotels on orange properties, investment costs are comparable while $J_1$'s advantage in rent money does not cover $J_2$'s salary. Eventually, $J_2$ will be able to proceed with development while $J_1$ hit a ceiling investment ; while $J_1$'s rent amounts to \$85 a turn, $J_2$'s one is \$104 a turn, insufficient to bankrupt $J_1$ but still winning the game with an infinite horizon.

Knowing after [41] that around 12% of games did not terminate after 500 turns, and assuming that luck determines the leading player at time 500, we believe that the owner of green properties has a winning probability undervalued by 6% in [17]'s model.

We recover these considerations in our study about the martingale parameters. When $J_2$ purchased orange properties, creating a drift and a variance term, purchasing green properties yields higher incomes (making $E(C) > 0$), while a reasonable variance term is added to $V(C)$ (as player's trajectories on the Monopoly board are close to be independent, there is no correlation effect between $J_1$'s and $J_2$'s incestment). We expect $J_1$ to have a higher martingale parameter thanks to his mean expectancy, which is especially favorable for high cash reserves : once again, we recover that heavy investment costs are justified when liquidity shortages are not an immediate concern. However, when $J_2$ has no color group, heavy investment is not justified unless $J_1$ owns a tremendous amount of cash ; indeed, when incomes are not guaranteed and investment creates a variance term while the previous one was small, the martingale parameter increases by a small quantity, indicating that the incentive is small : the higher the marginal investment costs, the higher the buying threshold. In Monopoly terms, the orange color group, with good return on investment times, is a better “first investment” than the green one ; however, the green one wins if players enter a long-run game. In other words, huge investment costs are acceptable when one expects no quick liquidity issues ; they are suitable to an optimistic view of the future.

# Chapitre 2

# Théorie de la ruine pour des processus stochastiques à environnement markovien

Nous nous intéressons ici au temps de faillite d'un actif dont le cours suit des variations temporellement dépendantes. Comme indiqué dans l'introduction générale, nous avons décidé de simuler cette dépendance temporelle par l'adjonction d'un processus markovien sous-jacent, indiquant l'état du marché générant ainsi les caractères dépendants des fluctuations du cours.

## 2.1 Résumé

Ce résumé présente les grandes lignes du travail, présentant les notions capitales et les méthodes utilisées pour parvenir au théorème principal de l'étude : l'approximation de Cramér-Lundberg pour des C-processus.

### 2.1.1 Modèle et outils

Nous commencerons par créer le modèle d'étude adéquat pour répondre à notre question, que l'on nommera C-processus. Il se compose à la fois du cours de l'actif lui-même, représenté par un processus stochastique $C$, et de la description du marché, codé par un processus markovien $M$. La donnée des lois de la dynamique du couple $(M, C)$ caractérisera donc le modèle construit. $M$ figurant sur un espace d'états $(A_i)_i$ désignant chacun un "état du marché", on définit $C$ via ses incréments successifs à l'aide de

$$C(t+1) - C(t) = D_{M(t)\to M(t+1)}(t+1)$$

où les paiements de transition $D_{i\to j}(t)$ forment des familles en les couples d'états $(i,j)$, dont les lois sont indépendantes et identiquement distribuées en $t$. Ainsi, $M$ est le seul responsable de la dépendance temporelle entre les transitions successives ; la loi de $C$ est donc intégralement définie par

— La matrice de transitions $P$ de $M$, dont les éléments sont notés $P_{i\to j}$ ;
— Les lois des incréments $D_{i\to j}(t)$ pour tous les états $A_i, A_j$, ces incréments étant nommés paiements de transition de $C$ ;
— Les conditions initiales $C(0)$ et $M(0)$.

Cette structure de C-processus sera d'abord étudiée d'un point de vue calculatoire : on exhibera ses caractéristiques, étendant ainsi de nombreux concepts naturels trouvés dans l'étude d'un seul processus de Lévy $L$. Ainsi : support, croissance, récurrence, et espérance de $L$ se traduisent respectivement par cycles, croissance globale, récurrence positive et espérance moyenne de $C$. Cette traduction permettra non seulement de mieux expliquer le concept de C-processus, mais aussi d'introduire des concepts nécessaires à l'établissement du résultat principal sur leur temps de faillite.

Pour évaluer les risques de faillite, nous souhaitons procéder comme dans les cas de processus de Lévy : évaluer l'exposant de Lundberg, puis montrer l'approximation de Cramér-Lundberg comme dans [7, 25]. Toutefois, le caractère markovien de $C$ empêche de résoudre directement l'équation de Lundberg

$$\mathbf{E}\left(e^{-\alpha(C(t+1)-C(t))}\right) = 1$$

impliquant la transformée de Laplace de l'incrément $C(t+1)-C(t)$ de $C$, parce que sa loi n'est pas stationnaire. Cette difficulté sera contournée par la création de la transformée de Laplace du processus $C$ dans son ensemble, dont la structure naturelle est celle d'une matrice, que l'on nommera transformée matricielle de Laplace de $C$ notée $L_C$, laquelle est définie en tout point $\alpha$ assurant sa convergence par

$$L_C(\alpha) = \left(P_{i\to j}\mathbf{E}\left(e^{-\alpha D_{i\to j}}\mathbf{1}_{D_{i\to j}<\infty}\right)\right)_{i,j}$$

On indiquera en particulier les propriétés essentielles de $L_C$, afin de s'apercevoir que l'équation de Lundberg prendra la forme vectorielle $L_C(\alpha(a))w = we^a$ (où $a$ est un paramètre positif libre) : la solution $\alpha(a)$ en sera nommée paramètre martingalisant de $C$, et $w^{(a)}$ son vecteur propre dominant au point $a$. On démontrera donc l'existence et l'unicité du paramètre martingalisant sous certaines conditions évoquées lors de l'analyse préliminaire, étendant donc la notion d'exposant de Lundberg aux C-processus. En particulier, on construira une martingale à l'aide de $\alpha$, $w$ et $C$ permettant d'en expliquer les rôles.

### 2.1.2 Théorème et preuve

L'objet de ce chapitre est d'évaluer la transformée de Laplace du temps de faillite de $C$ via $\alpha$ et $w$. On aura donc recours à un schéma de preuve similaire à l'approximation de Cramér-Lundberg pour des processus de Lévy (voir par exemple [25, 42, 7, 12]), avec toutefois des difficultés supplémentaires :

— Puisque $M$ influe sur $C$, il est obligatoire de traiter simultanément toutes les dispositions du couple $(M, C)$ au lieu du seul $C$. On y parvient via l'usage de $L_C$.

— Nous ne disposons d'aucune hypothèse sur la direction des sauts successifs de $C$. En particulier, les sauts "vers le haut" dans ce modèle à temps discret empêchent l'usage direct d'une idée à base d'équation de renouvellement, et un travail additionnel est requis pour n'avoir à traiter que des sauts "vers le bas".

— Nous ne sommes pas affranchis automatiquement des contraintes liées à la périodicité du support de $C$ comme en temps continu. C'est en particulier ici qu'un examen approfondi des cycles de $C$ se révèlera nécessaire, afin de mettre en évidence une hypothèse complémentaire indispensable au théorème désiré.

L'équation finalement obtenue étend naturellement celle de l'approximation bien connue de Cramér-Lundberg aux C-processus :

$$\mathbf{E}\left(e^{-aT}\right) \approx Z(a) w^{(a)}_{[M(0)]} e^{-\alpha(a) C(0)}$$

où $\alpha(a)$ est le paramètre martingalisant, $w^{(a)}$ est le vecteur propre dominant au point $a$, et $Z(a)$ est un facteur multiplicatif au point $a \in \mathbf{R}^+$ codant pour le taux d'escompte relatif au temps de faillite. On indiquera aussi la précision de cette approximation, montrant qu'elle est digne de foi dans les cas les plus naturels, et pourtant n'est pas améliorable en général.

Au final, on discutera sur la nature des équations et objets obtenus lors de l'étude, ce qui permettra d'interpréter les concepts de risque d'un C-processus. En particulier, on pourra caractériser un analogue de la variance d'un processus de Lévy, montrer comment les C-processus permettent de générer des phénomènes de bulles et crises économiques (ce dont les modèles à base de Lévy sont incapables faute de dépendance temporelle), et pourquoi ceux-ci nuisent à la stabilité du marché malgré des volumes d'échanges modestes.

# Ruin theory for Markovian-governed stochastic processes

We introduce a new class of processes aiming at modelling random fluctuations of an asset value more efficiently than traditional Lévy processes. In this study, we consider that the object value $C$ is a real discrete random process ($\mathbf{N} \to \mathbf{R}$), whose increments are subject to the present state of a "market", described by a Markovian process $M$ : as the successive market states are not pairwise independent, $C$'s fluctuations are not independent either, so $C$ cannot be assimilated as a Lévy process. We call this structure a C-process : we present methods to analyze it, mainly extending the notion of Lundberg's parameter of a diffusion Lévy process, to take $M$ into account during the computations. Once done, we aim more specifically at controlling $C$'s default time $T_0 = \min(\{t \in \mathbf{N} | C(t) < 0\})$ : we achieve it through its log-Laplace transform to get some of its properties, like $C$'s default probability.

## 2.2 Introduction

In this study, we address the question of ruin theory under some non-stationary behaviour of the market. We present an alternate process structure, whose aim is to take into account short-termed time dependency of successive random fluctuations. More specifically, we want to observe the effects of "momentum" behaviour of a stochastic process on its default risks and quantify them, then putting them in contrast with default risks of classical Lévy processes.

The problem of ruin theory has already received a lot of attention : starting from the classical Cramér-Lundberg model ([7]), it has been subsequently extended to the case when the risk process is a more general Lévy process (for instance, as given in [23]). More recently, [37] focused on ruin problems for a company investing capital in risky assets, raising the question of how the financial market influences the ruin probability. These models consider an agent in a purely speculative economy, whose assets are deemed to be an amount of cash $C$, randomly fluctuating over time : at time $t$, the assets amount to value $C(t)$. The aim of this study is to introduce a new model for $C$'s fluctuations and control $C$'s risks of default in it, for example its default probability ($C(t)$ falling down to negative values for some time $t$), or some properties about the distribution of its default time, given some properties of the process $C$. Specifically, we want to emphasize on correlation between $C$'s successive variations and an exogenous "market" $M$, whose configuration may influence $C$.

### 2.2.1 Motivation for a new model

After the basics of speculation theory came with Bachelier ([4]), introducing the notions of Brownian motions to model $C$, several refinements of Brownian motions have been suggested, eventually leading to the theory of Lévy processes used by most traditional models ([6]) : they introduce the notion of sudden jumps for $C$, as well as the ideas behind "fat tails" for the distribution of $C$'s increments, to improve the fit between the model and real observations. Hence, $C$ is commonly chosen to be a Lévy process (and in particular, a drifted Brownian motion or a variation thereof) with no need to define a market $M$, like in [43] ; indeed, default theory was introduced by the model of Cramér and Lundberg, and one may refer to [7] to investigate on the risks of bankruptcy for an insurance company. When $C$ is a Lévy process, ruin theory has already been investigated by [12], using Lundberg's exponent to evaluate the risks of default with accurate results ; as in [25], we shall extend this tool through scale functions to get results about the Laplace transform of the default time. For example, in McKean's selling problem ([34]), one eventually finds out that Lundberg's exponent governs the final result.

Unfortunately, models relying on Lévy processes are likely too restrictive, because they ignore the possible changes of the market environment (the market is not stationary, it may have momentum, etc.). In particular, independence of successive variations is not realistic, be it because of endogenous behaviour of investors towards momentum ([9, 21]), or even an exogenous consequence of price volatility ([27]). For example, speculative bubbles or busts are often the results of

- Positive feedback effects : one is encouraged to buy because prices are likely to increase, and more demand means higher prices ;
- Exogenous effects : financial markets may exhibit cycles due to some political decisions, or regulatory policies ; therefore, the modelled company may not be able to cover possible short-term losses by selling its assets because the market is suddenly becoming illiquid (or conversely, may benefit from these policies and should take them into account to extract maximum profits) ;
- States of the market : in a so-called "boom-bust" economy, the prices are

driven one way or the other depending on whether they were in a boom or bust phase, so one should always beware of the looming danger of the market suddenly turning around.

As we shall see during this study, estimations by means of Lévy processes severely underestimate the risks underlying behind positive correlation between fluctuations of the market, due to the persistence of an underlying market effect, which may eventually lead to disasters like the financial crisis of 2008. Hence, the aim of this study is to tackle the default problem for a broader class of discrete cash processes, that exhibits some persistence in $C$'s fluctuations.

### 2.2.2 Time dependency

To improve accuracy, we focused on taking into account some time dependency between fluctuations of $C$, as well as the "state of the market" affecting the random variables of its successive variations : hence, we need to define a new process $M$ indicating the market configuration, and whose successive states govern $C$'s variations. The most natural model allowing some short-term dependency is of course the Markovian process : information on the state of the market $M$ is required to be Markovian, and it governs the values of $C$'s fluctuations in turn.

More specifically, we want to exhibit the effects of market local trends on $C$ on its default risks, and interpret these results in order to help computing trading decisions in a speculative and volatile market, e.g. deciding whether to buy/sell a risky asset or not in order to avoid eventual bankruptcy. Well-known examples are investment dilemmas, where the investor must risk an immediate liquidity shortage to benefit from a long-term income, or McKean's problem ([34]), where one aims at optimizing the selling time of an option. For this purpose, we introduce a class of discrete Markov processes that extends the class of Lévy processes in discrete time, involving :

- An auxilliary Markovian process $M$, describing the state of the market ;
- Multiple possible random increments for $C$, being random variables whose distributions depend on the evolution of the market. The actual increment of $C$ between time $t$ and time $t+1$ will be the one associated to the market evolution, i.e. the states $M(t)$ and $M(t+1)$.

It should be noted that $M$ is exogenous and describes the whole market : whereas $M$ affects $C$'s increments, $C$ does not affect $M$ in return. This assumption may be backed up by the fact that $M$ alone suffices at describing the state of the market, so in a sense it "incorporates" $C$, or simply because the agent is deemed atomic and has negligible incidence on the market. Giving $M$ and $C$ leads to the definition of what will be called a C-process in this study.

### 2.2.3 Concept of the study

In particular, a similar example of risk analysis in such a model where $C$ describes the assets of an insurance company (under specific constraints on $C$'s increments) is named "Markov-modulated risk model" and has already been investigated, for example ([2, 42]). Despite yielding powerful results on default risks, we remarked that the Markov-modulated risk model failed to exhibit some specific facts of general C-processes, because corner stones of its behaviour are required to state the desired properties, that are not guaranteed in our work, like

- Continuous time : piecewise continuous trajectories of $C$ allow it to hit every positive value, which is a necessary hypothesis for the default asymptotics to work. Using discrete time allows us to introduce the issue of cycles and periodicity, as well as explaining why the given asymptotic is optimal.
- Purely negative jumps : in standard Markov-modulated risk models, $C$'s only jumps called "claims" are alwalys negative, which allows for a fairly easy study through the use of a renewal equation ([42]). In this study, we will not make assumptions on the jumps' direction. Consequently, one needs additional work to obtain such a renewal equation that gives the desired ruin probability estimation. Alternatively, one may refer to [45] if interested in jumps only.

We shall also focus on the interpretations of the main results in terms of econometrics, solving the elementary investment problem with comments on the corresponding solution, emphasizing on the differences between Lévy-like processes and C-processes in terms of default risks.

This study is thus organised as follows :

- Paragraph 2.3 presents the notion of C-processes and studies several of its features ;
- Paragraph 2.4 shows the theory : we extend the notion of Lundberg's exponent from Lévy processes to C-processes, allowing then to state the main theorems ;
- Paragraph 2.5 shows some examples of application, indroducing and using some links with Lévy-like risk processes ;
- Paragraph 2.6 discusses and concludes.

## 2.3 Model

This paragraph explains the model that will be used throughout this study. It features a Markovian environment $M$ governing a random process $C$, and will be central to our work.

### 2.3.1 Main definitions

We begin with the contents of our universe, i.e. the C-process itself and its default time, and then we shall introduce some useful tools to the analysis.

**Universe**

We consider the following discrete-time universe with a real-valued random process $C : (\mathbf{N} \to \mathbf{R})$.

**Model 2.1** *C-process*

*A C-process is the determination of*

— *A Markovian time-homogeneous process* $(M(t))_{t\in\boldsymbol{N}}$ *with a finite state space* $(A_i)_{i\leq A}$ *with* $A \in \boldsymbol{N}^*$*, whose transition probabilities are defined through*

$$\forall i, j \leq A, P_{i\to j} = \boldsymbol{P}(M(t+1) = A_j | M(t) = A_i)$$

*for any* $t \in \boldsymbol{N}$*. Moreover,* $M(0)$ *is deemed determinisitc ; the* $A_i$ *in* $M$*'s state space such that* $M(0) = A_i$ *almost surely is called* $M$*'s (or* $C$*'s) starting state.*

— *Random variables called transition payoffs, such that*

  — *For every* $i, j \leq A$*, there is a probability distribution over* $\boldsymbol{R} \cup \{+\infty\}$ *defining a random variable* $D_{i\to j}$ *with respect to this distribution ;*

  — *For every* $t \in \boldsymbol{N}^*$*, we define the family* $(D_{i\to j}(t))_{i,j\leq A}$ *to be an independent and identically distributed copy of the family* $(D_{i\to j})_{i,j\leq A}$*, with respect to the time variable* $t \in \boldsymbol{N}^*$*.*

*For every* $i, j \leq A$ *and* $t \in \boldsymbol{N}^*$*,* $D_{i\to j}(t)$ *is called the transition payoff between states* $A_i$ *and* $A_j$ *at time* $t$*.*

— *A process* $C$*, whose initial value* $C(0) = C_0 \in \boldsymbol{R}^+$ *is deterministic and called* $C$*'s starting point, and whose increments hold*

$$\forall t \in \boldsymbol{N}, C(t+1) = C(t) + D_{M(t)\to M(t+1)}(t+1)$$

*We say that* $C$ *is a C-process whose underlying Markovian process is* $M$*, and transition payoffs are (the distributions of) the random variables* $D_{i\to j}$*, for each state numbers* $i, j \leq A$*.*

When there is no possible confusion, we will commonly abbreviate for the sake of simplicity :

— For $i \leq A$, $i$ for $A_i$ or the reverse, especially when noting (vector) indices ;

— For $X$ a random variable identically equal to $x$ almost surely, $X$ for $x$ or the reverse ;

— When indicating $C$'s increment between time $t$ and time $t+1$, $D(t+1)$ for the transition payoff $D_{M(t)\to M(t+1)}(t+1)$, called $C$'s active increment at time $t+1$.

The set of transitions $(i \to j)$ happening with probability $P_{i\to j} > 0$ will be called $\Gamma \subseteq [|1, A|]^2$. We shall say that $C$ is

— Bounded iff $\forall (i,j) \in \Gamma$, $D_{i\to j}$ is an almost surely bounded random variable ;

— Strongly exponentially integrable (henceforth abbreviated as "sEI") iff

$$\forall (i,j) \in \Gamma, \forall \alpha \in \mathbf{R}, \mathbf{E}\left(e^{-\alpha D_{i\to j}}\right) < \infty$$

Our study aims at controlling $C$'s default time, i.e. the first time $t \in \mathbf{N}$ when $C(t) < 0$, noted by the random variable $T_0$.

**Definition 2.3.1** *Default time*

*The default time of a C-process $C$ is the random variable*

$$T_0 = \min\left(\{t \in \boldsymbol{N}; C(t) < 0\}\right)$$

*If this set is empty, we set $T_0 = \min(\emptyset) = +\infty$.*

Among several properties, we chose to look for its Laplace transform, because this simplifies the computations, and making $a = 0$ allows us to recover $C$'s default probability.

**Definition 2.3.2** *Laplace transforms*

*The Laplace transform of a random variable $X \in \boldsymbol{R} \cup \{+\infty\}$ is defined by*

$$L_X = \begin{pmatrix} \boldsymbol{R} & \to & \boldsymbol{R}^+ \cup \{+\infty\} \\ a & \to & \boldsymbol{E}\left(e^{-aX}\,\boldsymbol{1}_{X<+\infty}\right) \end{pmatrix}$$

*Its log-Laplace transform is*

$$\Lambda_X = \begin{pmatrix} \boldsymbol{R} & \to & \boldsymbol{R} \cup \{\pm\infty\} \\ a & \to & \ln\left(L_X(a)\right) \end{pmatrix}$$

*where* $\ln(0) = -\infty$.

Since a Laplace transform is a holomorphic function on its convergence domain, imposing $X < \infty$ in the definition has the same effect as considering its analytic continuation from $\mathbf{R}_+^*$ over $\mathbf{R}^-$ whenever possible.

### Characterization of the model

The first thing to notice regarding the model is that the only item preventing $C$ from being a Lévy process is the Markovian process $M$, introducing short-term dependency between successive increments of $C$. We are going to characterize this idea by the following proposition.

**Proposition 2.3.1** *Conditional independence*

*We consider*

- $C$ *a random discrete-time process, valued over* $\boldsymbol{R} \cup \{\infty\}$ *;*
- $M$ *a discrete-time, finite state space, time-homogeneous Markovian process, whose transition matrix is given by its entries* $P_{i \to j}$ *for* $i, j \leq A$ *;*
- *For every* $t \in \boldsymbol{N}$, $\boldsymbol{F}(t)$ *the natural filtration associated with the joint process* $(C, M)$ *up to time* $t$.

*The following statements are equivalent :*

1. $C$ *is a C-process whose underlying Markovian process is* $M$.
2. *We have simultaneously*
   - $C(0)$ *and* $M(0)$ *are deterministic ;*
   - *Markovian property : for every* $t \in \boldsymbol{N}$, $H$ *a mesurable subset of* $\boldsymbol{R}$ *and* $i \leq A$,

$$\begin{aligned} & \boldsymbol{P}(C(t+1) - C(t) \in H \wedge M(t+1) = A_i | \boldsymbol{F}(t)) \\ = \; & \boldsymbol{P}(C(t+1) - C(t) \in H \wedge M(t+1) = A_i | M(t)) \end{aligned}$$

   - *Time-homogeneous property : for every* $s, t \in \boldsymbol{N}$, $H$ *a mesurable subset of* $\boldsymbol{R}$ *and* $i, j \leq A$,

$$\begin{aligned} & \boldsymbol{P}(C(t+1) - C(t) \in H \wedge M(t+1) = A_j | M(t) = A_i) \\ = \; & P_{i \to j} \boldsymbol{P}(C(s+1) - C(s) \in H | M(s) = A_i \wedge M(s+1) = A_j) \end{aligned}$$

It follows from this property that C-processes are natural continuations of Lévy processes, since if $A = 1$ then $M$ is trivial and this definition is exactly the definition of a Lévy process, which means that there are no effects of an exogenous market, and increments have a stationary distribution. Conversely, a non-trivial $M$ allows to create hidden momentum effects on $C$'s increments. We get a simple example of this when building the following C-process :

- For some $p \in (0, 1)$, $M$ has 2 states, starts from $A_1$ and its transition matrix is
$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$
- Transition payoffs $D_{1 \to 1}$ and $D_{2 \to 1}$ are Gaussian of mean 1 and variance 1, while $D_{1 \to 2}$ and $D_{2 \to 2}$ are Gaussian of mean $-1$ and variance 1.

Observing the previous active increment $D(t)$ yields indications on $D(t+1)$. Assuming that $M(t)$ is in its stationary distribution, looking at $D(t)$ yields

$$\mathbf{P}\left(M(t) = A_1 | D(t)\right) = \frac{e^{-(D(t)-1)^2/2}}{e^{-(D(t)-1)^2/2} + e^{-(D(t)+1)^2/2}} = \frac{1}{1+e^{-2D(t)}}$$

which means in particular that the conditional expectancy of next increment given this observation may be decomposed depending on $M(t)$, eventually leading to

$$\mathbf{E}\left(D(t+1)|D(t)\right) = (2p-1)\frac{1-e^{-2D(t)}}{1+e^{-2D(t)}}$$

If $p > 1/2$, positive momentum effects are observed, as this expectancy keeps the sign of $D(t)$. Contrariwise, if $p < 1/2$, we have built a negative feedback process, whose successive increments tend to cancel out, while if $p = 1/2$ we actually have a Lévy process.

### Positive recurrence

Another useful hypothesis to ensure "permanent" behaviour of $C$ is positive recurrence of $M$. Indeed, the states $A_{i\leq A}$ of Markovian process $M$ we are going to analyze may split between several communicating classes, some of which are terminal (closed) ; as our goal lies in asymptotical considerations, we will often deem that $M$ itself already lies in some closed communicating class $A'$, which is tantamount to deem that $M$ is positive recurrent (over $A'$). Nevertheless, this is not sufficient to avoid issues of transience for $C$, because of the transition payoffs amounting to $+\infty$ that "push" $C$ to $+\infty$ without any possible return : $C$'s transition payoffs may be such that $C(t)$ is automatically driven to $+\infty$ if $M(t)$ hits some specific state $A_a$, voiding the default time ($T_0 = \infty$). These $+\infty$ payoffs translate to situations where the agent is safe from default : in real economy, they may stand for retirement from the market, or the fulfillment of some goal, or whatever is the final aim of the agent ; when looking at $C$'s default risks, a realization of such a payoff indicates that default will not ever happen. In this case, it is natural to say that $A_a$ is not "interestingly accessible" from the other states, and to remove $M$'s property of positive recurrence. This is done by the following definitions.

**Definition 2.3.3** *Positive recurrence*

*Let $C$ be a C-process, whose underlying Markovian process is $M$.*

1. *$M$ is said positive recurrent iff it holds the positive recurrence property for all states : for any $i, j \leq A$, there is $n \in \boldsymbol{N}$ such that for any $t$,*

$$\boldsymbol{P}\left(M(t+n) = A_j | M(t) = A_i\right) > 0$$

*Positive recurrence of $M$ implies existence and uniqueness of $M$'s invariant distribution without any coordinates equal to 0, that shall be noted*

$$\left(\mu_{[i]}\right)_{i\leq A} = \mu \in (0,1]^A$$

2. *$C$ is said positive recurrent iff one has the stronger condition*

$$\boldsymbol{P}\left(M(t+n) = A_j \wedge C(t+n) < \infty | M(t) = A_i \wedge C(t) < \infty\right) > 0$$

*Positive recurrence of $C$ implies positive recurrence of $M$ by construction.*

Finally, we will often deem $C$ to be sEI or even bounded throughout this study. In particular, we chose to avoid the concern of fat tails because
- Fat tails lead to integrability issues as well as wild behaviour for $C$'s trajectories, so one only gets poor results on $C$'s default time ;
- They do not suppress the concern about the dependency between successive increments of $C$ ;
- Similar behaviour to fat tails appears when considering a suitable $M$ yielding high momentum effects, as we shall see during the study.

Indeed, we will find out that apparently high moments of increments distributions (when looking at the market, or as prescribed in [29]) are actually a consequence of time dependency, so of $M$'s structure.

**Example**

Throughout this study, we shall illustrate this theory of C-processes by the following example process, hereby named "boom-bust process". To clarify the numeric values, let us deem that $C$ denotes a cash amount and the time step is one second.
- It has $A = 2$ states, where $A_1$ is named the "good" state and $A_2$ the "bad" state.
- The transition matrix governing $M$ is given by

$$P = \begin{pmatrix} .89 & .11 \\ .09 & .91 \end{pmatrix}$$

  where $M$'s starting state is $M(0) = A_1$.
- The transition payoffs have these distributions :
  - $D_{1\to 1}$ is Gaussian, of mean $3\$/s$ and variance $1\$^2/s$ ;
  - $D_{1\to 2}$ is Bernouilli, amounting to either 1 or $0\$/s$ with even probability ;
  - $D_{2\to 1}$ is a constant $-2\$/s$ ;

— $D_{2\to 2}$ is Gaussian, of mean $-2\$/s$ and variance $16\$^2/s$.
with $C$'s starting point to be defined in each example.

A typical random realization for this process $C$ with $C(0) = 6\$$ may be found on figure 2.1. The colors of the graph refer to the active transitions at present time.

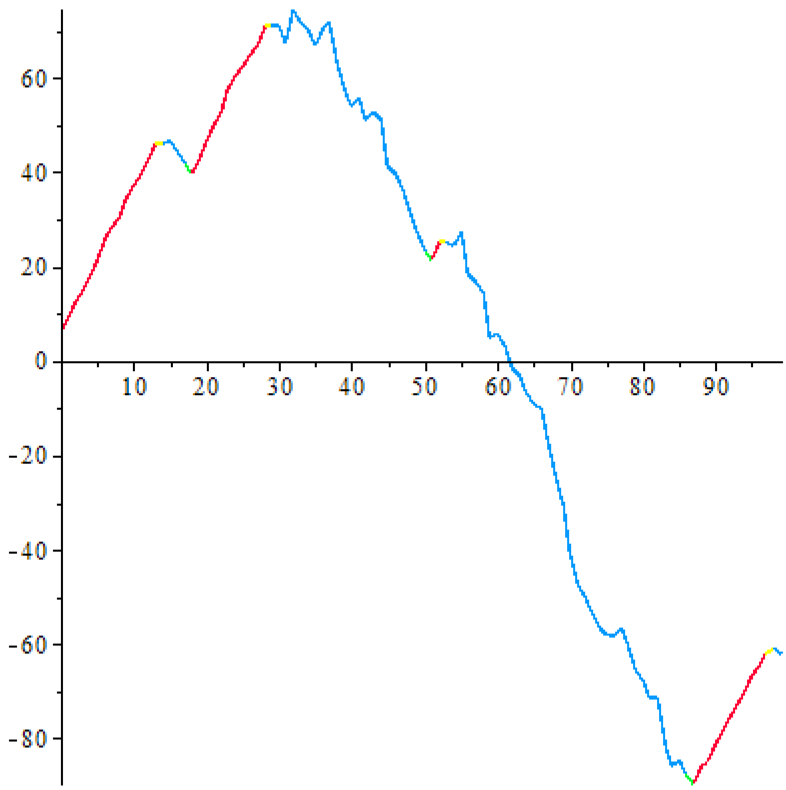


Figure 2.1 – Realization for $C$

It comes to no one's surprise that

— Long sequences of transitions between states $(1 \to 1)$ (red, "booms") and transitions $(2 \to 2)$ (blue, "busts") happen, as the respective transition probabilities are high. This is an observation of the desired momentum effects.

— These sequences are roughly drifted as stated by the expectancies of the associated transition payoffs, with the blue one being less "smooth" because of the higher variance.

In this example, the default time (first $t \in \mathbf{N}$ when $C(t)$ crosses 0) comes at $T_0 = 62s$. The aim for our study is to control (find bounds from above and below for) the Laplace transform of $T_0$.

### 2.3.2 Cycles of a C-process

Some Lévy processes are monotonous almost surely, which is equivalent to saying that their increments $D$ are either non-negative or non-positive almost surely, i.e. with supports $supp(D) \subseteq \mathbf{R}^+$ or $supp(D) \subseteq \mathbf{R}^-$. An analoguous definition holds for C-processes, but it requires analysis of cumulative increments over several time periods rather than one, to ensure that monotonicity is taken relatively to identical states of $M$ (one does not want the specificities of $M$'s states to hamper monotonicity properties) : hence, we consider the "possible" values of increments of a C-process. This leads us to look at the process defined as follows.

**Definition 2.3.4** *Restricted Lévy process*

*Let $C$ be a C-process, whose underlying Markovian process is $M$ deemed positive recurrent. For every $n \in \boldsymbol{N}$, let $T(n)$ be the random $n^{th}$ hitting time of $M(0)$ by $M$ (so $T(0) = 0$). Then the process given by*

$$C' = \begin{pmatrix} \boldsymbol{N} & \to & \boldsymbol{R} \\ n & \to & C(T(n)) \end{pmatrix}$$

*is a Lévy process, named $C$'s restricted Lévy process.*

Considerations on the increments of $C'$ will be of prime interest to state numerous properties of $C$.

#### Notions of paths and cycles

It turns out that $C$'s restricted Lévy process succeeds at characterizing the notions of monotonicity. However, direct considerations on $C$'s support lead to interesting and useful definitions for the proofs of the main result.

**Definition 2.3.5** *Paths and cycles*

*Let $C$ be a C-process whose underlying Markovian process is $M$, and $T \in \boldsymbol{N}$.*

- *A path of length $T$ is the determination of*
  - *Occupied state numbers $a_t \leq A$ for any $t \in [|0, T|]$, with $A_{a_0}$ named the starting state and $A_{a_T}$ named the finishing state ;*
  - *Payoffs values $x_t \in \boldsymbol{R}$ for any $t \in [|1, T|]$.*

  *These must obey the following constraints :*
  - *Possibility of transitions*

$$\forall t \in [|1, T|] \, , P_{a_{t-1} \to A_{a_t}} > 0$$

  - *Possibility of values*

$$\forall t \in [|1, T|] \, , x_t \in supp \left( D_{M(t-1) \to M(t)} \right)$$

*Its value is the sum of its payoffs values, i.e.* $\sum_{t=1}^{T} x_t$.

— *A cycle is a path whose starting and finishing states are identical.*

We define the cycle support of a C-process as the set of values that its cycles may take.

**Definition 2.3.6** *Cycle support of a C-process*

*Let* $C$ *be a positive recurrent C-process. For any* $n \in \boldsymbol{N}^*$*, let* $V_n(C) \subseteq \boldsymbol{R} \cup \{\infty\}$ *the set of all possible values for all cycles of length* $n$ *in* $C$*. The cycle support of* $C$ *is the set*

$$supp(C) = \bigcup_{n=1}^{\infty} V_n(C)$$

*We also define the minimum and maximum drifts* $\delta^-(C)$ *and* $\delta^+(C)$ *along cycles as the average increment by time period, i.e.*

$$\begin{aligned} \delta^-(C) &= \inf\left(\bigcup_{n=1}^{\infty} \frac{1}{n} V_n(C)\right) \\ \delta^+(C) &= \sup\left(\bigcup_{n=1}^{\infty} \frac{1}{n} V_n(C)\right) \end{aligned}$$

We may notice that

— $supp(C) \cap \mathbf{R}$ is never empty, because since $C$ is positive recurrent, there is $n \in \mathbf{N}^*$ such that

$$\mathbf{P}\left(M(n) = M(0) \wedge C(n) - C(0) < \infty\right) = p > 0$$

which means that there is a cycle of finite value going from $A_i$ to $A_i$ of length $n$.

— We excluded trivial paths of zero length (and zero value) from the definitions of $supp(C)$ and $\delta^{\pm}(C)$, so that each path lasts at least one time period.

— When $C$ is a Lévy process, $V_1(C)$ is the support of any of its increments, so this notion of cycle support extends the notion of support of a random variable, as $supp(C)$ is the support of all multi-period increments.

Defining paths and cycles will be useful both to

— Seize a probability of $C$ taking any "possible" specific path over a short amount of time (force $C$'s short-term behaviour) ;

— Assure that $C$ cannot drift faster than $\delta^{\pm}(C)$ over time (control $C$'s long-term behaviour).

**Monotonicity of a C-process**

The condition "existence of positive [negative] values in the support of increments" for Lévy processes translates to "existence of positive [negative] cycles" for C-processes. If a Lévy process only has non-negative [non-positive] increments, then it will be monotone non-decreasing [non-increasing] ; C-processes benefit from similar properties, as stated here.

**Definition 2.3.7** *Global monotonicity*

*A C-process $C$ is said to be*
- *Globally increasing iff supp*$(C) \subseteq \boldsymbol{R}^+ \cup \{\infty\}$ *;*
- *Globally decreasing iff supp*$(C) \subseteq \boldsymbol{R}^-$ *;*
- *Globally monotone iff either globally increasing or globally decreasing ;*
- *Globally constant iff both globally increasing and globally decreasing, i.e. when supp*$(C) = \{0\}$.

Globally monotone processes indeed hold properties of no-return.

**Proposition 2.3.2** *Consequences of global monotonicity*

*Let $C$ be a positive recurrent C-process. The following properties are equivalent :*

1. *$C$ is globally increasing.*
2. *$C$'s restricted Lévy process is almost surely non-decreasing.*
3. *There is $Q \in \boldsymbol{R}^+$ such that for any $s, t \in \boldsymbol{N}$ with $s < t$,*

$$\boldsymbol{P}\left(C(t) - C(s) \geq -Q\right) = 1$$

4. *There are C-processes $C^+$ and $C^=$, whose underlying Markovian processes are $C$'s one, with :*
   - *$C^+$ such that $C^+(0) = 0$ and almost surely non-decreasing ;*
   - *$C^=$ globally constant ;*
   - *$C$ rewrites as $C = C^+ + C^=$.*

We notice that
- For $C$ globally decreasing, the symmetric properties hold : the restricted Lévy process is non-increasing, $C(t) - C(s) \leq Q$ almost surely, and we may take $C^-$ non-increasing instead of $C^+$. However, we shall not need this symmetric lemma in the sequel.
- Positive recurrence of $C$ is required. A counter-example where $M$ and not $C$ is positive recurrent, yet we have statement 2 and not statement 1, is provided after the proof.

$C$ is actually said globally monotone because faults (values of $C(t) - C(s)$ of the wrong sign) in its monotonicity are always bounded by the value $Q$.

One expects globally constant C-processes to hold both bounds (by $-Q$ and $Q$) and be bounded over $\mathbf{N}$. It turns out that an even stronger control holds : a globally constant C-process is actually no more than its underlying Markov chain, as its values $C(t)$ are completely determined by $M(t)$.

**Proposition 2.3.3** *Globally constant C-processes*

*Let $C$ be a positive recurrent C-process. The following properties are equivalent :*

1. *$C$ is globally constant.*
2. *For each state number $i \leq A$, there is $c_i \in \boldsymbol{R}$ such that almost surely,*

$$\forall t \in \boldsymbol{N}, C(t) = c_{M(t)}$$

3. *There is $Q \in \boldsymbol{R}^+$ such that almost surely,*

$$\forall t \in \boldsymbol{N}, C(t) \in [C(0) \pm Q]$$

*Moreover, every globally constant and positive recurrent C-process is bounded.*

Looking at a random realization for a globally constant C-process yields the figure 2.2. In this graph, each color refers to the state hit by $M$ at present time. We see that points of a same color are at the same value for $C$, referring to the “height” of the corresponding state in the proof ; in particular, $C$ is stuck in an interval whose width is the maximal discrepancy between these heights. Likewise, a similar graph for globally increasing C-processes appears in figure 2.3 : points of a same color are oriented upwards, corresponding to non-negative values of cycles. The properties of global monotonicity will be used in the proof to recognize and exclude globally increasing processes easily from the study, because they void the definition of martingale parameters that we will see later.

### Periodicity of a C-process

In the proof, the key step ensuring the convenient control of $T_0$'s Laplace transform only works when $C$'s increments lie “evenly” on reals. It so happens that the only phenomenon preventing convergence to take place in the general case is the periodicity of $C$'s cycle support, described here.

**Definition 2.3.8** *Form of a cycle support*

*Let $C$ be a positive recurrent C-process, whose cycle support is supp$(C)$ deemed not contained in $\{0, \infty\}$.*

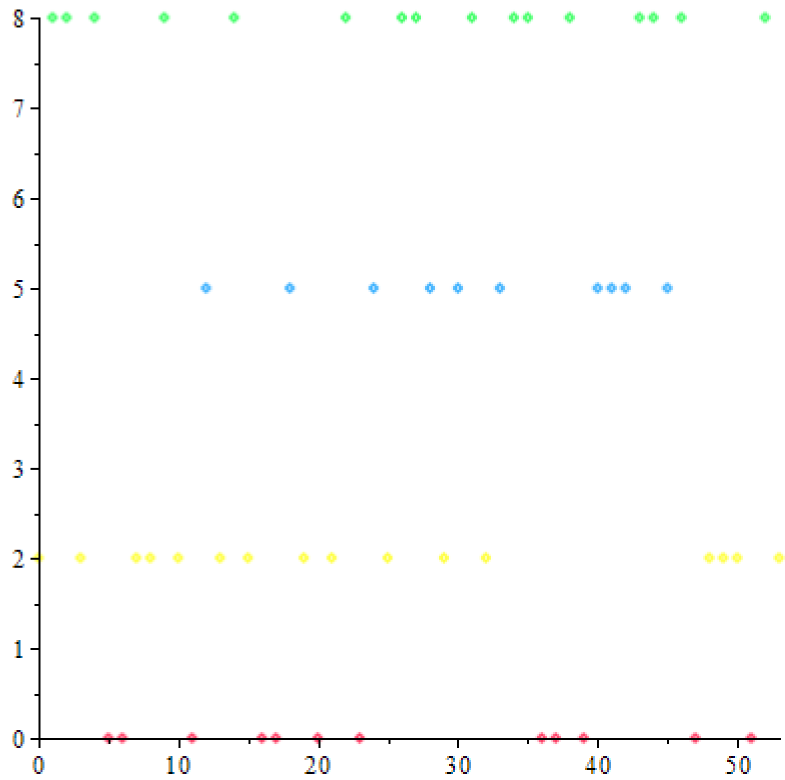


Figure 2.2 – Globally constant C-process

— *If there is $p \in \boldsymbol{R}_+^*$ such that supp$(C) \subseteq p\boldsymbol{Z} \cup \{\infty\}$, then $C$ will be said p-periodic. There is a largest such p holding this property, it will be called $C$'s fundamental period.*

— *If not, then $C$ will be said aperiodic.*

We should notice that :

— The case $C$ globally constant is excluded from this definition, as it would be $p$-periodic for any $p \in \mathbf{R}_+^*$, so there would not be a largest $p$.

— The set of values $q$ such that $C$ is $q$-periodic is $p/\mathbf{N}^*$ where $p$ is $C$'s fundamental period, and we may refer to them as harmonics of the fundamental frequency $(1/p)$.

The constraint of periodicity is very stringent to $C$'s transition payoffs, e.g. if $C$ is a Lévy, it means that its increments $D$ hold $D \in p\mathbf{Z}$ almost surely ; in particular, this holds for $C$'s restricted Lévy process. Since we shall consider bounded C-processes, then $D$ can take only finitely many values ; this idea will be used when solving the case where $C$ is periodic.

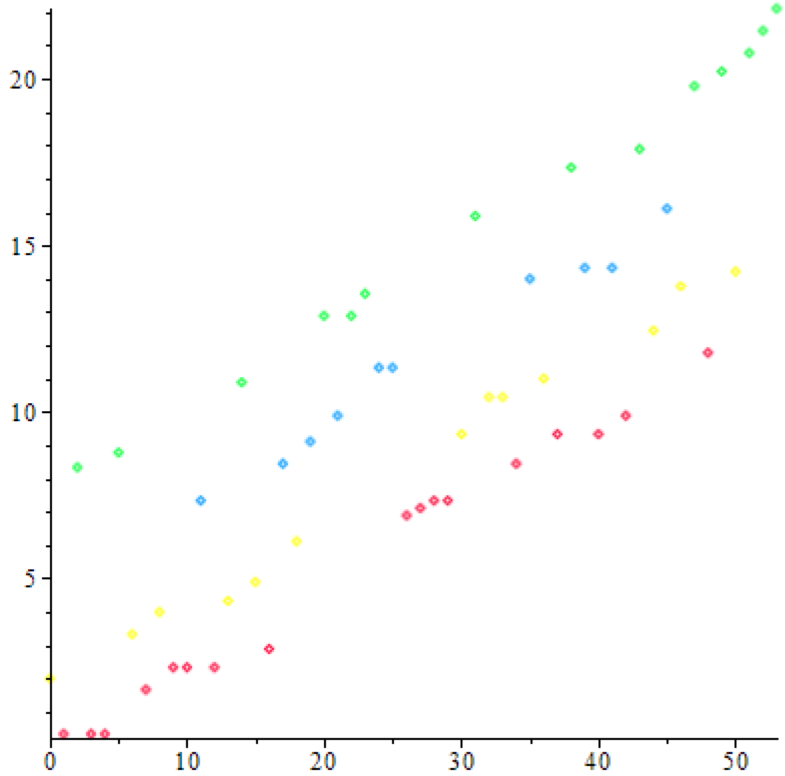


Figure 2.3 – Globally increasing C-process

### 2.3.3 Laplace matrix function

As we aim at controlling the Laplace transform of $C$'s default time $T_0$, it is natural to define Laplace transforms for the random variables governing it ; this is the purpose of this section.

#### Definition of the Laplace matrix function

While a Lévy process is characterized by the Laplace transform of its only increment, we chose to define $C$'s "Laplace transform" as a matrix of Laplace transforms for all increments. The reasons behind this definition are multiple, mainly because we will often need the structure of matrices to express main properties of a C-process.

**Definition 2.3.9** *Laplace matrix function*

*Let $C$ be a C-process. We define its Laplace matrix function as*

$$L_C = \left( \begin{array}{ccc} \boldsymbol{R} & \to & \boldsymbol{M}_A\left(\boldsymbol{R}^+\right) \\ \alpha & \to & \left(P_{i\to j}\boldsymbol{E}\left(e^{-\alpha D_{i\to j}}\boldsymbol{1}_{D_{i\to j}<\infty}\right)\right)_{i,j} \end{array} \right)$$

*whenever $\forall(i,j)\in\Gamma$, the expectancies converge. If $(i,j)\notin\Gamma$, then we set the entry $(L_C(\alpha))_{i,j}=0$ disregarding exponential integrability of $D_{i\to j}$.*

Requiring the condition $D_{i\to j}<\infty$ is tantamount to considering the analytical continuation of $D_{i\to j}$'s natural Laplace transform from $\mathbf{R}^+$ so that the values $+\infty$ are eliminated (this is done on purpose, as we shall see). This Laplace matrix function will play the part of the Laplace transform of $C$'s increments.

We know that giving the Laplace transform of a sEI random variable fully describes its distribution ; if $C$ is a Lévy process, for every $n\in\mathbf{N}^*$, the joint distribution of

$$(C(t))_{t\le n}\in\mathbf{R}^n$$

may be recovered using the Laplace transform of the increment $D$, as is gives $D$'s distribution function, thanks to

$$\mathbf{P}\left(\forall t\le n, C(t)-C(t-1)\le x_t\right)=\prod_{t=1}^{n}\mathbf{P}\left(D\le x_t\right)$$

So in a sense, one Laplace transform characterizes a whole Lévy process. Likewise, giving the Laplace matrix function of a C-process fully characterizes the distribution of its trajectories. As a matter of fact, it does *not* characterize the distribution of its underlying Markovian process in general, because of the cases where an increment is allowed to be $+\infty$.

**Proposition 2.3.4** *Characterization by the Laplace matrix function*

*Let $C_1$ and $C_2$ be two C-processes whose*

- *Underlying Markovian processes $M_1$ and $M_2$ share the common state space $(A_i)_{i\le A}$ ;*
- *Laplace matrix functions $L_1$ and $L_2$ coincide over an interval $I$ containing $0$ and a positive value ;*
- *Starting points and starting states are identical.*

*Then for every $n\in\boldsymbol{N}^*$, for every $x_t\in\boldsymbol{R}\cup\{\infty\}$ (with $t\le n$),*

$$\boldsymbol{P}\left(\forall t\le n, C_1(t)\le x_t\right)=\boldsymbol{P}\left(\forall t\le n, C_2(t)\le x_t\right)$$

*Moreover, if additionally $L_1(0)\left(\vec{1}\right)=\left(\vec{1}\right)$, the same holds for their underlying Markovian processes, i.e. for every $n\in\boldsymbol{N}^*$, for every $a_t\le A$ (with $t\le n$),*

$$\boldsymbol{P}\left(\forall t\le n, M_1(t)=A_{a_t}\wedge C_1(t)\le x_t\right)=\boldsymbol{P}\left(\forall t\le n, M_2(t)=A_{a_t}\wedge C_2(t)\le x_t\right)$$

*so the whole C-process is recovered.*

One might say that when $\mathbf{P}(D_{i\to j} = \infty) = p > 0$, there is a loss of $p$ between the sum of $L_1(0)$ over row number $i$ and the one for $M_1$ (which is always 1), and we do not know where it comes from in general. Even if one cannot determine for which $k \leq A$ the transition $(i \to k)$ allows for a probability $p/P_{i\to k}$ of being $+\infty$ looking only at $L_1$ (thus we cannot recover $M_1$), all possibilities have the same effect on $C$, i.e. driving it to $+\infty$ with probability $p$ (thus it is not an obstacle to the recovery of $C_1$).

### Concatenation of Laplace matrices

When $C$ is a Lévy process, the Laplace transform of the concatenation of $n$ increments, i.e. the random variable $C(t+n) - C(t)$, is simply given by the $n^{th}$ power of the original Laplace transform. When $C$ is a C-process, this multiplicative property does not hold as is because increments are not independent, but thanks to the construction of its Laplace matrix function, it translates to the matrix power. More specifically, let us take $C_{T_n}$ the process whose successive values are $C_{T_n}(t) = C(nt)$ ; as explained in the proof, it may be seen as a C-process whose

— Underlying Markovian process is $M_{T_n}$, such that $\forall t \in \mathbf{N}, M_{T_n}(t) = M(nt)$, with transition probabilities $P_{i \xrightarrow{n} j}$ ;

— Transition payoffs are $D_{i \xrightarrow{n} j}$ ;

— Starting point is $C(0)$.

Then $C_{T_n}$'s Laplace matrix function is $C$'s one to the $n^{th}$ power.

**Proposition 2.3.5** *Powers of $L_C(\alpha)$*

*Let $n \in \boldsymbol{N}^*$ and $\alpha \in \boldsymbol{R}$ such that $L_C(\alpha)$ is well-defined (converges). The $n^{th}$ power of the matrix $L_C(\alpha)$ corresponds to the Laplace matrix function of $C_{T_n}$ at point $\alpha$, i.e.*

$$\forall i, j \leq A, \left(L_C(\alpha)^n\right)_{i,j} = P_{i \xrightarrow{n} j} \boldsymbol{E}\left(e^{-\alpha D_{i \xrightarrow{n} j}}\right)$$

Incidentally, we notice that making $n = 0$ still works : the unit transition matrix is the identity, and empty transition payoffs are 0, so we still get

$$\left(P_{i \xrightarrow{0} j} \mathbf{E}\left(e^{-\alpha D_{i \xrightarrow{0} j}}\right)\right)_{i,j} = L_C(\alpha)^0 = Id$$

Nevertheless, the above property will become useful when considering long-range expectancies of $C$.

### Differentiation of a Laplace matrix function

A useful link between expectancies and Laplace transforms appears when computing their local derivatives at point 0 : for $X$ an exponentially integrable random variable, we have

$$-\mathbf{E}(X) = \frac{dL_X(\alpha)}{d\alpha}(\alpha = 0)$$

It is a well-known fact that the behaviour of a Lévy process tremendously depends on the expectancy and variance of each of its increments, but that they roughly suffice to describe its asymptotical behaviour. Indeed, the central-limit theorem proves that asymptotical computations on Lévy-kind processes "dump" any information on the distributions of increments but mean and variance. Some kind of variance term may be defined for $C$ a C-process, however it requires some deep analysis (as explained in paragraph 2.4.1), so we will focus on the notion of "average increment" for now, that also governs $C$'s asymptotical behaviour : we call it $C$'s mean expectancy. To get this average drift, we are going to translate the above characterization for a single random variable (the only increment of a Lévy process) to $C$'s Laplace matrix function, in order to get an analogous item for C-processes.

**Definition 2.3.10** *Diff-Laplace matrix function*

*Let $L_C$ be the Laplace matrix function of a C-process $C$, deemed well-defined over an opened interval $I \subseteq \boldsymbol{R}$. Its diff-Laplace matrix function is defined by*

$$R_C = \begin{pmatrix} I & \rightarrow & \boldsymbol{M}_A(\boldsymbol{R}) \\ \alpha & \rightarrow & \frac{-dL_C}{d\alpha}(\alpha) \end{pmatrix}$$

*When $X$ is a real random variable deemed exponentially integrable over $I$, we may define $R_X$ likewise.*

When $C$ is not sEI but only integrable, $R_C$ may not be defined anywhere. However, we shall still name $R_C(0)$ the matrix whose entries are given by

$$\forall i, j \leq A, (R_C(0))_{i,j} = P_{i \to j}\mathbf{E}(D_{i \to j})$$

as this is the expression we find when $C$ is sEI.

**Definition 2.3.11** *Mean expectancy*

*Let $C$ be a sEI C-process, whose underlying Markovian process is $M$ deemed positive recurrent, and transition payoffs are $D_{i \to j}$. By hypothesis, let $\mu$ be its invariant measure. The mean expectancy of $C$ is the value*

$$E(C) = \sum_{i=1}^{A}\sum_{j=1}^{A} \mu_{[i]} P_{i \to j} \boldsymbol{E}(D_{i \to j}) = \mu R_C(0)\left(\vec{1}\right)$$

A possible interpretation for the mean expectancy $E(C)$ is that it describes the average increment of $C$, as it is the mean of expected increments of transition payoffs, with weights being the expected amount of time $M$ will spend on each transition. Hence, the law of large numbers expects $C$ to drift along a line of slope $E(C)$. In particular, if $C$ is actually a Lévy process, then $E(C) = \mathbf{E}\left(C(t+1) - C(t)\right)$ for any $t \in \mathbf{N}$ is $C$'s drift.

## 2.4 Theory of C-processes

Now that the definitions are complete, we get on with controlling $\Lambda_{T_0}$ for a C-process $C$ : we eventually aim at proving an equivalent form of Schmidli's Cramér-Lundberg approximation from [42]

$$\mathbf{E}\left(e^{-aT_0}\right) = (Z + o(1))e^{-\alpha C(0)}$$

when $C$ is a discrete-time C-process. In this idea, we want to find an equivalent term to this $\alpha$ for a C-process ; however, we also expect the sought value to depend on $M$'s starting state $M(0)$, so we shall first define several characteristics of a C-process :

— An exponential parameter governing the decay of $C$'s default risks ;

— A vector indicating the specificity of each starting state on these risks.

As we know thanks to the proposition 2.3.4 that $L_C$ completely describes $C$'s distribution, we shall use $L_C$ to find these items. After this, we shall describe how they lead to the sought asymptotics of the default risks.

### 2.4.1 Martingale parameter

Specific calculations on default probabilities for Lévy processes (of random increments $D$) involve a parameter usually named "scale parameter" for diffusion processes (or Lundberg's exponent in [25]), the non-trivial value $\alpha \in \mathbf{R}$ for which

$$\mathbf{E}\left(e^{-\alpha D}\right) = 1$$

The scale parameter exists (when $D$ is sEI) iff neither $D \geq 0$ almost surely nor $D \leq 0$ almost surely, and controls the default probability thanks to the martingale property. Results about their default times derive from the fact that the exponential process defined below is a martingale :

$$\begin{pmatrix} \mathbf{N} & \to & \mathbf{R}^+ \\ t & \to & e^{-\alpha C(t)} \end{pmatrix}$$

C-processes have analoguous terms to scale parameters ; however, like in [25] we will extend this notion to other values for $\mathbf{E}\left(e^{-\alpha D}\right)$ rather than 1. Indeed, we shall look at an exponential equation like

$$\mathbf{E}\left(e^{-\alpha D}\right) = e^{a}$$

where $a \in \mathbf{R}_{+}^{*}$ (and after this, $a \in \mathbf{R}^{+}$) is a parameter, that we will use later to find the Laplace transform of $T_0$ at any point. When $C$ is a sEI Lévy process, this equation in $\alpha \in \mathbf{R}_{+}^{*}$ involves $D$'s Laplace transform ; for a sEI C-process, it becomes an equation involving "$C$'s Laplace transform", which is its Laplace matrix function $L_C$. This equation is then solved for its dominant eigenvalue, as it will be the one governing $C$'s behaviour.

### Main properties of the martingale parameter

A milestone for this study is the definition of $C$'s martingale parameter and dominant eigenvectors, as they govern its behaviour and will be required in the rest of this study.

**Proposition 2.4.1** *Martingale parameter*

*Let $C$ be a positive recurrent and sEI C-process, deemed not globally increasing. Let $a \in \boldsymbol{R}^{+}$ be a Laplace parameter.*

1. *Except perhaps $\alpha = 0$, there is a single $\alpha \in \boldsymbol{R}^{+}$ such that $e^{a}$ is the dominant (sometimes called "Perron-Frobenius") eigenvalue of $L_C(\alpha)$.*
2. *For this $\alpha$, the column eigenspace associated with the dominant eigenvalue $e^{a}$ of $L_C(\alpha)$ has dimension one, and may be directed by $w^{(a)}$ a positive vector.*
3. *The associated row eigenspace has dimension one, and may be directed by a positive vector $\mu^{(a)}$.*
4. *We may choose the scalings of $w^{(a)}$ and $\mu^{(a)}$ such that they hold*

$$\forall a \in \boldsymbol{R}^{+}, \mu^{(a)}\left(\vec{1}\right) = 1 \wedge \mu^{(a)} w^{(a)} = 1$$

   *These conditions, named "equations of scaling", ensure uniqueness of $w^{(a)}$ and $\mu^{(a)}$.*

5. *In particular, for $a = 0$, their limits exist and hold*
   - *(a) When $E(C) \leq 0$, we have $\alpha(0) = 0$, $w^{(0)} = \left(\vec{1}\right)$, and $\mu^{(0)} = \mu$ ($M$'s invariant distribution) ;*
   - *(b) When $E(C) > 0$, we have $\alpha(0) > 0$.*

*For every $a \in \boldsymbol{R}^+$, we name these items at point a :*

— *$\alpha(a)$ is C's (positive) martingale parameter ;*

— *$w^{(a)}$ is C's dominant (column) eigenvector ;*

— *$\mu^{(a)}$ is C's dominant row eigenvector.*

*These items have a smooth behaviour :*

1. *$\alpha(a)$ is an increasing and concave expression of $a \in \boldsymbol{R}^+$.*
2. *Viewed as functions of $a \in \boldsymbol{R}^+$, the expressions $\alpha(a)$, $w^{(a)}$ and $\mu^{(a)}$ are continuous over $\boldsymbol{R}^+$ and $C^\infty$ over $\boldsymbol{R}_+^*$.*

It is also possible to look for a negative value of $\alpha$ that solves the eigenvector equation. We name it the negative martingale parameter, and it holds the same properties as its positive counterpart.

**Proposition 2.4.2** *Negative martingale parameter*

*Let C be a positive recurrent and sEI C-process, deemed not globally decreasing.*

— *For every $a \in \boldsymbol{R}^+$, we name $\beta(a) \in \boldsymbol{R}^+$ the negative martingale parameter of C, defined as $-C$'s martingale parameter.*

— *In particular, for $a = 0$,*

1. *When $E(C) \geq 0$, we have $\beta(0) = 0$ ;*
2. *When $E(C) < 0$, we have $\beta(0) > 0$.*

We notice that the definition of $\alpha(a)$ fails when $C$ is globally increasing, while $\beta(a)$ fails when $C$ is globally decreasing. This is no surprise, since a Lévy process with non-negative or non-positive increments cannot have a scale parameter.

For now, we are going to investigate on what happens to $\alpha(a)$ and $\beta(a)$ when $a$ goes in the negative region. Throughout this study, this idea will be analyzed at some steps of the work, with their respective implications on its results.

— If $E > 0$, we find some martingale parameters $\alpha(a) \in \mathbf{R}_+^*$ when $a$ is not too large a negative value, and we will discuss about that in similar remarks. To get an idea, let us consider that $C$ is a Lévy whose increments are $D$. Recalling that $\alpha$ is the reciprocal function to $\Lambda_D$, it stops being defined when $a$ hits $\min(\Lambda_D)$, where $\alpha$ hits a double solution : indeed, we get $\alpha(a) = -\beta(a)$.

— If $E = 0$, $a$ cannot become negative, as in the proof we get that $D$ is a martingale and then $e^0 = 1$ is the minimal value of $D$'s Laplace transform.

— If $E < 0$, we find again some martingale parameters $\alpha(a) \in \mathbf{R}_-^*$ when $a$ is not too large a negative value ; the domain where $\alpha(a)$ may be continued is a finite interval (stopping at $\Lambda_D$'s minimum), unless $D \leq 0$ almost surely, which case means that $C$ is globally decreasing. As it turns out, we will find out later that $T_0$'s Laplace transform may be prolongated for these negative values of $a$, using the maringale parameter found this fashion.

### Martingale process

The usefulness of martingale parameters lies in the upcoming martingale property, as it transforms a C-process into a martingale.

**Definition 2.4.1** *Martingale process*

*Let $C$ be a positive recurrent and sEI C-process, deemed not globally increasing, and $a \in \boldsymbol{R}^+$. Let $\alpha(a)$ be its martingale parameter at point $a$ and $w^{(a)}$ its dominant eigenvector. We define the process :*

$$X_C^{(a)} = \begin{pmatrix} \boldsymbol{N} & \rightarrow & \boldsymbol{R}^+ \\ t & \rightarrow & w^{(a)}_{[M(t)]} e^{-\alpha(a)C(t)} e^{-at} \end{pmatrix}$$

*This process is a martingale, named $C$'s martingale process at point $a$.*

In contrast with [42], we recover Schmidli's lemma 9.3 where $g$ is the dominant eigenvector and $r$ the Laplace parameter, and the function $\theta$ given in there coincides with $\alpha^{-1}$. We notice that when $C$ is globally increasing, such exponential-based processes (for $\alpha > 0$) will also have a decreasing shape, so cannot be martingales (unless $C$ is globally constant and for $a = 0$, when $X_C^{(0)}$ becomes trivial).

Proceeding with the thoughts when $a$ becomes negative, we get :

- If $E > 0$, then $0 < \alpha(a) < \alpha(0)$. For $X_C^{(a)}$ to keep the martingale property, then $e^{-\alpha(a)C(t)}$ needs to be "smaller" than $e^{-\alpha(0)C(t)}$, which can be done to some extent ; in the cases of Lévy processes, it is until $\mathbf{E}\left(e^{-\alpha D}\right)$ encounters its positive minimum, where the martingale parameter stops being defined.
- If $E = 0$, the point $a = 0$ associated with $\alpha(0) = 0$ may be regarded as a "double" solution (the local derivative of $D$'s Laplace transform is zero and $\alpha(0) = \beta(0) = 0$). A common link between double solutions of a Laplace equation and solution functions appears : there is a martingale process whose form is a polynomial (affine) multiplied by an exponential, but we shall not discuss about it further.
- If $E < 0$, we get $\alpha(a) < \alpha(0) = 0$. For $X_C^{(a)}$ to keep the martingale property, then $e^{-\alpha(a)C(t)}$ again needs to be "smaller" than $e^{-\alpha(0)C(t)}$, which can be done to some extent. However, we could use this martingale property with the default time as stopping time, which leads to interesting results (negative values of $\alpha$ for which $\mathbf{E}\left(e^{-\alpha T_0}\right) < \infty$, indicating fast default events).

On a side note, since differentiation is a linear operator and the processes $X_C^{(a)}$ are martingales for every $a$, then differentiating them with respect to $a$ still yields a martingale by linear stability of martingales. This idea may be used to get a martingale equation involving $C$ itself as a linear term, but once again we shall not proceed in this way.

### Spread

The discrepancy between $w^{(a)}$'s coordinates measures the specificities of $M$'s states : as we shall see with the main theorem 2.1, higher values indicate that the corresponding state is detrimental to $C$ compared to other states, for an identical value of present $C(t)$. We may measure these discrepancies to quantify the magnitude of $M$'s effects on $C$'s default probability : this leads to the following definition.

**Definition 2.4.2** *Spread*

*Let $v \in \left(\boldsymbol{R}_+^*\right)^A$. Its spread is defined by*

$$\delta(v) = \ln\left(\max_{i,j\leq A}\left(\frac{v_{[i]}}{v_{[j]}}\right)\right)$$

A spread is always a non-negative real, being 0 iff $v$ is proportional to $\left(\vec{1}\right)$, so it is a satisfying measure of the discrepancies between $v$'s coordinates.

- If $w^{(a)}$ has a high spread, one should take extra care of $M$'s present state, as $C(t)$ and $\alpha$ may fail to represent default risks. Rewriting

$$w^{(a)}_{[M(t)]}e^{-\alpha(a)C(t)} = e^{-\alpha(a)\left(C(t)-\frac{\ln\left(w^{(a)}_{[M(t)]}\right)}{\alpha(a)}\right)}$$

  one sees that $w^{(a)}$'s spread may lead to an error of up to $\delta\left(w^{(a)}\right)/\alpha(a)$ on the "true value" of the cash reserves $C(t)$. In the state of the market $M(t)$, one should reevaluate the capital value $C(t)$ as described by this equation to correctly assess the default risks.
- Conversely, when $w^{(a)}$ has a low spread, $M$'s effects are limited and one can safely ignore it, simplifying the model to a Lévy process. In an extreme case, we have indeed the equivalence between $\delta\left(w^{(a)}\right) = 0$ and the fact that $C$ is actually a Lévy process, as explained below.

Thus, the maximal correction $\delta\left(w^{(a)}\right)/\alpha(a)$ is a measure of $C$'s distance from being a Lévy. In particular, having a zero spread means that $w^{(a)}$ is actually the unit vector thanks to the equations of scaling, and characterizes Lévy processes.

**Proposition 2.4.3** *Zero spread*

*Let $C$ be a positive recurrent, bounded, and not globally increasing C-process. For every $a \in \boldsymbol{R}_+^*$, let $w^{(a)}$ be $C$'s dominant eigenvector at point $a$. The following statements are equivalent :*

1. *For every $a \in \boldsymbol{R}_+^*$, $w^{(a)} = \left(\vec{1}\right)$.*

*2. There is a Lévy process $C'$ such that $\forall t \in \boldsymbol{N}, C(t) = C'(t)$ almost surely*

We remark that $M$'s states need not be equivalent for $C$ to have a Lévy-like distribution.

**Lévy processes and variance terms**

When $C$ is a Lévy process whose increments are $D$, the function $\alpha$ is actually reciprocal to $\Lambda_D$ over its domain. The above results now translate to simplified forms :

— $w^{(a)}$ and $\mu^{(a)}$ disappear, since they amount identically to the 1-dimensional vector whose only coordinate is 1.

— Their spread is identically zero, which characterizes Lévy processes after the proposition 2.4.3.

— The martingale process becomes

$$X_C^{(a)} = \left( \begin{array}{ccc} \mathbf{N} & \rightarrow & \mathbf{R}^+ \\ t & \rightarrow & e^{-\alpha(a)C(t)}e^{-at} \end{array} \right)$$

In particular, let us make $a = 0$ :

— When $E(C) > 0$, we get $\alpha(0) = \Lambda_D^{-1}(0)$ is $C$'s natural scale parameter, $\beta(0) = 0$, and

$$\frac{d\beta(a)}{da}(a=0) = \frac{1}{E(C)}$$

— When $E(C) < 0$, we get $\alpha(0) = 0$, $-\beta(0)$ is $C$'s natural scale parameter, and

$$\frac{d\alpha(a)}{da}(a=0) = \frac{-1}{E(C)}$$

Notice that this is not for Lévy processes only : differentiating the martingale equation for the martingale process around $a = 0$ yields

$$\begin{aligned} & \sum_{j=1}^{A} P_{i\to j} \frac{dw_{[j]}^{(a)}}{da}(a=0) - \frac{d\alpha(a)}{da}(a=0) \sum_{j=1}^{A} P_{i\to j} \mathbf{E}\left(D_{i\to j}\right) \frac{dw_{[j]}^{(a)}}{da}(a=0) \\ = \quad & \frac{dw_{[i]}^{(a)}}{da}(a=0) \end{aligned}$$

thus by definition of $E(C)$, one gets

$$\frac{dw_{[i]}^{(a)}}{da}(a=0)E(C)\left(\vec{1}\right) = (P - Id)\frac{dw^{(a)}}{da}(a=0)$$

and thanks to the definition of $\mu$, left multiplication by $\mu$ leads to the result. In particular, $E(C) = \mathbf{E}(D)$ when $C$ is a Lévy process.

Now, let us name $\gamma$ the one among $\alpha$ and $\beta$ that is 0 at point $a = 0$. It is $C^2$ around 0 as soon as $D$ is sEI, and one finds out that

$$-\frac{d^2\gamma(a)}{da^2}(a = 0) = \frac{\mathbf{V}(D)}{|\mathbf{E}(D)|^3}$$

This is, to some extend, linked with the relationship between moments of the random variable $D$ and its log-Laplace transform, as for a Lévy process $\alpha = \Lambda_D^{-1}$. An interesting idea might be to reverse this equality, getting some "variance term" for C-processes that we define as

$$V(C) = \frac{-\frac{d^2\gamma(a)}{da^2}(a = 0)}{\left(\frac{d\gamma(a)}{da}(a = 0)\right)^3}$$

As $\gamma$ is concave, one finds out that $V(C) \geq 0$ ; moreover, $V(C) = 0$ iff $\alpha$ has no concavity around zero. This definition is strenghtened by the fact that for every $T$ a suitable halting time such that $M(T) = M(0)$ almost surely, computations (we will not indicate them) lead to

$$V(C) = \frac{\mathbf{V}\left(C(T) - TE(C)\right)}{\mathbf{E}(T)}$$

However, using the notations $D_\tau$ and $\tau$ of the proof, $V(C) = 0$ leads to

$$\tau + 1 = \frac{D_\tau}{E(C)}$$

almost surely, which is tantamount to saying that there are a drift $E$ (it is $E(C)$) and a globally constant C-process $C^=$ such that $\forall t \in \mathbf{N}, C(t) = tE + C^=(t)$ almost surely. While a zero variance for Lévy processes is the translation of $C(t) = tE + C(0)$ almost surely, the term $C^=$ that plays the part of the constant $C(0)$ is now a globally constant C-process. In a sense, the "kernel" of Lévy processes is constituted by identically constant processes, while this notion translates to globally constant C-processes.

### 2.4.2 Main theorems

To simplify the proofs, we shall deem henceforth that $C$ is bounded and positive recurrent. However, we firmly believe that only $C$ sEI is required, even if we shall state our results only for $C$ bounded.

### Method

The natural method used to approximate $L_{T_0}(a)$ comes from the definition of the Laplace transform. Considering the general problem of finding $L_{T_0}(a)$ from any starting state $A_i$ and starting point $C_0$, named $L_{[i]}^{(a)}(C_0)$ hereafter, the proof follows these generic steps :

1. By virtue of of $T_0$'s definition and the fact that $C$ is a C-process, write down an equation describing the relationship between the expectancies
$$\mathbf{E}\left(L_{[M(u)]}^{(a)}(C(u))\right)$$
for $u = t$ and $u = t + 1$.
2. Use time concatenations to scale this equation in functions $L$ into an equation in auxiliary functions $K$, involving decreasing transition payoffs.
3. Prove that, when these functions $K$ behave "well", they converge to a common limit at infinity.
4. Get back to the initial question about $L_{T_0}(a)$, using all previous results.

This methods works for most bounded and positive recurrent C-processes, but fails in specific cases :

— When $C$ is globally increasing, it has no martingale parameter, and the Laplace equation cannot be scaled. However, the proposition 2.3.3 indicates that it cannot drop lower than the no-return property states ; as a consequence, there is $Q \in \mathbf{R}^+$ such that the default probability is exactly 0 once $C$ passes $Q$, so we are not interested in this case anymore.

— When $C$'s cycle support is not "evenly" distributed over $\mathbf{R}_-^*$, the functions $K$ do not converge. This is a serious failure (and not a flaw of our study), which leads to the issues of periodicity, dealt with in this section. On a side note, present litterature (e.g. [42, 45]) avoided these considerations thanks to the continuous nature of the premiums in Cramér and Lundberg's risk model ; however, we decided to emphasize on them because they will allow us to explain the optimality of $L_{[i]}^{(a)}(C_0)$'s asymptotics.

We first deal with the latter case, where the general method of proof fails. Hence, we deem that $C$ is periodic, and we shall note by $p \in \mathbf{R}_+^*$ its fundamental period.

### Statement for periodic C-processes

When $C$ is $p$-periodic, let us consider the support modulo $p$ of paths from $A_i$ to $A_j$ for $i, j \leq A$ (excluding values $+\infty$), herein named $S_{i\to j} \subseteq \mathbf{R}/p\mathbf{Z}$. Let us take a cycle of finite value for $C$, whose occupied state numbers are $k_n$ for $n \in [|0, T|]$.

By definition of a periodic C-process, we must have

$$\sum_{n=1}^{T} S_{k_{n-1}\to k_n} = \{0\}$$

This means that all sets $S_{k_{n-1}\to k_n}$ must be singletons, e.g. $\{p_{i,j}\}$ for some $p_{i,j} \in \mathbf{R}/p\mathbf{Z}$. When $C$ is positive recurrent, it is possible to link all values $p_{i,j}$ through a Chasles-like identity, creating values $p_k$ modulo $p$ such that $p_{i,j} = p_j - p_i$ for every $i, j$. If we choose $p_{M(0)} \equiv C(0)$ modulo $p$, it follows that for every $t \in \mathbf{N}$, we get $C(t) \equiv p_{M(t)}$ modulo $p$. To get a result when $C$ is $p$-periodic, our idea is to "split" the transition payoffs between

- A principal part in $p\mathbf{Z}$, indicating the main effects of transition payoffs in $C$ ;
- A residual part in $[0, p)$, indicating the residual effects of the transition payoffs, not hampering the study.

In particular, the residual part has no effect on $C$'s default. One may imagine that, for $p = 1$ in this example,

- The principal part is $C$'s integer part ;
- The residual part is $C$'s fractional part.

The idea behind this decomposition is that $C(t) < 0$ iff $\lfloor C(t) \rfloor < 0$, so we may dump the residual part and find out $C$'s default only observing its principal part, that lies in $\mathbf{Z}$.

**Definition 2.4.3** *Regular process*

*Let $C$ be a positive recurrent, periodic C-process. We assume that $C$ is not globally constant, so $p$ is its fundamental period.*

- *The values $(p_i)_{i\le A}$ such that $\forall t \in \boldsymbol{N}, C(t) \equiv p_{M(t)}$ modulo $p$ are called $C$'s natural offsets.*
- *The process $\tilde{C}$ defined by*

$$\tilde{C} = \begin{pmatrix} \boldsymbol{N} & \to & \boldsymbol{Z} \\ t & \to & \frac{1}{p}\left(C(t) - p_{M(t)}\right) \end{pmatrix}$$

*is well-defined in $\boldsymbol{Z}$, and is a C-process with integer increments, whose fundamental period is 1. It shall be named $C$'s regular process.*

Now, let us take $C$ a bounded, $p$-periodic C-process. As $\tilde{C}$ lies in $\mathbf{Z}$, the Laplace transform of its default time at point $a \in \mathbf{R}^+$ may be written through numerical sequences $(L_{k,c})_{k\le A, c\in\mathbf{Z}}$ solving induction-like equations

$$\forall k \le A, c \in \mathbf{N}, L_{k,c} = e^{-a} \sum_{j=1}^{A} \sum_{d=-\infty}^{\infty} P_{k\to j} \mathbf{P}\left(D_{k\to j} = d\right) L_{j,c+d}$$

where $L_{k,c} = 1$ whenever $c < 0$. This system may be solved analytically, which yields a solution for the process $\tilde{C}$, that eventually translates to a solution for $C$ itself.

**Proposition 2.4.4** *Laplace transform of $T_0$ for a periodic C-process*

*Let $C$ be a positive recurrent, bounded and not globally increasing C-process. We deem that $C$ is periodic, so it has $p \in \boldsymbol{R}_+^*$ as fundamental period. For every $a \in \boldsymbol{R}_+^*$, let $\alpha(a)$ be $C$'s martingale parameter at point $a$, and $w^{(a)}$ the associated column eigenvector. There is a function $K$ :*

*— Defined over $[0, p) \times \boldsymbol{R}^+$ onto $\boldsymbol{R}$ ;*

*— Being piecewise continuous over its domain ;*

*— Being continuous of its second variable at any fixed point for the first,*

*such that for every $i \leq A$, the log-Laplace transform of $T_0$ given $M(0) = A_i$ and $C(0) = C_0 \in p_0 + p\boldsymbol{N}$ (with $p_0 \in [0, p)$ being any congruence modulo $p$) holds*

$$-\Lambda_{T_0}(a) \in \left[\left(\alpha(a)C_0 - \ln\left(w_{[i]}^{(a)}\right) + K(p_0, a)\right) \pm e(C_0, a)\right]$$

*where $e$ is a non-negative error function, uniformly exponentially convergent to* 0 *over any compact subset of $\boldsymbol{R}^+$ in $a$ when $C_0$ goes to infinity. It means that for every $a_0 \in \boldsymbol{R}^+$,*

$$\exists e_1(a_0) \in \boldsymbol{R}^+, e_2(a_0) > 0, \forall C_0 \in \boldsymbol{R}^+, \forall a \leq a_0, e(C_0, a) < e_1(a_0)e^{-e_2(a_0)C_0}$$

As most remarks on this proposition are similar to observations on the main result of this study, we shall only focus on the differences between those.

— Because of $C$'s periodicity, the default times are piecewise constant of $C_0$.

— For this reason, since the martingale parameter $\alpha(a)$ has a linear (in $C_0$) effect on the main term in the theorem, the approximation of a constant function over a non-trivial interval by an affine (multiplicative factor $\alpha(a)$) function of $C_0$ on this interval yields an incompressible error, no matter how remote is $C_0$. It follows that the requirement stating $C_0 \in p_0 + p\mathbf{N}$ cannot be removed.

— The guaranteed convergence is exponential. The main idea behind this assertion comes from the fact that $p$ is $C$'s fundamental period, rather than an harmonic $np$ for some $n \in \mathbf{N}$ greater than 1, so $C$ has cycles whose value are $-kp$ for any $k \geq k_0$ for some large enough $k_0$. It follows that $L_C^{(a)}$'s "second" eigenvalue $\lambda_2(a)$ is strictly less than 1, so residual terms in the recursive scheme converge exponentially (at speed $o(\lambda^n)$ for every $\lambda > \lambda_2$). However, this idea does not work if $C$ is aperiodic, for reasons that we shall see in a later section ; indeed, the convergence may not be exponential.

### Main theorem

Now that the special cases have been removed from the study, we move on to the general case, where $C$ is not "simple" : neither globally increasing, nor restricted to a discrete subset of $\mathbf{Z}$. The main theorem in this study controls asymptotics of the $\Lambda_{T_0}$'s log-Laplace transform.

**Theorem 2.1** *Laplace transform of $T_0$ for an aperiodic C-process*

*Let $C$ be a positive recurrent, bounded and not globally increasing C-process, deemed aperiodic. For every $a \in \boldsymbol{R}^+$, let*

— *$\alpha(a)$ be $C$'s martingale parameter at point $a$,*

— *$w^{(a)}$ be the associated column eigenvector.*

*There is a continuous function $K : \left(\boldsymbol{R}^+ \to \boldsymbol{R}\right)$ such that for every $i \leq A$, the log-Laplace transform of $T_0$ giving $M(0) = A_i$ and $C(0) = C_0$ holds*

$$-\Lambda_{T_0}(a) \in \left[\left(\alpha(a)C_0 - \ln\left(w^{(a)}_{[i]}\right) + K(a)\right) \pm e(C_0, a)\right]$$

*where $e$ is a non-negative error function, uniformly convergent to $0$ over any compact subset of $\boldsymbol{R}^+$ in $a$, i.e.*

$$\forall a_0 \in \boldsymbol{R}^+, \forall \epsilon > 0, \exists C_{a_0} \in \boldsymbol{R}^+; \forall C_0 > C_{a_0}, \forall a \leq a_0, e(C_0, a) < \epsilon$$

*Moreover, if $C$ is a Lévy process, then $K(a) \in \boldsymbol{R}^*_+$.*

The strength of this theorem appears in the interpretation of its error terms.

— The error goes to 0 when $C_0$ increases. Indeed, the main idea in the proof of this main theorem is that $L_{T_0}$'s dependency on $C(T_0)$ and $M(T_0)$ eventually disappears when $C_0$ goes to infinity, because the distribution of $(C(T_0), M(T_0))$ conditionally to defaulting converges. This "limit" distribution, represented by the random couple $(C_f, M_f) \in \mathbf{R}^*_- \times \{A_i\}_{i \leq A}$, is used by the martingale property : neglecting dependency between $T_0$ and the final values $C(T_0)$ and $M(T_0)$, one should get

$$\begin{aligned} w^{(a)}_{[M(0)]} e^{-\alpha(a)C_0} &= \mathbf{E}\left(w^{(a)}_{[M(T_0)]} e^{-\alpha(a)C(T_0)} e^{-aT_0}\right) \\ &\approx \mathbf{E}\left(w^{(a)}_{[M_f]} e^{-\alpha(a)C_f}\right) \mathbf{E}\left(e^{-aT_0}\right) \end{aligned}$$

Hence, considering

$$K(a) \approx \ln\left(\mathbf{E}\left(w^{(a)}_{[M_f]} e^{-\alpha(a)C_f}\right)\right)$$

then one gets

$$\Lambda_{T_0}(a) \approx \ln\left(w^{(a)}_{[M(0)]} e^{-\alpha(a)C_0}\right) - K(a)$$

which is the term given by the theorem. The convolution equation in the proof "merges" uncertainty on the severity of default and the state of the market when defaulting into the term $K(a)$, and this holds independently of $C(0)$.

- The error term $e(C_0, a)$ comes from the "speed" of merging uncertainties, and $e(C_0, a)$ converges faster when the transition payoffs are evenly distributed among the convex hull of their supports. Let us take extreme examples for $C$ being a Lévy process, whose increments are represented by the random variable $D$ :
  - If $-D$ has an exponential distribution, then $C(T_0)$'s distribution will be exactly identical no matter $C_0$ and independent of $T_0$, because exponential distributions hold the "memory loss" property. The approximate computation above then holds rigourously and there is no error term. As a matter of fact, this corresponds to the special case of Cramér and Lundberg's risk model ([7]) with exponential claims, where we eventually get an exact ruin probability as $Ke^{-\alpha(0)C_0}$ ([40]).
  - If the Lebesgue measure is absolutely continuous with respect to $D$'s distribution over the convex hull of its negative support, for example if

    $$\exists y \in \mathbf{R}^+; \forall x \in (-Q, 0), \forall \eta \in (0, -x), \mathbf{P}\left(D \in (x, x+\eta)\right) \geq y\eta$$

    then the convergence will be at least exponential, with a better exponential parameter when $y$ is larger. This is because $D$ "greatly mixes" the possible values hit by $C$ through its way to default, so $(C(T_0), M(T_0))$ closes to $(C_f, M_f)$ faster, and the error term will decrease faster.
  - Contrariwise, if $C$ is periodic, the theorem fails because repetitions of $D$ do not mix the values hit by $C$, but concentrate them (e.g., if $C$ is a regular process, on integers). As stated before, the function associating $C_0$ with the sought Laplace transform at any fixed point $a$ will be piecewise constant, and cannot behave asymptotically like a non-trivial exponential.
  - If repetitions of $D$ hardly mix the values hit by $C$, we are driven back to periodicity problems, which happen with a severe impact if $t$ successive time concatenations of $D$ leave $C(t)$ "close to" a periodic set. Indeed, the upcoming proposition 2.4.6 creates such processes, whose error terms converge arbitrarily slowly.

For example, doing $a = 0$ leads us to the asymptotical default probability.

**Proposition 2.4.5** *Default probability*

*Let $C$ be a positive recurrent and bounded C-process, deemed aperiodic.*

1. *If $E \leq 0$, then $\boldsymbol{P}(T_0 < \infty) = 1$ unless $C$ is globally constant.*

*2. If $E > 0$, there is some constant $X_1 \in \boldsymbol{R}_+^*$ such that*

$$\boldsymbol{P}(T_0 < \infty) = X_1 w_{[i]}^{(0)} e^{-\alpha(0)C_0}(1 + o(1))$$

*where the $o(1)$ refers to $C(0)$ going to $+\infty$.*

**Slow or quick convergence**

The mixing involved in the proof constitutes the key step for the convergence speed of the error term $e(C_0, a)$ in the main theorem. The convergence may thus be

— Arbitrarily slow if this mixing is ill-distributed ;

— Exponentially fast if this mixing is well-distributed.

To enlighten this, we build the following Lévy processes, whose increments may only take two distinct values, chosen such that the mixing is slow.

**Definition 2.4.4** *Liouville processes*

*Let $f : (\boldsymbol{N} \to \boldsymbol{N})$ be an increasing function, with $f(0) = 0$. We define*

*— The Liouville number associated with $f$, named $f$-Liouville number, as*

$$L_f = \sum_{k=0}^{\infty} 10^{-f(k)}$$

*— The $f$-Liouville process is the Lévy process defined by its increments $D$ such that*

$$\boldsymbol{P}(D = -1) = \boldsymbol{P}(D = -L_f) = 1/2$$

*and $C(0) = C_0 \in \boldsymbol{R}^+$ is deterministic.*

Approximating at order $n \in \mathbf{N}^*$ gives the truncated $f$-Liouville number at order $n$

$$(L_f)_n = \sum_{k=0}^{n} 10^{-f(k)}$$

This states that $C$'s cycle support is "almost" $10^{-f(n)}$-periodic, as $L - L_n < 1.2 * 10^{-f(n+1)}$, but is aperiodic as soon as $L_f$ is not a rational number. In particular, if $\forall n \in \mathbf{N}^*, f(n) = n!$, then $L_f$ is the common Liouville number

$$L_f = 1 + \sum_{k=1}^{\infty} 10^{-k!}$$

Now, the mixing involved during the computations (descending process, convolution process) features a convolution equation like

$$K(x) = \sum_i p_i K(x - x_i)$$

with $\sum_i p_i = 1$ and values $x_i \in \mathbf{R}_+^*$. However, the construction of $L_f$ states that these values $x_i$ are concentrated, at order $n$, in intervals around points of $10^{-f(n)}\mathbf{N}$ whose length have order $10^{-f(n+1)}$. The terms $K(x - x_i)$ will not mix until the number of time concatenations exceeds something like $10^{f(n+1)-f(n)}$, which is the $C_0$ required to have a precision like $10^{-f(n)}$ on the error term : the convergence may be made arbitrarily slow.

**Proposition 2.4.6** *Slow convergence for the Liouville process*

*Let $g : \left(\boldsymbol{R}^+ \to \boldsymbol{R}^+\right)$ be any function that converges to 0 towards $\infty$. There is $f$ an increasing function ($\boldsymbol{N} \to \boldsymbol{N}$) with $f(0) = 0$, defining some aperiodic $f$-Liouville process $C$ such that any error function $e$ that suits the main theorem 2.1 for $C$ holds*

$$\forall a \in \boldsymbol{R}_+^*, \forall y \in \boldsymbol{R}_+^*, \exists x > y; e(x,a) > g(x)$$

*Moreover, if $E(C) > 0$, $x$ may be chosen uniformly :*

$$\forall y \in \boldsymbol{R}_+^*, \exists x > y; \forall a \in \boldsymbol{R}^+, e(x,a) > g(x)$$

Thus, convergence may be very slow compared with the “expected” exponential form found for periodic C-processes.

On the other hand, the standard Markov-modulated risk model holds exponential convergence thanks to the continuity of $C$'s premiums : this is illustrated by the following proposition.

**Proposition 2.4.7** *Quick convergence*

*Let $C$ be a positive recurrent and bounded C-process, whose underlying Markovian process is $M$, and $T$ be a stopping time for $C$ such that almost surely,*

*— Either $T = \infty$ ;*

*— Or $\forall t < T, C(T) < C(t)$.*

*We deem that there are $u \in \boldsymbol{R}_+^*$, $\eta > 0$ and $i, j \leq A$ such that for every $x < y$ in $[0, u]$,*

$$\boldsymbol{P}(T < \infty \wedge C(T) - C(0) \in [-y, -x] \wedge M(T) = A_j | M(0) = A_i) > (y - x)\eta$$

*Then the error term $e(C_0, a)$ from the theorem 2.1 is actually uniformly exponentially convergent in the sense of the proposition 2.4.4.*

We notice that periodicity voids this statement because the “mass” of $C(T)$ will be concentrated on $p_j + p\mathbf{Z}$ as given by the offsets. As a matter of fact, this proposition should also work with continuous-time processes, so it solves the Markov-modulated risk model (from [42]) ; however, looking at the same model in discrete time (provided that claim sizes are commensurable with the unit drift) yields periodicity, so one should take extra care with discretization.

## 2.5 Applications

In this paragraph, we shall use the previous main theorems to solve the example presented in paragraph 2.3.1, and discuss about how positive correlation of increments is dangerous to Lévy-like estimations of default risks.

### 2.5.1 Solution for the example process

The boom-bust C-process from paragraph 2.3.1 is sEI rather than bounded, however we shall admit that the theorems still work for it, since it is simple to present the ideas behind the main values when using it.

#### Laplace matrix function

When $C$ is the boom-bust process, computations lead to the values of $L_C(\alpha)$ and $R_C(\alpha)$ for every $\alpha \in \mathbf{R}$.

$$L_C(\alpha) = \frac{1}{200} \begin{pmatrix} 178e^{\alpha^2/2-2\alpha} & 11 + 11e^{-\alpha} \\ 18e^{2\alpha} & 182e^{8\alpha^2+2\alpha} \end{pmatrix}$$

In particular for $\alpha = 0$, one recovers $M$'s transition matrix in $L_C(0)$.

$$R_C(\alpha) = \frac{1}{200} \begin{pmatrix} 178(3-\alpha)e^{\alpha^2/2-2\alpha} & 11e^{-\alpha} \\ -36e^{2\alpha} & -182(16\alpha+2)e^{8\alpha^2+2\alpha} \end{pmatrix}$$

The matrix $R_C(0)$ indicates the weighted expectancies associated with each transition. Therefore, the good state is given positive terms while the bad state is given negative terms, which indicates that as expected by choice of the transition payoffs, the good state pulls $C$ upwards while the bad state pushes $C$ downwards. One also observes that diagonal terms are large relatively to other terms, as they have a greater effect because of the time spent by $M$ on these transitions.

#### Martingale parameter

We drew the solution set in $(a, \alpha)$ to the equation

$$\det\left(L_C(\alpha) - e^a Id\right) = 0$$

getting the martingale parameter for the boom-bust process in figure 2.4. This calls for some remarks :

— For a positive $a$, the martingale parameter $\alpha(a)$ is the first (here, only) point of the curve above the point $(a, 0)$. As it is increasing, this is indeed tantamount to looking for the rightmost (i.e. dominant) curve.

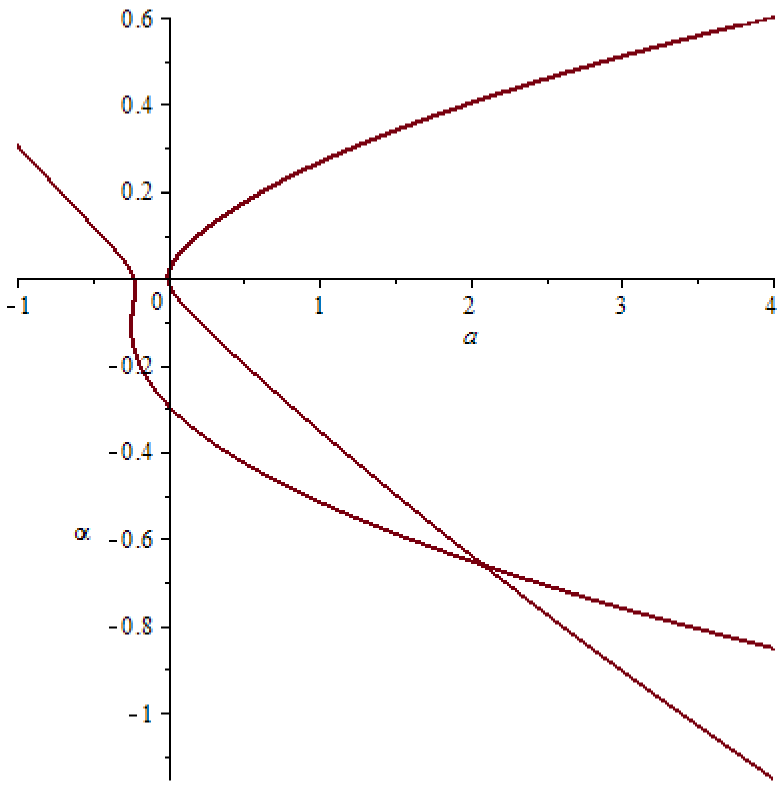


Figure 2.4 – Logarithmic eigenvalues for $L_C(\alpha)$

— Likewise, the negative martingale parameter $\beta(a)$ is the first portion of the curve below the horizontal axis. Other curves correspond to non-dominant eigenvalues.

— The tangent at point $(0, 0)$ is *not* vertical : the slope is $\alpha'(0) = -1/E(C)$ whenever $E(C) \neq 0$. This also means that $\alpha(0) > 0$ when $E(C) > 0$, and illustrates the existence of martingale parameters for negative values of $a$, when proceeding along the branches until hitting the minimum $a$, where a vertical slope is found.

— Despite the appearances, $f$ does *not* have any double solution on either dominant branch. The seemingly breaking slope is fairly common when looking at convex combinations of exponentials, like $\ln(e^x + e^{-x})$.

— For high values of $a$, the final slopes are roughly given by $-1/\delta^{\pm}(C)$, as these values come from the highest possible drifts for $C$. When $C$ is not bounded (as it is the case here), these slopes go to 0.

### Default probability

One finds for example at point $a = 0$ the values

$$\alpha(0) \approx .00398\$^{-1} \wedge \mu^{(0)} \approx \begin{pmatrix} .429 & .571 \end{pmatrix} \wedge w^{(0)} \approx \begin{pmatrix} .947 \\ 1.040 \end{pmatrix}$$

Hence, for a high starting point like $C(0) = 200\$$, one gets $\mathbf{P}(T_0 < \infty) \approx .427$. We remark that

— The martingale parameter is extremely low (regarding the relative values of the transition payoffs), so the Lévy-like estimation of $C$'s default probability through $e^{-\alpha(0)C(0)}$ decays really slowly to 0. This is a consequence of positive correlation between transition payoffs, increasing the risks of default : high cash levels are required to be prepared against a bad sequence of transitions (this translates to "a crisis" in econometrics).

— Despite $A_2$ appearing really worse than $A_1$ through the transition payoffs, $w^{(0)}$'s spread is only .094. However, we recall that the "corresponding correction" on $C$ amounts to $\delta\left(w^{(0)}\right)/\alpha(0) \approx 23.5\$$, a high value compared with $C$'s transition payoffs, so it is not safe to approximate $C$ as a Lévy process.

— The fact that $w^{(0)}_{[1]} < 1$ and $w^{(0)}_{[2]} > 1$ is not a surprise either : as $A_2$ is a bad state, the default probability is higher than normal when starting from it, and conversely for $A_1$.

Unfortunately, the exact default probability is extremely tedious to compute, requiring lookalikes to Pollaczek and Khintchine's formula ([25]). For this reason, we shall look at its behaviour later, through the case of a continuous-time C-process ; one might also refer to [14] for methods of numerical solving.

### Alternate variance term

Recalling that we "called" the variance of a C-process to be

$$V(C) = \frac{-\frac{d^2\alpha(a)}{da^2}(a=0)}{\left(\frac{d\alpha(a)}{da}(a=0)\right)^3}$$

in paragraph 2.4.1, we may compare $C$'s default probability to the default probability of a Brownian motion of drift $E(C)$ and variance $V(C)$. As we get $V(C) \approx 63.631\$^2/s$, we have

$$\frac{2E(C)}{V(C)} \approx .00397\$^{-1}$$

which comes remarkably close to the actual $\alpha(0)$. This stresses the accuracy of defining $V(C)$ as such, with the small discrepancy coming from the fact that $C$ is not a Brownian motion.

Interestingly, let us look at what happens if one computes $V(C)$ while neglecting time dependency, assuming that $C$ is a Lévy process. The random distribution of $C$'s increment becomes either $D_{i \to j}$ with probability given by the amount of time spent on each transition, i.e. $\mu_{[i]} P_{i \to j}$ : this yields the same expectancy $E(C)$ but now the variance becomes $V_l(C) \approx 14.222\$^2/s$, a largely underestimated value compared with $V(C)$. As an immediate consequence of this estimation error, the estimated martingale parameter is overvalued by a factor greater than 4, which means that the default probability is largely undervalued. Taking $C(0) = 200\$$, this leads to an estimated default probability around 2.87% instead of the correct 42.7%, illustrating that one cannot safely deem the transitions to be independent without risking huge errors while looking at market safety. In other words, momentum effects have a huge adverse effect on default safety, as they decrease the benefits of cash holdings by allowing long sequences of down-drifted momentum, which is illustrated by the downgrade of the martingale parameter.

### Quick default

When looking at a high value of $a$, e.g. $a = 9$, one gets

$$\alpha(9) \approx .9485 \wedge \mu^{(9)} \approx \begin{pmatrix} 7.4 * 10^{-5} & 1 - 7.4 * 10^{-5} \end{pmatrix} \wedge w^{(9)} \approx \begin{pmatrix} 9.417 * 10^{-6} \\ 1.00007 \end{pmatrix}$$

This time,

- The martingale parameter is low compared with the value $a = 9$, because of the simple possibility of $M$ getting stuck on the bad state : the probability of "quick" default is indeed headed by the worst sequences for $C$. Actually, when $C$ is bounded, it may in turn be related with the minimal drift $\delta^-(C)$, finding
$$\mathbf{E}\left(e^{-aT_0}\right) \approx e^{aC(0)/\delta^-(C)}$$
When $C$ is not bounded, thinking that $\delta^-(C) = -\infty$ is a rough explanation for the low $\alpha(a)$.
- Computations lead to a maximal correction on $C$ of $\delta\left(w^{(9)}\right)/\alpha(9) \approx 12.2\$$, which still indicates that $C$ does not look like a Lévy process. However, the Lévy-like estimation is almost correct for $M(0) = A_2$ with a correction of $\ln\left(w^{(9)}_{[2]}\right)/\alpha(9)$, below $10^{-4}\$$. Once again, this is because $A_2$ is the most likely state to trigger a default, while starting from $A_1$ "ensures" some time in this good state and delays the sequence of default for long enough to drop the probability of quick default.

### 2.5.2 Links with Lévy processes

As C-processes are a natural continuation of Lévy processes, one may find interesting links between these.

#### Hidden Lévy processes

First, if $A = 1$, i.e. the market always lies in the same state, the underlying Markovian process is trivial, the C-process becomes a pure Lévy process ; and reciprocally, any Lévy process is a C-process with $A = 1$. It follows that all results proved throughout this study also hold for Lévy processes, with a simplification : the dominant eigenvectors $w^{(a)}$ and $\mu^{(a)}$ amount to the constant 1 and disappear from the computations. We also proved earlier an equivalence between zero spread and $C$ being a Lévy : indeed, zero spread means $w^{(a)} \in \mathbf{R}_+^* \left(\vec{1}\right)$, and then the scaling equations for $w^{(a)}$ and $\mu^{(a)}$ lead to $w^{(a)} = \left(\vec{1}\right)$. Hence, this simplification holds again when $w^{(a)}$ has zero spread for every $a$ ; indeed, $w^{(a)}$'s influence disappears from the statement of the theorem.

An item used in the definition of the martingale parameter is the restricted Lévy process, being the process $C\left(\tau_k\right)_{k\in\mathbf{N}}$ where the times $\tau_k$ are the successive hitting times of $M$'s starting state, which is a Lévy thanks to the property of canonical time sequences. One may remark that this trick of computing increments on several time periods, to compare identical states of the market, is commonly used by economists for instance when calculating price variations on a year from and to identical calendar dates, as the market undergoes changes due to yearly periodicity : this corresponds to identical states of $M$.

#### Continuous-time Lévy processes

In this paragraph, we imagine some extension of the study when $C$ is a continuous time process. For example, a continuous-time C-process may be defined as such :

- $M$ is a continuous-time Markovian process, changing of states with exponential waitings (like a non-explosive Poisson process, similarly to the Markov-modulated risk model) ;
- Transition payoffs are continuous-time Lévy processes, whose parameters are given by the present state $M(t)$.

We notice that we may define additional random jumps for $C$ when $M$ changes of states (from $A_i$ to $A_j$), as a random variable $D_{i\to j}$ acting as a transition payoff between these states, but we will not in order to simplify the further explanations. As a consequence, the previous transition payoffs that were defined by means of transitions (between states) are now more accureately described as “state payoffs” (in a given state). For example, taking them to be the appropriate compounded Poisson processes, one recovers the Markov modulated risk model from [2].

It is possible to define martingale parameters for such C-processes, and we shall compare the built items with [42].

1. One takes an infinitesimal time period $dt$, and then computes the transition probabilities, giving $M$'s transition matrix $P^{dt}$ over this time interval

   $$\forall i, j \leq A, P^{dt}_{i,j} = \mathbf{P}\left(M(t+dt) = A_j | M(t) = A_i\right)$$

   Differentiation when $dt$ goes to 0 ultimately leads to an log-transition matrix $P \in \mathbf{M}_A\left(\mathbf{R}\right)$ such that

   $$\forall i, j \leq A, t \in \mathbf{R}^+, \mathbf{P}\left(M(t) = A_j | M(0) = A_i\right) = e^{tP}$$

   This matrix $P$ is called $\eta$ in [42].
2. Likewise, the Laplace matrix function of $C$ over a time interval $dt$ is reached, eventually getting a log-Laplace matrix function $\Lambda_C$ of $C$. This matrix is illustrated by $\Theta$ though the lemma 9.2 in [42].
3. The eigenvector equation should become

   $$\forall t \in \mathbf{R}^+, e^{t\Lambda_C(\alpha(a))} w^{(a)} = e^{ta} w^{(a)}$$

   yielding $\alpha(a)$ (called $R$ in [42]) and $w^{(a)}$ ($g(r)$ in [42], where $r$ plays $a$'s part).

Thus one obtains martingale parameters for continuous-time C-processes. One now gets a result similar to the main theorem 2.1 for C-processes whose jumps are bounded by $Q$ : assuming that $C$ is right-continuous, a descending process $\vec{C}$ is defined through the binary determination sequence $\rho$ given by

$$\rho(t) = 1 \Leftrightarrow (\forall s < t, \rho(s) = 1 \Rightarrow C(t) \leq C(s) - h)$$

where $h \in \mathbf{R}^*_+$ is an arbitrarily small parameter, ensuring that the time sequence $\tau$ defined by $\rho$ is discrete. One continues as in the proof for the main theorem, with special care taken when $\vec{C}$ hits the region $[0, h]$, as the convolution equation loses its validity here.

When $C$ is a non-jump process, another interisting idea is to build a discrete "grid" of interesting positions for $C$ (say, when $C(t) \in \mathbf{Z}$), then use that $C$'s trajectories are continuous to ensure that $C$ must cross $C_0 - 1$ on its way to default. This allows us to get a descending equation for the functions $K_i^{(a)}$, given by

$$\forall x \in \mathbf{N}, K_i^{(a)}(x+1) = \sum_{j=1}^{A} P^{(a)}_{i \to j} K_j^{(a)}(x)$$

where the terms $P^{(a)}_{i\to j}$ express the probability of having gone from $A_i$ to $A_j$ while $C$ loses one unit. As $P^{(a)}$ is ergodic as soon as $M$ is, we shall get an exponential convergence for the theorem 2.1. Finally, we notice that

— Problems related with periodicity do not appear here, as the values of $K_i^{(a)}$ in $(x, x+1)$ are controlled by the values of $K_j^{(a)}(x)$ because $C$ has continuous trajectories (which is false for discrete, periodic C-processes). This is the reason why the standard Markov-modulated risk model dodges periodicity issues.

— Incidentally, exact expressions for $K_i^{(a)}$ (thus $L_i^{(a)}$) are obtained once all values $K_i^{(a)}(0)$ are found, so one gets the exact log-Laplace values.

Therefore, there is a link between jumps and discrete time, as taking $C$ over a discrete-time universe creates its jumps (the values $C(t+1) - C(t)$). As a consequence, one is encouraged to use discrete C-processes whenever possible, unless continuous time allows $C$ to have continuous trajectories. We suggest referring to [42] or [2] for the main example of a Markov-modulated risk process for an insurance company.

**Example of default probability in continuous time**

Such a non-jump continuous-time C-process has a default probability shown in figure 2.5, and we shall admit that a typical discrete-time C-process yields a similar graph. The default probability of this process is given when starting from one of its 3 states (color), and the starting point $C(0)$ (x-axis), in a log-plot fashion.

— The (negated) asymptotical main slope of each graph indicates the martingale parameter ; we see that it is common to each state (parallel asymptotes).

— Existence of these asymptotes indicates that the residual term $(1 + o(1))$ is correct.

— The vertical distance between the asymptotes refers to the dominant eigenvector $w^{(0)}$ : in this example, the red state is associated with a coordinate of $w$ lower than 1, while the green one is higher. In particular, the vertical width containing all asymptotes is $w^{(0)}$'s spread.

— Likewise, the horizontal distance $\delta\left(w^{(0)}\right)/\alpha(0)$ is the maximal correction to $C(0)$ required to compare default probabilities between different states of $M$.

Notice that in general, with discrete time (or with jumps), the default probabilities are not necessarily continuous of $C(0)$, or well-ordered : there may be $x, y \in \mathbf{R}^+$ and $i, j \leq A$ such that

$$\mathbf{P}\left(T_0 < \infty | M(0) = A_i \wedge C(0) = x\right) < \mathbf{P}\left(T_0 < \infty | M(0) = A_j \wedge C(0) = x\right)$$

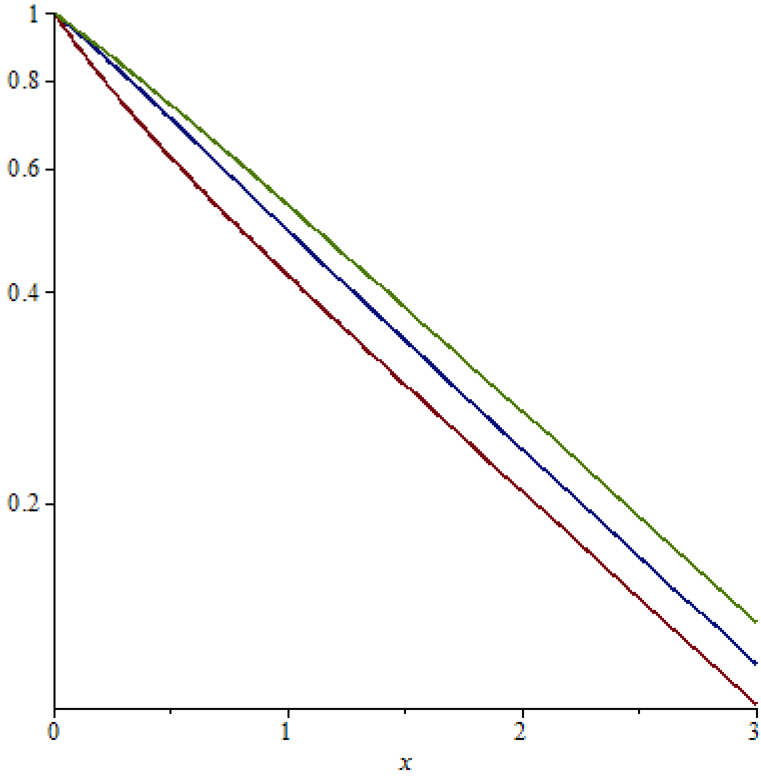


Figure 2.5 – Default probability

but still

$$\mathbf{P}\left(T_0 < \infty | M(0) = A_i \wedge C(0) = y\right) > \mathbf{P}\left(T_0 < \infty | M(0) = A_j \wedge C(0) = y\right)$$

However, asymptotical considerations still work under the hypotheses of the theorem 2.1.

### Links with Brownian motions

We briefly look at this example through a simple case, i.e. when $C$ itself is a non-jump Lévy process ($A = 1$ and $M$ is trivial). Infinite divisibility of a Lévy process means that one may decompose the time period into shorter and shorter subdivisions, so that the width of increments is progressively driven to 0. In particular, $C(T_0) = 0$, and as dominant eigenvectors of Lévy processes have no spread, the martingale process $X_C^{(a)}$ allows for the equation

$$\mathbf{E}\left(e^{-aT_0}\right) = e^{-\alpha(a)C_0}$$

We recall that the martingale parameter $\alpha(a)$ of a Lévy process of increments $D$ (with $\mathbf{P}(D < 0) > 0$) is defined by the only $\alpha \in \mathbf{R}_+^*$ solution to $L_D(\alpha) = e^a$, i.e.

$$\forall a \in \mathbf{R}_+^*, \alpha(a) = \Lambda_D^{-1}(a)$$

where $\Lambda_D^{-1}$ is the reciprocal function to the restriction of $\Lambda_D$ on the set where it is increasing. The log-Laplace transform of $T_0$ thus holds

$$\forall a \in \mathbf{R}_+^*, -\Lambda_{T_0}(a) = \Lambda_D^{-1}(a) C_0$$

However, when $C$ is a non-jump Lévy process with homogeneous increments, by Lévy-Ito decomposition it is also a Brownian motion (with a drift) ; setting $E \in \mathbf{R}$ its drift (that coincides with its mean expectancy) and $\sigma^2$ its variance, we know from the definition that

$$\forall w \in \mathbf{R}, \Lambda_D(w) = \frac{\sigma^2}{2}\left(w - \frac{E}{\sigma^2}\right)^2 - \frac{E^2}{2\sigma^2}$$

Then we get

$$\forall a \geq \frac{-E^2}{2\sigma^2}, \Lambda_D^{-1}(a) = \frac{E + \sqrt{2a\sigma^2 + E^2}}{\sigma^2}$$

So, the previous computation allows us to recover the formula for the log-Laplace transform of a drifted Brownian's default time from this expression :

$$\forall a \in \mathbf{R}_+^*, \Lambda_{T_0}(a) = -C(0)\frac{E + \sqrt{2a\sigma^2 + E^2}}{\sigma^2}$$

This allows us some observations :

- If $E \leq 0$, then $\Lambda_{T_0}(0) = 0$ and we recover the fact that $\mathbf{P}(T_0 < \infty) = 1$. If $E$ has a large negative value compared to $\sigma^2$, then first-order approximation leads to $\Lambda_{T_0}(a) \approx C(0)a/E$, which is indeed the expected behaviour of a deterministic process ($E$ dominates the variance).
- If $E > 0$, then $\Lambda_{T_0}(0) = -C(0)\left(2E/\sigma^2\right) < 0$. The above comparison allows us to recover here the characteristic parameter $2E/\sigma^2$ of the equivalent diffusion process' scale function.
- If $\sigma$ goes to 0, then the process behaves as deterministic : we get again $\Lambda_{T_0}(a) \approx C(0)a/E$ if $E < 0$, but now $\mathbf{P}(T_0 < \infty)$ goes to 0 if $E > 0$.
- If $\sigma$ goes to infinity, then on the contrary $\Lambda_{T_0}(a) \approx -C(0)\sqrt{2a}/\sigma$ goes to 0, because high volatility is detrimental to survival.

We ultimately recover the generic ideas about stochastic processes : drift enhances the suvival probability, while variance hampers it.

## 2.6 Conclusion

The model of C-processes provides interesting thoughts about apparent volatility of financial processes. For example,

— Fat tails of increments are not required to yield high risks of default ;

— Positive correlation however really is a concern ;

— The present "state of the market" has a quantifiable effect on default risks (provided that the market is known).

We present these ideas in this conclusion.

### 2.6.1 Discussion

In this section, we will discuss about and justify the hypotheses we assumed throughout this study, and the effects they have on the main theorem.

#### Need for positive recurrence of $M$

We deemed $M$ to be positive recurrent ; actually, the analysis only needs the starting state $M(0)$ to be positive recurrent, since if it is, then we only need to restrict $M$ to its accessible states, which leads to a positive recurrent Markov process. When the starting state is not recurrent, a problem arises when computing the successive reductions of the Markov chain in the proof : for example, if all recurrent states have merged into a single $A_k$, it is impossible to define transition payoffs "through" $A_k$ because there is no way out of $A_k$. This has an effect on the equivalence between real solutions $\alpha \in \mathbf{R}$ for $M$ and its reduced matrix, as now a viable real solution disappears (this corresponds to the case $(L_C(\alpha))_{k,k} = 1$ of the proof). The alternate solution we are going to present if $M$ is not positive recurrent has the effect of leading the study back to cases where it is. For the sake of simplicity, we shall only look at the case of the default probability ($a = 0$), and note $\alpha = \alpha(0)$, as the general case is similar.

It is possible to reorganize the matrix $L_C(\alpha)$ as to merge separately transient and groups of recurrent states (the "closed communicating classes" of the proof), and eventually getting a reduced $L_{C'}(\alpha)$ as a $(g+1)\times(g+1)$ matrix, with row and column 1 referring to the transient state, and rows and columns 2 to $g+1$ referring to the $g \in \mathbf{N}^*$ recurrent states : we shall get some sub-geometrical increments $D_{1,i}$ for any $i \leq g+1$ and $D_i$ for any $i$ from 2 to $g+1$ such that

$$L_{C'}(\alpha) = \begin{pmatrix} p_1 L_{D_1}(\alpha) & p_2 L_{D_{1,2}}(\alpha) & \cdots & p_{g+1} L_{D_{1,g+1}}(\alpha) \\ 0 & L_{D_2}(\alpha) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{D_{g+1}}(\alpha) \end{pmatrix}$$

with $p_i$ the corresponding transition probabilities of the reduced process $M'$. As $p_1 < 1$, either $D_1$ is nonnegative almost surely (in which case we set $\beta = \infty$), or it is not and then there is a single $\beta \in \mathbf{R}_+^*$ such that $L_{D_1}(\beta) = 1/p_1$. Now we get

— $\beta > 0$, which may be regarded as the martingale parameter of the probability of default before hitting the recurrent part of $M$ (i.e. at $a = 0$) ;
— Several values $\alpha_i$ for any $i$ from 2 to $g+1$, each one being the martingale parameter of the default probability in the closed communicating class number $i$ of $M$.

This case will not be analyzed any further, but one eventually gets that the smallest of values $\alpha_i$ and $\beta$ governs the exponential behaviour of the default probability : for instance, if it is zero, then the default probability converges to a positive constant when $C_0$ goes to infinity, correponding to the probability of $M$ eventually hitting a closed communicating class over which $C$'s mean expectancy is negative.

However, if one of $M$'s states fails to be positive recurrent, removing inaccessible states is mandatory, else the main result fails. For example with $C$ defined by the following $M$ and transition payoffs :

$$(P_{i \to j})_{i,j} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

hence $A_1$ is not positive recurrent ;

$$\begin{aligned} \mathbf{P}\left(D_{1\to 1} = -4\right) &= 1 \\ \mathbf{P}\left(D_{1\to 2} = 0\right) &= 1 \\ \mathbf{P}\left(D_{2\to 2} = \pi\right) = \mathbf{P}\left(D_{2\to 2} = -1\right) &= 1/2 \end{aligned}$$

Notice that we arbitrarily took $\pi$ as a value to avoid $C$ to be periodic, but any irrational number greater than 3 also worked. When $M(0) = A_1$, the estimate given by the martingale parameter is correct : the default probability is well-controlled by the probability of defaulting before leaving state $A_1$, which is $(1/2)^{1+\lfloor C_0/4 \rfloor}$, leading to a martingale parameter of $\beta = \ln(2)/4 \approx .173$. However, when $M(0) = A_2$, $C$ turns out to be a Lévy process, whose computations lead to $\alpha \approx .619$. This is because the "bad" state $A_1$ actually has no effect on $C$, while still contributing to decreasing $\alpha$, as it was not removed from the study.

### Need for $Q$

The transition payoffs are deemed to be bounded by $Q \in \mathbf{R}^+$ in the main theorem. The usefulness for this hypothesis lies

— When ensuring that $C$ is automatically positive recurrent provided once $M$ is ;

— When defining $C$'s Laplace matrix function, ensuring both that it will be well-defined and that the Laplace transform of the restricted payoffs explodes at its boundary ;
— To technical lemmata, ensuring that $C$'s descending process drops increments no larger than $Q$.

However, we expect the theorems to hold even when $C$ is not sEI, provided that $L_C$ explodes at the boundary $\alpha_0$ of its convergence domain : some reasons behind this idea lie in the construction of $\alpha$ and $L_C$, then of the functions $K_i^{(a)}$.

— A martingale parameter $\alpha(a)$ is still found for every value of $a \in \mathbf{R}^+$, with the additional property $\alpha < \alpha_0$ over $\mathbf{R}^+$.
— The definition of $\vec{C}$ is the same, and the convolution process $\Phi_C^{(a)}$ is still well-defined ;
— As exponential expectancies of $C$'s values only appear as

$$\mathbf{E}\left(e^{-wC(T)}\right)$$

for $T$ a hitting time and some $w \in \alpha\left(\mathbf{R}^+\right)$, which implies $w < \alpha_0$, the expectancies are always integrable.

When $L_C$ does not explode at its boundary, for example when $C$ is the Lévy process whose increments $D$ have a density $\psi$ defined by

$$\forall x < -1, \psi(x) = \frac{Ke^x}{x^2}$$

where $K$ is the normalization constant, the martingale parameter is no more defined once $a$ passes $a_0 = \Lambda_D(\alpha_0)$ (here computations lead to $\alpha_0 = 1$ and $a_0 = K$). The best we can do now is to use the desired property for $a_0$ and then say that $-\Lambda_{T_0}$'s derivative is bounded from below by 1, indeed

$$\frac{-d\Lambda_{T_0}(a)}{da} = \frac{\mathbf{E}\left(T_0 e^{-aT_0}\right)}{\mathbf{E}\left(e^{-aT_0}\right)}$$

and $T_0 \geq 1$ almost surely by construction, so we get

$$\forall x \in \mathbf{R}^+, -\Lambda_{T_0}(a) \geq \alpha_0 C(0) + (a - a_0)$$

However, we have the following interesting idea for the processes whose increments have fat tails (when $\alpha = \alpha_0$, one gets $\phi_{[\alpha]}^{(a)}$ with the quadratic tail) : default times for large values of $x$ are well-controlled by the probability of defaulting in one step. We know that for every random variable $X$ whose Laplace transform stops being defined at point $\alpha_0$, for every $\beta > \alpha_0$, there are arbitrarily large values of $x \in \mathbf{R}_+^*$ such that

$$\mathbf{P}\left(X < -x\right) > \frac{e^{-\beta x}}{x^2}$$

because the opposite would lead to well-defined values of $L_X$ for $x \in (\alpha_0, \beta)$ through integration over $(-\infty, -x)$. If $C(0)$ is one of these values of $x$, the default probability in one step is $\mathbf{P}(D < -C(0))$, which leads to

$$\mathbf{E}\left(e^{-aT_0}\right) \geq e^{-a}\mathbf{P}(T_0 = 1) \geq e^{-a}\frac{e^{-\beta C(0)}}{C(0)^2}$$

and so we get

$$-\Lambda_{T_0}(a) \leq a + \beta C(0) + 2\ln(C(0))$$

for these values. Noticing the similarity between this expression (when $\beta$ goes to $\alpha_0$) and its counterpart, we remark a strictly sub-linear discrepancy in $C(0)$. This means that

- We cannot improve further the quality of this control only through the change of linear terms (like the martingale parameter) ;
- More accurate knowledge of $D$ is required in the general case.

Finally, we may remark that this should also work without any assumption on $C$'s increments with $\alpha_0 = 0$ and $a_0 = 0$, but then the study is of little interest, except showing why processes with fat tails (like the Cauchy distribution) have a "wild" behaviour, with high default probabilities. Hence, for our Cauchy example, $-\Lambda_{T_0}$ is strictly sub-linear in $C(0)$.

### No fat tails

During the introduction, we chose to exclude the issues of fluctuations having fat tails or fractal distributions. Here are some lines of thought to explain why we could.

- Common sense : the world is finite, thus all modelled values should remain bounded.
- Discrete time : a common issue when looking at continuous-time processes lies in their nature, creating unbounded variations and jumps. This is not a concern here, as we only look at increments over well-defined, fixed intervals.
- In this model, *fat tails are an illusion*, which may look anticlimatic contrasted with "new" research ([29]). When underestimating the variance $V(C)$, it is common to assert that $C$ has fat tails ; however, it appears that this phenomenon comes solely from the time dependency between successive increments, creating long sequences of positive feedback and pushing $C$ far away from its mean expectancy.

To clarify this latter statement, let us take $C$ to be the process $(\mathbf{N} \to \mathbf{N})$ whose

- Transition probabilities are given by the transition matrix

$$P = \begin{pmatrix} 1-z & z \\ z & 1-z \end{pmatrix}$$

where $z < 1/2$ (positive feedback) ;

— Transition payoffs are deterministic $D_{1\to1} = 1$, $D_{1\to2} = 1$, $D_{2\to1} = -1$ and $D_{2\to2} = -1$.

We compute the martingale parameter, leading to the variance term being $1/z - 1 > 1$ when the feedback is positive (and lower than 1 when the feedback is negative), compared with $V_l = 1$ when dependency is omitted ($z = 1/2$). Worse yet, the variance is arbitrarily large when the dependency goes to $z = 0$. Although we could proceed with the definitions of higher moments of a C-process, the main effect is that the measurement of "tails" (moments) thanks to $C$'s martingale parameter is high despite the low width of each increment. For this reason, if this model is eventually applied to real data, it explains the observed fat tails by momentum effects rather than the nature of the transition payoffs, and assuming $C$ to be bounded should not be an issue.

#### Differences between periodicity and aperiodicity of $C$

Perhaps the most shocking observation about the main theorem 2.1 is that the convergence in $K_\infty^{(a)} + o(1)$ fails for the most natural C-processes, where the transition payoffs are expressed as commensurate quantities : one instead gets the other result, the proposition 2.4.4, specific to periodic C-processes, where the convergence of the error function does not hold on the whole $\mathbf{R}$. This is not a flaw of the study, as the default probabilty cannot be $K_\infty^{(a)} + o(1)$ everywhere, as we saw in the section dealing with periodicity issues.

On the bright side, the form $K_\infty^{(a)} + o(1)$ is recovered if one imposes $C$ to remain commensurable with its increments, which is the case if one follows $C$'s trajectory through transition payoffs, so the theorem may be applied "on-line", while analysing $C$'s expected behaviour with respect to time.

### 2.6.2 Interpretations

We shall end this study with several interpretations and examples of use for these results, like the effects of capital investment or liquidity issues.

#### Dimensional analysis

To understand the parameters given during this study, we give their respective units of measurement. For example,

— The time $t$ expresses in seconds $s$ ;

— The capital value $C$ expresses in dollars \$.

We chose to read the units of the continuous-time model, to avoid hidden terms like $\times 1s$ appearing in the equations involving transitions, as in $C(t+1) = C(t) +$

$D(t+1) \times 1s$.

- The transition payoffs $D_{i \to j}$ and the mean expectancy $E(C)$ express in $\$/s$ ;
- The Laplace parameter $a$, the differential transition matrix $P$ and the differential log-Laplace matrix function express in $1/s$ ;
- Probabilities and the dominant eigenvectors $w^{(a)}$ and $\mu^{(a)}$ express in 1 (without unit), as well as the maringale process $X_C^{(a)}$, whereas the differentiated martingale is in \$ ;
- The martingale parameter $\alpha$ expresses in 1/\$ ; in particular, its derivative $\alpha'$ is in $s/\$$ and its second derivative $\alpha''$ in $s^2/\$$, so the "variance" $-\alpha''(0)/\left(\alpha'(0)\right)^3$ is in $\$^2/s$.

These units explain the parameters :

- Items expressed in \$ are related to an amount of money, and refer to a quantity of cash related with them. For example, $\ln\left(w_{[i]}^{(a)}\right)/\alpha(a)$ is the corrective term one must substract to the cash $C(t)$ to get the "true" value of the assets when in state $A_i$, cancelling the effect of present-state overestimations.
- Items in $\$/s$ are cash flows (incomes), indicating the profit/loss balance affecting $C$ over a time period, and may be regarded as drifts. For instance, $E(C)$ is the main drift of $C$.
- Items in $1/s$ act as frequencies ; hence, the differential $P$ is the probability of $M$ switching states over a time period, and may be related with the exponential parameter of a Poisson process.
- Quantities with unit 1 are coefficients and indicate a ratio between a "specific" event and the "generic" configuration. Comparing $w^{(a)}$'s coordinates with 1 yields the correction between default event with and without knowledge of $M(t)$, e.g. for $X_C^{(0)}$ the probability of default, it indicates how the state $M(t)$ increases default risks.
- Being in 1/\$, $\alpha$ indicates the marginal effect of a single dollar on the default event. If one aims at avoiding default, looking at $\alpha(0)$ measures how being richer decreases the default probability.

On a side note, we recover that the "variance" term really expresses in $\$^2/s$, as well as the quadratic variation of a stochastic process, which is an additional reason to name it $C$'s variance.

### Nature of the result

We consider the main result of this study, discussing about its implications in terms of economy. The Laplace transform of the default time at point $a$ (and in particular the default probability, when $a = 0$) is written as

$$\mathbf{E}\left(e^{-aT_0}|M(0) = A_i \wedge C(0) = C_0\right) = e^{-K(a)} w_{[i]}^{(a)} e^{-\alpha(a)C_0}(1 + o(1))$$

One should notice that the martingale parameter $\alpha(a)$ does not depend on the state of the market and is an in-built parameter of the model. The only impact of the market state lies in the vector $w^{(a)}$ : high values of $w_{[i]}^{(a)}$ indicate that the market suffers from "bad" short-term situation, because of

— Incoming transition payoffs likely to decrease $C$ (adverse mean), thus shortening the default time ;
— By vertue of the martingale equation with $\alpha(a) > 0$, leading by convexity Jensen-like inequalities to a risk-adverse-shaped expected exponential default time, high uncertainty on the immediate future (adverse variance).

However, considering the market for a long-term analysis, then one should refer to the martingale parameter $\alpha(a)$, a better indicator of the expected outcome of the market than the "classical" mean-variance characterization, the latter being blind to the structure of a C-process.

In the main theorem, the constant term $K$ is related to the way $C$ may default : for additional considerations on the severity of default, one is encouraged to refer to [18]. Considering the martingale process

$$X_C^{(a)} = \begin{pmatrix} \mathbf{N} & \to & \mathbf{R}_+^* \\ t & \to & w_{[M(t)]}^{(a)} e^{-\alpha(a)C(t)} e^{-at} \end{pmatrix}$$

then we know thanks to the martingale property of proposition 2.4.1 that for every $a \geq 0$,

$$w_{[i]}^{(a)} e^{-\alpha(a)C_0} = \mathbf{E}\left(w_{[M(T_0)]}^{(a)} e^{-\alpha(a)C(T_0)} e^{-aT_0}|M(0) = A_i \wedge C(0) = C_0\right)$$

As we saw earlier, an interpretation of this is that successive time periods eventually "mix" the conditionnal probabilities $\mathbf{P}\left(M(t) = A_i|C(t) \in (x, x+\epsilon)\right)$ (for $i \leq A$) when $t$ increases, which is automatic as $C_0$ increases, regardless of the value $x$. So, the final value of $X_C^{(a)}$ will keep pretty much the same distribution for any (large) $C_0$ and any $M(0)$. Interestingly, we see that this mixing does not occur when $C$ has a period $p \in \mathbf{R}_+^*$, as for any $n \in \mathbf{N}$ and $p_k$ the natural offsets, then $\mathbf{P}\left(M(t) = A_i|C(t) = pn + p_k\right)$ is 1 iff $i = k$ and 0 otherwise (provided that several states $A_k$ do not share the same offset). This is yet another reason why the main theorem fails for periodic processes.

### Examples of use

Interpretations of the results found in this study are useful when computing some buying/selling decisions in markets where liquidity issues are a concern (see also McKean's problem, [34]). For example, with an intuitive model where investment of some cash enhances the transition payoffs (increases the martingale parameter $\alpha(a)$ to $\alpha'(a)$) at the expense of some illiquidity (decreases the cash level $C$ to $C' = C - I$ for $I \in \mathbf{R}^+$ the investment costs), one observes that investment is often beneficial for a rich enough buyer, as the negative logarithmic default probability is in $\alpha'(a)(C - I) > \alpha(a)C$, since $\alpha'(a) > \alpha(a)$, for $I/C$ small enough. We may also notice that the decrease in $\Lambda_{T_0}(a)$ should be lesser for high values of $a$, because $\Lambda_{T_0}(a)$ involves short-term default, whose probability is actually raised by the investment costs : if the market goes dramatically wrong for the investor, then the asset costs will cause liquidity shortage quicker (and before the increase in the transition payoffs amount to these costs). Conversely, we also find out that selling a valuable asset to avoid short-term default may be a viable strategy if liquidity issues are severe ($C$ close to 0), because the immediate risk of default outweights long-term profits.

As stated earlier, long-term behaviour is accurately expressed using the martingale parameter, which may lead to seemingly counter-intuitive facts. Hence, improving $C$'s mean expectancy does not automatically reduce the default probability, for example if the spread of increments now greatly exceeds the previous one. However, higher mean and lower variance do not guarantee a better martingale parameter in general, despite this being true specifically for Brownian motions ($\alpha(0) = 2E/\sigma^2$, obtained in the calculation of $\Lambda_{T_0}$ for a Brownian motion). This constatation may seem puzzling, as general theorems like the central-limit theorem state that long-term behaviour of a Lévy process only relies on mean and variance. An explanation for this is that the central-limit theorem only takes into account the asymptotical future of $C$, whereas its default time (if it arises) is likely to happen early compared with $C_0$. The main idea behind this previous sentence is that for large values of $C_0$ and $E > 0$, the law of large numbers ensures that time concatenation of increments greatly decrease the probability of a cumulative negative payoff ; then, for example, the Lévy $L$ with $\pm 1$ increments

$$\mathbf{P}\left(D = -1\right) = \epsilon = 1 - \mathbf{P}\left(D = 1\right)$$

has a probability of "early" (as soon as possible) default in

$$\mathbf{P}\left(T_0 = 1 + \lfloor C_0 \rfloor\right) = \epsilon^{1 + \lfloor C_0 \rfloor}$$

whereas the subsequent probabilities of default at posterior times are in a greater power of $\epsilon$, at least $2 + \lfloor C_0 \rfloor$ since $C$ went $+1$ somewhere. If $\epsilon$ is small, the central-limit theorem fails to represent $C$'s early defaults ; those being in a majority of defaults, it fails at accurately computing $C$'s default probability.

# Proofs

The rest of this content will be constituted by the proofs to the statements given during the study. Each paragraph title indicates the position of the statement to prove.

## 2.7 Preliminaries

This section is devoted to the proof for all statements and propositions used as basics to the definition of the martingale parameter of a C-process. We will require many tools during the proofs, mainly the notions of time concatenations and paths for a C-process, so we introduce them now. We will next use them and eventually prove all definitions and subsequent propositions provided in the beginning of the study. For this purpose, except when noted otherwise or when we aim at proving it, we consider throughout this part that $C$ is a C-process whose underlying Markovian process is $M$.

### 2.7.1 Proposition 2.3.1

We start with the properties of conditional independence, useful for further observations regarding independence between $C$'s increments.

#### Forward implication

We deem $C$ to be a C-process whose underlying Markovian process is $M$. As $M(0)$ and $C(0)$ are deterministic by definition, we want the Markovian property and the time-homogeneity property. To prove the Markovian property, let $t \in \mathbf{N}$, $H$ be a measurable subset of $\mathbf{R}$, and $i \leq A$. Let us compute

$$P_1(t, H, i) = \mathbf{P}\left(C(t+1) - C(t) \in H \wedge M(t+1) = A_i | \mathbf{F}(t)\right)$$

By definition of a C-process, since $C(t+1) - C(t)$ is actually the active increment $D_{M(t) \to M(t+1)}(t+1)$, then $P(t, H, i)$ is equal to

$$P_1(t, H, i) = \mathbf{P}\left(D_{M(t) \to i}(t+1) \in H \wedge M(t+1) = A_i | \mathbf{F}(t)\right)$$

However, $D_{M(t)\to i}(t+1)$ is independent of $\mathbf{F}(t)$ conditionally to $M(t)$, which is $\mathbf{F}(t)$-measurable, so we get the product

$$P_1(t,H,i) = \mathbf{P}\left(D_{M(t)\to i}(t+1) \in H | M(t)\right) \mathbf{P}\left(M(t+1) = A_i | \mathbf{F}(t)\right)$$

Since $M$ is Markovian, the rightmost probability is

$$\mathbf{P}\left(M(t+1) = A_i | \mathbf{F}(t)\right) = \mathbf{P}\left(M(t+1) = A_i | M(t)\right)$$

Finally, as $D_{M(t)\to i}(t+1)$ and $M(t+1)$ are independent conditionally to $M(t)$, the product yields

$$P_1(t,H,i) = \mathbf{P}\left(D_{M(t)\to i}(t+1) \in H \wedge M(t+1) = A_i | M(t)\right)$$

As $D_{M(t)\to i}(t+1)$ is $C(t+1) - C(t)$ when $M(t+1) = A_i$, we have the desired property. To prove time-homogeneity, we compute likewise for every $s,t \in \mathbf{N}$, $H$ a mesurable subset of $\mathbf{R}$ and $i,j \le A$,

$$P_2(t,H,i,j) = \mathbf{P}\left(C(t+1) - C(t) \in H \wedge M(t+1) = A_j | M(t) = A_i\right)$$

Like we saw before, $C(t+1) - C(t)$ is $D_{i\to j}(t+1)$ when $M(t) = A_i$ and $M(t+1) = A_j$, so $P_2(t,H,i,j)$ is by definition

$$\mathbf{P}\left(D_{i\to j}(t+1) \in H | M(t) = A_i \wedge M(t+1) = A_j\right) \mathbf{P}\left(M(t+1) = A_j | M(t) = A_i\right)$$

However, the family $\left(D_{i\to j}(t+1)\right)_{t\in\mathbf{N}}$ is independent and identically distributed, and independent of $M$, and $M$ is Markovian and time-homogeneous, so the terms in the product may rewrite, for any $s \in \mathbf{N}$, as

$$P_2(t,H,i,j) = \mathbf{P}\left(D_{i\to j}(s+1) \in H\right) P_{i\to j}$$

Since for any $s \in N$,

$$\mathbf{P}\left(D_{i\to j}(s+1) \in H\right) = \mathbf{P}\left(D_{i\to j}(s+1) \in H | M(s) = A_i \wedge M(s+1) = A_j\right)$$

once again, this ends the proof.

### Backward implication

This time, we know that $M$ is Markovian, is time-homogeneous, and $M(0)$ and $C(0)$ are deterministic by hypothesis. We want to create independent and identically distributed (with respect to $t \in \mathbf{N}$) families of random variables

$$\left(D_{i\to j}^{(t+1)}\right)_{i,j\le A}$$

such that for every $t \in \mathbf{N}$,

$$C(t+1) = C(t) + D^{(t+1)}_{M(t)\to M(t+1)}$$

If we succeed, then $C$ will be a C-process, whose underlying Markovian process is $M$ and transition payoffs are the (common) distributions of the created families. To prove that these families of transition payoffs are independent and identically distributed, let for every $i, j \leq A$, $H_{i,j} \subseteq \mathbf{R}$ be a measurable set, and

$$H = \prod_{i=1}^{A} \prod_{j=1}^{A} H_{i,j} \subseteq \mathbf{R}^{A^2}$$

We are going to compute, for $t \in \mathbf{N}$,

$$P(H,t) = \mathbf{P}\left( (D_{i\to j}(t+1))_{i,j\leq A} \in H | \mathbf{F}(t) \right)$$

and prove that this conditional probability is actually a constant that does not depend in $t$, which will lead to the result. Hence, let us consider the random variables

$$(D_{i\to j}(t+1))_{i,j\leq A, t\in\mathbf{N}}$$

defined by, for every $i, j \leq A, t \in \mathbf{N}$,

— If $M(t) = A_i$ and $M(t+1) = A_j$, then

$$D_{i\to j}(t+1) = C(t+1) - C(t)$$

— If not, then $D_{i\to j}(t+1)$ is a random variable called $Z_{i\to j}(t+1)$ whose distribution is the distribution of $C(s+1)-C(s)$ conditionally to $M(s) = A_i$ and $M(s+1) = A_j$ (no matter $s$, as ensured by hypothesis). All these random variables named with $Z$ are built to be mutually independent, and independent of all other random variables in this study.

We decompose $P(H,t)$ regarding $M(t)$ and $M(t+1)$ as follows :

$$P(H,t) = \mathbf{P}\left( \bigcup_{i_0=1}^{A} \bigcup_{j_0=1}^{A} \begin{pmatrix} M(t) = A_{i_0} \wedge M(t+1) = A_{j_0} \\ \wedge \forall i,j \leq A, D_{i\to j}(t+1) \in H_{i,j} \end{pmatrix} | \mathbf{F}(t) \right)$$

As the events $M(t) = A_{i_0} \wedge M(t+1) = A_{j_0}$ are pairwise exclusive, we get

$$P(H,t) = \sum_{i_0=1}^{A} \sum_{j_0=1}^{A} \mathbf{P}\left( \begin{pmatrix} M(t) = A_{i_0} \wedge M(t+1) = A_{j_0} \\ \wedge \forall i,j \leq A, D_{i\to j}(t+1) \in H_{i,j} \end{pmatrix} | \mathbf{F}(t) \right)$$

By definition of the payoffs, the conditions $M(t) = A_{i_0} \wedge M(t+1) = A_{j_0}$ allow us to rewrite the sought probability $P(H,t)$ as

$$\sum_{i_0=1}^{A} \sum_{j_0=1}^{A} \mathbf{P}\begin{pmatrix} M(t) = A_{i_0} \wedge M(t+1) = A_{j_0} \wedge C(t+1) - C(t) \in H_{i_0,j_0} \\ \wedge \forall (i,j) \neq (i_0,j_0), Z_{i\to j}(t+1) \in H_{i,j} | \mathbf{F}(t) \end{pmatrix}$$

The variables $Z_{i\to j}(t+1)$ are independent and independent of $\mathbf{F}(t)$ as well as $C$, thus they are factored. Additionally, $M(t)$ is $\mathbf{F}(t)$-measurable, so

$$P(H,t) = \sum_{i_0=1}^{A}\sum_{j_0=1}^{A}\left(\begin{array}{c}\mathbf{P}\left(M(t+1)=A_{j_0}\wedge C(t+1)-C(t)\in H_{i_0,j_0}|\mathbf{F}(t)\right)\\ \mathbf{1}_{M(t)=A_{i_0}}\prod\prod_{(i,j)\neq(i_0,j_0)}\mathbf{P}\left(Z_{i\to j}(t+1)\in H_{i,j}\right)\end{array}\right)$$

However, we know thanks to the Markovian property that

$$\begin{array}{rl} & \mathbf{P}\left(M(t+1)=A_{j_0}\wedge C(t+1)-C(t)\in H_{i_0,j_0}|\mathbf{F}(t)\right)\\ = & \mathbf{P}\left(M(t+1)=A_{j_0}\wedge C(t+1)-C(t)\in H_{i_0,j_0}|M(t)\right)\end{array}$$

and then thanks to the time-homogenous property, $\forall k\leq A, s\in\mathbf{N}$,

$$\begin{array}{rl} & \mathbf{P}\left(M(t+1)=A_{j_0}\wedge C(t+1)-C(t)\in H_{i_0,j_0}|M(t)=A_k\right)\\ = & P_{k\to j_0}\mathbf{P}\left(C(s+1)-C(s)\in H_{i_0,j_0}|M(s)=A_k\wedge M(s+1)=A_{j_0}\right)\end{array}$$

so we get for any $s\in\mathbf{N}$

$$\begin{array}{rl} & \mathbf{P}\left(M(t+1)=A_{j_0}\wedge C(t+1)-C(t)\in H_{i_0,j_0}|\mathbf{F}(t)\right)\\ = & \sum_{k=1}^{A}\mathbf{1}_{M(t)=A_k}P_{k\to j_0}\mathbf{P}\left(C(s+1)-C(s)\in H_{i_0,j_0}|M(s)=A_k\wedge M(s+1)=A_{j_0}\right)\end{array}$$

We also know, by definition of $Z_{i\to j}(t+1)$, that we have for any $s\in\mathbf{N}$

$$\mathbf{P}\left(Z_{i\to j}(t+1)\in H_{i,j}\right)=\mathbf{P}\left(C(s+1)-C(s)\in H_{i,j}|M(s)=A_i\wedge M(s+1)=A_j\right)$$

This leads to the expression of $P(H,t)$ being

$$\sum_{i_0=1}^{A}\sum_{j_0=1}^{A}\sum_{k=1}^{A}\left(\begin{array}{c}\mathbf{1}_{M(t)=A_k}\mathbf{1}_{M(t)=A_{i_0}}P_{k\to j_0}\\ \mathbf{P}\left(C(s+1)-C(s)\in H_{i_0,j_0}|M(s)=A_k\wedge M(s+1)=A_{j_0}\right)\\ \prod\prod_{(i,j)\neq(i_0,j_0)}\mathbf{P}\left(\begin{array}{c}C(s+1)-C(s)\in H_{i,j}\\ |M(s)=A_i\wedge M(s+1)=A_j\end{array}\right)\end{array}\right)$$

and it simplifies to

$$\sum_{i_0=1}^{A}\sum_{j_0=1}^{A}\mathbf{1}_{M(t)=A_{i_0}}P_{i_0\to j_0}\prod_{i=1}^{A}\prod_{j=1}^{A}\mathbf{P}\left(\begin{array}{c}C(s+1)-C(s)\in H_{i,j}\\ |M(s)=A_i\wedge M(s+1)=A_j\end{array}\right)$$

The double product does not depend on $i_0$ nor $j_0$, and is a constant term thanks to the time-homogenous property, that will be named $P_1(H)$ in the end of the proof. All that remains is now

$$P(H,t)=P_1(H)\sum_{i_0=1}^{A}\mathbf{1}_{M(t)=A_{i_0}}\sum_{j_0=1}^{A}P_{i_0\to j_0}$$

Hovever, the sum over $j_0$ amounts to 1 by definition of a stochastic matrix, and so does the sum over $i_0$, so $P(H,t) = P_1(H)$ does not depend on $t$. Since for every $t \in \mathbf{N}$,

$$D_{M(t)\to M(t+1)}(t+1) = C(t+1) - C(t)$$

then $C$ coincides with a C-process whose transition payoffs are given by the bulit families, which ends the proof.

### 2.7.2 Time concatenations

A running idea in this work consists in time concatenations, using increasing sequences of (possibly random) times $(\tau(k))_{k\in\mathbf{N}}$ ; we shall consider the C-process through its value $C(\tau(k))$ and its state $M(\tau(k))$. The trick of time concatenation allows us to define $C$-processes on the random subset of $\mathbf{N}$ described by a time sequence : as $M$ is Markovian and $C$'s transition payoffs are independent and identically distributed with respect to time, one may expect the process $(C(\tau(t)))_{t\in\mathbf{N}}$ to be a C-process.

#### Notions of concatenations

A concatenation is defined thanks to a time sequence, indicating the values of $C$ to be kept and observed. Most often, the successive times obey a condition making the concatenated process useful, defined as the binary determination.

**Definition 2.7.1** *Time sequence*

- *A binary determination sequence is a process of Bernouilli random variables, whose name is $\rho : (\boldsymbol{N} \to \{0,1\})$ such that*
  - $\rho(0) = 1$ *almost surely ;*
  - *For any $t \in \boldsymbol{N}^*$, $\rho(t)$ is $\boldsymbol{F}(t)$-measurable.*
- *The time sequence associated with $\rho$ is the increasing random process $\tau$ recursively defined by*

$$\tau = \begin{pmatrix} \boldsymbol{N} & \to & \boldsymbol{N} \cup \{\infty\} \\ t & \to & \min\left(\{u \in \boldsymbol{N}; (\rho(u) = 1 \wedge \forall s < t, u > \tau(s))\}\right) \end{pmatrix}$$

*If the set is empty, we set $\tau(t) = \min(\emptyset) = \infty$.*

Although problems arise when $\tau(t) = \infty$, as the definition $C(\tau(t))$ does not make sense there, it will be convenient to set for purposes of time concatenation :

- $C(\infty) = +\infty$ ;
- $M(\infty)$ is a new state, named $A_\infty$.

**Definition 2.7.2** *Concatenated processes*

*Let $C$ be a C-process whose underlying Markovian process is $M$, and $\tau$ be a time sequence.*

- *The concatenated Markovian process of $M$ associated with $\tau$ is the process $M_\tau$ defined by*

$$\forall t \in \boldsymbol{N}, M_\tau(t) = M\left(\tau(t)\right)$$

*If $\tau(t)$ is not finite, we define a new state $A_\infty$ and set $M_\tau(t) = M(\infty) = A_\infty$ almost surely.*

- *The concatenated process of $C$ associated with a time sequence $\tau$ is the process*

$$C_\tau = \begin{pmatrix} \boldsymbol{N} & \to & \boldsymbol{R} \cup \{\infty\} \\ t & \to & C\left(\tau(t)\right) \end{pmatrix}$$

*If $\tau(t)$ is not finite, we set $C_\tau(t) = +\infty$ almost surely.*

Some time sequences allow concatenation of a C-process into another C-process, as stated by the following fact ; they are called canonical time sequences.

**Definition 2.7.3** *Canonical time sequences*

*Let $M$ be a Markovian process and $\tau$ be a time sequence. Let $Z(s, H, i, \tau, t)$ be the event*

$$\tau(s+1) - \tau(s) = t \wedge C(\tau(s+1)) - C(\tau(s)) \in H \wedge M(\tau(s+1)) = A_i$$

*If $\tau$ holds the following properties :*

1. *Markovian concatenations : for every $s \in \boldsymbol{N}$, $t \in \boldsymbol{N}^*$, $i \leq A$, $H \subseteq \boldsymbol{R}$ measurable,*

$$\boldsymbol{P}\left(Z(s, H, i, \tau, t) | \boldsymbol{F}\left(\tau(s)\right)\right) = \boldsymbol{P}\left(Z(s, H, i, \tau, t) | M\left(\tau(s)\right)\right)$$

2. *Homogeneous concatenations : for every $r, s \in \boldsymbol{N}$, $t \in \boldsymbol{N}^*$, $i, j \leq A$,*

$$\boldsymbol{P}\left(Z(r, H, i, \tau, t) | M\left(\tau(r)\right) = A_i\right) = \boldsymbol{P}\left(Z(s, H, i, \tau, t) | M\left(\tau(s)\right) = A_i\right)$$

*then it is said to be a canonical time sequence to $M$.*

If a time sequence is canonical, then it allows the concatenated process to be a C-process.

**Lemma 2.7.1** *Concatenated C-process*

*Let $C$ be a C-process whose underlying Markovian process is $M$, and $\tau$ a canonical time sequence to $M$. Then $C_\tau$ is also a C-process, whose underlying Markovian process is $M_\tau$ the concatenated Markovian process of $M$ associated with $\tau$.*

N.B. : it may happen that concatenation drives some transition probabilities for $M_\tau$ to 0. If this happens for transition $(i \to j)$, then $C_\tau$'s transition payoff from $A_i$ to $A_j$ may be defined arbitrarily, as it will have no incidence on the sequel. The proof is provided in the next paragraph.

Time concatenations (with respect to canonical time sequences) will be useful in the sequel of this study. If one chooses an appropriate binary determination sequence, they lead to different forms of C-processes : setting $\rho(t) = 1$ iff

— $M(t) = M(0)$, we may build a Lévy process from $C$ (the restricted Lévy process), and use it to simplify the computations.

— $t \in n\mathbf{N}$, we "merge" multiple transitions at once (the $n$-periodically concatenated processes), thus considering $C$'s global trend rather than its individual increments.

— $C(t)$ is an all-time low, we may analyze $C$'s successive all-time lows (the descending process), forming a useful tool to get $C$'s default time.

All these will be used later in the proofs.

**Lemma 2.7.1**

We are going to prove that the concatenated process defined by the lemma 2.7.1 is a C-process. Hence, let $\rho$ be a binary determination sequence, and $\tau$ its time sequence, supposed canonical to $C$. We already know by hypothesis that

— $\rho(0) = 1$ by definition, so $\tau(0) = 0$ so $M_\tau(0) = M(0)$ and $C_\tau(0)$ are deterministic ;

— $M_\tau$ is Markovian and time homogeneous, because $\tau$ is canonical to $M$ ;

We aim at proving that $(C_\tau, M_\tau)$ holds

1. The Markovian property ;
2. The time-homogeneous property.

If we succeed, thanks to proposition 2.3.1, $C$ will be a C-process.

1. Let $s \in \mathbf{N}$, $H$ be a measurable subset of $\mathbf{R} \cup \{+\infty\}$, $i \leq A$, and $\mathbf{G}(s)$ be the natural filtration associated with $C_\tau, M_\tau$ up to time $s$. We want to compute

$$P(s, H, i, \tau) = \mathbf{P}\left(C_\tau(s+1) - C_\tau(s) \in H \wedge M_\tau(s+1) = A_i | \mathbf{G}(s)\right)$$

We decompose this probability considering the transition waiting time $\tau(s+1) - \tau(s)$ as

$$\sum_{t=1}^{\infty} \mathbf{P}\left(\left(\begin{array}{c} \tau(s+1) - \tau(s) = t \wedge C_\tau(s+1) - C_\tau(s) \in H \\ \wedge M_\tau(s+1) = A_i \end{array}\right) | \mathbf{G}(s)\right)$$

N.B. : we included $t = \infty$ in the previous sum. However,

— If $t = \infty$, then $M_\tau(s+1) = A_\infty$ automatically, so the sought probability is zero and has no effect on the sum whenever $i < \infty$, and $t = \infty$ may be removed.
— When considering $i = \infty$, we have $C_\tau(s+1) - C_\tau(s) = \infty$ automatically, so $P(s, H, i, \tau)$ is
  — Either 0 if $\infty \notin H$ ;
  — Or the complement of the probabilities $P(s, H, j, \tau)$ for $j < \infty$, so solving $i < \infty$ suffices to get this case too.

The $\sigma$-algebra $\mathbf{G}(s)$ is given by the random variables $M(u)$ and $\tau(u)$ for which there is $r \leq s$ such that $u = \tau(r)$, i.e. $\rho(u) = 1 \wedge u \leq \tau(s)$. It follows that
— $\mathbf{G}(s) \subseteq \mathbf{F}(\tau(s))$, as the latter is given by all random variables $M(u)$ and $\tau(u)$ for $u \leq \tau(s)$, and the event $\rho(u) = 1$ is $\mathbf{F}(u)$-measurable ;
— $\mathbf{G}(s) \supseteq \sigma(M(\tau(s)))$, as the former contains $M_\tau(s) = M(\tau(s))$.

Now, let us write (after the first inclusion)

$$\mathbf{P}(Z(s, H, i, \tau, t) | \mathbf{G}(s)) = \mathbf{E}(\mathbf{P}(Z(s, H, i, \tau, t) | \mathbf{F}(\tau(s))) | \mathbf{G}(s))$$

Thanks to the property of Markovian concatenations, we know that

$$\mathbf{P}(Z(s, H, i, \tau, t) | \mathbf{F}(\tau(s))) = \mathbf{P}(Z(s, H, i, \tau, t) | M(\tau(s)))$$

So we have

$$\mathbf{P}(Z(s, H, i, \tau, t) | \mathbf{G}(s)) = \mathbf{E}(\mathbf{P}(Z(s, H, i, \tau, t) | M(\tau(s))) | \mathbf{G}(s))$$

and the other inclusion yields

$$\mathbf{P}(Z(s, H, i, \tau, t) | \mathbf{G}(s)) = \mathbf{P}(Z(s, H, i, \tau, t) | M(\tau(s)))$$

which leads to the sought Markovian property once we sum these equalities for $t \in \mathbf{N}$.

2. The second proof is similar, relying on the fact that canonical time sequences hold homogeneous transitions : the main idea is that $Z(s, H, i, \tau, t)$ does not depend on $s$.

**Definition 2.3.4**

Hence, we may now prove that the restricted Lévy process from the definition 2.3.4 is actually a Lévy process. To get it, we set
— The binary determination sequence $\rho$

$$\forall t \in \mathbf{N}, \rho(t) = \mathbf{1}_{M(t) = A_i}$$

— The associated time sequence $\tau$, with finite terms almost surely by hypothesis ;
— The concatenated C-process $C_\tau$.

Then $C_\tau$ is $C$'s restricted Lévy process by construction. As a C-process whose state space is trivial is a Lévy process by definition (using the proposition 2.3.1), we are going to prove that

1. $\tau$ is canonical to $C$ ;
2. For every $t \in \mathbf{N}$, $M_\tau(t) = A_i$ almost surely.

This means that $C_\tau$ is a C-process, thanks to the proposition 2.7.1 ; as its underlying Markovian process is constant almost surely, it is a Lévy process.

1. First, $\tau$ is a time sequence by construction, indeed $M(0) = A_i$ so $\tau(0) = 0$. The concatenated transitions are Markovian, because $\rho$ derives from $M$'s present state only and the couple $(C, M)$ is a Markovian process ; they are time-homogenous, because $M$ itself is time-homogenous. So $\tau$ is canonical to $C$.
2. By construction, $M_\tau(t)$ cannot be any $A_j$ for any $j \neq i \leq A$. As $M$ is positive recurrent, it will not be $A_\infty$ almost surely either, so it will be $A_i$.

This proves that $C_\tau$ is a Lévy process.

### Periodical concatenations

Likewise, this other form of concatenation also yields a C-process.

**Definition 2.7.4** *Periodically concatenated C-process*

*Let $n \in \boldsymbol{N}^*$. The $n$-periodically concatenated C-process of $C$ is the C-process defined by concatenation of $C$ by the time sequence $T_n$ associated with the binary determination sequence $\forall t \in \boldsymbol{N}, \rho_n(t) = \boldsymbol{1}_{t \in n\boldsymbol{N}}$.*

— *Its underlying Markovian process is $M_{T_n}$ such that $\forall t \in \boldsymbol{N}, M_{T_n}(t) = M(nt)$. We name its transition probabilities as $P_{i \xrightarrow{n} j}$.*
— *We name its transition payoffs $D_{i \xrightarrow{n} j}(t)$, called $C$'s cumulative transition payoffs over $n$ time periods. For every $i, j \leq A$, they have the same distributions as new random variables, that we call $D_{i \xrightarrow{n} j}$ no matter $t \in \boldsymbol{N}$.*
— *Its starting point is $C(0)$.*

*N.B. : if $P_{i \xrightarrow{n} j} = 0$, then $D_{i \xrightarrow{n} j}(t)$ and $D_{i \xrightarrow{n} j}$ may be defined arbitrarily.*

The idea here is to "dilute" the state specificities, in the hope that for a large $n$, $C$ will spend an amount of time on each transition $(i \to j)$ proportional to $\mu_{[i]} P_{i \to j}$ no matter $M(t)$ at present time : discarding $M$ will simplify the study, and allow for a Lévy-like analysis. Finally, as $T_n$ is deterministic, the value of the concatenated C-process at time $t \in \mathbf{N}$ may be viewed equivalently as $C_{T_n}(t)$ or $C(nt)$.

### 2.7.3 Analysis of paths

The concept of paths allows us to seize the notion of “possible” values that $C$ may take during its trajectory. Not only it simplifies the study, but it also allows us to extend the notion of support for increments of Lévy processes to C-processes, eventually leading to the notion of cycles, global monotonicity, or periodicity.

#### Following paths

First, we show how to use paths, ensuring that every path for $C$ can be followed arbitrarily closely. The non-zero probability of being close to paths is essential to the proofs of the main theorems.

**Lemma 2.7.2** *Probability of paths*

1. *“Direct” property : let a path of length $T \in \boldsymbol{N}$, determined by*
   — *Its occupied state numbers $a_t \leq A$ for $t \in [|0, T|]$ ;*
   — *Its payoffs values $x_t \in \boldsymbol{R}$ for $t \in [|1, T|]$.*
   *We say that $(M, C)$ follows the said path with precision $\epsilon$ when*
   $$\forall t \in [|1, T|], \left(M(t) = A_{a_t} \wedge C(t) \in \left[\left(C(0) + \sum_{u=1}^{t} x_u\right) \pm \epsilon\right]\right)$$
   *We call this event $P(\epsilon)$. For every $\epsilon > 0$, the probability of following it with precision $\epsilon$ is positive :*
   $$\boldsymbol{P}\left(P(\epsilon) | M(0) = A_{a_0}\right) > 0$$
2. *“Inverse” property : let $T \in \boldsymbol{N}$, and families of*
   — *Occupied state numbers $a_t \leq A$ for $t \in [|0, T|]$,*
   — *Payoffs values $x_t \in \boldsymbol{R}$ for $t \in [|1, T|]$*
   *If for every $\epsilon > 0$, $P(\epsilon)$ is positive, then this determination of occupied state numbers and payoffs values defines a path of length $T$.*

We begin by the direct implication, as the other one comes from the definition of paths.

1. Let $C$ be a C-process whose underlying Markovian process is $M$, with transition probabilities given by the matrix by $P_{i \to j}$ for $i, j \leq A$. We consider such a path : by definition, the payoffs values must be possible at any precision, e.g. $\epsilon/T$ : therefore, for every $t \in [|1, T|]$,
   $$\begin{aligned} & \mathbf{P}\left(M(t) = A_{a_t} \wedge D_{a_{t-1} \to a_t}(t) \in [x_t \pm \epsilon/T] \,|M(t-1) = A_{a_{t-1}}\right) \\ = \; & P_{a_{t-1} \to a_t} \mathbf{P}\left(D_{a_{t-1} \to a_t}(t) \in [x_t \pm \epsilon/T]\right) \end{aligned}$$

because transition payoffs are independent of $M$, however by hypothesis both terms of the product on the right-hand side are positive. We decompose a subset of the event of following the path at precision $\epsilon$ by

$$\begin{aligned} & \mathbf{P}\left(\forall t \in [|1,T|]\,, M(t) = A_{a_t} \wedge D_{a_{t-1}\to a_t}(t) \in [x_t \pm \epsilon/T]\right) \\ = & \prod_{t=1}^{T} \mathbf{P}\left( \begin{array}{l} M(t) = A_{a_t} \wedge D_{a_{t-1}\to a_t}(t) \in [x_t \pm \epsilon/T] \mid \\ \forall s \in [|1,t-1|]\,, M(s) = A_{a_s} \wedge D_{a_{s-1}\to a_s}(s) \in [x_s \pm \epsilon/T] \end{array} \right) \end{aligned}$$

Since $M$ is Markovian and transition payoffs are independent and identically distributed, and are independent of $M$, then this product simplifies as

$$\prod_{t=1}^{T} \mathbf{P}\left(M(t) = A_{a_t} | M(t-1) = A_{a_{t-1}}\right) \mathbf{P}\left(D_{a_{t-1}\to a_t}(t) \in [x_t \pm \epsilon/T]\right)$$

which is a finite product of positive terms, so is positive. Finally, as the condition

$$\forall t \in [|1,T|]\,, D_{a_{t-1}\to a_t}(t) \in [x_t \pm \epsilon/T] \wedge M(t-1) = A_{a_{t-1}} \wedge M(t) = A_{a_t}$$

implies the condition

$$\forall t \in [|1,T|]\,, C(t) - C(0) \in \left[\sum_{u=1}^{t} x_t \pm \epsilon\right]$$

then the sought probability of $P(\epsilon)$ is also positive.

2. Considering any $\epsilon > 0$ and $t \leq T$, the given condition for $P(\epsilon/2)$ implies

$$\mathbf{P}\left( \begin{array}{cc} & M(t) = A_{a_t} \wedge M(t-1) = A_{a_{t-1}} \\ \wedge & C(t) \in \left[\left(C(0) + \sum_{u=1}^{t} x_u\right) \pm \epsilon/2\right] \\ \wedge & C(t-1) \in \left[\left(C(0) + \sum_{u=1}^{t-1} x_u\right) \pm \epsilon/2\right] \end{array} \right) > 0$$

The triangle inequality yields the implication

$$\begin{array}{ll} & \left( \begin{array}{cc} & C(t) \in \left[\left(C(0) + \sum_{u=1}^{t} x_u\right) \pm \epsilon/2\right] \\ \wedge & C(t-1) \in \left[\left(C(0) + \sum_{u=1}^{t-1} x_u\right) \pm \epsilon/2\right] \end{array} \right) \\ \Rightarrow & C(t) \in [(C(t-1) + x_t) \pm \epsilon] \end{array}$$

and leads to

$$\mathbf{P}\left(M(t) = A_{a_t} \wedge M(t-1) = A_{a_{t-1}} \wedge C(t) - C(t-1) \in [x_t \pm \epsilon]\right) > 0$$

Under these hypotheses for the states that $M$ hits, we know that $C(t) - C(t-1)$ is $D_{a_{t-1}\to a_t}(t)$, so we have both requirements for a path :

— All transitions are possible, because

$$\mathbf{P}\left(M(t) = A_{a_t} \wedge M(t-1) = A_{a_{t-1}}\right) = P_{a_{t-1}\to a_t}$$

is positive ;

— All payoffs values are possible, because

$$\mathbf{P}\left(D_{a_{t-1}\to a_t}(t) \in [x_t \pm \epsilon]\right) > 0$$

This being for every positive $\epsilon$, then $x_t$ belongs to $D_{a_{t-1}\to a_t}$'s support.

**Existence of paths**

When $M$ may eventually hit $A_j$ from $A_i$ with positive probability, we know that it is possible to find a finite sequence of states $A_{a_t}$ for $t \in [|0, T|]$ (with $T$ the length of this sequence) such that for any $s \in \mathbf{N}$,

$$\mathbf{P}\left(\forall t \in [|0,T|], M(s+t) = A_{a_t} | M(s) = A_{a_0}\right) = \prod_{t=1}^{T} P_{a_{t-1}\to a_t} > 0$$

We aim at extending this notion to paths of C-processes, consisting both in successive states occupied by $M$ and successive values taken by $C$.

**Lemma 2.7.3** *Existence of paths of a given value*

*Let $C$ be a C-process, whose underlying Markovian process is $M$, and $x \in \boldsymbol{R} \cup \{\infty\}$. We consider $i, j \leq A$ such that for any $s \in \boldsymbol{N}$,*

$$\forall \epsilon > 0, \boldsymbol{P}\left(\exists t \in \boldsymbol{N}^*; M(s+t) = A_j \wedge C(s+t) - C(s) \in [x \pm \epsilon] \,|M(s) = A_i\right) > 0$$

*Then for every $\epsilon > 0$, there are*

— *$T \in \boldsymbol{N}^*$ ;*

— *For every $t \in [|0, T|]$, intermediate state numbers $a_{t\leq T} \leq A$ such that $a_0 = i$, $a_T = j$, and*

$$\forall t \in [|1,T|], P_{a_{t-1}\to a_t} > 0$$

— *For every $t \in [|1, T|]$, intermediate payoffs*

$$x_{t\leq T} \in supp\left(D_{a_{t-1}\to a_t}\right)$$

*forming a non-trivial path whose starting state is $A_i$ and finishing state is $A_j$, of value*

$$\sum_{t=1}^{T} x_t \in [x \pm \epsilon]$$

Let us take $\epsilon > 0$. Decomposing the possibilities for $t$, then for $M(s+t)$ for $t \in \mathbf{N}$, there must be some $T \in \mathbf{N}$ and state numbers $a_t \leq A$ such that $a_0 = i$, $a_T = j$, and

$$\mathbf{P}\left(\forall t \in [|1, T|], M(s+t) = A_{a_t} \wedge C(s+T) - C(s) \in [x \pm \epsilon/2] \,|M(s) = A_i\right) > 0$$

Hence, as $[x \pm \epsilon/2]$ is a compact set, there is some value $y \in [x \pm \epsilon/2]$ in the support of $C(s+T) - C(s)$'s conditional distribution to $\forall t \in [|0, T|], M(s+t) = A_{a_t}$. Since this condition determines the active transition payoffs, we get

$$C(s+T) - C(s) = \sum_{i=1}^{T} D_{a_{t-1} \to a_t}(s+t)$$

Now, we shall use the fact that for any independent random variables $X_{t \leq T}$ valued on $\mathbf{R} \cup \{\infty\}$, the support of $X = \sum_{t=1}^{T} X_t$ is the topological closure of the sum of supports $supp(X_t)$ for $t \leq T$. Taking $X_t = D_{a_{t-1} \to a_t}(s+t)$, then there is some value

$$z \in [y \pm \epsilon/2] \cap \sum_{t=1}^{T} supp\left(D_{a_{t-1} \to a_t}(s+t)\right)$$

So there are values $x_t \in supp\left(D_{a_{t-1} \to a_t}\right)$ that sum to $z$; as $z$ is at distance at most $\epsilon/2 + \epsilon/2 = \epsilon$ of $x$, this ends the proof.

When there are no requirements on the value for the path, one can drop the assumption on $C(s+t) - C(s)$.

**Lemma 2.7.4** *Existence of paths of any value*

*Let $C$ be a C-process, whose underlying Markovian process is $M$. We consider $i, j \leq A$ such that for any $s \in \boldsymbol{N}$,*

$$\boldsymbol{P}\left(\exists t \in \boldsymbol{N}^*; M(s+t) = A_j | M(s) = A_i\right) > 0$$

*Then there are*

- *$T \in \boldsymbol{N}^*$ ;*
- *For every $t \in [|0, T|]$, intermediate state numbers $a_{t \leq T} \leq A$ such that $a_0 = i$, $a_T = j$, and*
$$\forall t \in [|1, T|], P_{a_{t-1} \to a_t} > 0$$
- *For every $t \in [|1, T|]$, intermediate payoffs*
$$x_{t \leq T} \in supp\left(D_{a_{t-1} \to a_t}\right)$$

*forming a non-trivial path whose starting state is $A_i$ and finishing state is $A_j$.*

To prove this lemma, one only needs to find $x$ that suits lemma 2.7.3. As $M$ may access to $A_j$ from $A_i$ through some intermediate state numbers $a_{t\leq T} \leq A$, taking

$$\forall t \leq T, x_t \in supp\left(D_{a_{t-1}\to a_t}\right)$$

and $x$ their sum, then

$$\begin{array}{rl} & \mathbf{P}\left(\exists t \in \mathbf{N}^*; M(s+t) = A_j \wedge C(s+t) - C(s) \in [x \pm \epsilon] \,|M(s) = A_i\right) \\ \geq & \mathbf{P}\left(\begin{array}{rl} & \forall t \in [|0,T|]\,, M(s+t) = A_{a_t} \\ \wedge & \forall t \in [|1,T|]\,, C(s+t) - C(s+t-1) \in [x_t \pm \epsilon/T] \end{array} |M(s) = A_i\right) \end{array}$$

Since $C(s+t)-C(s+t-1)$ is $D_{a_{t-1}\to a_t}(s+t)$ under these conditions, and transition payoffs are independent of $M$, then the latter term is

$$\left(\begin{array}{rl} & \mathbf{P}\left(\forall t \in [|0,T|]\,, M(s+t) = A_{a_t}|M(s) = A_i\right) \\ \times & \mathbf{P}\left(\forall t \in [|1,T|]\,, D_{a_{t-1}\to a_t}(s+t) \in [x_t \pm \epsilon/T]\right) \end{array}\right)$$

Thanks to the time-homogeneous property, this is

$$\left(\prod_{i=1}^{T} P_{a_{t-1}\to a_t}\right)\left(\prod_{i=1}^{T} \mathbf{P}\left(D_{a_{t-1}\to a_t} \in [x_t \pm \epsilon/T]\right)\right)$$

All terms are positive by construction of the transitions and the choice of $x_t$ in the support of transition payoffs, so this term is positive. This being for every $\epsilon > 0$, $x$ suits lemma 2.7.3.

Moreover, we notice that if $C$ is positive recurrent, one may choose the path such that $x < \infty$ by definition of positive recurrence, so we may enforce the requirement that $x < \infty$. This idea will be useful when considering paths of finite values, in particular to the next lemma.

### Paths and recurrence

The existence of paths provides a tool to characterize positive recurrence of C-processes, in a similar fashion to paths for a Markovian process.

**Lemma 2.7.5** *Existence of paths and positive recurrence*

*Let $C$ be a C-process, whose underlying Markovian process is $M$.*

1. *$M$ is positive recurrent iff for every state numbers $i, j \leq A$, there is a path from $A_i$ to $A_j$.*
2. *$C$ is positive recurrent iff for every state numbers $i, j \leq A$, there is a path of finite value from $A_i$ to $A_j$.*

As the statement of $M$ is a direct consequence of the lemma 2.7.4 and generic properties of positive recurrent Markovian processes, we will only focus on the statement for $C$.

- The direct implication is a consequence of the definition 2.3.3 and lemma 2.7.4 for paths of finite values.
- The reverse implication comes from lemma 2.7.2 that leads to a suitable lower bound of probability in the definition 2.3.3.

This lemma will be useful when translating the property of positive recurrence into terms of paths.

### Concatenation of paths

Another useful idea when following paths of C-processes happens when considering successive paths : if $C$ runs on a path, then proceeds with another path, it is natural that the concatenation of both paths forms a longer (and possible) path.

**Definition 2.7.5** *Concatenation of paths*

*Let us consider two paths of respective lengths $T_a, T_b \in \boldsymbol{N}$ and values $x, y \in \boldsymbol{R} \cup \{\infty\}$, defined respectively by*

- *Occupied state numbers $a_t \leq A$ for any $t \in [|0, T_a|]$, and $b_t \leq A$ for any $t \in [|0, T_b|]$ ;*
- *Payoffs values $x_t \in \boldsymbol{R}$ for any $t \in [|1, T_b|]$, and $y_t \in \boldsymbol{R}$ for any $t \in [|1, T_b|]$.*

*If $a_{T_a} = b_0$, then we may define a path as*

- *Occupied state numbers $c_t \leq A$ for any $t \in [|0, T_a + T_b|]$, such that*

$$\forall t \in [|0, T_a|], c_t = a_t \wedge \forall t \in [|T_a + 1, T_a + T_b|], c_t = b_{t-T_a}$$

- *Payoffs values $z_t \in \boldsymbol{R}$ for any $t \in [|0, T_a + T_b|]$, such that*

$$\forall t \in [|1, T_a|], z_t = x_t \wedge \forall t \in [|T_a + 1, T_a + T_b|], z_t = y_{t-T_a}$$

*It is a path from $A_{a_0}$ to $A_{b_{T_b}}$, whose value is $x + y$.*

This property is the consequence of the definition of paths. Let $\epsilon > 0$, and consider two such paths. Let us consider the events $H_1$ and $H_2$ of following respectively the first and the second path at precision $\epsilon/2$ : thanks to lemma 2.7.2, we know that $\mathbf{P}(H_1|M(0) = A_{a_0}) > 0$ and $\mathbf{P}(H_2|M(0) = A_{b_0}) > 0$. However, we know that

$$\mathbf{P}(H_1 \wedge H_2|M(0) = A_{a_0}) = \mathbf{P}(H_1|M(0) = A_{a_0}) \, \mathbf{P}(H_2|H_1 \wedge M(0) = A_{a_0})$$

The rightmost probability is $\mathbf{P}\left(H_2|M(T_a) = A_{a_{T_a}} = A_{b_0}\right)$ by definition thanks to $C$'s Markovian property (proposition 2.3.1). Hence, $\mathbf{P}(H_1 \wedge H_2|M(0) = A_{a_0}) > 0$, and the triangle inequality states that $H_1 \wedge H_2$ implies following the concatenation at precision $\epsilon$, so the previous lemma 2.7.2 implies that it forms a path. The fact that its value is $x + y$ comes from the definiton of values for a path.

### Rotation of cycles

As cycles begin and end on the same state, we may be able to start them anytime during their executions, looping to the initial state when reaching the final one as they are the same. This leads to the definition of a rotated cycle, as given below.

**Definition 2.7.6** *Rotations of a cycle*

*Let us consider a cycle of $C$ of length $T \in \boldsymbol{N}^*$, whose*

- *Occupied state numbers are $a_t \leq A$ for $t \in [|0, T|]$ ;*
- *Payoffs values are $x_t \leq A$ for $t \in [|1, T|]$.*

*For every $s \leq T$, there is a cycle of the same length whose*

- *Occupied state numbers are first $b_t = a_{(t+s)}$ for $t \in [|0, T - s|]$, and then $b_t = a_{(t+s-T)}$ for $t \in [|T - s + 1, T|]$ ;*
- *Payoffs values are first $y_t = x_{(t+s)}$ for $t \in [|1, T - s|]$, and then $y_t = x_{(t+s-T)}$ for $t \in [|T - s + 1, T|]$.*

*It is named a Rotated cycle (of the initial cycle) to the state number $b_0 = a_s$, or to the state $A_{b_0} = A_{a_s}$, and has the same value as the initial cycle.*

This lemma holds because

- Transitions are still possible, as the loop when hitting $A_{a_T} = A_{a_0}$ allows starting the initial cycle over ;
- Addition is a commutative operation, so the value of a cycle is preserved by rotation.

### Universal cycles

Among the cycles for $C$, when $M$ is positive recurrent, some occupy every $A_{i \leq A}$ through their executions. They are called universal cycles, and allow some simplifications of the proofs.

**Definition 2.7.7** *Universal cycles*

*Let $C$ be a C-process.*

- *Let us consider a cycle of $C$ of length $T \in \boldsymbol{N}^*$, whose occupied state numbers are $a_t \leq A$ for $t \in [|0, T|]$. It is said universal iff for every state number $i \leq A$, there is $t \in [|0, T|]$ such that $a_t = i$.*
- *For any $n \in \boldsymbol{N}^*$, the set of all possible values for all universal cycles of $C$ is called $U_n(C) \subseteq \boldsymbol{R} \cup \{\infty\}$.*
- *The universal cycle support of $C$ is the set*

$$ucs(C) = \bigcup_{n=1}^{\infty} U_n(C)$$

Universal cycles may be rotated to any state, since they run through every state.

**Lemma 2.7.6** *Rotations of a universal cycle*

*For every $i \leq A$, every universal cycle of $C$ has a rotated cycle to $A_i$, that is still universal.*

This property came from the definition of a universal cycle, as one may choose $s$ so that $A_s$ is any arbitrarily chosen state. It leads to the following lemma, linked to the positive recurrence of $C$'s underlying Markovian process : the statements given by the lemma 2.7.5 have a counterpart when considering $ucs(C)$ instead of $supp(C)$.

**Lemma 2.7.7** *Properties of the universal cycle support*

*Let $C$ be a C-process.*

1. *$ucs(C) \neq \emptyset$ iff $M$ is positive recurrent.*
2. *$ucs(C)$ contains a finite value iff $C$ is positive recurrent.*
3. *If $C$ is positive recurrent, then $ucs(C) + supp(C) \subseteq ucs(C)$.*

The proof relies on the previous lemma 2.7.5 and the lemma 2.7.5.

1. If $M$ is positive recurrent, then there are paths joining every state $A_i$ to every state $A_j$. Hence, starting from the state $A_{i \leq A-1}$ there is a path $P_i$ going to the state $A_{i+1}$, then there is a path $P_A$ going from $A_A$ to $A_1$. The lemma 2.7.5 states that the concatenation of these paths yields a path ; it runs through every state, and is a cycle by construction, so it has a value $v \in ucs(C)$.

   Let $i, j \leq A$. If $C$ has a universal cycle, then we rotate it to $A_i$ ; as it runs through $A_j$ for some time $t \in \mathbf{N}$ because it is universal, the path given by its restriction for time up to $t$ links $A_i$ to $A_j$. This being for every $i, j \leq A$, $C$ is positive recurrent.
2. If $C$ is positive recurrent, we use the same method, this time involving paths of finite values as given by the lemma 2.7.5.
3. Let $x \in ucs(C)$ and $y \in supp(C)$. By construction, there are $m, n \in \mathbf{N}^*$ such that $x \in U_m(C)$ and $y \in V_n(C)$, so there are both a universal cycle $P_1$ of length $m$, occupied state numbers $a_{t \in [|0,m|]}$, and value $x$, and a cycle $P_2$ of length $n$, occupied state numbers $b_{t \in [|0,n|]}$, and value $y$. As $P_1$ is universal, we rotate it to $A_{b_0}$, getting a universal cycle $P_3$ of identical value $x$. Concatenating $P_3$ and $P_2$ yields a path by lemma 2.7.5 because $P_3$'s finishing state is $P_2$'s starting state ; it is a universal cycle because $P_3$ is universal ; its value is $x + y$ by addition. So, $x + y$ must belong to $ucs(C)$, which ends the proof.

### Remote density

We present a result stating that, provided that $supp(C)$ contains two "close" values, then one gets a density-like property for $ucs(C)$, i.e. large negative values of $x$ are approximated by values in $ucs(C)$. It will be used to create paths of $C$ going to (approximations of) low enough arbitrary values.

**Lemma 2.7.8** *Remote density*
*Let $C$ be a positive recurrent C-process, deemed not globally increasing, whose universal cycle support is $ucs(C)$. We assume the existence of two values $x_1, x_2 \in supp(C)$ such that $x_1 - x_2 = \epsilon > 0$. Then there is $X^- \in \boldsymbol{R}^-$ such that for every $x \leq X^-$, $[x \pm \epsilon/2] \cap ucs(C) \neq \emptyset$.*

First, as $ucs(C)$ is stable by addition (lemma 2.7.7) and contains a negative value $z < 0$ (else $C$ would be globally increasing), then by some number $n$ of additions of $z$ to $x_1$ and $x_2$, we may find $z_1 = x_1 + nz$ and $z_2 = x_2 + nz$ in $ucs(C) \cap \mathbf{R}_-^*$ with discrepancy $\epsilon$. Now, let $x \leq z_1$. For every $i \in [|0, \lfloor x/z_1 \rfloor |]$, we consider

$$y_i = z_1 \left( \lfloor x/z_1 \rfloor - i \right) + z_2 i$$

As $ucs(C)$ is stable by addition and $z_1, z_2 \in ucs(C)$, then $y_i \in ucs(C)$. We know that the discrepancy between two successive values for $y_i$ is less than $\epsilon$, and that $y_0 \geq x$. Hence, if the last $y_i$ (for $i = \lfloor x/z_1 \rfloor$) obeys $y_i \leq x$, then it follows that one among the values $y_k$ will be at distance $|y_k - x| \leq \epsilon/2$. This will be the case as soon as

$$z_2 \left( \left\lfloor \frac{x}{z_1} \right\rfloor \right) \leq x \Leftarrow z_2 \left( \frac{x}{z_1} - 1 \right) \leq x$$

As $z_1$ and $z_2$ have been taken negative, then this will hold whenever

$$x \leq \frac{-z_1 z_2}{|z_2 - z_1|}$$

As this bound is lower than $z_1$, then $X^- = \frac{-z_1 z_2}{|z_2 - z_1|}$ suits the property.

### Density alternative

To use the lemma 2.7.8, one aims at finding two "close" values $z_1$ and $z_2$ in $ucs(C)$ being less than $\epsilon$ apart. This is where $C$'s periodicity is an issue :

- If $C$ is aperiodic, the additivity of $ucs(C)$ will allow us to find these values no matter $\epsilon$, so we eventually get a "strong" result on $ucs(C)$'s remote density.
- If $C$'s fundamental period is $p \in \mathbf{R}_+^*$, one cannot beat $\epsilon = p$, and the result will be substantially weaker.

What we find is the following lemma.

**Lemma 2.7.9** *Density alternative*

*Let $C$ be a positive recurrent C-process, deemed not globally increasing, whose universal cycle support is $ucs(C)$.*

1. *If $C$ is aperiodic, then for every $\epsilon > 0$ there is $X^-(\epsilon)$ such that for every $x \leq X^-(\epsilon)$, we have*
$$ucs(C) \cap [x \pm \epsilon/2] \neq \emptyset$$
2. *If $C$'s fundamental period is $p$, then there is $X^- \in \boldsymbol{R}$ such that, for every $x \in p\boldsymbol{Z}$ no higher than $X^-$, $x$ belongs to $ucs(C)$.*

In this proof, we shall use the set

$$G = \{x_2 - x_1; x_1, x_2 \in ucs(C) \backslash \{\infty\}\} \subseteq \mathbf{R}$$

To verify that it is an additive subgroup of $\mathbf{R}$, one only verifies that

— $0 \in G$, because $C$ is positive recurrent by the lemma 2.7.7, so $ucs(C)$ has a non-trivial value ;

— As $ucs(C)$ is additive, then $G$ is additive ;

— $G$ is symmetrical.

So, we know that $G$ is either

— $\{0\}$, which means that $C$ is globally increasing and is excluded ;

— A discrete group, whose form is $G = q\mathbf{Z}$ for some $q \in \mathbf{R}_+^*$ ;

— A dense subgroup of $\mathbf{R}$.

However, we know that

— If $G = q\mathbf{Z}$, then let $x \in ucs(C) \backslash \{\infty\}$ be any value of an universal cycle. By definition of $G$, it follows that

$$ucs(C) \subseteq (x + q\mathbf{Z}) \cup \{\infty\}$$

Now, as $ucs(C)$ is additive, then writing $x = qn+z$ with $n \in \mathbf{Z}$ and $z \in [0, q)$ yields

$$x + x = 2qn + 2z \in q\mathbf{Z} + z$$

and this implies that $z = 0$, so $C$ is $q$-periodic.

— If $C$ is $q$-periodic, then $G$ will be included in $q\mathbf{Z}$. As a consequence, if $G$ is dense, then $C$ must be aperiodic.

We proved that $C$ is aperiodic iff $G$ is dense, and this leads to the lemma the following way :

1. If $C$ is aperiodic, then $G$ is dense, which means by definition that we may find arbitrarily close values in $ucs(C)$. The lemma 2.7.8 concludes.

2. If $C$'s fundamental period is $p \in \mathbf{R}_+^*$, then $G \subseteq p\mathbf{Z}$ is an additive subgroup of $\mathbf{R}$, so $G$ cannot be dense and there is $q \in \mathbf{R}_+^*$ such that $G = q\mathbf{Z}$, so
   (a) There is $n \in \mathbf{N}^*$ such that $q = pn$ ;
   (b) Like before, $C$ is $q$-periodic.

   However, as $p$ is $C$'s fundamental period, then $q \leq p$, which is possible only if $p = q$. Hence, there must by $x_1$ and $x_2$ in $ucs(C)$ such that $x_2 - x_1 = p$. Finally, applying the lemma 2.7.8 to $x_1$ and $x_2$ yields that

$$\exists X^- \in \mathbf{R}^-; \forall x \leq X^-, [x \pm p/2] \cap ucs(C) \neq \emptyset$$

   However, as $ucs(C) \subseteq p\mathbf{Z}$, we get

$$[x \pm p/2] \cap ucs(C) \cap p\mathbf{Z} \neq \emptyset$$

   Taking every $x \in p\mathbf{Z}$ not above $X^-$ yields $\{x\} \cap ucs(C) \neq \emptyset$, thus such an $x$ must belong to $ucs(C)$, which ends the proof.

### 2.7.4 Propositions 2.3.2 and 2.3.3

Thanks to the lemmata about paths and concatenations, we are now able to prove the propositions involving globally monotone C-processes.

#### Globally increasing C-processes

This paragraph aims at proving the proposition 2.3.2. First we note that if $C$ is positive recurrent, the lemma 2.7.3 may be applied to $C$, considering any starting and finishing states, with some finite given value $x$ because $C$ is positive recurrent.

— (1 implies 2) Let us assume that $C$ is globally increasing. We consider its restricted Lévy process $C_\tau$, and we want to prove that $\forall t \in \mathbf{N}, C_\tau(t+1) \geq C_\tau(t)$.
  — If $C_\tau(t) = \infty$, we know by construction of $C_\tau$ that $C_\tau(t+1) = \infty$ automatically, be it because $\tau(t) = \infty$ or $C(\tau(t)) = \infty$. Hence this case is solved.
  — If $C_\tau(t+1) = \infty$, the case is immediately solved.
  — So we will deem that $C_\tau(t) < \infty$ and $C_\tau(t+1) < \infty$ in the rest of this proof. In particular, this implies $\tau(t) < \infty$ and $\tau(t+1) < \infty$.

  To compute the probability

$$\mathbf{P}\left(C_\tau(t+1) < C_\tau(t) | \tau(t) < \infty \wedge \tau(t+1) < \infty\right)$$

  we make use of the canonical time sequence $\tau$ of $C_\tau$. As $A_i$ is accessible from itself using a path of finite value (because $C$ is positive recurrent), there is $x \in \mathbf{R}$ such that

$$\forall \epsilon > 0, \mathbf{P}\left(C_\tau(t+1) - C_\tau(t) \in [x \pm \epsilon] \,|C_\tau(t) < \infty \wedge C_\tau(t+1) < \infty\right) > 0$$

So, for every $\epsilon > 0$, there are $s_1$ (a finite value for $\tau(t)$) and $s_2$ (a finite value for $\tau(t+1)$) such that for every $\epsilon > 0$, the probability

$$\mathbf{P}\begin{pmatrix} \tau(t+1) = s_2 \wedge \tau(t) = s_1 \wedge C(s_2) - C(s_1) \in [x \pm \epsilon] \\ |C_\tau(t) < \infty \wedge C_\tau(t+1) < \infty \end{pmatrix}$$

is positive. The events describing $\tau$ indicate that $M(s_1) = M(s_2) = A_i$, hence

$$\mathbf{P}\left(\begin{pmatrix} M(s_1) = A_i \wedge M(s_2) = A_i \wedge C(s_2) - C(s_1) \in [x \pm \epsilon] \\ |C_\tau(t) < \infty \wedge C_\tau(t+1) < \infty \end{pmatrix}\right) > 0$$

As the event $C_\tau(t) < \infty \wedge C_\tau(t+1) < \infty$ has a non-zero probability because $C$ is positive recurrent, we get

$$\mathbf{P}\left(M(s_1) = A_i \wedge M(s_2) = A_i \wedge C(s_2) - C(s_1) \in [x \pm \epsilon]\right) > 0$$

Thanks to the lemma 2.7.3, for every $\epsilon > 0$, this means that there is a non-trivial path from $A_i$ to $A_i$ whose value belongs to $[x \pm \epsilon]$. As a cycle of a globally increasing C-process must have a non-negative value, then $x$ cannot be negative (else take $\epsilon = -x/2$). So we proved that if

$$\forall \epsilon > 0, \mathbf{P}\left(C_\tau(t+1) - C_\tau(t) \in [x \pm \epsilon]\right) > 0$$

i.e. $x$ belongs to the support of one of $C_\tau$'s increments, then it is non-negative : this means that $C_\tau$'s increments are non-negative almost surely, so $C_\tau$ is increasing almost surely.

— (2 implies 1) Let us assume that $C$ has a cycle $P_0$ of negative value $v_0 \in \mathbf{R}_-^*$, whose starting and finishing states are both $A_{j \leq A}$. As $C$ is positive recurrent, lemma 2.7.5 builds a path $P_1$ of finite value $v_1$ from $M(0)$ to $A_j$, and another path $P_2$ of finite value $v_2$ from $A_j$ to $M(0)$. Concatenating $P_1$, then $n \in \mathbf{N}$ times $P_0$, then $P_2$, yields a cycle whose value is $v_1 + nv_0 + v_2$ ; since $v_1 + v_2 < \infty$ and $v_0 < 0$, there is $n \in \mathbf{N}$ large enough such that this cycle has a negative value $v < 0$. Following it thanks to lemma 2.7.2 at precision $-v/2$ yields that we eventually hit $A_i$ at a time $t$ such that $C(t) \leq C(0) - v/2$ with positive probability, so there is $s \in \mathbf{N}$ such that $\mathbf{P}\left(C_\tau(s) < C_\tau(0)\right) > 0$.

— (1 implies 4) For this proof, we consider the starting state $M(0)$ to be our reference state, and we define the height $h_i \in \mathbf{R}$ of any state $A_i$ to be $h_{M(0)} = 0$ and

$$\forall i \leq A, h_i = \sup\left(\{x \in \mathbf{R}; \forall t \in \mathbf{N}, \mathbf{P}\left(C(t) \leq x \wedge M(t) = A_i\right) = 0\}\right)$$

Hence, stating that $A_i$ has height $h_i$ means that paths going from $M(0)$ to $A_i$ have values no less than $h_i$, this being ensured by lemma 2.7.2.

- No $h_i$ may be $-\infty$, because then lemma 2.7.3 would create paths of arbitrarily low values from $M(0)$ to $A_i$ ; as lemma 2.7.5 creates a path of finite value from $A_i$ to $M(0)$, then their concatenation may have a negative value, which is excluded.
- No $h_i$ may be $+\infty$ either, because then $C$ would not be positive recurrent because of the state $A_i$, that would be inaccessible using finite transition payoffs.

So, we may consider the C-process $C^=$ defined by $C^=(0) = C(0)$ and transitions from $A_i$ to $A_j$ being $h_j - h_i \in \mathbf{R}$ almost surely, and $C^+$ defined by $\forall t \in \mathbf{N}, C^+(t) = C(t) - C^=(t)$.

- For every $t \in \mathbf{N}$, we have

$$\begin{aligned} C^+(t+1) - C^+(t) &= (C(t+1) - C(t)) - (C^=(t+1) - C^=(t)) \\ &= D_{M(t)\to M(t+1)}(t+1) - \left(h_{M(t+1)} - h_{M(t)}\right) \end{aligned}$$

  However, by definition of heights,

  - For every $\epsilon > 0$, $i \leq A$, there is a path $P$ from $M(0)$ to $A_i$ whose value belongs to

  $$[h_i, h_i + \epsilon]$$

  thanks to lemma 2.7.3 ;
  - There is no path from $M(0)$ to $A_j$ whose value is less than $h_j$ thanks to lemma 2.7.2 (no matter $j \leq A$).

  Thus, if there is $x > 0$ such that

  $$\mathbf{P}\left(D_{M(t)\to M(t+1)}(t+1) \leq \left(h_{M(t+1)} - h_{M(t)}\right) - x\right) > 0$$

  then we get some states $A_i$ (a possibility for $M(t)$) and $A_j$ (a possibility for $M(t+1)$) such that

  $$\mathbf{P}\left(M(t) = A_i \wedge M(t+1) = A_j \wedge D_{i\to j}(t+1) \leq (h_j - h_i) - x\right) > 0$$

  It follows that there is a one-step path from $A_i$ to $A_j$ whose value is at most $(h_j - h_i) - x$, and concatenation of it after $P$ yields a path from $M(0)$ to $A_j$ whose value is at most $h_j + \epsilon - x$. Taking $\epsilon = x/2$ makes this contradictory to $h_j$'s value. So, for every $x > 0$,

  $$\mathbf{P}\left(D_{M(t)\to M(t+1)}(t+1) \leq \left(h_{M(t+1)} - h_{M(t)}\right) - x\right) = 0$$

  It follows from this that almost surely

  $$D_{M(t)\to M(t+1)}(t+1) \geq \left(h_{M(t+1)} - h_{M(t)}\right)$$

and then $C^+(t+1) - C^+(t) \geq 0$ almost surely, so that $C^+$ is almost surely non-decreasing.
- $C^=$ is globally constant, because for every $s < t \in \mathbf{N}$, $C^=(t) - C^=(s)$ is the telescopic sum of height differences between successive states hit by $M$, i.e. $h_{M(t)} - h_{M(s)}$. If $M(t) = M(s)$, then $C^=(t) - C^=(s) = 0$, so the only value in $C^=$'s cycle support is 0, which is the definition for a globally constant C-process.

As $C^=$ is a C-process by construction and is finite almost surely, then $C - C^=$ is a C-process, so this ends the proof.
- (4 implies 3) Considering that $C = C^+ + C^=$, let us deem that for every $Q \in \mathbf{R}$, there are $s < t \in \mathbf{N}$ such that $\mathbf{P}(C(t) < C(s) - Q) > 0$. By the lemma 2.7.3, $C$ must have paths of arbitrarily low values ; as $M$'s state space is finite, there are two states $A_i$ and $A_j$ such that there are paths of arbitrarily low values from $A_i$ to $A_j$. The lemma 2.7.5 now creates a path of finite value from $A_j$ to $A_i$, so their concatenation yields cycles of arbitrarily low values for $C$. However, $C = C^+ + C^=$ by hypothesis, $C^+$ is non-decreasing so must have a non-negative cycle support, and $C^=$ is globally constant so has $\{0\}$ as a cycle support : by addition, it is impossible to get cycles of arbitrarily low values. This ends the proof.
- (3 implies 1) If $C$ has a cycle $P_0$ of length $t \in \mathbf{N}$ and negative value $v_0 \in \mathbf{R}_-^*$, whose starting and finishing states are both $A_{j \leq A}$, then as $C$ is positive recurrent, lemma 2.7.5 builds a path $P_1$ of length $u \in \mathbf{N}$ and finite value $v_1$ from $M(0)$ to $A_j$, which leads by concatenation of $P_1$ with $n \in \mathbf{N}$ times $P_0$ to a path of length $u + nv$ and value $v_1 + nv_0$ from $M(0)$ to $A_j$. As $v_1 + nv_0$ can be made arbitrarily low, then with positive probability, $C(u+nv) - C(0)$ may be arbitrarily low, so no $Q$ of statement 3 may exist.

To see why positive recurrence of $C$ is required, let us take the C-process defined by
- $A = 2$ states, $A_1$ being the starting state ;
- $\forall i, j \leq A$, $P_{i \to j} = 1/2$ ;
- The transition payoffs are $D_{1 \to 1} = 1$, $D_{1 \to 2} = D_{2 \to 2} = -1$, and $D_{2 \to 1} = +\infty$ almost surely.

Then the return to $A_1$ starting from $A_1$ means either a cycle going only through $A_1$ (so it has a positive value), or going through the transition $(A_2 \to A_1)$ if having gone through $A_2$ (so it has a value of $+\infty$) ; hence $C$'s Lévy process is non-decreasing. However, as $C$ allows for the cycle going $(A_2 \to A_2)$ of value $-1$, it is not globally increasing.

### Globally constant C-processes

This paragraph aims at proving the proposition 2.3.3.

— (1 implies 2) We consider the heights $h_i$ of $M$'s states like in the step (1 implies 4) of the previous proof, for the proposition 2.3.2 : they still cannot be $\pm\infty$, as $C$ globally increasing implies in both cases that there is a transition payoff whose value is $+\infty$ almost surely like before, and so $supp(C) \ni \infty$, which is impossible since $C$ is globally constant. For the same reasons as before, there are $C^=$ such that $\forall t \in \mathbf{N}, C^=(t) = C(0) + h_{M(t)}$, and $C^+$ non-decreasing, with $C = C^= + C^+$. It follows that $C$ and $C^=$ are globally constant, so $C^+ = C - C^=$ is globally constant, non-decreasing, and starts from $C(0) - C^=(0) = 0$. As $C^+$ is non-decreasing, then every path of $C^+$ must have a non-negative value, but as $C^+$ is globally constant, concatenations of paths into cycles must have zero values. This is possible only if every path of $C^+$ has zero value, and so $C^+ \equiv 0$ almost surely, so $C = C^=$ which satisfies the statement 2 with $\forall i \leq A, c_i = C(0) + h_i$.

— (2 implies 3) This is because $Q = \max_i (c_i) - \min_i (c_i)$ works.

— (3 implies 1) As $C$ is a bounded process (bounded by $2Q$), then $-C$ is also a C-process. We use the implication (3 implies 1) of the previous lemma 2.3.2 to get that both $C$ and $-C$ are globally increasing, so $C$ is both globally increasing and globally decreasing.

Finally, if $C$ is positive recurrent and globally constant, then it is bounded by $2Q$ thanks to statement 3.

### 2.7.5 Properties of the Laplace matrix function

Before working on $C$'s martingale parameter, we focus on its Laplace matrix function, the extension of the Laplace transform of a single random variable. Indeed, the proposition 2.4.1 involves a matrix $L_C(\alpha)$, the "Laplace transform" of the C-process $C$.

#### Proposition 2.3.4

We aim at characterizing C-processes using only their Laplace matrix functions. To do this, we will build a third C-process $C$, whose underlying Markovian process has a state space being

$$\{A_i; i \leq A\} \cup \{A_\infty\}$$

where $A_\infty$ is a new state, reserved to transition payoffs amounting to $+\infty$. We shall

1. Build $C$ and $M$ using only $L_1 (= L_2)$ ;
2. Verify that $C$ has the same distribution as $C_1$ (and thus $C_2$ because we used only $L_1$, which ends the proof of the first part) ;

3. Verify that when no transitions amount to $+\infty$, $A_\infty$ is useless so $M$ has the same distribution as $M_1$ (and thus $M_2$).

Hence, let us start from $C_1$, whose transition probabilities $P_{1,i\to j}$ and transition payoffs $D_{1,i\to j}$ are used in its Laplace matrix function $L_1$.

1. By definition of the Laplace transform, for every $i,j \le A$ we have

$$(L_1(0))_{i,j} = P_{1,i\to j}\mathbf{P}\left(D_{1,i\to j} < \infty\right)$$

We define $P$ ($M$'s transition matrix) to be the $(A+1)$-dimensional matrix, whose entries are named after the states $A_i$ and $A_\infty$ and defined as follows :

— Over the "old" states,

$$\forall i,j \le A, P_{i,j} = (L_1(0))_{i,j}$$

— We go from any old state $A_{i\le A}$ to $A_\infty$ with the remaining probability

$$P_{i,\infty} = 1 - \sum_{j=1}^{A} (L_1(0))_{i,j} = 1 - \left(L_1(0)\left(\vec{1}\right)\right)_{[i]}$$

— $A_\infty$ is absorbing, i.e. $P_{\infty,\infty} = 1$ and $\forall j \le A, P_{\infty,j} = 0$.

We verify that $P$ is a stochastic matrix, because $\forall i,j \le A, 0 \le P_{i,j} \le P_{1,i\to j}$, so $P_{i,\infty}$ is not an issue. We define the random transition payoffs of $C$ as follows :

— For $i,j \le A$ with $P_{1,i\to j} > 0$, we divide the entry number $(i,j)$ of $L_1$ by $P_{1,i\to j}$, yielding the expression

$$\mathbf{E}\left(e^{-\alpha D_{1,i\to j}}\right)$$

for every $\alpha \in I$ (where $I$ is the convergence domain of $L_1$). As one knows, this is sufficient to recover $D_{1,i\to j}$'s distribution, after which we create a transition payoff $D_{i\to j}$ of the same distribution between $A_i$ and $A_j$. Thanks to

$$\mathbf{E}\left(e^{-\alpha D_{1,i\to j}}\right) = \mathbf{P}\left(D_{1,i\to j} < \infty\right)\mathbf{E}\left(e^{-\alpha D_{1,i\to j}} | D_{1,i\to j} < \infty\right)$$

this is tantamount to computing the distribution of $D_{1,i\to j}$ conditionally to being finite.

— For $i,j \le A$ with $P_{1,i\to j} = 0$, we set arbitrarily $D_{i\to j} = 0$.

— For $i = \infty$ or $j = \infty$, we set $D_{i\to j} = +\infty$.

Finally, we set $C$'s starting point and state as given for $C_1$ and $C_2$. This definition has been done only with items common to $C_1$ and $C_2$, so if $C$ and $C_1$ share the same distribution, the first proposition is solved.

2. Let $n \in \mathbf{N}^*$ and $x_t \in \mathbf{R} \cup \{\infty\}$ for $t \leq n$, defining events

$$X = \{\forall t \leq n, C(t) \leq x_t\} \wedge X_1 = \{\forall t \leq n, C_1(t) \leq x_t\}$$

We want to compare $q = \mathbf{P}(X)$ with $q_1 = \mathbf{P}(X_1)$, which is done by decomposition over the states $M$ may hit over the first $n$ time periods. Hence, let us take a family of state numbers

$$a = (a_i)_{i \leq A} \in ([|1, A|] \cup \{\infty\})^n$$

For every such $a$, we name $Z_a$ the event of $M$ following the states described by $a$ successively, i.e.

$$Z_a = \{\forall t \leq n, M(t) = a_t\}$$

Its probability $p_a$ is obtained thanks to the Markovian property

$$p_a = \prod_{t=1}^{n} P_{a_{t-1} \to a_t}$$

whereas the probability of event $X$ conditionally to event $Z_a$ is

$$\prod_{t=1}^{n} \mathbf{P}\left(D_{a_{t-1} \to a_t} + C(t-1) \leq x_t\right)$$

Now, let us distinguish the cases for infinite values of $x_t$.

— If there are $t_1 < t_2 \leq n$ with $x_{t_1} = \infty$ and $x_{t_2} < \infty$, then $q = 0$ as $C$ gets stuck on $+\infty$ after $t_1$, while $q_1 = 0$ as well.

— If there is $t_0 \leq n-1$ with $\forall t \leq t_0, x_t < \infty$ and $\forall t > t_0, x_t = \infty$, then $q$ is by definition $P_{a_{t_0} \to a_\infty}$ times

$$\underbrace{\sum_{a_1=1}^{A} \cdots \sum_{a_{t_0}=1}^{A}}_{t_0} \prod_{t=1}^{t_0} \left( \begin{array}{l} P_{1,a_{t-1} \to a_t} \mathbf{P}\left(D_{1,a_{t-1} \to a_t} < \infty\right) \\ \mathbf{P}\left(D_{a_{t-1} \to a_t} + C(t-1) \leq x_t\right) \end{array} \right)$$

However, $D_{a_{t-1} \to a_t}$ is $D_{1,a_{t-1} \to a_t}$ conditionally to being finite whenever $P_{1,a_{t-1} \to a_t} > 0$, and the terms with $P_{1,a_{t-1} \to a_t} = 0$ have no influence whatsoever. As it is independent of $C(t-1)$, we recognize

$$\left( \begin{array}{ll} & \left( \underbrace{\sum_{a_1=1}^{A} \cdots \sum_{a_{t_0}=1}^{A}}_{t_0} \prod_{t=1}^{t_0} P_{1,a_{t-1} \to a_t} \mathbf{P}\left(D_{1,a_{t-1} \to a_t} + C(t-1) \leq x_t\right) \right) \\ \times & \mathbf{P}\left(C_1(t_0+1) = +\infty\right) \end{array} \right)$$

which is $q_1$ by construction.

— If $\forall t \leq n, x_t < \infty$, the same idea works without the final probability describing infinite transitions $\mathbf{P}\left(C_1(t_0+1) = +\infty\right)$.

In all cases, the distributions of $C$ and $C_1$ coincide, which solves this part.

3. Under the given condition, the probabilities $P_{i\to\infty}$ are all zero. It follows that the sub-matrix of rows and columns 1 to $A$ of $P$ defines a stochastic matrix, thus $L_1(0)$ is a stochastic matrix, which must be $M_1$'s transition matrix. As a consequence, we recovered the whole C-process.

As we recovered the initial terms using only $L_1$, this ends the proof.

### Propostion 2.3.5

We are going to do a recursion on $n$.

— If $n = 1$, then $T_1$ is the identity ($\forall t \in \mathbf{N}, T_1(t) = t$), so $C_{T_1} = C$ and the equality is a tautology ;

— Admitting equality for $n$, then for every $i, j \leq A$, the entry number $(i, j)$ of $\left(L_C(\alpha)\right)^{n+1}$ is

$$\left(L_C(\alpha)\right)^{n+1}_{i,j} = \sum_{k=1}^{A} \left(\left(L_C(\alpha)\right)^n\right)_{i,k} \left(L_C(\alpha)\right)_{k,j}$$

By recursion hypothesis, this comes down to

$$\left(\left(L_C(\alpha)\right)^{n+1}\right)_{i,j} = \sum_{k=1}^{A} P_{i\xrightarrow{n}k} \mathbf{E}\left(e^{-\alpha D_{i\xrightarrow{n}k}}\right) P_{k\to j} \mathbf{E}\left(e^{-\alpha D_{k\to j}}\right)$$

However, the above random variables $D_{i\xrightarrow{n}k}$ and $D_{k\to j}$ are independent, and by definition of $D_{i\xrightarrow{n+1}j}$, we recognize the expression of

$$\sum_{k=1}^{A} P_{i\xrightarrow{n}k} \mathbf{E}\left(e^{-\alpha D_{i\xrightarrow{n}k}}\right) P_{k\to j} \mathbf{E}\left(e^{-\alpha D_{k\to j}}\right) = P_{i\xrightarrow{n+1}j} \mathbf{E}\left(e^{-\alpha D_{i\xrightarrow{n+1}j}}\right)$$

when conditionning on the $n^{th}$ step, which ends the proof.

### Positive recurrent matrices

Throughout the proofs, we shall use Perron-Frobenius' theorem as a basis tool granting properties of $L_C(\alpha)$'s dominant eigenvalue. Interestingly, the proposition 2.4.1 breaks down when $C$ is not positive recurrent, which is related to $L_C(\alpha)$'s failure to hold the hypothesis of "positive recurrence" defined below that voids Perron-Frobenius' result.

**Definition 2.7.8** *Positive recurrent matrix*

*Let $n \in \boldsymbol{N}^*$, and $L \in \boldsymbol{M}_n\left(\boldsymbol{R}^+\right)$ be a non-negative matrix. We say that $L$ is positive recurrent iff*

$$\forall i, j \leq n, \exists k \in \boldsymbol{N}^*; \left(L^k\right)_{i,j} > 0$$

In particular,

— For $M$ a time-homogeneous Markovian process, its transition matrix $L$ is positive recurrent iff $M$ itself is positive recurrent.

— If $M$ is positive recurrent and aperiodic, it is possible to select the same $k$ large enough for all couples $(i, j)$, and $L^k$ will be a positive matrix.

Since Perron-Frobenius' theorem works for positive recurrent matrices, and we shall apply it to $C$'s Laplace matrix function, we look for the Laplace matrix functions holding positive recurrence.

**Lemma 2.7.10** *Positive recurrent Laplace matrix functions*

*Let $C$ be a sEI C-process. $C$ is positive recurrent iff for every $\alpha \in \boldsymbol{R}^+$, $L_C(\alpha)$ is a positive recurrent matrix.*

To prove this lemma, we take $C$ a C-process.

1. Let us deem $C$ positive recurrent, and take $\alpha \in \mathbf{R}^+$ and $i, j \leq A$. By virtue of the lemma 2.7.5, there is $k \in \mathbf{N}^*$ allowing for a path of length $k$ of finite value between $A_i$ and $A_j$. We characterize this path by
   — Its occupied state numbers $A_{a_u}$, for $u \in [|0, k|]$ and $a_u \leq A$ ;
   — Its payoffs values $x_u \in \mathbf{R}$, for $u \in [|u, k|]$ : they are finite because $C$ is positive recurrent.

   Thanks to this path choice, the Laplace transforms of the involved transition payoffs are positive and the transition probabilities are positive. For every $u \leq k$, we note

   $$L_{D_{a_{u-1} \to a_u}}(\alpha) = \mathbf{E}\left(e^{-\alpha D_{a_{u-1} \to a_u}}\right) = l_u(\alpha) > 0$$

   The $k$-periodically concatenated transition matrix (whose entries are $P_{i \xrightarrow{k} j}$) is the $k^{th}$ power of $M$'s transition matrix, and the entry number $(i, j)$ of $(L_C(\alpha))^k$ is no less than

   $$\prod_{u=1}^{k} P_{A_{a_{u-1}} \to A_{a_u}} l_u(\alpha)$$

   by construction of the matrix product, which is positive because all terms in this product are positive. This being for every $i, j \leq A$, $L_C(\alpha)$ is positive recurrent.

2. If $C$ is not positive recurrent, there are states $i, j \leq A$ such that for every $k \in \mathbf{N}^*$,

$$\mathbf{P}\left(M(t+k) = A_j \wedge C(t+k) < \infty | M(t) = A_i \wedge C(t) < \infty\right) = 0$$

so this rewrites as

$$\begin{pmatrix} \mathbf{P}\left(M(t+k) = A_j | M(t) = A_i \wedge C(t) < \infty\right) \\ \mathbf{P}\left(C(t+k) < \infty | M(t) = A_i \wedge M(t+k) = A_j \wedge C(t) < \infty\right) \end{pmatrix} = 0$$

However, $M$ is Markovian, and one recognizes the conditional probability of $D_{i \xrightarrow{k} j}$ being finite in the rightmost term, so

$$P_{i \xrightarrow{k} j} \mathbf{P}\left(D_{i \xrightarrow{k} j} < \infty\right) = 0$$

Hence, for every $k \in \mathbf{N}^*$, the entry $(i, j)$ of $(L_C(\alpha))^k$ must be 0 by definition, so $L_C(\alpha)$ is not positive recurrent.

This ends the proof.

## 2.8 Proposition 2.4.1

This section aims at proving all statements about the martingale parameter of a C-process $C$ : existence, uniqueness, behaviour, properties. By definition, its value at point $a \in \mathbf{R}_+^*$ should allow for the following equation, hereafter called the "eigenvector equation" in vector $w^{(a)} \in \mathbf{R}^A$, to have a non-zero solution :

$$L_C(\alpha(a)) w^{(a)} = w^{(a)} e^a$$

This section is divided in multiple steps :

1. Introduce some preliminary notions ;
2. Introduce the trick of "reduced processes" ;
3. Prove that these reduced processes allow for the dominant eigenvalue to stand through reductions ;
4. Find the correct value for $\alpha(a)$, leading to the proposition 2.4.1 for $a \in \mathbf{R}_+^*$ ;
5. Prove the other properties of $C$'s martingale parameter and extend them to $a \in \mathbf{R}^+$.

### 2.8.1 Preliminary explanations

Let us start with $C$'s Laplace matrix function $L_C$ and its eigenvector equation. We are going to ensure that there really is a solution to the eigenvector equation for every value of $a \in \mathbf{R}_+^*$ when $C$ is sEI, positive recurrent, and not globally increasing. We also explain why there is no solution when $C$ is globally increasing.

**Existence of the martingale parameter**

First, we prove that for every $a \in \mathbf{R}_+^*$, a martingale parameter $\alpha(a)$ exists.

**Lemma 2.8.1** *Existence of a martingale parameter*
*Let $C$ be a positive recurrent, sEI, not globally increasing C-process. For every $a \in \boldsymbol{R}_+^*$, there is $\alpha(a) \in \boldsymbol{R}_+^*$ such that $e^a$ is the dominant eigenvalue of $L_C(\alpha(a))$.*

To ensure this statement, we start with Perron-Frobenius' theorem, stating that every (square) non-negative and positive recurrent matrix $L$ has a single non-negative dominant eigenvalue $f(L)$ such that

- Every eigenvalue $\lambda$ of $L$ holds $|\lambda| \leq f(L)$ ;
- The associated eigenspace has dimension one ($f(L)$ is a single root of $L$'s characteristic polynomial), directed by a vector $w(L)$ whose coordinates are positive.

Let us consider this function $f$. Tyrtyshnikov's result about continuity of eigenvalues indicates that $f$ is continuous, so let us consider the function $g$ defined by

$$g = \begin{pmatrix} \mathbf{R}^+ & \to & \mathbf{R}^+ \\ \alpha & \to & f\left(L_C(\alpha)\right) \end{pmatrix}$$

As $L_C$ is continuous over $\mathbf{R}^+$ by construction because $C$ is sEI, then $g$ is a continuous function. We find its limits on the boundaries :

- At point $\alpha = 0$, $g(0)$ is the dominant eigenvalue of the matrix

  $$\left(P_{i \to j} \mathbf{P}\left(D_{i \to j} < \infty\right)\right)_{i,j}$$

  This matrix is thus dominated by $M$'s transition matrix, so $g(0) \leq 1$.
- To analyze when $\alpha$ goes to infinity, we use the hypothesis under which $C$ is not globally increasing. It follows from this assumption and lemma 2.7.2 that there are $i \leq A$ and $n \in \mathbf{N}^*$ such that there is a cycle going from $A_i$ to $A_i$ whose value is negative, so $P_{i \xrightarrow{n} i} > 0$ and there is $x > 0$ such that $\mathbf{P}\left(D_{i \xrightarrow{n} i} \leq -x\right) = p > 0$. Hence, noting $C_{T_n}$ as $C$'s $n$-periodically concatenated C-process, the coefficient $(i,i)$ of $L_{C_{T_n}}(\alpha)$ is at least $pe^{\alpha x}$. Since lemma 2.3.5 indicates that

  $$L_{C_{T_n}}(\alpha) = \left(L_C(\alpha)\right)^n$$

and $L_C(\alpha)$ is a non-negative matrix, its dominant eigenvalue is at least

$$p^{(1/n)}e^{\alpha x/n}$$

so $g$ must go to infinity when $\alpha$ goes to infinity.

It follows from these facts and the intermediate value theorem that there is (at least) one real solution to $g(\alpha) = e^a$ in $\mathbf{R}_+^*$, no matter $a \in \mathbf{R}_+^*$.

— In the sequel of this proof, we define $\alpha(a)$ as any such solution to this equation.

— We might also define a "negative martingale parameter" when $C$ is not globally decreasing, as a negative solution to $g(\alpha) = e^a$. Even if this definition comes useful when computing $C$'s differentiated process, we shall not discuss it as it is symmetrical to the study of $C$'s natural martingale parameter.

### Case of a globally increasing C-process

To explain why the study fails when $C$ is globally increasing, let us assume for now that it is a Lévy process : it means that its increments $D$ are non-negative almost surely. It follows that the Laplace transform of $D$ (over $\mathbf{R}_+^*$) will not hit any value in $(1, \infty)$, so the eigenvector equation will have no positive solution in $\alpha$. The same phenomenon happens in the general case for $C$ globally increasing, although some of its increments may be negative.

**Lemma 2.8.2** *Global increase and dominant eigenvalue*

*Let $C$ be a positive recurrent C-process, whose Laplace matrix function is $L_C$. The following assertions are equivalent :*

1. *$C$ is globally increasing.*
2. *$L_C$ is well-defined over $\boldsymbol{R}_+^*$, and $L_C(\alpha)$'s dominant eigenvalue is both non-increasing and bounded by $1$ when $\alpha$ goes to infinity.*
3. *$L_C$ is well-defined over $\boldsymbol{R}_+^*$, and $L_C(\alpha)$'s dominant eigenvalue is bounded when $\alpha$ goes to infinity.*

1. If $C$ is globally increasing, we know after the proposition 2.3.2 that it rewrites as $C^+ + C^=$ with $C^+$ non-decreasing almost surely and $C^=$ globally constant. Thanks to the proposition 2.3.3, we rewrite $C^=$'s increments from $A_i$ to $A_j$ as $c_j - c_i \in \mathbf{R}$, so $C$'s Laplace matrix function at point $\alpha \in \mathbf{R}_+^*$ is

$$L_C(\alpha) = P_{i \to j}\mathbf{E}\left(e^{-\alpha D_{i \to j}^+}\right) e^{-\alpha(c_j - c_i)}$$

This expression is non-increasing and converges because $c_j - c_i$ is finite and $C^+$'s increments $D_{i \to j}^+$ are non-negative almost surely by hypothesis.

It follows from it that $L_C$'s dominant eigenvalue, being a non-decreasing function of its coefficients, is no more than the one of $L_{C^=}(\alpha)$ ; however, setting $D(\alpha)$ as the diagonal matrix whose coefficents are $e^{-c_i\alpha}$ and $P$ as $C$'s transition matrix, we have

$$L_{C^=}(\alpha) = (D(\alpha))^{-1} PD(\alpha)$$

As a consequence, $L_{C^=}(\alpha)$'s eigenvalues are the same as $P$'s ones, so are dominated by 1, which ends the proof.

2. If $C$ is not globally increasing, then thanks to the lemma 2.8.1, for every $a \in \mathbf{R}_+^*$, there is a martingale parameter $\alpha(a)$ that can make $L_C(\alpha(a))$'s dominant eigenvalue arbitrarily large. Since $L_C$ is continuous, divergence may only happen when $\alpha(a)$ goes to infinity, which ends the proof.

When $C$ is, so to speak, "strictly" globally increasing, a similar property holds.

**Lemma 2.8.3** *Strict global increase and dominant eigenvalue*

*Let $C$ be a positive recurrent C-process, whose Laplace matrix function is $L_C$. The following assertions are equivalent :*

1. *$C$ is globally increasing and not globally constant.*
2. *$L_C$ is well-defined over $\mathbf{R}_+^*$, and the dominant eigenvalue $d(\alpha)$ of $L_C(\alpha)$ holds $\forall\alpha \in \mathbf{R}_+^*, d(\alpha) < 1$.*

1. Let us assume $C$ globally increasing and not globally constant. Thanks to the proposition 2.3.2, we rewrite it as $C = C^+ + C^=$ with $C^+$ non-decreasing and $C^=$ globally constant. We know that $C^+$ has a cycle of positive value, as else it would be identically 0 so $C = C^=$ would be globally constant : this cycle has
   — A length $T \in \mathbf{N}^*$ ;
   — A starting state $A_k$ with $k \leq A$ ;
   — A positive value $v \in \mathbf{R}_+^*$
   First, considering $C^=$'s increments from $A_i$ to $A_j$ as $c_j - c_i \in \mathbf{R}$ thanks to the proposition 2.3.3, and noting by $\Delta(\alpha)$ the diagonal matrix whose entries are
   $$\forall\alpha \in \mathbf{R}, i \leq A, (\Delta(\alpha))_{i,i} = e^{-\alpha c_i}$$
   the Laplace matrix function of $C$ rewrites as
   $$\forall\alpha \in \mathbf{R}_+^*, L_C(\alpha) = (\Delta(\alpha))^{-1} L_{C^+}(\alpha)\Delta(\alpha)$$
   so $L_C(\alpha)$ and $L_{C^+}(\alpha)$ have the same eigenvalues, so we shall look at the dominant eigenvalue of $L_{C^+}(\alpha)$. Hence, let us take $\alpha$ and look at the dominant eigenvalue $\lambda \in \mathbf{R}^+$ of $L_{C^+}(\alpha)$ and its dominant eigenvector $w$ : thanks

to the concatenation property, we have after $T$ time periods

$$(L_{C^+}(\alpha))^T w = \lambda^T w$$

However, this matrix is the Laplace matrix function of $C^+$'s $T$-periodically concatenated C-process thanks to the proposition 2.3.5, so its entry number $(i,j)$ is

$$P_{i \xrightarrow{T} j} \mathbf{E}\left(e^{-\alpha D_{i \xrightarrow{T} j}}\right)$$

Thanks to the cycle of positive value, we know that its entry number $(k,k)$ is strictly less than $P_{k \xrightarrow{T} k}$. As the other entries hold the large inequality because transition payoffs are non-negative, there is a non-negative matrix $X$, whose entry number $(k,k)$ is positive, such that

$$(L_{C^+}(\alpha))^T = P^T - X$$

where $P$ is $M$'s transition matrix. It follows that

$$\lambda^T w = (L_{C^+}(\alpha))^T w = \left(P^T - X\right) w$$

so $P^T w = \left(\lambda^T Id + X\right) w$. Multiplying by $\mu$ being $M$'s invariant measure ($P$'s row eigenvector), this leads to $\mu w = \lambda^T \mu w + \mu X w$ ; as $\mu X w > 0$, this is possible only if $\lambda < 1$, which ends the proof.

2. We already know after the lemma 2.8.2 that this condition implies $C$ globally increasing. However, if $C$ is a globally constant process, rewriting it as $C^+ + C^=$ yields that $L_C(\alpha)$'s dominant eigenvalue is $L_{C^+}(\alpha)$'s one ; as $C^+ \equiv 0$, $L_{C^+}(\alpha) = P$ and thus the dominant eigenvalue must be 1.

### Drifted C-processes

In the incoming work, we shall use a trick consisting in "drifting" C-processes by a fixed trend. This consists in adding a constant value $d \in \mathbf{R}$ to each of its transitions, so that the value of $C$'s drifted process at time $t \in \mathbf{N}$ will be $C(t) + td$.

**Definition 2.8.1** *Drifted C-process*

*Let $C$ be a C-process whose underlying Markovian process is $M$. For every $d \in \boldsymbol{R}$, $C$'s drifted process by $d$ is the process $C^{[d]}$, whose*

— *Underlying Markovian process is the same $M$ ;*

— *Transition payoffs are given through $C$'s ones $D_{i \to j}$ by*

$$\forall i, j \leq A, D^{[d]}_{i \to j} = D_{i \to j} + d$$

— *Starting point is $C(0)$.*

*It follows from this that*

$$\forall t \in \boldsymbol{N}, C^{[d]}(t) = C(t) + dt$$

This definition is useful when looking at $C^{[d]}$'s Laplace matrix function, given by

$$\forall \alpha \in \mathbf{R}, L_{C^{[d]}}(\alpha) = L_C(\alpha)\, e^{-\alpha d}$$

This property will allow us several simplifications during the incoming work.

### 2.8.2 Reductions

The next step in our study is to enable the method of "reduction" of a C-process. The main idea is to turn the C-process into its restricted Lévy process by elimination of all its states but one, getting "reduced C-processes" in the work.

#### Notions of reduction

The main idea is to eliminate one of $M$'s states $A_k \neq M(0)$, considering that the time steps when $M$ hits $A_k$ are skipped by the C-process : this is tantamount to looking at the concatenated process $M_\tau$ where $\tau$ comes from the binary determination sequence $\rho$ given by

$$\forall t \in \mathbf{N}, \rho(t) = \mathbf{1}_{M(t) \neq A_k}$$

In other words, eliminating a state $A_{k \leq A}$ is skipping all times $t \in \mathbf{N}$ for which $M(t) = A_k$, concatenating the "previous" and the "next" transitions

$$(M(t-1) \to A_k), (A_k \to M(t+1))$$

to get a single transition

$$(M(t-1) \to M(t+1))$$

However, for computations purposes, we allow the reductions to take into account some drifts for $C$ : indeed, the reduced C-process will be obtained as follows.

1. Drift all transition payoffs by $d \in \mathbf{R}$, yielding $C^{[d]}$ ;
2. Concatenate $C^{[d]}$ with the above time sequence ;
3. "Un-drift" the resuling transition payoffs, drifting them by $-d$.

Concatenated transition probabilities, transition payoffs and C-processes then arise like told below.

**Definition 2.8.2** *Reduced C-process*

*Let $C$ be a positive recurrent, sEI C-process and $A_k \neq M(0)$ a state to eliminate.*

- *The time sequence for reduction with respect to state $A_k$ is $\tau_k$, whose binary determination sequence is $\rho_k$ such that*

$$\forall t \in \boldsymbol{N}, \rho_k(t) = \boldsymbol{1}_{M(t) \neq A_k}$$

*We define $M_{-k}$ the reduced Markovian process with respect to the time sequence $\tau_k$.*

- *For every $d \in \boldsymbol{R}$, the reduced C-process with respect to state $A_k$ and drift $d$ is obtained by*

$$\forall t \in \boldsymbol{N}, C_{(-k,d)}(t) = C^{[d]}\left(\tau_k(t)\right) - td$$

This definition yields a C-process because $\tau_k$ is a canonical time sequence to $C^{[d]}$, which follows from the definition of canonical time sequences because $\rho_k$ relies only on present time. We may now compute the transition probabilities and transition payoffs of $C_{-k}$.

**Lemma 2.8.4** *Distributions of reduced processes*

*Let $C$ be a positive recurrent C-process whose underlying Markovian process is $M$, $A_k \neq M(0)$ be a state of $M$, and $d \in \boldsymbol{R}$ be a drift. Let $M_{-k}$ and $C_{(-k,d)}$ be the reduced Markovian process and C-process of $C$ with respect to state $A_k$ and drift $d$.*

1. *The transition probabilities of $M_{-k}$ are*

$$\forall i, j \neq k, P'_{-k,i\to j} = P_{i\to j} + \frac{P_{i\to k}P_{k\to j}}{1 - P_{k\to k}}$$

2. *The transition payoffs of $C_{(-k,d)}$ are $D_{(-k,d),i\to j}$ such that there are independent random variables*
   - *$D_{i\to j}$, $D_{i\to k}$, $D^{(i)}_{k\to k}$ (the latter ones being independent and identically distributed copies of $D_{k\to k}$ for $i \in \boldsymbol{N}^*$) and $D_{k\to j}$,*
   - *$X_1$ a Bernouilli random variable indicating the "direct" transition going $(A_i \to A_j)$, with*

$$\boldsymbol{P}(X_1 = 1) = \frac{P_{i\to j}}{P_{-k,i\to j}}$$

   - *$X_2$ a geometric random variable indicating the number of loops in state $A_k$, of parameter $P_{k\to k}$,*

   *such that the transition payoff from $A_i$ to $A_j$ is*

$$D_{(-k,d),i\to j} = \boldsymbol{1}_{X_1=1}D_{i\to j} + \boldsymbol{1}_{X_1=0}\left(D_{i\to k} + \sum_{n=1}^{X_2} D^{(i)}_{k\to k} + D_{k\to j} + (X_2+1)d\right)$$

*Moreover, $C_{(-k,d)}$ is also a positive recurrent C-process over the state space*

$$\{A_i; i \leq A \wedge i \neq k\}$$

We note that

— $P_{k \to k} < 1$ because $M$ is positive recurrent and $A \geq 2$ (the states $M(0)$ and $A_k$ being distinct) ;

— If $P_{-k,i \to j} = 0$, then $D_{(-k,d),i \to j}$ may be defined arbitrarily, as it will have almost surely no subsequent effects on $C$.

As the proof of these statements are similar, we will only explain the construction of the transition probabilities of $M_{-k}$, the reduced Markovian process with respect to $A_{k \leq A}$.

— Previous transition probabilities going from $A_i$ to $A_{j \neq k}$ are not modified, so transfer to $M_{-k}$. This case corresponds to $X_1 = 1$.

— With probability $P_{i \to k}$, one gets in $A_k$, this case corresponding to $X_1 = 0$. Then the probability of going out of $A_k$ by state $A_j$ after exactly $n$ loops (corresponding to $X_2 = i$) of the transition $(A_k \to A_k)$ is

$$P^n_{k \to k} P_{k \to j}$$

So, the probability of going out of $A_k$ by state $A_j$ is

$$\sum_{n=0}^{\infty} P^n_{k \to k} P_{k \to j} = \frac{P_{k \to j}}{1 - P_{k \to k}}$$

It follows that

— The transition probabilities of process $M_{-k}$ are increased to

$$P_{-k,i \to j} = P_{i \to j} + \frac{P_{i \to k} P_{k \to j}}{1 - P_{k \to k}}$$

— $C_{(-k,d)}$ is positive recurrent, as for every states $A_i, A_j \neq M(0)$, if there is a path

$$(A_i \to A_{a_1} \to \dots A_j)$$

of finite value going through $A_k$, then the sequence given by skipping steps in $A_k$ still forms a path of finite value. Indeed,

$$i, j \neq k \Rightarrow P_{-k,i \to j} \geq P_{i \to j} > 0$$

and “skipped” times correspond to some positive term

$$P_{i \to k} \left(P_{k \to k}\right)^n P_{k \to j}$$

that appears in $P_{-k,i \to j}$ too.

For transition payoffs, the same method and the fact that $C$ is a C-process lead to the result a similar way, after disjunction with respect to values taken by $X_1$ and $X_2$. The term $(X_2+1)d$ itself is the consequence of the drift ; for some value of $X_2$, the concatenated path takes a time of $1+X_2+1$, leading to the remaining drift after multiplication.

### Successive reductions and restricted Lévy process

We proved that the reduction of a positive recurrent C-process over $A \geq 2$ states exists and is a positive recurrent C-process over $A-1$ states. Hence, we may apply the reduction again, until we finally hit a C-process over one single state, i.e. a Lévy process. It turns out that the result of these successive reductions coincides with $C$'s restricted Lévy process, as we only select times $t \in \mathbf{N}$ holding $M(t) = M(0)$.

**Lemma 2.8.5** *Reductions up to the restricted Lévy process*
*Let $C$ be a positive recurrent C-process, and $(k_i)_{i \leq A}$ be an enumeration of the state numbers with $A_{k_1} = M(0)$. Let us take $d \in \boldsymbol{R}$ a drift, and build the sequence of successive reduced C-processes recursively, with $C_A = C$ and*

$$\forall u \leq A-1, C_u = (C_{u+1})_{(-k_{u+1},d)}$$

*Let $C_*$ be $C$'s restricted Lévy process and $\tau_*$ be its time sequence. Then*

$$\forall t \in \boldsymbol{N}, C_1(t) = C_*(t) + (\tau_*(t) - t)\, d$$

*We call it $C$'s d-restricted Lévy process.*

We are going to iterate reductions, choosing successively state numbers $k_A$ to $k_2$ to eliminate (other than $M(0)$).

- First, these successive reductions are possible, as we proved that the reduction of a positive recurrent, non-degenerated C-process (i.e. $A \geq 2$) is still a C-process in lemma 2.8.4. As reductions each eliminate one state, we correctly do $A-1$ of them so only one state is left.
- For every $u \leq A$, we look at
  - $M_u$, being $C_u$'s underlying Markovian process (so $M_A = M$) ;
  - $\rho_u$, being $C_u$'s binary determination sequence, as being a reduction of $C_{u+1}$ when $u \leq A-1$ ;
  - $\tau_u$, being $\rho_u$'s associated time sequence when $u \leq A-1$.

  We also define chained time sequences $\upsilon_u$ by $\forall s \in \mathbf{N}, \upsilon_1(s) = s$ and

  $$\forall s \in \mathbf{N}, \upsilon_u(s) = \tau_{u-1}(\upsilon_{u-1}(s))$$

  In particular, we want to prove that $\upsilon_A$ is $\tau_*$, the time sequence of $C$'s restricted Lévy process.

— We know that (for any $s \in \mathbf{N}$) $\rho_u(s) = 1$ iff $M_u(s) \neq A_{k_u}$. As $C$ is positive recurrent, return to $M(0)$ is almost certain, so no $\tau_u(s)$ amounts to $+\infty$ almost surely, and we have

$$\forall s \in \mathbf{N}, M_u(s) = M_{u+1}\left(\tau_u(s)\right)$$

so by induction on $u$, we get

$$\forall s \in \mathbf{N}, M_1(s) = M_A\left(\upsilon_A(s)\right)$$

We prove that $\forall s \in \mathbf{N}, M_u\left(\upsilon_u(s)\right) = M(0)$. This is done by induction on $u$ : when $u = 1$, $M_1$ has $M(0)$ as only state, and we know that if $M_u\left(\upsilon_u(s)\right) = M(0)$, then

$$M_{u+1}\left(\upsilon_{u+1}(s)\right) = M_{u+1}\left(\tau_u\left(\upsilon_u(s)\right)\right) = M_u\left(\upsilon_u(s)\right)$$

by definition of $M_u$. Hence, since $M = M_A$, we proved that $\upsilon_A$ only hits states equal to $M(0)$.

— Finally, we prove that all values of $t \in \mathbf{N}$ such that $M(t) = M(0)$ are hit by $\upsilon_A$, i.e. there is $s \in \mathbf{N}$ with $\upsilon_A(s) = t$. To do this, let us assume that $M(t) = M(0)$ and let us start with $t_A = t$. For $u$ from $A-1$ down to 1, we do the following loop ($A-1$ times) :
  - — We verify that $M_{u+1}(t_{u+1}) = M(0) \neq A_{k_{u+1}}$. Indeed, this is true for $u = A-1$ because $A \geq 2$, $M(0) = A_{k_1}$ and by hypothesis, and for further steps thanks to the last part of the previous loop.
  - — $\rho_u$ indicates times $x \in \mathbf{N}$ such that $M_{u+1}(x) \neq A_{k_{u+1}}$. In particular, this holds for $x = t_{u+1}$, so there is $t_u \in \mathbf{N}$ such that $t_{u+1} = \tau_u\left(t_u\right)$.
  - — We select such a $t_u$, and we have by construction of $M_u$

    $$M_u(t_u) = M_{u+1}\left(\tau_u(t_u)\right) = M_{u+1}\left(t_{u+1}\right) = M(0)$$

    and provided that $u \neq 1$, this is not $A_{k_u}$, which enables the verification for the next loop (and when $u$ hits 1, looping stops here).

  This gets us a sequence $\left(t_u\right)_{u \leq A}$ such that $\forall u \leq A-1, t_{u+1} = \tau_u\left(t_u\right)$. In particular, we get $t = t_A = \upsilon_A(t_1)$, so $s = t_1$ solves this part.

Hence, there is $s \in \mathbf{N}$ such that $\upsilon_A(s) = t$ iff $M(t) = M(0)$. As $\upsilon_A$ is increasing ($\mathbf{N} \to \mathbf{N}$) (composition of increasing functions $\tau_u$), then it must coincide with $\tau_*$. Now, to get the value of $C_1(t)$, we make use for every $u \leq A-1$ of

$$C_u(t) = \left(C_{u+1}\right)_{(-k_{u+1},d)}(t) = \left(C_{u+1}^{[d]}\right)\left(\tau_u(t)\right) - td = \left(C_{u+1}\right)\left(\tau_u(t)\right) + \left(\tau_u(t) - t\right)d$$

This leads by induction to

$$C_1(t) = C_A\left(\upsilon_A(t)\right) + \sum_{u=1}^{A-1}\left(\upsilon_{u+1}(t) - \upsilon_u(t)\right)d$$

which simplifies to

$$C_1(t) = C_A\left(\tau_*(t)\right) + \left(\tau_*(t) - t\right) d$$

This ends the proof.

### Mean expectancy and reductions

When $C$ is positive recurrent and integrable, we aim at evaluating the mean expectancy of its reduced C-process by virtue of the following lemma. In particular, we shall be interested only in the case $d = 0$, so we shall abbreviate
— $C_{(-k,0)}$ by $C_{-k}$ ;
— $D_{(-k,0),i\to j}$ by $D_{-k,i\to j}$.
in the next paragraphs.

**Lemma 2.8.6** *Scaling of mean expectancies*
*Let $C$ be a positive recurrent, integrable C-process ; let $A_k \neq M(0)$, and $C_{-k}$ be $C$'s reduced process with respect to $A_k$ and drift $0$.*

1. *$C_{-k}$ is positive recurrent and integrable.*
2. *$C_{-k}$'s invariant distribution $\mu_{-k}$ is given by*

$$\forall i \neq a, \mu_{-k,[i]} = \frac{\mu[i]}{1-\mu[k]}$$

3. *$C_{-k}$'s mean expectancy is*

$$E\left(C_{-k}\right) = \frac{E(C)}{1-\mu_{[k]}}$$

4. *In particular, the mean expectancy of $C$'s restricted Lévy process $C_\tau$ is given by*

$$E\left(C_\tau\right) = \frac{E(C)}{\mu_{[M(0)]}}$$

Noting by $\mu$ the row-vector of $M$'s invariant probabilities, $E$ may be expressed by definition as

$$E = \sum_{i=1}^{A}\sum_{j=1}^{A} \mu_{[i]} P_{i\to j} \mathbf{E}\left(D_{i\to j}\right) = \mu R_C(0)\left(\vec{1}\right)$$

Considering the matrix $R_{C_{-k}}(0)$, the statements to prove are dealt with as follows :

1. Its entries converge, because $C_{-k}$'s transition payoffs are sub-geometrical. Indeed, let us take $i, j \leq A$ : using $D_{-k,i\to j}$ as written in lemma 2.8.4,

$$\begin{aligned} & \mathbf{E}\left(|D_{-k,i\to j}|\right) \\ \leq\ & \mathbf{P}\left(X_1 = 1\right)\mathbf{E}\left(|D_{i\to j}|\right) \\ +\ & \mathbf{P}\left(X_1 = 0\right)\left(\mathbf{E}\left(|D_{i\to k}|\right) + \sum_{n=0}^{\infty}\mathbf{P}\left(X_2 = n\right)n\mathbf{E}\left(|D_{k\to k}|\right) + \mathbf{E}\left(|D_{k\to j}|\right)\right) \end{aligned}$$

   Now, we know that
   - If $\mathbf{E}\left(|D_{i\to j}|\right) = \infty$, then $(i, j) \notin \Gamma$ (else $C$ would not be integrable), so one must have $\mathbf{P}\left(X_1 = 1\right) = 0$. It follows that the first term of the sum is integrable, be it naturally or because it is irrelevant.
   - For the same reason, if either one of the transition payoffs $D_{i\to k}$ or $D_{k\to j}$ is not integrable, then the corresponding $(i, k)$ or $(k, j)$ would not be in $\Gamma$ so $\mathbf{P}\left(X_1 = 0\right) = 0$.
   - Finally, if $D_{k\to k}$ is not integrable, then $(k, k) \notin \Gamma$, which implies that $X_2 = 0$ almost surely and the effect of $D_{k\to k}$ is nullified.

   Hence, the only needed verification is that the series converges. We know that

$$\sum_{n=0}^{\infty}\mathbf{P}\left(X_2 = n\right)n = \mathbf{E}\left(X_2\right) = \frac{P_{k\to k}}{1 - P_{k\to k}} < \infty$$

   because $P_{k\to k} < 1$ since $C$ is positive recurrent, then all terms are integrable so $R_{C_{-k}}(0)$ is well-defined.
2. Let $M_{-k}$ be the reduced Markov chain ; as it is still positive recurrent, it has a single invariant distribution. We note by $P$ and $P_{-k}$ the transition matrices of $M$ and $M_{-k}$ ; we prove that $\mu_{-k}$ as given is $M_{-k}$'s invariant distribution by testing if

$$\mu_{-k}\left(\vec{1}\right) = 1 \wedge \mu_{-k}P_{-k} = \mu_{-k}$$

   The first constraint comes from the identity

$$\sum_{i\neq k}\mu_{[i]} = 1 - \mu_{[k]}$$

   To get the other one, we evaluate for every $j \neq k$

$$\left(\mu_{-k}P_{-k}\right)_{[j]} = \sum_{i\neq k}\mu_{-k,[i]}\left(P_{-k}\right)_{i,j}$$

   Thanks to the lemma 2.8.4, this rewrites as

$$\left(\mu_{-k}P_{-k}\right)_{[j]} = \sum_{i\neq k}\frac{\mu_{[i]}}{1 - \mu_{[k]}}\left(P_{i,j} + \frac{P_{i,k}P_{k,j}}{1 - P_{k,k}}\right)$$

Decomposing between "direct" and "indirect" transitions, we have

$$\sum_{i\neq k}\frac{\mu_{[i]}}{1-\mu_{[k]}}\left(P_{i,j}+\frac{P_{i,k}P_{k,j}}{1-P_{k,k}}\right)=\sum_{i\neq k}\frac{\mu_{[i]}}{1-\mu_{[k]}}P_{i,j}+\sum_{i\neq j}\frac{\mu_{[i]}}{1-\mu_{[k]}}\frac{P_{i,k}P_{k,j}}{1-P_{k,k}}$$

However, since

$$\sum_{i\neq k}\mu_{[i]}P_{i,j}=\mu_{[j]}-\mu_{[k]}P_{k,j}$$

then this sum rewrites as

$$\sum_{i\neq k}\frac{\mu_{[i]}}{1-\mu_{[k]}}P_{i,j}+\sum_{i\neq j}\frac{\mu_{[i]}}{1-\mu_{[k]}}\frac{P_{i,k}P_{k,j}}{1-P_{k,k}}$$

which is

$$\frac{\mu_{[j]}-\mu_{[k]}P_{k,j}}{1-\mu_{[k]}}+\frac{\mu_{[k]}-\mu_{[k]}P_{k,k}}{\left(1-\mu_{[k]}\right)(1-P_{k,k})}P_{k,j}$$

that simplifies to

$$\frac{\mu_{[j]}-\mu_{[k]}P_{k,j}}{1-\mu_{[k]}}+\mu_{[k]}\frac{1-P_{k,k}}{\left(1-\mu_{[k]}\right)(1-P_{k,k})}P_{k,j}=\frac{\mu_{[j]}}{1-\mu_{[k]}}=\mu_{-k,[i]}$$

So, $\mu_{-k}$ is the sought invariant distribution.

3. We aim at finding

$$E_{-k}=E(C_{-k})=\mu_{-k}R_{C_{-k}}(0)\left(\vec{1}\right)$$

The definition of $C_{-k}$'s transition payoffs leads to

$$E_{-k}=\sum_{i\neq k}\sum_{j\neq k}\left(\frac{\mu_{[i]}}{1-\mu_{[k]}}\right)\left(\begin{array}{c}P_{i\to j}\mathbf{E}\left(D_{i\to j}\right)\\ +P_{i\to k}P_{k\to j}\sum_{n=0}^{\infty}P_{k\to k}^{n}\left(\begin{array}{c}\mathbf{E}\left(D_{i\to k}\right)\\ +n\mathbf{E}\left(D_{k\to k}\right)\\ +\mathbf{E}\left(D_{k\to j}\right)\end{array}\right)\end{array}\right)$$

We focus on the rightmost sum

$$x=\sum_{i\neq k}\sum_{j\neq k}\sum_{n=0}^{\infty}\mu_{[i]}P_{i\to k}P_{k\to j}P_{k\to k}^{n}\left(\mathbf{E}\left(D_{i\to k}\right)+n\mathbf{E}\left(D_{k\to k}\right)+\mathbf{E}\left(D_{k\to j}\right)\right)$$

We split it into the three relevant terms

$$\begin{aligned}x &= \sum_{i\neq k}\mu_{[i]}P_{i\to k}\mathbf{E}\left(D_{i\to k}\right)\sum_{j\neq k}P_{k\to j}\sum_{n=0}^{\infty}P_{k\to k}^{n}\\ &+ \sum_{j\neq k}P_{k\to j}\mathbf{E}\left(D_{k\to j}\right)\sum_{i\neq k}\mu_{[i]}P_{i\to k}\sum_{n=0}^{\infty}P_{k\to k}^{n}\\ &+ \sum_{n=0}^{\infty}P_{k\to k}^{n}n\mathbf{E}\left(D_{k\to k}\right)\sum_{i\neq k}\mu_{[i]}P_{i\to k}\sum_{j\neq k}P_{k\to j}\end{aligned}$$

The terms now simplify as follows :

— For the first term, we know that

$$\sum_{j\neq k} P_{k\to j} = 1 - P_{k\to k} = \frac{1}{\sum_{n=0}^{\infty} P_{k\to k}^n}$$

It means that the first term of $x$ is

$$\sum_{i\neq k} \mu_{[i]} P_{i\to k} \mathbf{E}\left(D_{i\to k}\right)$$

— For the second term, since $\mu$ is the invariant distribution

$$\sum_{i\neq k} \mu_{[i]} P_{i\to k} = \mu_{[k]} - \mu_{[k]} P_{k\to k}$$

It means that the second term of $x$ is

$$\sum_{j\neq k} P_{k\to j} \mathbf{E}\left(D_{k\to j}\right) \mu_{[k]}$$

— For the third term, we know that

$$\sum_{n=0}^{\infty} P_{k\to k}^n n = \frac{P_{k\to k}}{\left(1 - P_{k\to k}\right)^2}$$

It means that the third term of $x$ is

$$\frac{P_{k\to k}}{\left(1 - P_{k\to k}\right)^2} \mathbf{E}\left(D_{k\to k}\right)\left(1 - P_{k\to k}\right)\left(1 - P_{k\to k}\right) = P_{k\to k} \mathbf{E}\left(D_{k\to k}\right)$$

Hence,

$$E_{-k} = \frac{1}{1-\mu_{[k]}} \left( \begin{array}{ll} & \sum_{i\neq k} \sum_{j\neq k} \mu_{[i]} P_{i\to j} \mathbf{E}\left(D_{i\to j}\right) \\ + & \sum_{i\neq k} \mu_{[i]} P_{i\to k} \mathbf{E}\left(D_{i\to k}\right) \\ + & \sum_{j\neq k} P_{k\to j} \mathbf{E}\left(D_{k\to j}\right) \mu_{[k]} \\ + & P_{k\to k} \mathbf{E}\left(D_{k\to k}\right) \end{array} \right)$$

This finally simplifies to

$$E_{-k} = \frac{1}{1-\mu_{[k]}} \sum_{i=1}^{A} \sum_{j=1}^{A} \mu_{[i]} P_{i\to j} \mathbf{E}\left(D_{i\to j}\right) = \frac{E}{1-\mu_{[k]}}$$

4. We are going to use the successive reductions from lemma 2.8.5. According to its notations, we name
   — The state to be removed is $A_{k_u} \neq M(0)$ for $u \in [|2, A|]$ ; for convenience, we shall note $A_{k_1} = M(0)$ the last state (the starting state) that will not be removed.

— The reduced processes before reduction number $u$, for $u \in [|1, A|]$, is

$$C_u = \left((C_{-k_A})_{-k_{A-1}}\right)_{\ldots -k_{u+1}}$$

— The invariant measure of its underlying Markovian process is

$$\mu^u \in \mathbf{R}^u$$

In particular, $\mu^A = \mu$.

Our idea is to prove by induction that for every $u \leq A$,

$$E\left(C_u\right) = \frac{E\left(C\right)}{\sum_{x=1}^{u} \mu_{[k_x]}}$$

and

$$\forall x \leq u, \mu^u_{[k_x]} = \frac{\mu_{[k_x]}}{\sum_{x=1}^{u} \mu_{[k_x]}}$$

— At $u = A$, we have tautologically $E\left(C_A\right) = E(C)$, $\mu^A = \mu$, and $\mu$ sums to 1 by definition.

— Taking $C_u$ into account for $u \in [|2, A|]$, we have

  — For every $x \leq u - 1$, we know that

$$\mu^{u-1}_{[k_x]} = \frac{\mu^u_{[k_x]}}{1 - \mu^u_{[k_u]}}$$

  By induction hypothesis, this amounts to

$$\mu^{u-1}_{[k_x]} = \frac{\frac{\mu_{[k_x]}}{\sum_{i=1}^{u} \mu_{[k_i]}}}{1 - \frac{\mu_{[k_u]}}{\sum_{i=1}^{u} \mu_{[k_i]}}} = \frac{\mu_{[k_x]}}{\sum_{i=1}^{u} \mu_{[k_i]} - \mu_{[k_u]}}$$

  that simplifies to

$$\mu^{u-1}_{[k_x]} = \frac{\mu_{[k_x]}}{\sum_{i=1}^{u-1} \mu_{[k_i]}}$$

  — The property of reduction yields

$$E\left(C_{u-1}\right) = \frac{E\left(C_u\right)}{1 - \mu^u_{[k_u]}}$$

  By virtue of the induction hypothesis and the value of $\mu_u[k_u]$,

$$E\left(C_{u-1}\right) = \frac{\frac{E(C)}{\sum_{i=1}^{u} \mu_{[k_i]}}}{1 - \frac{\mu[k_u]}{\sum_{i=1}^{u} \mu_{[k_i]}}} = \frac{E(C)}{\sum_{i=1}^{u-1} \mu_{[k_i]}}$$

It follows from this that, taking $u = 1$,

$$E(C_1) = \frac{E(C)}{\mu_{[M(0)]}}$$

which ends the proof.

This result is not surprising, as the mean expectancy of a C-process is its expected variation divided by the required amount of time to get it : a reduction means an acceleration of time by cancellation of the steps $t \in \mathbf{N}$ where $M(t) = A_k$, which happens roughly at a fraction $\mu_{[k]}$ of time, which leads to an acceleration of $E$ by a factor $1/(1 - \mu_{[k]})$.

### Conservation of global monotonicity

When discarding drifts (i.e. $d = 0$), global monotonicity is conserved upon reductions.

**Lemma 2.8.7** *Global monotonicity and reductions*

*Let $C$ be a positive recurrent C-process, and $C_{-k}$ be $C$'s reduced process with respect to a state $A_k \neq M(0)$ and drift $0$. $C$ is globally increasing [decreasing] iff $C_{-k}$ is globally increasing [decreasing].*

As the case $C$ globally decreasing is symmetrical to the other one, we shall only consider the case of globally increasing C-processes. We may note that this symmetry holds because $C$ is positive recurrent, so $C_{-k}$ cannot produce a "by default" (no further realization of $M(t) \neq A_k$) increment $+\infty$, because $\tau$ is almost surely finite. The main idea is to remark that $C$'s and $C_{-k}$'s restricted Lévy processes are one and the same : as the binary determination sequence of the restricted Lévy process retains only times $t \in \mathbf{N}$ such that $M(t) = M(0)$, and none of them are removed by reduction with respect to a state $A_k \neq M(0)$, then their restricted Lévy processes will coincide. The lemma 2.3.2 ends the proof.

### Integrability of a reduced Laplace matrix function

Since we computed the transition probabilities and payoffs of $C$'s reduced process $C_{(-k,d)}$ in lemma 2.8.4, we may calculate its Laplace matrix function. However, this must be done with care, since $C_{(-k,d)}$ may not be exponentially integrable on the same domain as $C$. We provide a lemma indicating the domain of integrability of $C$'s $d$-restricted Lévy process, as defined by lemma 2.8.5.

**Lemma 2.8.8** *Domain of integrability*

*Let $C$ be a positive recurrent, sEI C-process, deemed not globally increasing. Let $d \in \boldsymbol{R}$ be a drift and $C^{(d)}$ be $C$'s d-restricted Lévy process. We define the function*

$$f = \begin{pmatrix} \left(\boldsymbol{R}_+^* \times \boldsymbol{R}\right) & \to & \boldsymbol{R}^+ \cup \{\infty\} \\ (\alpha, d) & \to & \lambda\left(L_{C^{(d/\alpha)}}(\alpha)\right) \end{pmatrix}$$

*where $\lambda(M)$ is $M$'s dominant eigenvalue if $M$ is well-defined, and $+\infty$ otherwise. It holds these properties :*

1. *The subset $S$ of $\left(\boldsymbol{R}_+^* \times \boldsymbol{R}\right)$ defined by $S = f^{-1}(\boldsymbol{R})$ is opened and non-empty.*
2. *Let $(\alpha, d) \in S$. For every $\alpha' \in (0, \alpha]$ and $d' \in [d, \infty)$, we also have $(\alpha', d') \in S$.*
3. *$f$ is $C^\infty$ and convex over $S$.*
4. *For every $d \in \boldsymbol{R}$, let us name*

   $$z(d) = \sup\left(\left\{\alpha \in \boldsymbol{R}_+^*; (\alpha, d) \in S\right\}\right) \in \boldsymbol{R}^+ \cup \{\infty\}$$

   *Then for every $d \in \boldsymbol{R}$,*

   $$\lim_{\alpha \to z(d)} (f(\alpha, d)) = \infty$$

To prove this lemma, we shall use the successive reductions leading to $C^{(d)}$ : reusing the notations from lemma 2.8.5, we shall name them $C_u^{(d)}$ for $u$ from $A$ down to 1. Defining $g$, the function given by

$$g = \begin{pmatrix} \mathbf{R}_+^* \times \mathbf{R} & \to & \mathbf{R}_+^* \times \mathbf{R} \\ (\alpha, d) & \to & (\alpha, d/\alpha) \end{pmatrix}$$

we are going to prove some translation of the properties for $C_u^{(d)}$ :

1. The subset $S_u$ of $\left(\mathbf{R}_+^* \times \mathbf{R}\right)$ defined by

   $$S_u = \left\{(\alpha, d) \in \mathbf{R}_+^* \times \mathbf{R}; \forall i, j \in \{k_x; x \leq u\}, \left(L_{C_u^{(d)}}(\alpha)\right)_{i,j} < \infty\right\}$$

   is opened ;
2. For every $d \in \mathbf{R}$,

   $$S_u \cap \left(\mathbf{R}_+^* \times \{d\}\right) \neq \emptyset$$

3. The entries of $L_{C_u^{(d)}}(\alpha)$ are rational fractions of $e^{\alpha d}$ and entries of $L_C(\alpha)$, being well-defined over $S_u$.
4. For every $(\alpha_1, d_1) \in g^{-1}(S_u)$, if $\alpha_2 \in (0, \alpha_1]$ and $d_2 \geq d_1$, then $(\alpha_2, d_2) \in g^{-1}(S_u)$.

We do a proof by induction on $u$ from $A$ down to 1. First, we verify the properties for $u = A$ :

— By definition, $C_A^{(d)} = C$ is sEI, so $S_A = \mathbf{R}_+^* \times \mathbf{R}$ is an opened set.

— The entries of $L_C(\alpha)$ constitute the sought rational fractions themselves.

Now, we assume the properties to be true at step $u+1$ and prove them for step $u$.

1. Let us take $u \leq A$ and $i, j \in \{k_x; x \leq u\}$. For $\alpha \in \mathbf{R}_+^*$ and $d \in \mathbf{R}$, let us note by $P_{u,i\to j}$ and $D_{u,d,i\to j}$ the transition probability and payoffs of $C_u^{(d)}$ ; we compute the Laplace matrix function thanks to the lemma 2.8.4, by

$$\begin{aligned}
&\left(L_{C_u^{(d)}}(\alpha)\right)_{i,j} \\
= \;& P_{u+1,i\to j}\mathbf{E}\left(e^{-\alpha D_{u+1,d,i\to j}}\right) \\
+ \;& \left(\begin{array}{c} P_{u+1,i\to k_{u+1}}\mathbf{E}\left(e^{-\alpha D_{u+1,d,i\to k_{u+1}}}\right) \\ \sum_{n=0}^{\infty} P_{u+1,k_{u+1}\to k_{u+1}}^n \left(\mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to k_{u+1}}}\right)\right)^n e^{-\alpha(n+1)d} \\ P_{u+1,k_{u+1}\to j}\mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to j}}\right) \end{array}\right)
\end{aligned}$$

If $(\alpha, d) \notin S_{u+1}$, there are $i, j \in \{k_x; x \leq u+1\}$ such that $\left(L_{C_{u+1}^{(d)}}(\alpha)\right)_{i,j} = \infty$.

— If $i, j \neq k_{u+1}$, then the term

$$P_{u+1,i\to j}\mathbf{E}\left(e^{-\alpha D_{u+1,d,i\to j}}\right) = \left(L_{C_{u+1}^{(d)}}(\alpha)\right)_{i,j} = \infty$$

appears in $\left(L_{C_u^{(d)}}(\alpha)\right)_{i,j}$, so $(\alpha, d) \notin S_u$.

— If $i = k_{u+1}$ and $j \neq k_{u+1}$, we use the fact $C$ is positive recurrent, so there is $i' \in \{k_x; x \leq u\}$ such that

$$P_{u+1,i'\to k_{u+1}}\mathbf{E}\left(e^{-\alpha D_{u+1,d,i\to k_{u+1}}}\right)$$

is bounded from below by some $v > 0$. This time, it is the term

$$P_{u+1,i'\to k_{u+1}}P_{u+1,k_{u+1}\to j}\mathbf{E}\left(e^{-\alpha D_{u+1,d,i'\to k_{u+1}}}\right)\mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to j}}\right)$$

that amounts to $\infty$ and appears in $\left(L_{C_u^{(d)}}(\alpha)\right)_{i',j}$, so $(\alpha, d) \notin S_u$.

— If $i \neq k_{u+1}$ and $j = k_{u+1}$, we find $j' \in \{k_x; x \leq u\}$ such that

$$P_{u+1,k_{u+1}\to j'}\mathbf{E}\left(e^{-\alpha D_{u+1,-d,k_{u+1}\to j'}}\right)$$

is bounded from below by $v > 0$ the same way, so a convenient term appears in $\left(L_{C_u^{(d)}}(\alpha)\right)_{i,j'}$.

— If $i = k_{u+1}$ and $j = k_{u+1}$, we find both $i'$ and $j'$ the same way.
Hence, we proved that if $(\alpha, d) \notin S_{u+1}$, then $(\alpha, d) \notin S_u$, from which follows that $S_u \subseteq S_{u+1}$. Now, let us take $(\alpha, d) \in S_{u+1}$. Since all entries of $L_{C^{(d)}_{u+1}}(\alpha)$ are finite by hypothesis, the only possibility of divergence comes from the sum over $n$. Hence, let us define the function $h_u$ as

$$h_u = \begin{pmatrix} S_{u+1} & \to & \mathbf{R}^+ \\ (\alpha, d) & \to & P_{u+1,k_{u+1}\to k_{u+1}} \mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to k_{u+1}}}\right) e^{-\alpha d} \end{pmatrix}$$

As $h_u$ is non-negative over $S_{u+1}$, determining whether or not $(\alpha, d) \in S_u$ is equivalent to comparing $h_u(\alpha, d)$ with 1. Here, we recall that

$$\left(L_{C^{(d)}_{u+1}}(\alpha)\right)_{k_{u+1},k_{u+1}}$$

is a rational fraction of $e^{\alpha d}$ and $L_C(\alpha)$'s entries, so the set

$$S'_{u+1} = h_u^{-1}\left((-\infty, 1)\right)$$

is opened, which means that the domain of integrability $S_u$ is $S_{u+1} \cap S'_{u+1}$, which is opened.

2. It suffices to prove that the property holds for $S'_{u+1}$. We recall that over $S_{u+1}$, $h_u(\alpha, d)$ is a rational fraction of $e^{\alpha d}$ and $L_C(\alpha)$'s entries by definition, and thanks to the induction property there is $\alpha$ such that

$$k(\alpha) = \mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to k_{u+1}}}\right) < \infty$$

$k$ being a Laplace transform, it is continuous over $(0, \alpha)$ and has a limit no higher than 1 at point 0. It follows that there is $\alpha' \in (0, \alpha)$ such that $h_u(\alpha', d) < 1$, because since $C$ is positive recurrent we have

$$P_{u+1,k_{u+1}\to k_{u+1}} < 1$$

and this $\alpha'$ solves the statement.

3. In the set $S_u$, the sum in the expression of $L_{C^{(d)}_u}(\alpha)$ converges by definition of $S'_{u+1}$. Hence, the entry number $(i, j)$ converges to

$$\begin{aligned} & \left(L_{C^{(d)}_u}(\alpha)\right)_{i,j} \\ = \; & P_{u+1,i\to j} \mathbf{E}\left(e^{-\alpha D_{u+1,d,i\to j}}\right) \\ + \; & \frac{P_{u+1,i\to k_{u+1}} \mathbf{E}\left(e^{-\alpha D_{u+1,d,i\to k_{u+1}}}\right) P_{u+1,k_{u+1}\to j} \mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to j}}\right)}{e^{\alpha d} - P_{u+1,k_{u+1}\to k_{u+1}} \mathbf{E}\left(e^{-\alpha D_{u+1,d,k_{u+1}\to k_{u+1}}}\right)} \end{aligned}$$

that rewrites to

$$\left(L_{C_u^{(d)}}(\alpha)\right)_{i,j} = \left(L_{C_{u+1}^{(d)}}(\alpha)\right)_{i,j} + \frac{\left(L_{C_{u+1}^{(d)}}(\alpha)\right)_{i,k_{u+1}}\left(L_{C_{u+1}^{(d)}}(\alpha)\right)_{k_{u+1},j}}{e^{a} - \left(L_{C_{u+1}^{(d)}}(\alpha)\right)_{k_{u+1},k_{u+1}}}$$

so it is a rational fraction of the desired form, well-defined by construction of $S_u$.

4. Let us take $(\alpha_1, d_1) \in g^{-1}(S_u)$, $\alpha_2 \in (0, \alpha_1]$ and $d_2 \geq d_1$. By construction, the matrix

$$L_{C_u^{(d_1/\alpha_1)}}(\alpha_1)$$

converges. As it is the Laplace matrix function of a concatenated process with a drift $d_1/\alpha_1$, its entries rewrite as

$$\forall i, j \in \{k_x; x \leq u\}, \left(L_{C_u^{(d_1/\alpha_1)}}(\alpha_1)\right)_{i,j} = P_{u,i\to j}\mathbf{E}\left(e^{-\alpha_1(D_{u,i\to j}+(d_1/\alpha_1)T_{u,i\to j})}\right)$$

where $T_{u,i\to j}$ is a random non-negative variable indicating the number of concatenated steps between hitting $A_i$ and $A_j$. Hence,

$$\forall i, j \in \{k_x; x \leq u\}, \left(L_{C_u^{(d_1/\alpha_1)}}(\alpha_1)\right)_{i,j} = P_{u,i\to j}\mathbf{E}\left(e^{-\alpha_1 D_{u,i\to j}}e^{-d_1 T_{u,i\to j}}\right)$$

As $T_{u,i\to j}$ is always non-negative, and Laplace transforms are convex, we have $\forall \alpha_2 \in (0, \alpha_1], d_2 \in [d_1, \infty)$,

$$\begin{aligned} & \left(L_{C_u^{(d_2/\alpha_2)}}(\alpha_2)\right)_{i,j} \\ \leq\ & P_{u,i\to j}\mathbf{E}\left(\left(1 + e^{-\alpha_1 D_{u,i\to j}}\right)e^{-d_1 T_{u,i\to j}}\right) \leq 1 + \left(L_{C_u^{(d_1/\alpha_1)}}(\alpha_1)\right)_{i,j} \\ <\ & \infty \end{aligned}$$

so $g(\alpha_2, d_2) \in S_u$.

This induction scheme allows us to state the properties for $u = 1$ :

1. The subset $S_1$ of $\left(\mathbf{R}_+^* \times \mathbf{R}\right)$ defined by

$$S_1 = \bigcap_{u=2}^{A} S'_u = \left\{(\alpha, d) \in \mathbf{R}_+^* \times \mathbf{R}; L_{C_1^{(d)}}(\alpha) < \infty\right\}$$

is opened ;
2. For every $d \in \mathbf{R}$, there is $\alpha > 0$ such that $(\alpha, d) \in S_1$ ;
3. The only entry of $L_{C_1^{(d)}}(\alpha)$ is a rational fraction of $e^{\alpha d}$ and entries of $L_C(\alpha)$, being well-defined over $S_1$.

4. If $(\alpha_1, d_1) \in g^{-1}(S_1)$, $\alpha_2 \in (0, \alpha_1]$ and $d_2 \geq d_1$, then $(\alpha_2, d_2) \in g^{-1}(S_1)$.

However, we know thanks to the lemma 2.8.5 that $L_{C_1^{(d)}}(\alpha)$ is given by $C$'s $d$-restricted Lévy process : noting by $\tau$ the random waiting time of return to $M(0)$ and $D_\tau$ the increment of $C$'s restricted Lévy process, we have

$$C_1(1) - C_1(0) = C(\tau) - C(0) = D_\tau + \tau d$$

so the function $f$ defined by the lemma is actually given by

$$\forall (\alpha, d) \in S_1, f(\alpha, d) = L_{C_1^{(d/\alpha)}}(\alpha) = \mathbf{E}\left(e^{-\alpha D_\tau} e^{-\tau d}\right)$$

We are now able to prove the given statements.

1. $S$ is given by $g^{-1}(S_1)$ ; $g$ being continuous over its domain, $S$ is an opened set.
2. This is a direct consequence of $S = g^{-1}(S_1)$ and the property obtained at step 1.
3. As $S$ is opened, for every $(\alpha, d) \in S$, we can select $(\alpha', d') \in S$ with $\alpha' > \alpha$ and $d' < d$. The use of

$$\frac{\partial^{n_1+n_2} f}{\partial 1^{n_1} \partial 2^{n_2}}(\alpha', d') = \mathbf{E}\left((-D_\tau)^{n_1} e^{-\alpha' D_\tau} (-\tau)^{n_2} e^{-\tau d'}\right)$$

allows Leibniz's integral rule to work over some opened set containing $(\alpha, d)$. In particular, with $n_1 + n_2 = 2$, one gets the Hessian matrix, whose determinant amounts to

$$\begin{aligned} H(\alpha, d) &= \det \begin{pmatrix} \frac{\partial^2 f}{\partial 1^2} & \frac{\partial^2 f}{\partial 1 \partial 2} \\ \frac{\partial^2 f}{\partial 1 \partial 2} & \frac{\partial^2 f}{\partial 2^2} \end{pmatrix} (\alpha, d) \\ &= \mathbf{E}\left(D_\tau^2 e^{-\alpha D_\tau} e^{-\tau d}\right) \mathbf{E}\left(\tau^2 e^{-\alpha D_\tau} e^{-\tau d}\right) - \mathbf{E}\left(D_\tau \tau e^{-\alpha D_\tau} e^{-\tau d}\right)^2 \end{aligned}$$

To simplify matters, let us define $(D'_\tau, \tau')$ to be an independent copy of $(D_\tau, \tau)$. We get that $H(\alpha, d)$ is

$$\begin{aligned} & \frac{1}{2}\left(\mathbf{E}\left(D_\tau^2 e^{-\alpha D_\tau} e^{-\tau d} (\tau')^2 e^{-\alpha D'_\tau} e^{-\tau' d}\right)\right) \\ + & \frac{1}{2}\left(\mathbf{E}\left((D'_\tau)^2 e^{-\alpha D'_\tau} e^{-\tau' d} \tau^2 e^{-\alpha D_\tau} e^{-\tau d}\right)\right) \\ - & \mathbf{E}\left(D_\tau \tau D'_\tau \tau' e^{-\alpha D_\tau} e^{-\tau d} e^{-\alpha D'_\tau} e^{-\tau' d}\right) \end{aligned}$$

and this simplifies to

$$2H(\alpha, d) = \mathbf{E}\left((D_\tau \tau' - D'_\tau \tau)^2 e^{-\alpha D_\tau} e^{-\tau d} e^{-\alpha D'_\tau} e^{-\tau' d}\right) \geq 0$$

It follows that, as both the trace and the determinant of $f$'s Hessian matrix are non-negative, $f$ is convex.

4. For every $u$ and $d$, let us consider

$$J_u(d) = \left\{\alpha \in \mathbf{R}_+^*; g(\alpha, d) \in S_u\right\}$$

We verify that

— These sets $J_u(d)$ are opened : indeed, if $\alpha \in J_u(d)$, then $(\alpha, d/\alpha) \in S_u$ which is opened, so one may find an opened subset of $S_u$ containing $(\alpha, d/\alpha)$ ; as $g$ is continuous, its inverse image by $g$ yields an opened subset of $J_u(d)$ containing $\alpha$.

— Let us take $\alpha_1 > \alpha_2 \in \mathbf{R}_+^*$. If $\alpha_1 \in J_u(d)$, thenwe get that $(\alpha_1, d) \in g^{-1}(S_u)$, so thanks to the above property $(\alpha_2, d) \in g^{-1}(S_u)$ and $\alpha_2 \in J_u(d)$.

— The sets $J_u(d)$ are not empty thanks to $\forall d \in \mathbf{R}, \exists \alpha \in \mathbf{R}_+^*; g(\alpha, d) \in S_u$.

It follows that for every $u$ and $d$, there is $x_u(d) \in \mathbf{R}_+^* \cup \{\infty\}$ such that

$$J_u(d) = (0, x_u(d))$$

As the sets $S_u$ are ordered by inclusion, the sequence $(x_u(d))_u$ is non-decreasing of $u$ and

$$J_1 = \bigcap_{u=1}^{A} J_u$$

By definition of $z(d)$,

$$z(d) = \sup(J_1) = \min_{u \leq A} \sup(J_u)$$

This minimum is attained for some value of $u \leq A$, and we select the largest one among them. This means that for this $u$,

$$z(d) = \sup(J_u) = x_u(d) < \sup(J_{u+1}) = x_{u+1}(d)$$

— If $z(d) < \infty$, this means that $z(d) \notin J_u$ but $z(d) \in J_{u+1}$, so $g(z(d), d) \notin S_u$ but belongs to $S_{u+1}$, which implies that it cannot belong to $S'_{u+1}$. By definition of $S'_{u+1}$, we get that

$$h_{u+1}(g(z(d), d)) \geq 1$$

but as $z(d) = \sup(J_u)$, we must have

$$\forall x < z(d), h_{u+1}(g(x, d)) < 1$$

As $h_{u+1} \circ g$ is continuous around $(z(d), d)$, then $h_{u+1}(g(z(d), d)) = 1$. Now, like when proving that an entry of $L_{C_u^{(d)}}$ diverges, we find $i', j' \in \{k_x; x \leq u\}$ such that

$$P_{u+1,i\to k_{u+1}} \mathbf{E}\left(e^{-\alpha D_{u+1,d/\alpha,i\to k_{u+1}}}\right) P_{u+1,k_{u+1}\to j'} \mathbf{E}\left(e^{-\alpha D_{u+1,d/\alpha,k_{u+1}\to j'}}\right)$$

is bounded from below when $\alpha$ goes to $z(d)$ (this is true because $z(d) < \infty$), so multiplication by

$$\sum_{n=0}^{\infty} P^n_{u+1,k_{u+1}\to k_{u+1}} \mathbf{E}\left(e^{-\alpha D_{u+1,d/\alpha,k_{u+1}\to k_{u+1}}}\right)^n e^{-nd} = \frac{1}{1 - h_u(g(\alpha, d))}$$

will yield an entry of $L_{C_u^{(d/\alpha)}}(\alpha)$ that diverges around $z(d)$. Once again, as $z(d) < \infty$, further reductions still allow for an entry that diverges here (and $L_{C_1^{(d/\alpha)}}(\alpha)$ is still well-defined for every $\alpha < z(d)$ because $z(d) = \sup(J_1)$).

— If $z(d) = \infty$, we want to prove that

$$\lim_{\alpha\to\infty} \left(\mathbf{E}\left(e^{-\alpha D_\tau} e^{-\tau d}\right)\right) = \infty$$

Recalling that $C$ is not globally increasing, $C$ has a cycle of negative value and thus there are $n \in \mathbf{N}$, $x > 0$ and $p > 0$ such that

$$\mathbf{P}\left(D_\tau \leq -x \wedge \tau = n\right) = p > 0$$

so we get

$$\mathbf{E}\left(e^{-\alpha D_\tau} e^{-\tau d}\right) \geq p e^{\alpha x} e^{-nd}$$

When $\alpha$ goes to $\infty$ at a fixed $d$, $x > 0$ solves this case.

This ends the proof.

In particular, as $f$ is continuous over $S$, and for every $d \in \mathbf{R}$ and $x > 1$, there is $\alpha > 0$ such that $(\alpha, d) \in S$ and $f(\alpha, d) < x$, the use of the intermediate value theorem proves that for every $d \in \mathbf{R}$ and $a \in \mathbf{R}_+^*$, there is $\alpha \in \mathbf{R}_+^*$ such that $(\alpha, a) \in S$ and

$$f(\alpha, d) = \mathbf{E}\left(e^{-\alpha D_\tau} e^{-\tau d}\right) = e^a$$

This property shall be used later, especially when making $d = a$.

### 2.8.3 Solution

When $d = a$, the latter property exhibits a value $\alpha$ that solves

$$f(\alpha, a) = e^a$$

We recall that $f$ is convex and goes to a limit no larger than 1 when $\alpha$ goes to 0. Hence, there may not be any other solution $\alpha$ to this equation, so we define the function $\alpha$ that maps $a \in \mathbf{R}_+^*$ to the single corresponding solution. Our next step is to ensure that

1. If $\beta \in \mathbf{R}_+^*$ solves the proposition 2.4.1, then $\beta = \alpha(a)$ ;

2. This value of $\alpha(a)$ solves the proposition 2.4.1.

So, in this paragraph, we shall set $a \in \mathbf{R}_+^*$ and assume that $\beta \in \mathbf{R}_+^*$ solves the proposition 2.4.1. Our idea is to reduce the matrix $L_C(\beta)$ until it becomes a single element, and then identify $\beta$ with $\alpha(a)$, because the solution to

$$f(\beta, a) = e^a$$

is single.

### Reduced Laplace matrix function

Now that issues of integrability have been taken care of, we may define the Laplace matrix function of a reduced process.

**Definition 2.8.3** *Reduced matrix*

*Let $L \in \boldsymbol{M}_A\left(\boldsymbol{R}^+\right)$ be a positive recurrent matrix. We define its reduced matrix with respect to a dimension number $k \leq A$ and a parameter $x > \ln\left(L_{k,k}\right)$ as the matrix $L_{(-k,x)}$, whose rows and columns are indexed by $[|1, A|] \setminus \{k\}$ in the natural order, and whose entry number $(i, j)$ for $i, j \neq k$ is*

$$\left(L_{(-k,x)}\right)_{i,j} = L_{i,j} + \frac{L_{i,k} L_{k,j}}{e^x - L_{k,k}}$$

For the sake of simplicity, we henceforth note a matrix $L \in \mathbf{M}_n(\mathbf{C})$ whose row number $r \leq n$ and column number $c \leq n$ are removed as

$$\widehat{L}^{r,c} = \left(L_{i,j}\right)_{i \in [|1,A|] \setminus \{r\}, j \in [|1,A|] \setminus \{c\}} \in \mathbf{M}_{n-1}(\mathbf{C})$$

We may now rewrite the reduction of $C$'s Laplace matrix function as the Laplace matrix function of $C$'s reduced process.

**Lemma 2.8.9** *Reduced Laplace matrix function*

*Let $C$ be a positive recurrent C-process, whose Laplace matrix function is $L_C$. Let $C_{(-k,d)}$ be $C$'s reduced process with respect to $A_k \neq M(0)$ a state to eliminate and a drift $d \in \boldsymbol{R}$. Let $\beta \in \boldsymbol{R}_+^*$ such that*

$$\left(L_C(\beta)\right)_{k,k} < e^{\beta d}$$

*Thanks to the construction from the lemma 2.8.8, $L_{C_{(-k,d)}}(\beta)$ is well-defined.*

1. *For every $(\beta, d) \in S_k$, $L_{C_{(-k,d)}}(\beta)$ is the reduced matrix of $L_C(\beta)$ with respect to the dimension $k$ and the parameter $\beta d$, i.e.*

$$L_{C_{(-k,d)}}(\beta) = \left(L_C(\beta)\right)_{(-k,\beta d)}$$

2. *Noting by $H$ the following row-vector and $V$ the following column-vector :*

$$H = \left((L_C(\beta))_{k,j}\right)_{j\neq k} \wedge V = \left((L_C(\beta))_{i,k}\right)_{i\neq k}$$

*the previous sentence rewrites as*

$$L_{C_{(-k,d)}}(\beta) = \widehat{L_C(\beta)}^{k,k} + \frac{VH}{e^{\beta d} - (L_C(\beta))_{k,k}}$$

This is a consequence of the formula

$$\begin{aligned} & \left(L_{C_{(-k,d)}}(\beta)\right)_{i,j} \\ = \ & P_{i\to j}\mathbf{E}\left(e^{-\beta D_{i\to j}}\right) \\ + \ & (P_{i\to k}P_{k\to j})\sum_{n=0}^{\infty} P_{k\to k}^n \mathbf{E}\left(e^{-\beta D_{i\to k}}\right)\left(\mathbf{E}\left(e^{-wD_{k\to k}}\right)\right)^n \mathbf{E}\left(e^{-wD_{k\to j}}\right)e^{-\beta(n+1)d} \end{aligned}$$

and the fact that by definition of $L_{C[d]}(\beta)$,

$$\sum_{n=0}^{\infty}\left(P_{k\to k}\mathbf{E}\left(e^{-\beta D_{k\to k}}\right)e^{-\beta d}\right)^n = \frac{1}{e^{\beta d} - e^{-\beta d}\left(L_{C[d]}(\beta)\right)_{k,k}}$$

The idea is to state that as $e^a$ is the dominant eigenvalue of $C$'s Laplace matrix function at point $\beta \in \mathbf{R}_+^*$ by hypothesis, it is still the dominant eigenvalue of $C_{(-k,d)}$'s Laplace matrix function at the same point if one chooses $d = a/\beta$.

**Lemma 2.8.10** *Conservation of the dominant eigenvalue*

*Let $C$ be a positive recurrent C-process, and $a \in \boldsymbol{R}_+^*$, $\beta \in \boldsymbol{R}_+^*$ such that $L_C(\beta)$ is well-defined and its dominant eigenvalue is $e^a$. Let $A_k \neq M(0)$ be a state of $C$'s underlying Markovian process $M$ and $C_{(-k,a/\beta)}$ be $C$'s reduced C-process with respect to $A_k$ and drift $a/\beta$.*

1. *Its Laplace matrix function at point $\beta$ is well-defined.*
2. *$e^a$ is also the dominant eigenvalue of $L_{C_{(-k,a/\beta)}}(\beta)$.*

To prove this lemma, we will separately prove that

1. $L_{C_{(-k,a/\beta)}}(\beta)$ is well-defined ;
2. $e^a$ remains a eigenvalue for it ;
3. If $L_{C_{(-k,a/\beta)}}(\beta)$ has an eigenvalue greater than $e^a$, so does $L_C(\beta)$.

If all properties are true, the lemma 2.8.10 will follow.

#### Conservation of the dominant eigenvalue

First we verify that $L_{C_{(-k,a/\beta)}}(\beta)$ is well-defined. We know by hypothesis that $L_C(\beta)$'s dominant eigenvalue is $e^a$, so as $L_C(\beta)$ is positive recurrent, its diagonal entries must be less than $e^a$. Hence, the corresponding term $h_u(\beta, a/\beta)$ from the proof is lower than 1 ; as it was the only condition for integrability, the reduced matrix is well-defined. To prove that every $\beta \in \mathbf{R}$ such that $e^a$ is the dominant eigenvalue of $L_C(\beta)$ also gives $e^a$ as an eigenvalue of $L_{C_{(-k,a/\beta)}}(\beta)$, we are going to state this intermediate result.

**Lemma 2.8.11** *Determinant of a reduced matrix*
*Let $L \in \boldsymbol{M}_A\left(\boldsymbol{R}^+\right)$ be a positive recurrent matrix. Let $k \leq A$ be a state number, with $A \geq 2$. For every $x > \ln(L_{k,k})$,*

$$\det\left(e^x Id - L\right) = \det\left(e^x Id - L_{(-k,x)}\right)\left(e^x - L_{k,k}\right)$$

We write down $L_{(-k,x)}$ using the vectors $H$ and $V$ denoting $L$'s removed row and column, like in lemma 2.8.9. We develop the determinant with respect to row number $k$, which leads to

$$\begin{aligned}\det\left(e^x Id - L\right) &= \left(e^x - L_{k,k}\right)\det\left(e^x Id - \widehat{L_C(\beta)}^{k,k}\right) \\ &+ \sum_{u\neq k}(-1)^{u-k} H_u \det\left(e^x \widehat{Id}^{k,u} - \widehat{L_C(\beta)}^{k,u}\right)\end{aligned}$$

As the determinant is multilinear and anti-symmetric, swapping the column $k$ with the skipped column $u$ in the rightmost matrix has a signature $(-1)^{k-u+1}$. Let us note by $X_u$ the matrix $\widehat{L}^{k,k}$ modified as follows :

— Its column number $u$ is replaced by $V$ ;
— After this, we add $e^x$ to its entry number $(u,u)$, to “cancel” the entry $(u,u)$ of the incoming identity matrix.

This gives

$$\det\left(e^x Id - L\right) = \left(e^x - L_{k,k}\right)\det\left(e^x Id - \widehat{L}^{k,k}\right) - \sum_{u\neq k} H_u \det\left(e^x Id - X_u\right)$$

Now, we deal with $L_{(-k,x)}$.

— First, $L_{k,k} < e^x$ by hypothesis.
— The determinant is multilinear and anti-symmetric, and columns proportionnal to $V$ are added to $\widehat{L}^{k,k}$ to get $L_{(-k,x)}$.

So, we get

$$\det\left(e^x Id - \widehat{L}^{k,k} - \frac{VH}{e^x - L_{k,k}}\right) = \det\left(e^x Id - \widehat{L}^{k,k}\right) - \sum_{u\neq k}\frac{H_u}{e^x - L_{k,k}}\det\left(e^x Id - X_u\right)$$

The lemma 2.8.11 follows. The idea is now to apply this lemma to $L = L_C(\beta)$. Since $e^a$ is its dominant eigenvalue, its entry number $(k,k)$ is lower than $e^a$ because $C$ is positive recurrent, thus

$$\det\left(e^a Id - L_C(\beta)\right) = \det\left(e^a Id - \left(L_C(\beta)\right)_{(-k,a)}\right)\left(e^a - L_{k,k}\right)$$

Since the determinant is 0 by hypothesis and $e^a > L_{k,k}$, so

$$\det\left(e^a Id - \left(L_C(\beta)\right)_{(-k,a)}\right) = 0$$

However, we recall after the lemma 2.8.9 that

$$L_{C_{(-k,d)}}(\beta) = \left(L_C(\beta)\right)_{(-k,\beta d)}$$

so setting $d = a/\beta$ yields

$$\det\left(e^a Id - L_{C_{(-k,a/\beta)}}(\beta)\right) = 0$$

which means that $e^a$ is an eigenvalue for $L_{C_{(-k,a/\beta)}}(\beta)$.

### Dominant eigenvalue

We want to ensure that the eigenvalue $e^a$ is still dominant for $L_{C_{(-k,a/\beta)}}(\beta)$. As Perron-Frobenius' theorem ensures that its dominant eigenvalue is non-negative, we only need to prove that $L_{C_{(-k,a/\beta)}}(\beta)$ has no eigenvalue $\lambda > e^a$. By contradiction, we shall deem that $w'$ is an eigenvector of $L_{C_{(-k,a/\beta)}}(\beta)$ with an associated eigenvalue $\lambda > e^a$, and use it to build a vector $w \in \mathbf{R}^A$ such that the sequence

$$\left(\left(e^{-a} L_C(\beta)\right)^n w\right)_{n\in\mathbf{N}}$$

geometrically diverges, which will indicate that $L_C(\beta)$ has an eigenvalue greater than $e^a$. First, as $C_{(-k,a/\beta)}$ is still positive recurrent by construction (lemma 2.8.4), then $L_{C_{(-k,a/\beta)}}(\beta)$ is a positive recurrent matrix whenever defined, thanks to the lemma 2.7.10, and Perron-Frobenius' theorem states that we may take $w'$ with all coordinates being positive. We start with the eigenvector equation $L_{C_{(-k,a/\beta)}}(\beta)w' = \lambda w'$. By definition of the reduced matrix, we get

$$\widehat{L_C(\beta)}^{k,k} w' + \frac{1}{e^a - \left(L_C(\beta)\right)_{k,k}} V H w' = \lambda w'$$

Let $w$ be the vector $w'$ with the additionnal coordinate

$$w_{[k]} = z = \frac{Hw'}{e^a - \left(L_C(\beta)\right)_{k,k}}$$

Then we rewrite $L_C(\beta)w$ decomposing the product between terms issued from column number $k$ and other columns :

— At row number $k$, we get

$$(L_C(\beta)w)_{[k]} = Hw' + (L_C(\beta))_{k,k}\, z = ze^a$$

— At other rows, considered as a whole as a vector,

$$\left((L_C(\beta)w)_{[i]}\right)_{i\neq k} = \widehat{L_C(\beta)}^{k,k} w' + Vz$$

However, we know by construction that

$$\widehat{L_C(\beta)}^{k,k} w' = L_{C_{(-k,a/\beta)}}(\beta)w' - \frac{1}{e^a - (L_C(\beta))_{k,k}} VHw' = L_{C_{(-k,a/\beta)}}(\beta)w' - Vz$$

So, the eigenvalue equation leads to

$$\left((L_C(\beta)w)_{[i]}\right)_{i\neq a} = L_{C_{(-k,a/\beta)}}(\beta)w' = \lambda w'$$

Since $\lambda > e^a$, we get that $L_C(\beta)w$ is $we^a$ plus a non-zero vector $y \in \left(\mathbf{R}^+\right)^A$, which rewrites as

$$L_C(\beta)w = we^a + y$$

Now, let $n \in \mathbf{N}$, so the previous equation leads to

$$(L_C(\beta))^n w = we^{na} + \sum_{k=0}^{n-1} (L_C(\beta))^k y$$

Since $y \neq 0$ is non-negative, it has a $q \leq A$ such that $y_{[q]} > 0$. However, we recall that $C$ is positive recurrent, so thanks to lemma 2.7.10, for any $i, j \leq A$, there is $n_{i,j} \in \mathbf{N}^*$ such that

$$((L_C(\beta))^{n_{i,j}})_{i,j} > 0$$

Hence, set any $i \leq A$ and $j = q$ : the product

$$((L_C(\beta))^{n_{i,j}})_{i,q}\, y$$

will yield a nonnegative vector whose coordinate number $i$ is positive. Hence, setting

$$n = 1 + \max\left(\{n_{i,q}; i \leq A\}\right)$$

the above sum will evaluate to a vector $y' \in \left(\mathbf{R}_+^*\right)^A$, which leads in turn to $c > 0$ and a nonnegative vector $y''$ such that

$$(L_C(\beta))^n w = (e^{na} + c)w + y''$$

Finally, we find out that the sequence

$$((L_C(\beta))^{nu}\, w)_{u\in\mathbf{N}}$$

grows at least at speed $\left(\sqrt[n]{e^{na} + c}\right)^u$, so the dominant eigenvalue of $L_C(\beta)$ is greater than $e^a$. By contradiction, we proved that the spectral radius of $L_{C_{(-k,a/\beta)}}(\beta)$ cannot be greater than $e^a$, which means that $e^a$ is still the its dominant eigenvalue.

**Result of successive reductions**

Assuming that $e^a$ was $L_C(\beta)$'s dominant eigenvalue, we found out that the term $L_{C_{(-k,a/\beta)}}(\beta)$ is well-defined and $e^a$ is its dominant eigenvalue. It follows that we may start over with $C_{(-k,a/\beta)}$ instead of $C$, as it is still positive recurrent thanks to the lemma 2.8.4. We eliminate successively all states of $M$ other than $M(0)$ : starting from $C$, whose Laplace matrix function has $e^a$ as a dominant eigenvalue at point $\beta$, we

1. Select any state $A_k$ of its underlying Markovian process other than $M(0)$ ;
2. Build its reduced C-process $C_{(-k,a/\beta)}$ with respect to $A_k$ : thanks to the above remark, it is still positive recurrent and its Laplace matrix function has $e^\beta$ as a dominant eigenvalue ;
3. Start again, until only the state $M(0)$ remains.

The final reduced C-process is thus $C$'s $(a/\beta)$-restricted Lévy process $C^{(a/\beta)}$ thanks to the lemma 2.8.5, whose increments are written as

$$D_\tau + (a/\beta)\tau$$

like above. Thanks to the lemma 2.8.10, the term

$$\mathbf{E}\left(e^{-\beta D_\tau}e^{-a\tau}\right)$$

is well-defined and amounts to $e^\beta$, so $\beta$ solves $f(\beta, a) = e^a$ ; as $\alpha(a)$ is the single solution to this equation, then $\beta = \alpha(a)$. Hence, we proved that the eigenvector equation has at most one solution $\alpha(a)$ ; as the lemma 2.8.1 assures the existence, we proved that $\alpha(a)$ is the one and only solution, which ends the proof of the first statement of the proposition 2.4.1 for $a \in \mathbf{R}_+^*$.

**Eigenspaces of the martingale parameter**

Now that the martingale parameter of a C-process has been defined, we focus on its properties as given through the study. To prove the proposition 2.4.1, we focus on the eigenspaces spanned by the dominant eigenvalue $e^a$ of the Laplace matrix function $L_C(\alpha(a))$ at point $a \in \mathbf{R}_+^*$.

— Dimension of the column eigenspace : since $C$ is positive recurrent, the lemma 2.7.10 ensures that $L_C(\alpha)$ is a positive recurrent matrix no matter $\alpha \in \mathbf{R}_+^*$. It follows from Perron-Frobenius' theorem that the dominant eigenspace is one-dimensional and directed by a positive vector.

— The row eigenspace holds the same properties a similar way.

— The scaling may be chosen as desired provided that the dot products of vectors in the considered eigenspaces are not 0, which is true because these eigenvectors may be chosen positive.

Finally, as the eigenspaces are one-dimensional and the affine equations of scaling are not collinear with them as proved above, the solutions $w^{(a)}$ and $\mu^{(a)}$ are unique. Hence, we proved the existence of the items from the proposition 2.4.1 for $a > 0$, and the case $a = 0$ will be discussed during the paragraph 2.8.5.

**Martingale process**

To prove that the martingale process $X_C^{(a)}$ from definition 2.4.1 really is a martingale for $a \in \mathbf{R}_+^*$., we may now compute, for any $t \in \mathbf{N}$,

$$\mathbf{E}\left(X_C^{(a)}(t+1)|\mathbf{F}(t)\right)$$

By construction of $X_C^{(a)}$, this is

$$\mathbf{E}\left(X_C^{(a)}(t+1)|\mathbf{F}(t)\right) = \mathbf{E}\left(w_{[M(t+1)]}^{(a)} e^{-\alpha(a)C(t+1)}|\mathbf{F}(t)\right) e^{-(t+1)a}$$

By definition of a C-process, we get (conditionning over $M(t+1)$)

$$\mathbf{E}\left(X_C^{(a)}(t+1)|\mathbf{F}(t)\right) = \sum_{k=1}^{A} P_{M(t)\to k} w_{[k]}^{(a)} \mathbf{E}\left(e^{-\alpha(a)\left(C(t)+D_{M(t)\to k}\right)} e^{-at}|\mathbf{F}(t)\right) e^{-a}$$

that simplifies to

$$\mathbf{E}\left(X_C^{(a)}(t+1)|\mathbf{F}(t)\right) = e^{-at} e^{-\alpha(a)C(t)} \sum_{k=1}^{A} P_{M(t)\to k} \mathbf{E}\left(e^{-\alpha(a)D_{M(t)\to k}}\right) w_{[k]}^{(a)} e^{-a}$$

We recognize the sum as the entry number $M(t)$ of the vector $L_C(\alpha(a))w^{(a)}$. However, since $w^{(a)}$ is an eigenvector of $L_C(\alpha(a))$ associated with the eigenvalue $e^a$ by definition of $\alpha(a)$, this equation simplifies to

$$\mathbf{E}\left(X_C^{(a)}(t+1)|\mathbf{F}(t)\right) = e^{-at} e^{-\alpha(a)C(t)} w_{[M(t)]}^{(a)} = X_C^{(a)}(t)$$

This ends the proof for $a > 0$, while the case $a = 0$ will be discussed during the paragraph 2.8.5.

### 2.8.4 Regularity of the dominant eigenvectors

In this paragraph, we prove that the martingale parameter $\alpha$ and the dominant eigenvectors of a C-process are $C^\infty$ functions over $\mathbf{R}_+^*$. The method we shall use is

1. Verify that $\alpha$ is continuous, as an inverse function of $L_C(\alpha)$'s dominant eigenvalue ;

2. Use $\alpha$'s continuity to apply the implicit functions theorem, expressing it through the equation governing $C$'s restricted Lévy processes, so $\alpha$ is $C^\infty$ ;
3. Verify that $\mu^{(a)}$ and $w^{(a)}$, viewed as functions of $a$, are locally bounded ;
4. Use this fact to get continuity over $\mathbf{R}_+^*$ ;
5. Like previously, apply the implicit functions theorem and get that they are $C^\infty$.

In particular, we shall use $\alpha$'s second derivative to verify that is is concave.

**Dominant eigenvalue**

For a non-negative matrix $L \in \mathbf{M}_n\left(\mathbf{R}^+\right)$, let us consider $L$'s dominant eigenvalue : it is a continuous expression of $L$'s entries (in $\mathbf{R}^+$), and it is positive as soon as $L$ is positive recurrent. For every $\beta \in \mathbf{R}_+^*$, $L_C(\beta)$ is a positive recurrent matrix thanks to the lemma 2.7.10, so we may define $\lambda(\beta)$ to be the logarithm of $L_C(\beta)$'s dominant eigenvalue : $\lambda$ is a continuous function $\left(\mathbf{R}_+^* \to \mathbf{R}\right)$. Let us look at the set $X = \lambda^{-1}\left(\mathbf{R}_+^*\right) \subseteq \mathbf{R}_+^*$. We know that $\lambda$ is bijective over $X$ thanks to the proposition 2.4.1, which indicates that

— $X$ is a convex set : indeed, if $a < b \in X$, with e.g. $\lambda(a) \leq \lambda(b)$, and $c \in (a, b) \setminus X$, the intermediate value theorem creates a value $x \in [c, b]$ such that $\lambda(x) = \lambda(a) > 0$, which contradicts $\lambda$'s injectivity.

— $X$ is an opened set, being the inverse image of $\mathbf{R}_+^*$ by the continuous function $\lambda$ ;

— $\lambda$ is not bounded from above (this is the lemma 2.8.1), so $X$ must contain arbitrarily large values and is a non-empty interval.

It follows that there is $\alpha_0 \in \mathbf{R}^+$ such that $X = (\alpha_0, \infty)$, so $\lambda$ is bijective over this set onto $\mathbf{R}_+^*$ ; being continuous, and not decreasing (else we would get $\lambda\left(\alpha_0\right) = \infty$, which is incompatible with $X$'s definition and $C$ sEI), it must be increasing. Therefore, the martingale parameter $\alpha$, being $\lambda$'s inverse function $\left(\mathbf{R}_+^* \to X\right)$, is also continuous and increasing.

**Implicit function**

We recall that $\alpha(a)$ is the only solution to the equation $h(a, \alpha(a)) = 0$, where $f$ is the function defined by

$$f = \begin{pmatrix} S & \to & \mathbf{R} \\ (a, \beta) & \to & \mathbf{E}\left(e^{-\beta D_\tau} e^{-\tau a}\right) \end{pmatrix}$$

$h$ is the function defined by

$$h = \begin{pmatrix} S & \to & \mathbf{R} \\ (a, \beta) & \to & f\left(a, \beta\right) - e^a \end{pmatrix}$$

and $S$ is the opened set coming from the lemma 2.8.8. We also know that $h$ is $C^\infty$ over $S$, which allows us to apply the implicit function theorem as follows.

— Let us start from $a \in \mathbf{R}_+^*$. We know by the proposition 2.4.1 that there is a single $\alpha(a) \in \mathbf{R}_+^*$, with $(a, \alpha(a)) \in S$, such that $h(a, \alpha(a)) = 0$.
— We compute $\partial h / \partial 2$ at this point $(a, \alpha(a))$.

$$\forall a \in \mathbf{R}_+^*, \frac{\partial h}{\partial 2}(a, \alpha(a)) = \frac{\partial f}{\partial 2}(a, \alpha(a)) = \mathbf{E}\left(-D_\tau e^{-\alpha(a) D_\tau} e^{-\tau a}\right)$$

However, at a fixed $a \in \mathbf{R}_+^*$, we know that
— $f$ is a convex function so $\partial f / \partial 2$ must be non-decreasing of $\alpha$ ;
— $f$ goes to a limit no higher than 1 when $\beta$ goes to 0, and $f(a, \alpha(a)) = e^a$ is larger than 1.

The mean value theorem indicates that $\partial f / \partial 2$ must be positive somewhere in the interval $(0, \alpha(a))$, so

$$\frac{\partial h}{\partial 2}(a, \alpha(a)) > 0$$

Hence, the implicit function theorem states that there are
— Opened sets $U_a \ni a$, $V_a \ni \alpha(a)$ ;
— A single function $h_a \in C^\infty(U_a \to V_a)$,

such that the following subsets of $U_a \times V_a$ coincide :

$$\{(x, y) \in U_a \times V_a; h(x, y) = 0\} = \{(x, h_a(x)); x \in U_a\}$$

We recall that $\alpha$ is continuous and solves $h(a, \alpha(a)) = 0$, so there is an opened set $U'_a \ni a$ such that $\alpha(a) \subseteq V_a$. Hence,

$$\{(x, \alpha(x)); x \in U'_a\} \subseteq \{(x, h_a(x)); x \in U_a\}$$

so $\alpha(x)$ and $h_a(x)$ must coincide for $x \in U'_a$. As $h_a$ is $C^\infty$, it follows that $\alpha$ is $C^\infty$ around $a$ ; this being for every $a \in \mathbf{R}_+^*$, $\alpha$ is $C^\infty$ over $\mathbf{R}_+^*$.

### Spread of $w^{(a)}$

We want to prove that $\mu^{(a)}$ and $w^{(a)}$ are locally bounded. We know that $\mu^{(a)}$'s coordinates are bounded by 1 thanks to the first equation of scaling ; to get a similar property for $w^{(a)}$, we make use of its spread introduced by the definition 2.4.2.

**Lemma 2.8.12** *Spread of C's dominant eigenvector*

*Let $C$ be a positive recurrent, sEI, not globally ingreasing C-process. For every $a \in \boldsymbol{R}^+$, we define $C$'s dominant eigenvector $w^{(a)}$ as in proposition 2.4.1. There are constants $c \in \boldsymbol{R}^+$ and $n \in \boldsymbol{N}$ such that*

$$\forall a \in \boldsymbol{R}^+, \delta\left(w^{(a)}\right) \leq c + na$$

*As a consequence, $w^{(a)}$'s coordinates are bounded by $e^{c+na}$.*

Let us start from the eigenvector equation

$$L_C\left(\alpha(a)\right) w^{(a)} = e^a w^{(a)}$$

Thanks to the proposition 2.3.5, the $n^{th}$ power of $C$'s Laplace matrix function yields

$$\forall i \leq A, \sum_{j=1}^{A} P_{i \xrightarrow{n} j} \mathbf{E}\left(e^{-\alpha(a) D_{i \xrightarrow{n} j}}\right) w^{(a)}_{[j]} = e^{na} w^{(a)}_{[i]}$$

In particular, we get that

$$\forall i, j \leq A, n \in \mathbf{N}, \frac{w^{(a)}_{[i]}}{w^{(a)}_{[j]}} \geq e^{-na} P_{i \xrightarrow{n} j} \mathbf{E}\left(e^{-\alpha(a) D_{i \xrightarrow{n} j}}\right)$$

Since $C$ is not globally increasing, we know that for every $i, j \leq A$, there is a path from $A_i$ to $A_j$ whose value is $-v < 0$ and length is $n_{i,j} \in \mathbf{N}$ ; noting by $p$ the probability of following it at precision $v$, we have $p > 0$ and

$$\forall a \in \mathbf{R}^+, P_{i \xrightarrow{n_{i,j}} j} \mathbf{E}\left(e^{-\alpha(a) D_{i \xrightarrow{n_{i,j}} j}}\right) \geq P_{i \xrightarrow{n_{i,j}} j} \mathbf{E}\left(e^{-\alpha(a) D_{i \xrightarrow{n_{i,j}} j}} \mathbf{1}_{D_{i \xrightarrow{n_{i,j}} j} \leq 0}\right) \geq p$$

Thus we have

$$\forall i, j \leq A, n \in \mathbf{N}, \frac{w^{(a)}_{[i]}}{w^{(a)}_{[j]}} \geq p e^{-a n_{i,j}}$$

and this rewrites as

$$\forall i, j \leq A, n \in \mathbf{N}, \ln\left(\frac{w^{(a)}_{[j]}}{w^{(a)}_{[i]}}\right) \leq -\ln(p) + a n_{i,j}$$

Taking $c = -\ln(p)$ and

$$n = \max_{i,j \leq A}\left(n_{i,j}\right)$$

solves this lemma. Now, to bound $w^{(a)}$, we know that there is always $i \leq A$ such that $w^{(a)}_{[i]} \leq 1$, as else

$$\mu^{(a)}_{[i]} w^{(a)}_{[i]} > \mu^{(a)}_{[i]}\left(\vec{1}\right) = 1$$

By definition of the spread, since we have

$$\forall j \leq A, \ln\left(\frac{w_{[j]}^{(a)}}{w_{[i]}^{(a)}}\right) \leq c + an$$

then we get the desired inequality for every $w_{[j]}^{(a)}$, which ends the proof.

**Continuity**

Let $a, b \in \mathbf{R}_+^*$. We start with the equality

$$\left(\mu^{(b)} - \mu^{(a)}\right)(L_C(\alpha(a)) - e^a Id) = \mu^{(b)}\left(e^b - e^a\right) - \mu^{(b)}\left(L_C(\alpha(a)) - L_C(\alpha(b))\right)$$

that one may verify, using

$$\mu^{(a)}\left(L_C(\alpha(a)) - e^a Id\right) = \mu^{(b)}\left(L_C(\alpha(b)) - e^b Id\right) = 0$$

by construction. When $b$ converges to $a$, the terms $e^b - e^a$ and $L_C(\alpha(a)) - L_C(\alpha(b))$ go to zero because $\alpha$ and $L_C$ are continuous ; as $\mu^{(b)}$ is locally bounded (by 1), we get

$$\lim_{b\to a}\left(\left(\mu^{(b)} - \mu^{(a)}\right)(L_C(\alpha(a)) - e^a Id)\right) = 0$$

We also know that

$$\left(\mu^{(b)} - \mu^{(a)}\right)\left(\vec{1}\right) = 0$$

by the equation of scaling. Hence, let us look at the linear function

$$F_a = \left(\begin{array}{ccc} \mathbf{R}^A & \to & \mathbf{R}^A \times \mathbf{R} \\ x & \to & \left(x\left(L_C(\alpha(a)) - e^a Id\right), x\left(\vec{1}\right)\right) \end{array}\right)$$

It is injective, because if $x \in \mathbf{R}^A$ is such that

$$x\left(L_C(\alpha(a)) - e^a Id\right) = 0$$

then $\exists k \in \mathbf{R}; x = k\mu^{(a)}$, so the second part of $F_a(x)$ amounts to $k\mu^{(a)}\left(\vec{1}\right) = k$ by the first equation of scaling : if it is also 0, then $x = 0$. However, we proved that

$$\lim_{b\to a}\left(F_a\left(\mu^{(b)} - \mu^{(a)}\right)\right) = 0$$

so $\mu^{(b)}$ must go to $\mu^{(a)}$ ; this being for every $a \in \mathbf{R}_+^*$, $\mu$ as a function is continuous over $\mathbf{R}_+^*$. The similar property for $w$ comes from the equality

$$\left(L_C(\alpha(a)) - e^a Id\right)\left(w^{(b)} - w^{(a)}\right) = \left(e^b - e^a\right) w^{(b)} - \left(L_C(\alpha(a)) - L_C(\alpha(b))\right) w^{(b)}$$

that one may verify, using

$$(L_C(\alpha(a)) - e^a Id)\, w^{(a)} = \left(L_C(\alpha(b)) - e^b Id\right) w^{(b)} = 0$$

by construction. Likewise, as $w^{(b)}$ is locally bounded (as proved above), we get

$$\lim_{b \to a} \left((L_C(\alpha(a)) - e^a Id) \left(w^{(b)} - w^{(a)}\right)\right) = 0$$

We also know that

$$\mu^{(a)} \left(w^{(b)} - w^{(a)}\right) = \left(\mu^{(a)} - \mu^{(b)}\right) w^{(b)}$$

so it goes to 0 when $b$ goes to $a$, and this time we use the function

$$G_a = \left( \begin{array}{ccc} \mathbf{R}^A & \to & \mathbf{R}^A \times \mathbf{R} \\ x & \to & \left((L_C(\alpha(a)) - e^a Id)\, x, \mu^{(a)} x\right) \end{array} \right)$$

that is injective thanks to the second equation of scaling, on $x = w^{(b)} - w^{(a)}$, with a similar ending.

**Differentiability**

We introduce the function

$$f = \left( \begin{array}{ccc} \mathbf{R}_+^* \times \mathbf{R}^A \times \mathbf{R}^A & \to & \mathbf{R}^A \times \mathbf{R}^A \\ (a, x, y) & \to & \left(x L_C(\alpha(a)) - x \left(\vec{1}\right) x e^a, L_C(\alpha(a)) y - y x y e^a\right) \end{array} \right)$$

N.B.: $x$ is a row vector and $y$ is a column vector in this definition. As $\alpha$ is $C^\infty$ as we proved before, $f$ itself is $C^\infty$ over its domain, so we consider $f$'s Jacobian matrix : we shall note by $\partial f_x/\partial x$, $\partial f_x/\partial y$, $\partial f_y/\partial x$ and $\partial f_y/\partial y$ its sub-matrices $(A \times A)$ related to

- For $f_x$, the first $A$ coordinates of $f$, and $f_y$ are the other ones ;
- For $\partial x$, differentiation with respect to the $A$ coordinates of $x$, and $\partial y$ for $y$'s ones.

Computations lead to the block related with $x$

$$\forall x, y \in \mathbf{R}^A, \frac{\partial f_x}{\partial x}(x, y) = (L_C(\alpha(a)))^* - x \left(\vec{1}\right) e^a Id - \left(\left(\vec{1}\right) x e^a\right)^*$$

the block related with $y$

$$\forall x, y \in \mathbf{R}^A, \frac{\partial f_y}{\partial y}(x, y) = L_C(\alpha(a)) - x y e^a Id - y x e^a$$

but also to

$$\forall x, y \in \mathbf{R}^A, \frac{\partial f_x}{\partial y}(x,y) = 0$$

which means that the restriction of $f$'s Jacobian matrix related to derivatives with respect to $(x, y)$ will be invertible as soon as both matrices $\partial f_x/\partial x$ and $\partial f_y/\partial y$ are. Taking them at a point $x = \mu^{(a)}$ and $y = w^{(a)}$ leads to

$$\frac{\partial f_x}{\partial x}\left(\mu^{(a)}, w^{(a)}\right) = \left(L_C(\alpha(a))\right)^* - \mu^{(a)}\left(\vec{1}\right)e^a Id - \left(\left(\vec{1}\right)\mu^{(a)}e^a\right)^*$$

and

$$\frac{\partial f_y}{\partial y}\left(\mu^{(a)}, w^{(a)}\right) = L_C(\alpha(a)) - \mu^{(a)}w^{(a)}e^a Id - w^{(a)}\mu^{(a)}e^a$$

However, the equations of scaling simplify the previous equations to

$$\begin{aligned} & \frac{\partial f_x}{\partial x}\left(\mu^{(a)}, w^{(a)}\right) = \left(L_C(\alpha(a)) - e^a Id - \left(\vec{1}\right)\mu^{(a)}e^a\right)^* \\ \wedge \quad & \frac{\partial f_y}{\partial y}\left(\mu^{(a)}, w^{(a)}\right) = L_C(\alpha(a)) - e^a Id - w^{(a)}\mu^{(a)}e^a \end{aligned}$$

Now, we prove that the matrix

$$L_C(\alpha(a)) - e^a Id - \left(\vec{1}\right)\mu^{(a)}e^a$$

is injective, thus invertible. If there is a row vector $v \in \mathbf{R}^A$ such that

$$v\left(L_C(\alpha(a)) - e^a Id\right) = e^a v\left(\vec{1}\right)\mu^{(a)}$$

then we have

— Either $v\left(\vec{1}\right) = 0$, thus $vL_C(\alpha(a)) = e^a v$ so $v$ belongs to $L_C(\alpha(a))$'s dominant row eigenspace (which is $\mu^{(a)}\mathbf{R}$), but then $v\left(\vec{1}\right) = 0$ implies $v = 0$ because $\mu^{(a)}\left(\vec{1}\right) \neq 0$.

— Or there is $v' \in \mathbf{R}^A$ such that

$$\mu^{(a)} = v'\left(L_C(\alpha(a)) - e^a Id\right)$$

and by $\mu^{(a)}$'s definition of an eigenvector, we get by right multiplication that

$$v'\left(L_C(\alpha(a)) - e^a Id\right)^2 = 0$$

As $e^a$'s is an eigenvalue of order 1, this is possible only if already

$$v'\left(L_C(\alpha(a)) - e^a Id\right) = 0$$

so $\mu^{(a)} = 0$, which is impossible.

Likewise, the case of the matrix

$$L_C(\alpha(a)) - e^a Id - w^{(a)}\mu^{(a)}e^a$$

is solved when looking for a right kernel, through the use of the other equation of scaling. Hence, the restriction of $f$'s Jacobian matrix related to derivatives with respect to $(x, y)$ at point $\left(\mu^{(a)}, w^{(a)}\right)$ is invertible : as

$$f\left(a, \mu^{(a)}, w^{(a)}\right) = 0$$

by definition of $\mu^{(a)}$ and $w^{(a)}$, one may apply the implicit functions theorem : there are

— Opened sets $U_a \subseteq \mathbf{R}_+^*$ containing $a$, $V_a \subseteq \mathbf{R}^A$ containing $\mu^{(a)}$, and $W_a \subseteq \mathbf{R}^A$ containing $w^{(a)}$ ;

— A single function $g_a$ being $C^\infty(U_a \to V_a)$ and a single function $h_a$ being $C^1(U_a \to W_a)$

such that

$$\{(z, g_a(z), h_a(z))\,; z \in U_a\} = \{(z, x, y) \in U_a \times V_a \times W_a; f(z, x, y) = 0\}$$

Once again, as $\left(a, \mu^{(a)}, w^{(a)}\right)$ cancels $f$, and $\mu^{(a)}$ and $w^{(a)}$ are continuous of $a$ (as we proved above), there is an opened set $U'_a$ such that $\mu^{(a)}(U'_a) \subseteq V_a$ and $w^{(a)}(U'_a) \subseteq W_a$. It follows that

$$\left\{\left(z, \mu^{(z)}, w^{(z)}\right); z \in U'_a\right\} \subseteq \{(z, g_a(z), h_a(z))\,; z \in U_a\}$$

so $\mu^{(z)}$ and $g_a(z)$, as well as $w^{(z)}$ and $h_a(z)$, coincide over $U'_a$ that contains $a$ ; as $g_a$ and $h_a$ are $C^\infty$ over $U'_a$, it follows that $\mu$ and $w$ are $C^1$ around $a$. This being for every $a \in \mathbf{R}_+^*$, this ends the proof.

### Concavity

Finally, we prove that $\alpha$ is concave over $\mathbf{R}_+^*$, which requires no more than defining

$$f = \begin{pmatrix} S & \to & \mathbf{R} \\ (a, \beta) & \to & \mathbf{E}\left(e^{-\beta D_\tau} e^{-a(\tau+1)}\right) \end{pmatrix}$$

and computing the second derivative of the equation $f(a, \alpha(a)) = 1$.

— The first derivative yields

$$\mathbf{E}\left(-\left(\frac{d\alpha(a)}{da} D_\tau + \tau + 1\right) e^{-\alpha(a) D_\tau} e^{-a(\tau+1)}\right) = 0$$

and from this we get that

$$\frac{d\alpha(a)}{da} = \frac{\mathbf{E}\left((\tau+1)e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right)}{\mathbf{E}\left(-D_\tau e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right)}$$

and we recall that this term is well-defined and non-negative, thus positive because $C$ is positive recurrent.

— The second derivative yields

$$\begin{aligned} 0 &= \mathbf{E}\left(-\left(\frac{d^2\alpha(a)}{da^2}D_\tau\right)e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right) \\ &+ \mathbf{E}\left(\left(\frac{d\alpha(a)}{da}D_\tau+\tau+1\right)^2 e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right) \end{aligned}$$

Thanks to the previous equation, the first term of this sum simplifies, leading to

$$\begin{aligned} &\frac{d^2\alpha(a)}{da^2}\frac{-\mathbf{E}\left((\tau+1)e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right)}{\frac{d\alpha(a)}{da}} \\ &= \mathbf{E}\left(\left(\frac{d\alpha(a)}{da}D_\tau+\tau+1\right)^2 e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right) \end{aligned}$$

The rightmost term is non-negative, and as $d\alpha(a)/da > 0$ and $C$ is positive recurrent, $\alpha$'s second derivative is non-positive, which ends the proof.

### 2.8.5 Limit at point zero

We aim at proving the propositions 2.4.1 for $a = 0$, leading later to properties about $C$'s default time. Throughout this paragraph, we shall name $f_C$ the function defined by

$$f_C = \left(\begin{array}{ccc} \mathbf{R}^+ \times \mathbf{R}^+ & \to & \mathbf{R}\cup\{\infty\} \\ (a,\beta) & \to & \mathbf{E}\left(\mathbf{1}_{D_\tau<\infty}e^{-\beta D_\tau}e^{-a(\tau+1)}\right) \end{array}\right)$$

so that $f_C(a,\alpha(a)) = 1$ no matter $a \in \mathbf{R}_+^*$ : we are going to control this function $f_C$ to get the desired results when $a$ goes to 0.

#### Integrability

We aim at finding $\beta_0 \in \mathbf{R}_+^*$ such that $f_C(0,\beta_0) < \infty$. If we succeed, then as we know that for every $a \in \mathbf{R}^+$ and $\beta \in [0,\beta_0]$, $f_C(a,\beta)$ will be bounded by $f_C(a,\beta_0)$, the dominated convergence theorem indicates that

— $f_C$ is continuous over the whole set $\mathbf{R}^+ \times [0,\beta_0]$ ;

— $f_C$ is $C^\infty$ over $\mathbf{R}_+^* \times (0, \beta_0)$ ;
— In particular, $f_C(0,0) = \mathbf{P}\left(D_\tau < \infty\right)$.

We remember that the expression of $L_{C^{(d)}}(\beta)$, for $d \in \mathbf{R}$, was obtained as a rational fraction of $L_C(\beta)$'s entries and terms $e^{\beta d}$. When $d = 0$, the successive constraints we find on the successive reductions come to

$$\forall u \in [|2, A|]\,, P_{u,k_u \to k_u} \mathbf{E}\left(e^{-\beta D_{u,k_u \to k_u}}\right) < 1$$

where $D_{u,k_u \to k_u}$ is exponentially integrable over some opened set $(0, \beta_u)$ with $\beta_u > 0$ thanks to the induction hypothesis. As $P_{u,k_u \to k_u} < 1$ because $u \geq 2$, the induction loop follows. Finally, for $u = 1$, we get some $\beta_1 > 0$, and taking any $\beta_0 \in (0, \beta_1)$ works.

**Limit of eigenvectors**

We prove that $\alpha$, $\mu$ and $w$ are continuous at point $a = 0$.

— Let us start with $\alpha$. As it is increasing and bounded from below by 0, it converges to a non-negative limit $l$ at point 0. Hence, we may extend $\alpha$'s domain to 0 by $\alpha(0) = l$ ; as $\alpha$ was already continuous over $\mathbf{R}_+^*$, it is now continuous over $\mathbf{R}^+$.

— The Laplace matrix function $L_C$ may also be continued by a limit $L$ at point 0, being the matrix whose entry $(i, j)$ is given by

$$L_{i,j} = P_{i \to j} \mathbf{P}\left(D_{i \to j} < \infty\right)$$

It follows that $L_C\left(\alpha(0)\right)$ is a well-defined limit, whether $\alpha(0) = 0$ or not, and is still a positive recurrent matrix whenever $C$ is positive recurrent. As the function

$$\left(\begin{array}{ccc} \mathbf{R}^+ & \to & \mathbf{R} \\ a & \to & \det\left(L_C(\alpha(a)) - e^a Id\right) \end{array}\right)$$

is then well-defined and continuous over $\mathbf{R}^+$, and identically zero over $\mathbf{R}_+^*$, then it is also zero at point 0, so 1 is an eigenvalue for $L_C(\alpha(0))$. This unit eigenvalue must be dominant, because (complex) eigenvalues of $L_C(\alpha(a))$ are continuous functions of $a$ and dominated by $e^a$ over $\mathbf{R}_+^*$ by construction of $\alpha$. Hence, we may note by

— $\mu_0 \in \left(\mathbf{R}_+^*\right)^A$ a row eigenvector of $L_C(\alpha(0))$ such that $\mu_0\left(\vec{1}\right) = 1$ ;
— $w_0 \in \left(\mathbf{R}_+^*\right)^A$ a column eigenvector of $L_C(\alpha(0))$ such that $\mu_0 w = 1$.

— We rewrite the eigenvector equation as

$$\left(\begin{array}{cc} & \mu^{(a)}\left(L_C(\alpha(a)) - L_C(\alpha(0))\right) \\ + & \left(\mu^{(a)} - \mu_0\right) L_C(\alpha(0)) \\ + & \mu_0 L_C(\alpha(0)) \end{array}\right) = \left(\begin{array}{cc} & \mu^{(a)}\left(e^a - 1\right) \\ + & \left(\mu^{(a)} - \mu_0\right) \\ + & \mu_0 \end{array}\right)$$

After simplifications, this leads to

$$\left(\mu^{(a)} - \mu_0\right)\left(L_C(\alpha(0)) - Id\right) = \mu^{(a)}\left(L_C(\alpha(a)) - L_C(\alpha(0)) - (e^a - 1)Id\right)$$

However, we know that

— $\mu^{(a)}$'s coordinates are bounded by 1 because they are non-negative and $\mu^{(a)}\left(\vec{1}\right) = 1$ ;

— $L_C(\alpha(a)) - L_C(\alpha(0))$ converges to 0 when $a$ goes to 0 because $\alpha$ is continuous at point 0 and $L_C$ is continuous at point $\alpha(0)$.

We note by $F$ the linear function defined by

$$F = \left( \begin{array}{ccc} \mathbf{R}^A & \rightarrow & \mathbf{R}^{A+1} \\ x & \rightarrow & \left(x\left(L_C(\alpha(0)) - Id\right), x\left(\vec{1}\right)\right) \end{array} \right)$$

$F$ is injective : indeed, if $xL_C(\alpha(0)) = x$, then there is $k \in \mathbf{R}$ such that $x = k\mu_0$ (thanks to Perron-Frobenius' theorem), and then $k = 0$ thanks to the first equation of scaling. As we proved that $F\left(\mu^{(a)} - \mu_0\right)$ goes to zero when $a$ goes to zero, this implies that we may continue the function $\mu$ by $\mu^{(0)} = \mu_0$, and $\mu$ is now continuous over $\mathbf{R}^+$.

— A similar property holds for $w$ : we use the same line of thought, requiring only (when using the scaling properties) that
  — $\mu$ is continuous at point 0, which was just proved ;
  — $w$ is locally bounded around 0, which is true because $\mu$ being continuous around 0, its lowest coordinates hold

$$\forall i \leq A, \exists x \in [0,1] ; \inf_{y\in[0,1]} \left(\mu_{[i]}^{(y)}\right) = \mu_{[i]}^{(x)} > 0$$

  so there is a positive real $r > 0$ such that $\forall i \leq A, \forall x \in [0,1], \mu_{[i]}^{(x)} \geq r$, which implies that $w$'s coordinates are bounded bt $1/r$ thanks to the second equation of scaling.

It follows that we may define $\alpha(0)$, $\mu^{(0)}$ and $w^{(0)}$.

Henceforth, we shall rename $\alpha$, $\mu$ and $w$ their respective continuations at point $a = 0$.

**Case** $E(C) = \infty$

This paragraph and the following one aim at proving that $\alpha(0) = 0$ iff $E(C) \leq 0$ ; we begin here by the case $E(C) = \infty$. There is a transition $(i \rightarrow j)$ such that

$$\mu_{[i]} P_{i\rightarrow j} \mathbf{E}\left(D_{i\rightarrow j}\right) = \infty$$

As $C$ is positive recurrent, we use it to build a cycle going from $M(0)$ to $M(0)$ of infinite value, thus $\mathbf{P}\left(D_\tau = \infty\right) > 0$. Let us assume that $\alpha(0) = 0$. As $\alpha$ is continuous and increasing, there is $a_0 \in \mathbf{R}_+^*$ such that $\forall a \in (0, a_0), \alpha(a) \in (0, \beta_0)$ where $\beta_0$ was defined earlier. However, as $f_C$ is continuous over $\mathbf{R}^+ \times [0, \beta_0]$, then

$$f_C(0, \alpha(0)) = \lim_{a \to 0} \left( f_C(a, \alpha(a)) \right) = 1$$

whereas $f_C(0,0) = \mathbf{P}(D_\tau < \infty) < 1$. It follows that $\alpha(0)$ cannot be 0 ; as it is non-negative, it is positive.

**Case $E(C) > 0$ and finite**

As $C$ is positive recurrent and sEI, the negative part of its increments is integrable, so $E(C) < \infty$ indicates that $C$'s increments are all integrable. Let us assume that $\alpha(0) = 0$. Once again, there is $a_0 \in \mathbf{R}_+^*$ such that $\forall a \in (0, a_0), \alpha(a) \in (0, \beta_0')$ where $\beta_0' < \beta_0$ that was defined earlier. Differentiation of the equation $f_C(a, \alpha(a)) = 1$ at such a point yields

$$\frac{d\alpha(a)}{da} = \frac{\mathbf{E}\left((\tau+1)e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right)}{\mathbf{E}\left(-D_\tau e^{-\alpha(a)D_\tau}e^{-a(\tau+1)}\right)}$$

However, when $a$ goes to 0, we analyze the numerator :

— We know that $\mathbf{E}(\tau) < \infty$, because $\tau + 1$ is $M$'s return time to $M(0)$, so holds a sub-geometric distribution ;
— As a consequence, the dominated convergence theorem indicates that the numerator converges to a positive value (because $D_\tau$ is not $\infty$ almost surely, as $C$ is positive recurrent) named $x_1 \in \mathbf{R}_+^*$.

Now we analyze the denominator :

— The positive part of the denominator is controlled by

$$\mathbf{E}\left(-D_\tau e^{-\beta_0' D_\tau}\mathbf{1}_{D_\tau < 0}\right)$$

However, this is bounded by

$$\frac{1}{\beta_0 - \beta_0'}\mathbf{E}\left(e^{-\beta_0 D_\tau}\right) < \infty$$

so the positive part is bounded ;
— The negative part of the denominator is controlled by

$$\mathbf{E}\left(D_\tau \mathbf{1}_{D_\tau \geq 0}\right)$$

This value is finite, because thanks to the lemma 2.8.6, as $C$ is integrable, $D_\tau$ is integrable.

— The theorem of dominated convergence thus yields

$$x_2 = \lim_{a\to 0}\left(\mathbf{E}\left(-D_\tau e^{-\alpha(a)D_\tau} e^{-a(\tau+1)}\right)\right) = -\mathbf{E}\left(D_\tau\right)$$

and thanks to the lemma 2.8.6, this is

$$x_2 = -\frac{E(C)}{\mu_{[M(0)]}} \in \mathbf{R}_-^*$$

Hence, we get that

$$\lim_{a\to 0}\left(\frac{d\alpha(a)}{da}\right) = \frac{x_1}{x_2} < 0$$

This is impossible because $\alpha$ is increasing over $\mathbf{R}_+^*$, so once again $\alpha(0) > 0$.

**Case** $E(C) \leq 0$

This time we assume that $\alpha(0) > 0$. First, as by definition 1 is the dominant eigenvalue of $L_C(\alpha(0))$ and $C$ is positive recurrent, the same construction as in the lemma 2.8.8 enables successive reductions of $C$ up to its 0-restricted Lévy process, i.e. its restricted Lévy process. It follows that $f_C(\alpha(0), 0) = 1$ ; noting by $g$ the function defined by

$$g = \begin{pmatrix} [0, \alpha(0)] & \to & \mathbf{R} \\ \beta & \to & f_C(\beta, 0) \end{pmatrix}$$

then $g$ is continuous over $[0, \alpha(0)]$ and $C^2$ over $(0, \alpha(0))$. Moreover, we know that

$$\forall \beta \in (0, \alpha(0)), \frac{d^2 g(\beta)}{d\beta^2} \geq 0$$

However, when $\beta$ goes to 0, we find out that

$$g(0) = \mathbf{P}\left(D_\tau < \infty\right) = 1$$

because the lemma 2.8.6 indicates that $C$ is integrable. So, $g$ is a convex function that evaluates to 1 at both points 0 and $\alpha(0)$, but we also know that

$$\lim_{\beta\to 0}\left(\frac{dg(\beta)}{d\beta}\right) = \mathbf{E}\left(-D_\tau \mathbf{1}_{D_\tau<\infty}\right)$$

and this term is non-negative thanks to the lemma 2.8.6. Hence, $g$ must be constant over $[0, \alpha(0)]$, which is possible only if $D_\tau$ is either 0 or $\infty$ almost surely, which is excluded as $C$ is not globally increasing. So, we proved that $\alpha(0) = 0$ ; to find the dominant eigenvectors, we shall verify that

$$\forall i, j \leq A, \left(L_C(0)\right)_{i,j} = P_{i\to j}$$

indeed if $D_{i\to j} = \infty$, then $P_{i\to j} = 0$ else one would get $E(C) = \infty$. As the dominant eigenspaces have unit dimension, we only need to exhibit any non-zero eigenvector of $L_C(0)$, then scale it to find $\mu^{(0)}$ and $w^{(0)}$ ; by definition of $M$'s transition matrix, $\mu^{(0)} = \mu$ and $w^{(0)} = \left(\vec{1}\right)$ work, which ends the proof for the limit terms at $a = 0$.

### 2.8.6 Proposition 2.4.3

We end this paragraph by the characterization of Lévy processes through having no spread.

#### Forward implication

Let us assume that $C$'s dominant eigenvector is identically $\left(\vec{1}\right)$. Thanks to the eigenvector equation, we get at point $a$

$$\forall i \leq A, \sum_{j=1}^{A} P_{i\to j} \mathbf{E}\left(e^{-\alpha(a) D_{i\to j}}\right) = e^a$$

It follows that the Laplace transform $L_i$ of the random variable $C(t+1) - C(t)$ knowing $M(t) = A_i$ holds (no matter $t$)

$$\forall i \leq A, \forall a \in \mathbf{R}_+^*, L_i\left(\alpha(a)\right) = e^a$$

Since the martingale parameter is defined over $\mathbf{R}_+^*$ and its image is a non-trivial interval, the functions $L_i$ coincide over a non-trivial interval, which suffices to imply that all such random variables are identically distributed. As the increment $C(t+1) - C(t)$ is one among them, then the increments $C(t+1) - C(t)$ are identically distributed. Finally, we know by definition of a C-process (proposition 2.3.1) that $C(t+1) - C(t)$ is independent of the filtration $\mathbf{F}(t)$ conditionally to $M(t)$. However, as it is also independent of $M(t)$ by equality of functions $L_i$, then it is independent of $\mathbf{F}(t)$, so $C$'s increments are independent and identically distributed.

#### Backward implication

Let us assume that $C$ is actually a Lévy process. As it is positive recurrent, conditioning to $M(t) = A_i$ is non-empty for some $t \in \mathbf{N}$, yielding a conditional distribution of $C(t+1) - C(t)$, that has a Laplace transform $L_i$. However, by hypothesis all functions $L_i$ must coincide (calling by $L$ the common one), so by definition of conditioned transition payoffs we get the matrix equation

$$\forall \alpha \in \mathbf{R}_+^*, L_C(\alpha)\left(\vec{1}\right) = L(\alpha)\left(\vec{1}\right)$$

Thus, $\left(\vec{1}\right)$ is an eigenvector of the positive recurrent matrix $L_C(\alpha)$ at every point $\alpha \in \mathbf{R}_+^*$, so this holds for the matrix $L_C(\alpha(a))$ at every point $a \in \mathbf{R}_+^*$. As the only positive eigenspace of a positive recurrent matrix is its dominant eigenspace (Perron-Frobenius), then $\left(\vec{1}\right)$ must be the dominant eigenvector, which ends the proof.

# 2.9 Proposition 2.4.4

In this part, we take $C$ as a C-process whose underlying Markovian process is $M$. We aim at finding what happens when $C$ does not fall in the scope of the main method, mainly because it is periodic : the Laplace transforms end up not being of the desired form. Indeed, when $C$ is globally increasing, it will never default provided that its starting point is high enough (when $C_0$ amounts at least to the $Q$ from the proposition 2.3.2). As a consequence, we shall hereby assume that $C$ is aperiodic, and has $p \in \mathbf{R}_+^*$ as a fundamental period.

## 2.9.1 Definition 2.4.3

We start by proving the correctness of the definition 2.4.3, describing properties of $C$'s offsets and how to use them towards proving the proposition 2.4.4.

### Values for the offsets

First, we introduce the offsets as given along with the regular process from the definition 2.4.3. Let $i, j \leq A$ ; as $C$ is positive recurrent, we create some paths of finite value, thanks to the lemma 2.7.5 and the requirement for $C$ to be positive recurrent.

— From $A_i$ to $A_j$, we define $S_{i \to j} \subseteq \mathbf{R}$ the set of finite values for all such paths. $S_{i \to j}$ is not empty thanks to the lemma 2.7.5, so we may choose any $v_{i,j} \in S_{i \to j}$ ;

— From $A_j$ to $A_i$, we choose a path of finite value named $x \in \mathbf{R}$ (this is possible thanks to lemma 2.7.5 again).

The concatenation of the path $(A_i \to A_j)$ of value $v_{i,j}$ and the path $(A_j \to A_i)$ of value $x$ yields a cycle of value $v_{i,j} + x$ that must belong to $p\mathbf{Z}$ by hypothesis. It follows that

$$\exists x \in \mathbf{R}; \forall v_{i,j} \in S_{i \to j}, v_{i,j} + x \in p\mathbf{Z}$$

So, for every $i, j \leq A$, there is $p_{i,j}$ (it may be thought of as $-x$ modulo $p$) such that

$$S_{i \to j} \subseteq p_{i,j} + p\mathbf{Z}$$

$C$ being positive recurrent, all states are accessible using paths of finite values, and concatenation of paths leads to

$$\forall i, j, k \leq A, S_{i \to j} + S_{j \to k} \subseteq S_{i \to k}$$

Since the sets are not empty, then $p_{i,j} + p_{j,k} - p_{i,k} \in p\mathbf{Z}$. We set e.g. for $i = M(0)$ and some $x \in \mathbf{R}$ the values $p_j = p_{M(0),j} + x$, which leads to the condition

$$\forall j, k \leq A, p_{j,k} \in p_k - p_j + p\mathbf{Z}$$

For any fixed value of $x$ leading to $p_{M(0)} = x$, this condition leads to a single solution vector $(p_i)_{i \leq A} \in ([0, p))^A$, which creates offsets once choosing $x \equiv C(0)$ modulo $p$.

### Regular process

To prove that the definition 2.4.3 is correct, we take $C$ a C-process as given, and prove by induction on $t$ that $\forall t \in \mathbf{N}, \tilde{C}(t) \in \mathbf{Z} \cup \{\infty\}$ almost surely.

— For $t = 0$,

$$\tilde{C}(0) = \frac{C(0) - p_{M(0)}}{p}$$

is an integer by hypothesis on $p_{M(0)}$.

— For every $t \in \mathbf{N}$, let us assume that $\tilde{C}(t) \in \mathbf{Z} \cup \{\infty\}$ almost surely and write

$$C(t) = p\tilde{C}(t) + p_{M(t)}$$

In the equation $C(t+1) = C(t) + D_{M(t) \to M(t+1)}$, we know that $D_{M(t) \to M(t+1)}$ belongs to $p_{M(t+1)} - p_{M(t)} + p\mathbf{Z}$ almost surely thanks to the definition of the offsets, so $C(t+1)$ may be written with a random integer increment $Z$ (maybe $Z = \infty$) as

$$C(t+1) = C(t) + p_{M(t+1)} - p_{M(t)} + pZ$$

So, this leads to

$$C(t+1) = p\left(\tilde{C}(t) + Z\right) + p_{M(t+1)}$$

As $\tilde{C}(t)$ is either an integer or $\infty$ by induction hypothesis, then $C(t+1)$ is of the right form, so $\tilde{C}(t+1)$ is either an integer or $\infty$.

Finally, since $C$'s fundamental period is $p \in \mathbf{R}_+^*$, then after scaling by $1/p$, $\tilde{C}$'s one will be 1.

**Transfer of properties**

We are going to express the martingale parameter for $\tilde{C}$, along with its dominant eigenvectors.

1. Let us start with $\tilde{C}$'s properties, ensuring the existence of a martingale parameter for $\tilde{C}$.

   (a) If $C$ is positive recurrent, then by lemma 2.7.5, for every states $A_i$ and $A_j$, there is a path of length $T \in \mathbf{N}$ and finite value $v \in \mathbf{R}$ from $A_i$ to $A_j$, defined by its
   — Occupied state numbers $a_t \leq A$ for $t \in [|0, T|]$, with $a_0 = i$ and $a_T = j$ ;
   — Payoffs values $x_t \in \mathbf{R}$ for $t \in [|1, T|]$.
   By construction of $\tilde{C}$'s transition payoffs, the following determinations define a path of finite value for $\tilde{C}$ :
   — Same occupied state numbers $a_t \leq A$ ;
   — Payoffs values defined by

   $$\forall t \in [|1, T|], y_t = \frac{x_t - p_{a_t} + p_{a_{t-1}}}{p}$$

   This being for every state numbers $i, j \leq A$, $\tilde{C}$ is positive recurrent.

   (b) We prove that $C$ and $\tilde{C}$ have the same cycle support (up to a scaling by $p$), so that the lemma 2.3.7 will lead to the equivalence between global monotonicity of one and the other. Let us start with a cycle of finite value $x \in \mathbf{R}$ for $C$, whose successive payoffs values are $x_t \in \mathbf{R}$. Like previously, we find a cycle for $\tilde{C}$ whose successive payoffs values are

   $$\forall t \in [|1, T|], y_t = \frac{x_t - p_{a_t} + p_{a_{t-1}}}{p}$$

   Its value $y$ amounts to

   $$\sum_{t=1}^{T} \frac{x_t - p_{a_t} + p_{a_{t-1}}}{p} = \frac{x - p_{a_T} + p_{a_0}}{p}$$

   However, $a_0 = a_T$ by definition of a cycle, so $y = x/p$, and it follows that

   $$\mathbf{R} \cap supp\,(C) \subseteq \mathbf{R} \cap \left(supp\left(\tilde{C}\right)\right) p$$

   As this works also the other way around, we get the desired property. In particular, if $C$ is not globally increasing, then $\tilde{C}$ cannot be either.

(c) We prove that $C$ and $\tilde{C}$ have the same default time. Let $t \in \mathbf{N}$. If $C(t) < 0$, then $\tilde{C}(t) < 0$ because the offsets are nonnegative. Conversely, if $\tilde{C}(t) < 0$, then $\tilde{C}(t) \leq -1$ because $\tilde{C}(t) \in \mathbf{Z}$ almost surely, so $C(t) \leq p_{M(t)} - p < 0$ because the offsets are lower than $p$. It follows that $C(t) < 0$ iff $\tilde{C}(t) < 0$, which ends the proof.

(d) Finally, if $C$ is sEI (or bounded), then $\tilde{C}$ is still sEI (or bounded), because

— The "relevant" transitions (i.e. $(i, j) \in \Gamma$) are the same for both processes because $M$ is still $\tilde{C}$'s Markovian process (in particular, if $C$ is positive recurrent, then $\tilde{C}$ is also positive recurrent) ;

— Adding a bounded (by $p$) constant to the relevant transition payoffs does not change their integrability properties.

Hence, if the conditions to the existence of a martingale parameter are satified for $C$, they are for $\tilde{C}$.

2. We compare the Laplace matrix functions of $C$ and $\tilde{C}$. By definition of $\tilde{C}$'s transition payoffs, we have

$$\forall i, j \leq A, \tilde{D}_{i \to j} = \frac{D_{i \to j} - p_j + p_i}{p}$$

In particular, for every $\alpha \in \mathbf{R}$, using the diagonal change-of-basis matrix $\Delta(\alpha)$ defined by

$$\forall i \leq A, (\Delta(\alpha))_{i,i} = e^{-\alpha p_i}$$

then by definition of Laplace matrix functions,

$$\forall \alpha \in \mathbf{R}, L_C(\alpha) = (\Delta(\alpha))^{-1} L_{\tilde{C}}(p\alpha)\Delta(\alpha)$$

As we recall the proposition 2.4.1 defining the martingale parameter $\alpha(a)$ and the dominant eigenspaces $w^{(a)}$, we get that

(a) The eigenvalues of $L_C(\alpha)$ and $L_{\tilde{C}}(p\alpha)$ being identical, the martingale parameter $\tilde{\alpha}(a)$ must be given by $p\alpha(a)$ ;

(b) If $w \in \mathbf{R}^A$ is a dominant eigenvector for $L_C(\alpha)$, then using the change-of-basis matrix $\Delta(\alpha)$ and the above equation, $v = \Delta(\alpha)w$ is a dominant eigenvector for $L_{\tilde{C}}(p\alpha)$. Likewise, if $\nu \in \mathbf{R}^A$ is a dominant row eigenvector for $L_C(\alpha)$, then $\nu\,(\Delta(\alpha))^{-1}$ is a dominant row eigenvector for $L_{\tilde{C}}(p\alpha)$.

#### Prerequisites to the proof for periodic C-processes

We aim at proving the proposition 2.4.4 for a C-process $C$ responding to the hypotheses. Since its default time is identical to the one of its regular process $\tilde{C}$

thanks to the above work, we are going to prove a similar property for $\tilde{C}$. Let us note respectively

- — $\tilde{\alpha}$ is $\tilde{C}$'s martingale parameter (so $\forall a \in \mathbf{R}^+, \tilde{\alpha}(a) = p\alpha(a)$) ;
- — $\tilde{C}_0$ is $\tilde{C}$'s starting point, i.e.

$$\tilde{C}_0 = \left\lfloor \frac{C_0}{p} \right\rfloor \in \mathbf{Z}$$

  by definition of the regular process ;
- — $\tilde{w}^{(a)}$ is $\tilde{C}$'s dominant eigenvector, defined previously.

We want to prove this lemma :

**Lemma 2.9.1** *Convergence for the regular process*

*Let $C$ be a C-process whose underlying Markovian process is $M$. We assume that*

- — *$C$ is positive recurrent, bounded and not globally increasing ;*
- — *For every $t \in \boldsymbol{N}$, $C(t) \in \boldsymbol{Z}$ almost surely ;*
- — *$C$'s fundamental period is 1.*

*Such a C-process will be called a regular C-process. Let $a \in \boldsymbol{R}^+$ and $s \leq A$. We set*

- — *Its random default time as $T \in \boldsymbol{N} \cup \{\infty\}$ ;*
- — *$T$'s log-Laplace at point $a$ starting from $M(0) = A_s$ and $C(0) = x \in \boldsymbol{N}$ as*

$$\Lambda_s^{(a)} = \left( \begin{array}{ccc} \boldsymbol{N} & \rightarrow & \boldsymbol{R} \\ x & \rightarrow & \ln\left(\boldsymbol{E}\left(e^{-aT}\boldsymbol{1}_{T<\infty}|M(0) = A_s \wedge C(0) = x\right)\right) \end{array} \right)$$

- — *$C$'s martingale parameter at point $a$ as $\alpha(a)$ ;*
- — *$C$'s dominant eigenvector at point $a$ as $w^{(a)}$.*

*For every $a \in \boldsymbol{R}^+$, there is $K'(a) \in \boldsymbol{R}$ such that*

$$\forall x \in \boldsymbol{N}, -\Lambda_s^{(a)}(x) \in \left[\left(\alpha(a)x - \ln\left(w_{[s]}^{(a)}\right) + K'(a)\right) \pm e\left(x, a\right)\right]$$

*where $e$ is an error function $\left(\left(\boldsymbol{N} \times \boldsymbol{R}_+^*\right) \rightarrow \boldsymbol{R}^+\right)$, uniformly exponentially convergent to 0*

- — *Over any subset of the form $a \in [0, b]$ with $b \in \boldsymbol{R}^+$ ;*
- — *When $x$ goes to $+\infty$ (keeping integer values).*

*Moreover,*

1. *$K'(a)$ is a continuous expression of $a$ ;*
2. *If $C$ is a Lévy process, then $\forall a \in \boldsymbol{R}^+, K'(a) \geq \alpha(a)$.*

The proof for this lemma itself is postponed until the next part, as it relies on the definitions of descending processes.

### 2.9.2 Proof of the theorem for periodic C-processes

In this paragraph, we assume the previous lemma 2.9.1 to be granted, and we aim at proving the proposition 2.4.4. More specifically, we want to build the functions $K$ and $e$, using the error functions given by this lemma 2.9.1. Henceforth, we shall note

- For every $x \in \mathbf{R}^+$, $\tilde{C}^{[s,x]}$ is $C$'s regular process where $C$ starts from $C(0) = x$ and $M(0) = A_s$ ;
- For every $i \leq A$, $p_i^{[s]}$ is the natural offset of the state $A_i$ when $C$ starts from $M(0) = A_s$ and any $C(0) \in p\mathbf{N}$ ;
- For $x \in \mathbf{R}$ and $y \in \mathbf{R}_+^*$, $\overbrace{x}^{y}$ is the only element of $x + y\mathbf{Z} \cap [0, y)$ (so-called "$x$ modulo $y$") ;

For example, it follows from these notations that the natural offset of the state $A_i$ when $C$ starts from $M(0) = A_s$ and $C(0) = x \in \mathbf{R}^+$ is

$$\overbrace{p_i^{[s]} + x}^{p}$$

#### Breaking points

Our basis is to remark that $\Lambda_i^{(a)}$ is piecewise constant over $\mathbf{R}^+$ in a $p$-periodic fashion.

**Definition 2.9.1** *Breaking points*

*Let $C$ be a positive recurrent, periodic C-process, whose fundamental period is $p \in \boldsymbol{R}_+^*$ and random default time is $T$. Let $a \in \boldsymbol{R}^+$ and $s \leq A$, so that we set $\Lambda_s^{(a)}$ as in lemma 2.9.1. There is a family*

$$\left(q_i^{[s]}\right)_{i \in [|0, A-1|]} \in [0, p)^A$$

*sorted by ascending order with $q_0^{[s]} = 0$, such that for every $n \in \boldsymbol{N}$,*

- *$\Lambda_s^{(a)}$ is constant over $[np + q_{A-1}^{[s]}, (n+1)p)$ ;*
- *For every $i \in [|1, A-1|]$, $\Lambda_s^{(a)}$ is constant over $[np + q_{i-1}^{[s]}, np + q_i^{[s]})$.*

*We shall call such points $q_i^{[s]}$ breaking points of $\Lambda_s^{(a)}$. They do not depend on $a \in \boldsymbol{R}^+$.*

We prove the correctness of this definition making use of $C$'s regular process $\tilde{C}^{[s,x]}$ for several starting points $C(0) = x \in \mathbf{R}^+$, with the same starting state $A_s$. The main idea is to remark that $\tilde{C}^{[s,x]}$ is the same C-process for "close" values of $x$, as long as no offset of $C$ loops back from $p$ to 0.

— Let us start with $x = 0$. We get $C$'s natural offsets $p_i^{[s]}$, and as they are in a finite number, some value

$$p_{\max}^{[s]} = \max_{i \leq A} \left( p_i^{[s]} \right) < p$$

— As long as $x < p - p_{\max}^{[s]}$ called $q_1^{[s]}$, then the family of offsets $p_i^{[s]} + x$ keeps defining offsets of $C$ that satisfy the condition for natural offsets (because $p_i^{[s]} + x < 1$). As natural offsets are unique, they are the natural offsets of $C$ starting from this $x$ ; by definition of $\tilde{C}^{[s,x]}$, $x$ cancels out in the computations of the values of the regular process, so $\tilde{C}^{[s,x]}$ is the same process for every $x \in [0, q_1^{[s]})$.
— Then we start again with $x = q_1^{[s]}$, getting some value $p_{\max'}^{[s]}$ (the second maximum among the offsets $p_i^{[s]}$) and $q_2^{[s]} = p - p_{\max'}^{[s]}$, such that $\tilde{C}^{[s,x]}$ is the same process for every $x \in [q_1^{[s]}, q_2^{[s]})$, and so on.
— We proceed until $A - 1$ states have been reached. As the last maximum is the minimum $p_s^{[s]} = 0$ by definition of $\tilde{C}^{[s,0]}$, then $\tilde{C}^{[s,x]}$ is the same process for every $x \in [q_{A-1}^{[s]}, p)$.
— After that, the whole thing restarts with the same successive values $q_k^{[s]}$ (shifted of $p$) because $C$'s natural offsets are $p$-periodic in $C(0)$ by construction.

It follows by induction that the family $\left(q_i^{[s]}\right)_i$ (with $q_0^{[s]} = 0$) forms breaking points for $\Lambda_s^{(a)}$.

### Identification of the terms for breaking points

Let $C$ be a C-process, deemed positive recurrent, bounded and not globally increasing, starting from $M(0) = A_s$ and $C(0) = x \in \mathbf{R}^+$, whose fundamental period is $p \in \mathbf{R}_+^*$. We aim at finding suitable terms $K$ and $e$ for the proposition 2.4.4, for now only when $x \in q_i^{[s]} + p\mathbf{N}$ where $q_i^{[s]}$ is some breaking point for $\Lambda_s^{(a)}$. To do that, we set any $a \in \mathbf{R}^+$, as the choice of $a$ is irrelevant to the breaking points. The main idea is that $\tilde{C}^{[s,x]}$'s transition payoffs do not change when $x$ stays in $q_i^{[s]} + p\mathbf{N}$, so only the starting point changes for $\tilde{C}^{[s,x]}$, which means that we may apply the lemma 2.9.1 and get, for every $i \leq A$, suitable values of $K'$ and errors $e$ that do not depend on $s$ ; we note them respectively $K_i'$ and $e_i$. However, we recall that since $x \in q_i^{[s]} + p\mathbf{N}$,

— $\tilde{C}^{[s,x]}$ martingale parameter is $p\alpha(a)$, and its starting point is

$$z = \frac{x - q_i^{[s]}}{p}$$

— For every $j \leq A$, let

$$p_j^{[s]}(x) = \overbrace{p_j^{[s]} + x}^{p}$$

be $A_j$'s natural offset of $C$ starting from $x$ and $M(0) = A_s$, viewed as a $p$-periodic function of $x$. When $x \in q_i^{[s]} + p\mathbf{N}$, it keeps a single value, noted

$$p_{j,[i]}^{[s]} = \overbrace{p_j^{[s]} + q_i^{[s]}}^{p}$$

Let $\Delta^{[s]}(i, \alpha)$ be the diagonal matrix whose non-zero entries are these offsets

$$\forall j \leq A, \left(\Delta^{[s]}(i, \alpha)\right)_{j,j} = e^{-\alpha p_{j,[i]}^{[s]}}$$

We know that there is $k^{[s,i]}(a) \in \mathbf{R}_+^*$ such that $\tilde{C}^{[s,x]}$'s dominant eigenvector is

$$\tilde{w}^{[s,i],(a)} = k^{[s,i]}(a)\Delta^{[s]}(i, \alpha(a))w^{(a)}$$

The statement from lemma 2.9.1 applied to $\tilde{C}^{[s,x]}$ yields (for the Laplace transform $\tilde{\Lambda}_s^{(a)}$ of $\tilde{C}^{[s,x]}$'s default time) that for every $z \in \mathbf{N}$

$$-\tilde{\Lambda}_s^{(a)}(z) \in \left[\left(p\alpha(a)z - \ln\left(k^{[s,i]}(a)\left(\Delta^{[s]}(i, \alpha(a))\right)_{s,s} w_{[s]}^{(a)}\right) + K_i'(a)\right) \pm e_i\left(z, a\right)\right]$$

Hence, when $x = q_i^{[s]} + pz$, we know that :

— By construction of the regular process,

$$\tilde{\Lambda}_s^{(a)}(z) = \Lambda_s^{(a)}(x)$$

— For every $\alpha \in \mathbf{R}^+$,

$$\left(\Delta^{[s]}(i, \alpha)\right)_{s,s} = e^{-\alpha p_{s,[i]}^{[s]}} = e^{-\alpha \overbrace{p_s^{[s]} + q_i^{[s]}}^{p}}$$

— As $p_s^{[s]} = 0$ by definition of the starting state,

$$\overbrace{p_s^{[s]} + q_i^{[s]}}^{p} = q_i^{[s]}$$

it follows that the expression in the above control simplifies to

$$-\Lambda_s^{(a)}(x) \in \left[\left(\alpha(a)x - \ln\left(w_{[s]}^{(a)}\right) + K_i'(a) - \ln\left(k^{[s,i]}(a)\right)\right) \pm e_i\left(\frac{x - q_i^{[s]}}{p}, a\right)\right]$$

Hence, one may set

$$\forall x \in q_i^{[s]} + p\mathbf{N}, \forall a \in \mathbf{R}^+, K(x,a) = K_i'(a) - \ln\left(k^{[s,i]}(a)\right)$$

and

$$\forall x \in q_i^{[s]} + p\mathbf{N}, \forall a \in \mathbf{R}^+, e(x,a) = e_j\left(\left\lfloor \frac{x}{p} \right\rfloor, a\right)$$

N.B. : if $q_{i-1}^{[s]} = q_i^{[s]}$ for some $i$, these definitions not are contradictory because the regular processes $\tilde{C}^{[s,x]}$ would coincide : so, $K_{i-1}' = K_i'$, $k_{i-1} = k_i$ and $e_{i-1} = e_i$, and the definitions for $x \in q_{i-1}^{[s]} + p\mathbf{N}$ and $x \in q_i^{[s]} + p\mathbf{N}$ coincide.

**Extension to $\mathbf{R}^+ \times \mathbf{R}^+$**

Now that the cases $x \in q_i^{[s]} + p\mathbf{N}$ are settled, we use the breaking points given by definition 2.9.1 to extend $K$ and $e$ to $\mathbf{R}^+ \times \mathbf{R}^+$. For conveniency, we shall note

— The "boundary" breaking points of $\Lambda_s^{(a)}$ are $q_0^{[s]} = 0$ and $q_A^{[s]} = p$ ;

— The intervals where $\Lambda_s^{(a)}$ is constant are, for every $n \in \mathbf{N}$ and $i \leq A$,

$$I_{n,i}^{[s]} = [q_{i-1}^{[s]} + pn, q_i^{[s]} + pn)$$

We should begin with the remark

$$\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{A} I_{n,i}^{[s]} = \mathbf{R}^+$$

where the sets $I_{n,i}^{[s]}$ are pairwise disjoint, so with every $x \in \mathbf{R}^+$ one may associate a single $(n,i)$ such that $x \in I_{n,i}^{[s]}$. Now, let $n \in \mathbf{N}$ and $i \leq A$. $\Lambda_s^{(a)}$ is constant over $I_{n,i}^{[s]}$, and we already know that for $x \in q_{i-1} + p\mathbf{N}$, $a \in \mathbf{R}^+$, and $i \leq A$, we have

$$-\Lambda_s^{(a)}(x) \in \left[\left(\alpha(a)x - \ln\left(w_{[s]}^{(a)}\right) + K_{i-1}'(a) - \ln\left(k^{[s,i-1]}(a)\right)\right) \pm e_{i-1}\left(\left\lfloor \frac{x}{p} \right\rfloor, a\right)\right]$$

Hence, setting for $x \in I_{n,i}^{[s]}$ the function $K$ as

$$K(x,a) = K\left(q_{i-1}^{[s]} + pn, a\right) - \left(x - \left(q_{i-1}^{[s]} + pn\right)\right)\alpha(a)$$

and the function $e$ as

$$e(x,a) = e_{i-1}(n,a)$$

then the control

$$\forall a \in \mathbf{R}^+, s \leq A, -\Lambda_s^{(a)}(x) \in \left[\left(\alpha(a)x - \ln\left(w_{[s]}^{(a)}\right) + K(x,a)\right) \pm e(x,a)\right]$$

will still hold over $x \in I_{n,i}^{[s]}$ : doing this for every $i \leq A$ and $n \in \mathbf{N}$ thus yields the desired control over the whole $\mathbf{R}^+$. Finally, we verify that this function $K$ is $p$-periodic of its first variable. Let us take $x \in I_{n,i}^{[s]}$ and $a \in \mathbf{R}^+$, and look for $K(x+p,a)$ : as the sets $I_{n,i}^{[s]}$ are built $p$-periodically, then $x+p \in I_{n+1,i}^{[s]}$, so

$$K(x+p,a) = K\left(q_{i-1}^{[s]} + (n+1)p, a\right) - \left((x+p) - \left(q_{i-1}^{[s]} + (n+1)p\right)\right)\alpha(a)$$

$p$ cancels out in the rightmost term, and we recall that

$$K\left(q_{i-1}^{[s]} + (n+1)p, a\right) = K_i'(a) - \ln\left(k^{[s,i]}(a)\right)$$

that does not depend on $n$, which ends the proof. As a consequence, we will

— Redefine the function $K$ as being $\Big([0,p) \times \mathbf{R}^+ \to \mathbf{R}\Big)$, as given by its periodicity ;

— Keep the function $e$,

in the subsequent paragraphs.

#### Continuity of $K$

We verify that $K$ is piecewise continuous as required by the theorem 2.4.4. When $x \in I_{n,i}^{[s]}$, the terms $n$ and $i$ are constant, and so is $y = q_{i-1}^{[s]} + pn$. We recall that

$$\forall x \in I_{n,i}^{[s]}, K(x,a) = K(y,a) - (x-y)\,\alpha(a)$$

However, $\alpha$ is continuous and $K(y,a)$ is defined by

$$\forall a \in \mathbf{R}^+, K(y,a) = K_i'(a) - \ln\left(k^{[s,i]}(a)\right)$$

$K_i'$ is continuous thanks to the lemma 2.9.1. To get this property for $k^{[s,i]}(a)$, we use the fact that it is the normalization constant of $\tilde{w}^{[s,i],(a)}$ : noting $\tilde{\mu}^{[s,i],(a)}$ the dominant row eigenvector of $\tilde{C}^{[s,i]}$, one has

$$1 = \tilde{\mu}^{[s,i],(a)}\tilde{w}^{[s,i],(a)} = k^{[s,i]}(a)\tilde{\mu}^{[s,i],(a)}\Delta^{[s]}(i,\alpha(a))w^{(a)}$$

however $\alpha(a)$, $\tilde{\mu}^{[s,i],(a)}$, and $\tilde{w}^{[s,i],(a)}$ are continuous of $a$, with the vectors being positive and the diagonal matrix having positive entries, so $k^{[s,i]}$ is positive and continuous, which ends the proof of $K$'s continuity in $a$. It follows that

— $K$ is continuous in $a$ no matter $x$ (fixed) ;

— This being for every $i \leq A$, $K$ is continuous over every $I_{0,i}^{[s]} \times \mathbf{R}^+$, thus piecewise continuous over its domain.

**Sign of $K$**

We want to ensure that when $C$ is a Lévy process, $K(x,a) > 0$ for every $x \in \mathbf{R}^+$ and $a \in \mathbf{R}_+^*$. As Lévy means $A = 1$, let us take $n \in \mathbf{N}$ such that $x \in I_{n,1}^{[s]}$ ; according to the previous notations, we have

$$K(x,a) = K_0(a) - \ln\left(k^{[1,0]}(a)\right) - \left(x - \left(q_0^{[1]} + pn\right)\right)\alpha(a)$$

We know that $K_0(a) \geq p\alpha(a)$ thanks to the lemma 2.9.1 and the fact that the regular process has a martingale parameter $p\alpha(a)$ at point $a$. To evaluate $\ln\left(k^{[1,0]}(a)\right)$, we use the equations of scaling : the dominant row eigenvector $\tilde{\mu}^{(a)}$ of $\tilde{C}$ becomes the unit $\left(\vec{1}\right)^*$, and it follows from this that

$$k^{[1,0]}(a) = \left(\Delta^{[1]}(0, \alpha(a))\right)^{-1}$$

We recall that, by definition of $\Delta^{[s]}(i-1, \alpha(a))$, its only term amounts to

$$\left(\Delta^{[1]}(0,\alpha(a))\right)_{1,1} = e^{-\alpha(a)\overbrace{p_1^{[1]} + q_0^{[1]}}^{p}}$$

However, $p_1^{[1]} = 0$ by construction and $q_0^{[1]}$ is also 0 modulo $p$, so everything simplifies and finally

$$k^{[1,0]}(a) = 1$$

As a consequence, for $x \in I_{n,1}^{[1]} = [np, (n+1)p)$, we have

$$K(x,a) = K_0(a) - (x - np)\,\alpha(a) > p\alpha(a) - ((n+1)p - np)\,\alpha(a) = 0$$

which ends the proof.

**Exponential convergence**

We may now verify that $e$ converges uniformly exponentially to 0. To do that, let us set $a_{\max} \in \mathbf{R}_+^*$ and look at the functions $e_i$ for $i < A$. By hypothesis on them given by lemma 2.9.1, for each $i$ there are $B_i \in \mathbf{R}_+^*$ and $\beta_i > 0$ such that

$$\forall n \in \mathbf{N}, \forall a \leq a_{\max}, e_i(n,a) < B_i e^{-n\beta_i}$$

When looking for $n \in \mathbf{N}$ and $i$ such that $x \in I_{n,i}^{[s]}$, we get

$$n > \frac{x}{p} - 1$$

Hence, setting

$$B = \max_{i<A} \left( B_i e^{\beta_i} \right) \in \mathbf{R}_+^*$$

and

$$\beta = \min_{i<A} \left( \frac{\beta_i}{p} \right) > 0$$

one gets by construction of $e$ that

$$\forall x \in \mathbf{R}^+, \forall a \leq a_{\max}, e(x,a) < Be^{-x\beta}$$

which ends the proof of the proposition 2.4.4.

## 2.10 Convolution processes

In this paragraph, we shall deem that $C$ is a bounded and non-globally increasing C-process, and we eventually aim at proving the main theorem 2.1, for now only when $a \in \mathbf{R}_+^*$. We shall note the Laplace transform of its default time starting from the state $M(0) = A_{i \leq A}$ and the point $C(0) = x \in \mathbf{R}^+$ at point $a \in \mathbf{R}_+^*$ by

$$L_i^{(a)}(x) = \mathbf{E} \left( e^{-aT_0} | M(0) = A_i \wedge C(0) = x \right)$$

The core of the proof for $\Lambda_{T_0}$'s asymptotical behaviour for large values of $C_0$ lies in the properties governing the Laplace transform. Indeed, by definition of the default time and the Markovian, time-homogeneous behaviour of $C$, we have

$$\forall i \leq A, x \in \mathbf{R}_+, a \in \mathbf{R}_+^*, L_i^{(a)}(x) = \sum_{k=1}^{A} P_{i \to k} \mathbf{E} \left( L_k^{(a)} \left( x + D_{i \to k} \right) \right)$$

To simplify the future study, we want to assume the transition payoffs to be negative almost surely, for a reason that will appear below. The aim of this part is to build a new C-process, named "Convolution process", whose $\Lambda_{T_0}$ is preserved but whose increments are negative almost surely and easier to deal with.

### 2.10.1 Goal : the convolution equation

For the sake of simplicity, let us consider that $C$ is a Lévy process for the time being, and that we seek its default probability when starting from the point $x \in \mathbf{R}^+$, defined by

$$\forall x \in \mathbf{R}^+, P(x) = \mathbf{P} \left( T_0 < \infty | C(0) = x \right) = L^{(0)}(x)$$

The naive analysis of $C$'s default time will yield, by the Markovian property, an equation like

$$\forall x \in \mathbf{R}^+, P(x) = \int_{y=-\infty}^{\infty} P(x+y)\sigma(y)dy$$

where $\sigma$ is the distribution of $C$'s transition payoff $D$. Unfortunately, as $\sigma$ takes both positive and negative support values, trying to use the Laplace transform on this equation does not work ; this is somewhat equivalent to saying that the equation is not in "solved" form, with $P(x)$ expressed as a combination of previous values $P(y)$ for $y < x$. For instance, when $C$ is periodic, its regular process is $\tilde{C}$, that expresses $P(x)$ for $x \in \mathbf{N}$ through a recurrence equation that is not in solved form.

### Convolution equation

For this reason, the next idea is transforming the previous equation in another equation where $D$'s support is contained in $\mathbf{R}_-^*$.

**Lemma 2.10.1** *Convolution equation*
*Let $C$ be a bounded (by $Q \in \boldsymbol{R}_+^*$), non-globally increasing C-process and $a \in \boldsymbol{R}_+^*$, allowing us to define*
— *$C$'s martingale parameter at point $a$ as $\alpha(a) \in \boldsymbol{R}^+$ ;*
— *$C$'s dominant eigenvector at point $a$ as $w^{(a)} \in \left(\boldsymbol{R}_+^*\right)^A$.*
*Then for every $i, j \leq A$, there are*
— *Random variables $G_{i\to j}^{(a)} \in (0, Q]$ ;*
— *Constants named $P_{i\to j}^{(a)} \in [0, 1]$ ;*
— *Measurable functions $K_i^{(a)} : \left(\boldsymbol{R} \to \boldsymbol{R}^+\right)$*
*such that*

1. *The sought Laplace transforms hold*

$$\forall i \leq A, x \in \boldsymbol{R}^+, L_i^{(a)}(x) = e^{-\alpha(a)x} w_{[i]}^{(a)} K_i^{(a)}(x)$$

2. *The coefficients $P_{i\to j}^{(a)}$ form convex combinations, i.e.*

$$\forall i \leq A, \left(\left(\forall j \leq A, P_{i\to j}^{(a)} \geq 0\right) \wedge \sum_{j=1}^{A} P_{i\to j}^{(a)} = 1\right)$$

3. *The random variables $G_{i\to j}^{(a)}$ hold*

$$\forall i \leq A, x \in \boldsymbol{R}^+, K_i^{(a)}(x) = \sum_{j=1}^{A} P_{i\to j}^{(a)} \boldsymbol{E}\left(K_j^{(a)}\left(x - G_{i\to j}^{(a)}\right)\right)$$

*We shall name respectively*

— $G_{i\to j}^{(a)}$ *the convolution random variables, and*

— $P_{i\to j}^{(a)}$ *the convolution transition probabilities*

*of the convolution equation.*

An alternate formulation of the descending equation may be written as follows.

**Lemma 2.10.2** *Convolution process*

*Let $C$ be a bounded (by $Q \in \boldsymbol{R}_+^*$), non-globally increasing C-process and $a \in \boldsymbol{R}_+^*$ as above (lemma 2.10.1). We define the convolution random variables $G_{i\to j}^{(a)}$ and transition probabilities $P_{i\to j}^{(a)}$ of the convolution equation as above. Then the C-process $\Phi^{(a)}$ defined by*

— *Its underlying Markovian process $M^{(a)}$, holding $M^{(a)}(0) = M(0)$ almost surely and whose transition probabilities are, for any $t \in \boldsymbol{N}$,*

$$\forall i, j \le A, \boldsymbol{P}\left(M^{(a)}(t+1) = A_j | M^{(a)}(t) = A_i\right) = P_{i\to j}^{(a)}$$

*This is a correct definition, as the constants $P_{i\to j}^{(a)}$ sum to 1 by the previous lemma 2.10.1 ;*

— *Its transition payoffs between $A_i$ and $A_j$, being $-G_{i\to j}^{(a)}$ ;*

— *Its starting point $\Phi^{(a)}(0) = C(0)$.*

*Then the process $Y^{(a)}$ defined by*

$$Y^{(a)} = \begin{pmatrix} \boldsymbol{N} & \to & \boldsymbol{R}^+ \\ t & \to & K_{M^{(a)}(t)}^{(a)}\left(\Phi^{(a)}(t)\right) \end{pmatrix}$$

*is a martingale.*

Admitting lemma 2.10.1, we have indeed

$$\mathbf{E}\left(Y^{(a)}(t+1)|\mathbf{F}(t)\right) = \sum_{j=1}^{A} P_{M^{(a)}(t)\to j}\mathbf{E}\left(K_j^{(a)}\left(Y^{(a)}(t) - G_{M_a(t)\to j}^{(a)}\right)|\mathbf{F}(t)\right)$$

This is $K_{M^{(a)}(t)}^{(a)}\left(Y^{(a)}(t)\right)$ by lemma 2.10.1, which proves the statement.

### Introduction to the descending process

To prove the lemma 2.10.1, we will use another concatenation trick, skipping times $t \in \mathbf{N}^*$ for which $C(t)$ is not less than its previous minimum

$$\min_{u<t}\left(\{C(u)\}\right)$$

For some value of $C(t)$, we wait until $C$ hits an inferior value, which is equivalent to considering

$$\min\left(\left\{u \in \mathbf{N}; \sum_{k=1}^{u} D_{M(t+k-1)\to M(t+k)} < 0\right\}\right)$$

If there is no such $u$, then we set $u = \min(\emptyset) = \infty$ and we also know that (provided that $C(t) \geq 0$) $C$ will never default, so $T_0 = \infty$. The descending process of $C$ is defined as the process of successive all-time low values and associated wating times.

**Definition 2.10.1** *Descending process*

*Let $C$ be a C-process. We define its sequence $\tau$ of all-time lows recursively as $\tau(0) = 0$ and*

$$\forall u \in \boldsymbol{N}, \tau(u+1) = \min\left(\{t > \tau(u); C(t) < C(\tau(u))\}\right)$$

*$C$'s descending process is the concatenated process of $C$ associated with this time sequence $\tau$.*

- *It is noted $\vec{C}$ ;*
- *Its underlying Markovian process is noted $\vec{M}$.*

To verify that $\vec{C}$ is a C-process, one uses the proposition 2.7.1 only requiring $\tau$ to be canonical. It is, because the time increment $\tau(u+1) - \tau(u)$ only relies on the moves of $M$ and $C$ between time $\tau(u)$ and time $\tau(u+1)$, and thanks to the Markovian property. As a consequence, we may define its transition payoffs and transition probabilities. However, we are also interested in the time increment $n = \tau(u+1) - \tau(u)$ taken by every transition : for this reason, we shall decompose $\vec{C}$ with respect to $n$.

**Definition 2.10.2** *Distribution of $\vec{C}$'s increments*

*Let $C$ be a C-process, $\vec{C}$ be its descending process, $i, j \leq A$ and $n \in \boldsymbol{N}^*$. For any $t \in \boldsymbol{N}$,*

- *The transition probability of $\vec{M}$ going from $A_i$ to $A_j$ while $M$ goes through exactly $n$ steps is noted*

$$Q_{i \xrightarrow{n} j} = \boldsymbol{P}\left(\tau(t+1) - \tau(t) = n \wedge \vec{M}(t+1) = A_j | \vec{M}(t) = A_i\right)$$

- *The distribution of $\vec{C}(t+1) - \vec{C}(t)$ conditionally to the former transition is called*

$$\sigma_{i \xrightarrow{n} j}$$

  *Its support lies in $[-Q, 0)$ by construction.*

— *Its mirror distribution is called*

$$\forall x \in \boldsymbol{R}, d\nu_{i \xrightarrow{n} j}(x) = d\sigma_{i \xrightarrow{n} j}(-x)$$

*We shall name* $F_{i \xrightarrow{n} j}$ *a random variable whose distribution is* $\nu_{i \xrightarrow{n} j}$. *In particular,* $F_{i \xrightarrow{n} j} > 0$ *almost surely.*

*When considering the transitions going to the "new" state* $A_\infty$ *(when the binary determination sequence of the concatenation gets stuck on* $0$*), we have*

— *When* $i \neq \infty$*, then for every* $n \in \boldsymbol{N}$*, we have* $Q_{i \xrightarrow{n} \infty} = 0$ *and we set for any* $t \in \boldsymbol{N}$

$$Q_{i \xrightarrow{\infty} \infty} = \boldsymbol{P}\left(\tau(t+1) = \infty | \tau(t) < \infty \wedge \vec{M}(t) = A_i\right)$$

$\vec{C}(t+1) - \vec{C}(t)$ *is then* $+\infty$ *almost surely by definition.*

— *When* $i = \infty$*, we need not define* $Q_{\infty \xrightarrow{n} \infty}$*, as we may assume* $\vec{C}(t+1) - \vec{C}(t) = +\infty$ *almost surely.*

We note that when the conditions are empty because $P_{i \xrightarrow{n} j} = 0$, then these definitions may be taken arbitrarily, as they have no effect on $\vec{C}$.

### Solved convex combination

We still aim at controlling the function $L_{T_0}$ : this is tantamount to controlling the value of one among the Laplace transforms $L_i^{(a)}$. Here, we want $L_{T_0}(a) = L_{M(0)}^{(a)}(C(0))$. The expression of $L_i^{(a)}(x)$ as a convex combination of previous values may now be calculated thanks to the previous $Q_{i \xrightarrow{n} j}$ and $\nu_{i \xrightarrow{n} j}$.

**Lemma 2.10.3** *Convolution equation for* $L_i^{(a)}$

*Let* $C$ *be a C-process,* $\vec{C}$ *be its descending process,* $a \in \boldsymbol{R}_+^*$*, and* $i, j \leq A$. *Thanks to the lemma 2.10.2, we define*

— $Q_{i \xrightarrow{n} j}$ *the conditional transition probabilities of* $\vec{C}$ *;*

— $\nu_{i \xrightarrow{n} j}$ *the mirror conditional transition payoffs of* $\vec{C}$.

*Let us note*

$$\nu_{i \to j}^{(a)} = \sum_{n=1}^{\infty} Q_{i \xrightarrow{n} j} e^{-an} \nu_{i \xrightarrow{n} j}$$

*These are nonnegative measures of finite masses, that solve*

$$\forall i \leq A, a \in \boldsymbol{R}_+^*, x \in \boldsymbol{R}^+, L_i^{(a)}(x) = \sum_{j=1}^{A} \left(L_j^{(a)} * \nu_{i \to j}^{(a)}\right)(x)$$

Noting by $Q_{i \to j}$ the sum of probabilities $Q_{i \xrightarrow{n} j}$ for $n \in \mathbf{N}$, we notice that

— These measures have finite masses because $a > 0$, $Q_{i\to j} \leq 1$, and the measures $\nu_{i\xrightarrow{n}j}$ have unit masses ;
— When $a$ goes to 0, we get

$$\nu^{(0)}_{i\to j} = \left(\sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j}\nu_{i\xrightarrow{n}j}\right)$$

whose mass is $Q_{i\to j}$ ; when it is non-zero, dividing the measure $\nu^{(0)}_{i\to j}$ by its mass yields the distribution of $(-1)\left(\vec{C}(t+1)-\vec{C}(t)\right)$ conditionally to $\vec{M}(t) = A_i$ and $\vec{M}(t) = A_j$.

Now we prove this lemma 2.10.3. To get $L_{T_0}$ using $\vec{C}$, we use the fact that $C$'s default time must be an all-time low for $C$, so must be recorded by $\vec{C}$ : we get $C$'s default time, reading $\vec{C}$'s default time $T_{\vec{C}} \in \mathbf{N}\cup\{\infty\}$ and computing the total time

$$T_0 = \tau\left(T_{\vec{C}}\right) = \sum_{t=1}^{T_{\vec{C}}}\left(\tau(t)-\tau(t-1)\right)$$

As $\vec{C}$ is a C-process, then the values $L^{(a)}_i(x)$ decomposes following the possibilities of immediate future for $\vec{C}$ as follows.

1. A random determination of both a transition (starting from $A_i$) and a descent time is drawn, with respect to $\vec{C}$'s transition probabilities

$$\left(Q_{i\xrightarrow{n}j}\right)_{i,j\leq A;n\in\mathbf{N}^*}$$

This has a multiplicative effect of $Q_{i\xrightarrow{n}j}$ on the following.
2. If $j \neq \infty$, after $n$ time periods, $M$ goes to $A_j$ and $C$ goes to $x - F_{i\xrightarrow{n}j}$ : waiting $n$ time periods has a multiplicative effect of $e^{-na}$ on the conditionnal expected value of the Laplace transform we get there

$$\mathbf{E}\left(e^{-aT_0}|C(0) = x - F_{i\xrightarrow{n}j} \wedge M(0) = A_j\right) = L^{(a)}_j\left(x - F_{i\xrightarrow{n}j}\right)$$

If $j = \infty$, then $T_0 = \infty$ almost surely, so the value we get there is 0.

So we finally find out that

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, L^{(a)}_i(x) = \sum_{j=1}^{A}\sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j}e^{-an}\mathbf{E}\left(L^{(a)}_j\left(x - F_{i\xrightarrow{n}j}\right)\right)$$

Rewriting this equation as a convolution equation using the distributions $\nu_{i\xrightarrow{n}j}$, one gets

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, L^{(a)}_i(x) = \sum_{j=1}^{A}\left(L^{(a)}_j * \left(\sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j}e^{-an}\nu_{i\xrightarrow{n}j}\right)\right)(x)$$

By definition of $\nu^{(a)}_{i\to j}$, this ends the proof.

**Masses**

This lemma 2.10.3 will be especially useful when the distributions $\nu^{(a)}_{i\to j}$, having a nonnegative support, have a unit mass, meaning that the values $L^{(a)}_i(x)$ will be convex combinations of previous values $L^{(a)}_i(y)$ for $y \leq x$. Our next step is now to twist these distributions to get the convex combinations. Hence, let $\beta \in \mathbf{R}$, and set

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, d\nu^{(a)}_{i\to j,[\beta]}(x) = d\nu^{(a)}_{i\to j}(x)e^{\beta x}$$

Then $\nu^{(a)}_{i\to j,[\beta]}$ is a nonnegative measure whose mass is

$$m^{(a)}_{i\to j,[\beta]} = \int_{x=0}^{\infty} e^{\beta x} d\nu^{(a)}_{i\to j}(x)$$

By definition of $\nu^{(a)}_{i\to j,[\beta]}$, the mass rewrites as

$$m^{(a)}_{i\to j,[\beta]} = \sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j} e^{-an}\mathbf{E}\left(e^{\beta F_{i\xrightarrow{n}j}}\right)$$

Recalling that $F_{i\xrightarrow{n}j}$ is upper bounded by $Q$ by construction and positive almost surely, then for $a \in \mathbf{R}^*_+$ and $\beta \in \mathbf{R}$, this mass is finite. Moreover, it is

— Non-zero whenever $Q_{i\to j} > 0$ ;

— 0 when $Q_{i\to j} = 0$, but this is no issue as it will have no effect on further computations.

After properties of the convolution product, and defining the twisted Laplace transforms

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, L^{(a)}_{i,[\beta]}(x) = L^{(a)}_i(x)e^{\beta x}$$

then lemma 2.10.3 rewrites as

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, L^{(a)}_{i,[\beta]}(x) = \sum_{j=1}^{A} \left(L^{(a)}_{j,[\beta]} * \nu^{(a)}_{i\to j,[\beta]}\right)(x)$$

If the measures $\nu^{(a)}_{i\to j,[\beta]}$ have non-zero masses, we may divide them by their finite masses, getting probability measures that will be noted $\phi^{(a)}_{i\to j,[\beta]}$ ; if they have zero mass, then we set $\phi^{(a)}_{i\to j,[\beta]}$ arbitrarily, as they will have no effect on the sequel. This leads to

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, L^{(a)}_{i,[\beta]}(x) = \sum_{j=1}^{A} m^{(a)}_{i\to j,[\beta]} \left(L^{(a)}_{j,[\beta]} * \phi^{(a)}_{i\to j,[\beta]}\right)(x)$$

Since the measures $\phi^{(a)}_{i\to j,[\beta]}$ are probability measures (with support in $(0,Q]$), they define random variables called $G^{(a)}_{i\to j,[\beta]}$ on the same support. The previous equation may hence be rewritten as

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, L^{(a)}_{i,[\beta]}(x) = \sum_{j=1}^{A} m^{(a)}_{i\to j,[\beta]} \mathbf{E}\left(L^{(a)}_{j,[\beta]}\left(x - G^{(a)}_{i\to j,[\beta]}\right)\right)$$

We shall use this equation in the next paragraph.

### Scaling

The previous paragraph ensured the measures $\phi^{(a)}_{i\to j,[\beta]}$ to be of unit masses, but at the expense of parameters $m^{(a)}_{i\to j,[\beta]}$ appearing in the equation. Now, we aim at scaling these parameters to get $L^{(a)}_{i,[\beta]}(x)$ as a convex combination of expectancies

$$\mathbf{E}\left(L^{(a)}_{j,[\beta]}\left(x - G^{(a)}_{i\to j,[\beta]}\right)\right)$$

To do that, we allow the functions $L^{(a)}_{i,[\beta]}$ to be scaled by multiplicative positive constants $\left(r_{[i]}\right)_{i\leq A}$ :

$$\forall i \leq A, L^{(a)}_{i,[\beta]} = r_{[i]} K^{(a)}_{i,[\beta]}$$

After this scaling, the convolution equation becomes

$$\forall i \leq A, a \in \mathbf{R}^*_+, x \in \mathbf{R}^+, r_{[i]} K^{(a)}_{i,[\beta]}(x) = \sum_{j=1}^{A} m^{(a)}_{i\to j,[\beta]} r_{[j]} \mathbf{E}\left(K^{(a)}_{j,[\beta]}\left(x - G^{(a)}_{i\to j,[\beta]}\right)\right)$$

Hence, we shall get the desired result whenever

$$\forall i \leq A, r_{[i]} = \sum_{j=1}^{A} m^{(a)}_{i\to j,[\beta]} r_{[j]}$$

which happens when $r$ is an eigenvector associated to eigenvalue 1 of the “mass” matrix $m^{(a),[\beta]}$, whose entries are

$$\forall i,j \leq A, m^{(a),[\beta]}_{i,j} = m^{(a)}_{i\to j,[\beta]}$$

The goal is now to adjust $\beta$ to $a$, so that $m^{(a),[\beta]}$ will have 1 as an eigenvalue.

### Martingale equation

We recall, after paragraph 2.10.1, that

$$\forall i,j \leq A, m_{i,j}^{(a),[\beta]} = m_{i\to j,[\beta]}^{(a)} = \sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j} e^{-an} \mathbf{E}\left(e^{\beta F_{i\xrightarrow{n}j}}\right)$$

Let $\tau$ be the waiting time for $C$'s first descent below its starting point, so that $\vec{C}(1) = C(\tau)$. For $i,j \leq A$, $a \in \mathbf{R}^+$, and $\beta \in \mathbf{R}^+$, we define the function

$$g_{i,j} = \begin{pmatrix} \mathbf{R}^+ \times \mathbf{R}^+ & \to & \mathbf{R}^+ \cup \{\infty\} \\ (a,\beta) & \to & \mathbf{E}\left(e^{-\beta(C(\tau)-C(0))} e^{-a\tau} \mathbf{1}_{\tau<\infty \wedge M(\tau)=A_j} | M(0) = A_i\right) \end{pmatrix}$$

Incidentally, we notice how close this definition comes to the function $f_C$, where the concatenation time $\tau$ is now given by the descent time instead of the return time. The computation of $g_{i,j}$ is obtained by disjunction on $C$'s future :

- $C$'s first descent happens at time $n \in \mathbf{N}^*$, with $M(n) = A_j$ : this happens with probability $Q_{i\xrightarrow{n}j}$, and yields an expectancy of

$$\mathbf{E}\left(e^{\beta F_{i\xrightarrow{n}j}} e^{-an} | M(0) = A_i\right) = e^{-an} \mathbf{E}\left(e^{\beta F_{i\xrightarrow{n}j}}\right)$$

- $C$'s first descent happens at time $n \in \mathbf{N}^*$, but with $M(n) = A_k \neq A_j$ : this happens with probability $Q_{i\xrightarrow{n}k}$, and yields an expectancy of 0.
- $C$ does not descend, which happens with probability $Q_{i\to\infty}$ and yields an expectancy of 0 by construction.

It follows that

$$g_{i,j}(a,\beta) = \sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j} e^{-an} \mathbf{E}\left(e^{\beta F_{i\xrightarrow{n}j}}\right) = m_{i,j}^{(a),[\beta]}$$

Hence, looking for the vector $r$ like above, one gets the equation

$$\mathbf{E}\left(r_{[M(\tau)]} e^{-\beta(C(\tau)-C(0))} e^{-a\tau} \mathbf{1}_{\tau<\infty} | M(0) = A_i\right) = \sum_{j=1}^{A} m_{i,j}^{(a),[\beta]} r_{[j]}$$

so the question is to find $\beta$ and $r$ such that

$$\forall i \leq A, r_{[i]} = \mathbf{E}\left(r_{[M(\tau)]} e^{-\beta(C(\tau)-C(0))} e^{-a\tau} \mathbf{1}_{\tau<\infty} | M(0) = A_i\right)$$

### Choice of $\beta$

We want to prove that $\beta = \alpha(a)$ and $r = w^{(a)}$ work for this equation. Thanks to the proposition 2.4.1, we know that the process $X_C^{(a)}$ is a martingale, so we build a stopping time for it, using parameters

— $T_1, T_2 \in \mathbf{N}^*$ (deterministic times) ;
— $X \in \mathbf{R}^+$ a barrier for $C$ ;

defining stopping times through

— $\tau_0$ the descent time of $C$, as before ;
— $\tau_1(X, T_1)$ such that

$$\tau_1(X, T_1) = \min\left(\{t > T_1; C(t) - C(0) > X\}\right)$$

which is a stopping time ;
— The stopping time that we shall use is

$$\tau(X, T_1, T_2) = \min\left(\tau_0, \tau_1(X, T_1), T_2\right)$$

so it is bounded by $T_2$.

Thanks to the martingale property for $X_C^{(a)}$, we have

$$\mathbf{E}\left(w^{(a)}_{[M(\tau(X,T_1,T_2))]} e^{-\alpha(a)(C(\tau(X,T_1,T_2))-C(0))} e^{-a\tau(X,T_1,T_2)}\right) = w^{(a)}_{M(0)}$$

and this decomposes to

$$\begin{aligned}
w^{(a)}_{M(0)} &= \mathbf{E}\left(w^{(a)}_{[M(\tau_0)]} e^{-\alpha(a)(C(\tau_0)-C(0))} e^{-a\tau_0} \mathbf{1}_{\tau(X,T_1,T_2)=\tau_0}\right) \\
&+ \mathbf{E}\left(w^{(a)}_{[M(\tau_1(X,T_1))]} e^{-\alpha(a)(C(\tau_1(X,T_1))-C(0))} e^{-a\tau_1(X,T_1)} \mathbf{1}_{\tau(X,T_1,T_2)=\tau_1(X,T_1)<\tau_0}\right) \\
&+ \mathbf{E}\left(w^{(a)}_{[M(T_2)]} e^{-\alpha(a)(C(T_2)-C(0))} e^{-aT_2} \mathbf{1}_{\tau(X,T_1,T_2)=T_2<\tau_1(X,T_1),\tau_0}\right)
\end{aligned}$$

We deal with these terms separately.

1. When $T_1$ and $T_2$ go to $\infty$ (no matter $X$), the monotone convergence theorem indicates that the first term converges to

$$\mathbf{E}\left(w^{(a)}_{[M(\tau_0)]} e^{-\alpha(a)(C(\tau_0)-C(0))} e^{-a\tau_0} \mathbf{1}_{\tau_0<\infty}\right)$$

which is the desired term in the equality to prove ;

2. Recalling that $w^{(a)}$'s coordinates are bounded through its spread, because it must have a coordinate no larger than 1 thanks to the second equation of scaling, the second term is bounded by

$$e^{\delta\left(w^{(a)}\right)} e^{-\alpha(a)X} \mathbf{P}\left(\tau(X, T_1, T_2) = \tau_1(X, T_1) < \tau_0\right)$$

3. Finally, the third term is bounded by

$$e^{\delta\left(w^{(a)}\right)} \mathbf{P}\left(\tau(X, T_1, T_2) = T_2 < \tau_1(X, T_1), \tau_0\right)$$

Now, we control the second and the third term, looking for values of $X, T_1, T_2$ such that they become arbitrarily small. As $\alpha(a) > 0$ because $a > 0$, one chooses $X$ such that $e^{-\alpha(a)X}$ is as small as wanted, and for this value of $X$, $C - C(0)$ has a zero probability of staying eternally in the interval $[0, X]$ (because it is true for its restricted Lévy process once $C$ is not globally constant), so one may choose $T_2$ such that the probability involved in the third term is as small as wanted. Hence, we proved that

$$\mathbf{E}\left(w^{(a)}_{[M(\tau_0)]} e^{-\alpha(a)(C(\tau_0)-C(0))} e^{-a\tau_0} \mathbf{1}_{\tau_0<\infty}\right) = w^{(a)}_{[M(0)]}$$

So, the values of functions $K^{(a)}_{i,[\alpha(a)]}(x)$ will be averages of previous values $K^{(a)}_{j,[\alpha(a)]}(y)$ for $y < x$, which means that the functions $K^{(a)}_{i,[\alpha(a)]}$ solve this lemma 2.10.1 for

$$P^{(a)}_{i\to j} = m^{(a)}_{i\to j,[\alpha(a)]} \frac{w^{(a)}_{[j]}}{w^{(a)}_{[i]}}$$

This leads to the desired result by definition of the functions $K^{(a)}_{j,[\alpha(a)]}$, which ends the proof of the lemma 2.10.1.

### 2.10.2 Properties of the convolution process

In this paragraph, we take $C$ as a positive recurrent, not globally increasing, bounded C-process. For $a \in \mathbf{R}^+$, let us define

— $\Phi^{(a)}_C$ its convolution process ;

— $M^{(a)}$ the underlying Markovian process of $\Phi^{(a)}_C$.

Our proof method for the proposition 2.4.4 and the theorem 2.1 transforms $C$ into $\Phi^{(a)}_C$. As a consequence, we want to ensure that $C$'s useful properties (positive recurrence, periodicity) are not broken by this transformation.

#### Regularity of the convolution processes

We want to state some properties for the regularity of $\Phi^{(a)}_C$'s parameters, especially when $a$ goes to 0 so that we will be able to extend the analysis to $a = 0$.

**Lemma 2.10.4** *Regularity of $\Phi^{(a)}_C$'s transitions*

*Let $C$ be as in lemma 2.10.1.*

1. *Viewed as a function of $a \in \boldsymbol{R}^+$, the probabilities $P^{(a)}_{i\to j}$ are continuous.*
2. *Viewed as functions of $a \in \boldsymbol{R}^+$ the random variables $G^{(a)}_{i\to j}$ such that $Q_{i\to j} > 0$ are continuous in probabilities, i.e. for every $i, j \le A$ with $Q_{i\to j} > 0$, for every measurable set $S \subseteq \boldsymbol{R}$,*

$$\lim_{b\to a}\left(\boldsymbol{P}\left(G^{(b)}_{i\to j} \in S\right)\right) = \boldsymbol{P}\left(G^{(a)}_{i\to j} \in S\right)$$

*Hence, we may define a convolution process $\Phi_C^{(0)}$ when $a = 0$ from the probabilities $P_{i\to j}^{(0)}$ and the random variables $G_{i\to j}^{(0)}$, that still solves the convolution property from the lemma 2.10.2.*

We prove this lemma 2.10.4 considering how $\Phi_C^{(a)}$'s parameters were built.

1. This comes from their definitions as

$$P_{i\to j}^{(a)} = m_{i\to j,[\alpha(a)]}^{(a)} \frac{w_{[j]}^{(a)}}{w_{[i]}^{(a)}}$$

As the dominant eigenvectors are known to be continuous, we only have to verify that the mass matrix is continuous, which is the consequence of $\alpha$'s continuity and

$$\forall a \in \mathbf{R}^+, \beta \in \mathbf{R}^+, m_{i\to j,[\beta]}^{(a)} = \sum_{n=1}^{\infty} Q_{i\xrightarrow{n}j} e^{-na} \mathbf{E}\left(e^{\beta F_{i\xrightarrow{n}j}}\right)$$

The random variables $F_{i\xrightarrow{n}j}$ are bounded by $Q$ and the probabilities $Q_{i\xrightarrow{n}j}$ sum to $Q_{i\to j} \leq 1$, so the continuity follows by monotonous convergence.

2. When $Q_{i\to j} > 0$, let us recall that the measures defined by

$$\nu_{i\to j}^{(a)} = \sum_{i=1}^{\infty} Q_{i\xrightarrow{n}j} e^{-an} \nu_{i\xrightarrow{n}j}$$

have non-zero masses. Let $S$ be any measurable subset of $\mathbf{R}:$ by definition,

$$\mathbf{P}\left(G_{i\to j}^{(a)} \in S\right) = \phi_{i\to j,[\alpha(a)]}^{(a)}(S) = \frac{\nu_{i\to j,[\alpha(a)]}^{(a)}(S)}{m_{i\to j,[\alpha(a)]}^{(a)}}$$

However, we have

$$\nu_{i\to j,[\alpha(a)]}^{(a)}(S) = \int_{x\in S} e^{x\alpha(a)} d\nu_{i\to j}^{(a)}(x)$$

so finally

$$\mathbf{P}\left(G_{i\to j}^{(a)} \in S\right) = \frac{\sum_{i=1}^{\infty} Q_{i\xrightarrow{n}j} e^{-an} \int_{x\in S} e^{x\alpha(a)} d\nu_{i\xrightarrow{n}j}}{m_{i\to j,[\alpha(a)]}^{(a)}}$$

Thus, it is a continuous expression of $a$, which ends the proof.

### Irrelevance of $a$ to the existence of paths

We mention here that $\Phi_C^{(a)}$ has a path of value $z \in \mathbf{R}$ for any $a \in \mathbf{R}^+$, then this path exists for all values of $a$ simultaneously.

**Lemma 2.10.5** *Simultaneous existence of paths*

*Let $a \in \boldsymbol{R}^+$. We assume that $P$ is a path for $\Phi_C^{(a)}$, and consider for every $\epsilon > 0$ the probability $p^{(a)}(\epsilon)$ of following $P$ at precision $\epsilon$, as in lemma 2.7.2. Then for every $b \in \boldsymbol{R}^+$,*

- *$P$ is a also a path for $\Phi_C^{(b)}$,*
- *Its probability $p^{(b)}(\epsilon)$ of being followed at precision $\epsilon$ converges to $p^{(a)}(\epsilon)$ when $b$ goes to $a$.*

We shall use the lemma 2.7.2 here. Let $P$ be a path of length $T \in \mathbf{N}$, we note

- $a_{t\in[|0,T|]} \leq A$ its successive occupied state numbers ;
- $v_{t\in[|0,T|]} \in \mathbf{R}$ its successive partial values (sums of the first $t$ payoffs values).

The probability of $\Phi_C^{(a)}$ following $P$ is

$$\prod_{t=1}^{T} P^{(a)}_{a_{t-1}\to a_t} \mathbf{P}\left(\forall t \leq T, \sum_{t=1}^{T} G^{(a),t}_{a_{t-1}\to a_t} \in [v_t \pm \epsilon]\right)$$

The notation $G^{(a),t}_{a_{t-1}\to a_t}$ rather than $G^{(a)}_{a_{t-1}\to a_t}$ highlights that these random variables are independent. Now,

- Since the distribution of the partial sums are given through convolution equations and convolution preserves continuity with respect to probabilities, then this probability is continuous of $a$.
- By construction, $P^{(a)}_{a_{t-1}\to a_t}$ will be non-zero iff the descending process $\vec{M}$ yields a non-zero transition from $A_{a_{t-1}}$ to $A_{a_t}$ ; since $P$ is a path, we get $Q_{a_{t-1}\to a_t} > 0$, so for every $b \in \mathbf{R}^+$ we still have $P^{(b)}_{a_{t-1}\to a_t} > 0$.
- As all random variables $G^{(b)}_{a_{t-1}\to a_t}$ have the same support, it follows that $P$ is still a path for $\Phi_C^{(b)}$.

This ends the proof.

### Closed communicating classes

As $M^{(a)}$ was defined thanks to a time concatenation through $\vec{C}$, we may have lost $M$'s positive recurrence during the concatenation. However, as we shall see here, this is no issue.

**Definition 2.10.3** *Descending class*

*There is $A' \subseteq [|1, A|]$ such that, for every $a \in \boldsymbol{R}^+$, $M^{(a)}$ has $A'$ as only closed communicating class. We name it $C$'s descending class.*

To ensure that this definition is correct, we start by proving that $a$ plays no part. By construction the increments $G^{(a)}_{i\to j}$ have the same support no matter $a$, and both zero and non-zero transition probabilities are conserved when $a$ changes by construction of $P^{(a)}_{i\to j}$ (like above, when paths were conservated). Now, let $i, j \leq A$ be state numbers indicating states $A_i$ and $A_j$, belonging to closed communicating classes $A'$ and $A''$ of $M^{(a)}$. As $C$ is positive recurrent and not globally increasing, there is a universal cycle from $A_i$ to itself whose value is $v < 0$.

— We concatenate this cycle an arbitrarily large number of times, and consider the successive all-time lows hit by this concatenation, at successive times $t_k \in \mathbf{N}$, whose values are named $v'(t_k)$ : the sequence $(t_k)_{k\in\mathbf{N}}$ is infinite because $v < 0$.

— We do the same thing but with analysis starting at the first hitting time of $A_j$ (that exists because the cycle is universal), and get a sequence $(u_k)_{k\in\mathbf{N}}$ describing all-time lows named $v''(u_k)$.

Once $v'(t_k)$ and $v''(t_k)$ are both negative, which happens because $v < 0$, then the times $u_k$ and $v_k$ will coincide (maybe with a shift in $k$), because they will describe the same all-time lows. Hence, let $t \in \mathbf{N}$ that is simultaneously a $t_k$ and a $u_k$. As $A'$ is closed, then any all-time low $v'(t_k)$ hit by a process starting from $A_i$ must hold $M(t_k) \in A'$. The same idea for $A''$ leads to $M(u_k) \in A''$, so $M(t) \in A' \cap A''$. However, as closed communicating classes are either disjoint or identical, this is possible only if $A' = A''$, which proves that there is at most one closed communicating class.

**Periodicity issues**

The property of periodicity is not impaired when changing $C$ into $\Phi^{(a)}_C$ either.

**Lemma 2.10.6** *Aperiodicity of C's convolution process*
*Let $a \in \boldsymbol{R}^+$.*

1. *If $C$ is aperiodic, then the restriction of $\Phi^{(a)}_C$ to its descending class $A'$ is aperiodic.*
2. *If $C$'s fundamental period is $p \in \boldsymbol{R}^*_+$, then the restriction of $\Phi^{(a)}_C$ to $A'$ is periodic, and its fundamental period is the same $p$.*

1. First, we shall deal with the case where $C$ is aperiodic. We need to recall $\Phi^{(a)}_C$'s cycle support by definition of its increments $-G^{(a)}_{i\to j}$ : for $-x \in [-Q, 0)$, we want to find a sufficient condition that will imply $x \in supp\left(G^{(a)}_{i\to j}\right)$. Let $i, j \in A'$ such that $Q_{i\to j} > 0$ : we recall that the random variables $G^{(a)}_{i\to j}$ were defined as the variables $G^{(a)}_{i\to j,[\alpha(a)]}$. By definition, they have the same support as $\phi^{(a)}_{i\to j,[\alpha(a)]}$, then as $\nu^{(a)}_{i\to j,[\alpha(a)]}$ because $Q_{i\to j} > 0$. As a consequence,

every $x \in (0, Q]$ will belong to $G^{(a)}_{i \to j}$'s support as soon as there is $n \in \mathbf{N}^*$ such that $Q_{i \xrightarrow{n} j} > 0$ and $x \in supp\left(\nu_{i \xrightarrow{n} j}\right)$. Hence, let us assume that there is a path from $A_i$ to $A_j$, of length $n \in \mathbf{N}^*$ such that, for some $\epsilon > 0$,

— Its final value $v$ is lower than $-\epsilon$ (then negative) ;
— All of its partial values are higher than $\epsilon$ (then positive).

Then with positive probability, a C-process starting from $A_i$ will follow this path, taking positive partial values higher than $\epsilon/2$ and a negative final value lower than $-\epsilon/2$, which means that it hits an all-time low at time $n$ : hence, $v$ must be in $supp\left(\sigma_{i \xrightarrow{n} j}\right)$. Now, let $A_{a_{i \in [|0,T|]}}$ be a cycle of length $T \in \mathbf{N}$ and value $v < 0$. We "turn around" this cycle, as in the definition 2.7.6, so that $A_{a_T}$ is an all-time low. We rename by $A_{a_{i \in [|0,T|]}}$ this cycle with $z(T) = v$ is its (single) all-time low. To prove that $v$ belongs to $\Phi_a$'s cycle support, it suffices to prove that it is the value of a cycle $A_{b_{i \in [|0,U|]}}$ of $M^{(a)}$ for some $U \in \mathbf{N}^*$. Let us consider $A_{a_{i \in [|0,T|]}}$ as a cycle for $M$, whose successive all-time lows are hit at times $t_k \leq T$, for $k \leq U$. We consider the restriction of this path over $[|t_{k-1}, t_k|]$ (with $t_0 = 0$). For the sake of simplicity, we will deem that none of its partial values $z_k(t)$ for $t \in [|t_{k-1}+1, t_k|]$ are exactly zero, a case that will be discussed later. Then its value $x_k$ holds

$$x_k \in supp\left(\sigma_{A_{t_{k-1}} \xrightarrow{n} A_{t_k}}\right)$$

thanks to the previous analysis, so $x_k$ belongs to $-G^{(a)}_{A_{t_{k-1}} \to A_{t_k}}$'s support. By concatenation, it follows that the sum of such values $x_k$ for $k \leq U$ (which amounts to $v$) defines a cycle for $\Phi_a$ whose value is $v$.

The case with partial sums (at $t \in [|t_{k-1}+1, t_k - 1|]$) equal to zero is a bit more tedious, as $C(t)$ may or may not be lesser than $C(t_{k-1})$. However, this is no issue, as one may proceed to the following disjunction :

— If $\mathbf{P}\left(C(t) - C(t_{k-1}) < 0\right) = 0$, then we just "skip" $t$, as it cannot be an all-time low for paths of $C$.
— If not, then we can consider in the above some $\epsilon > 0$ such that

$$\mathbf{P}\left(C(t) - C(t_{k-1}) < -\epsilon\right) > 0$$

and only paths for which $C(t) - C(t_{k-1}) < -\epsilon$, as this does not drive the sought probability to 0 ; then we introduce $t$ as an intermediate time like $t_{k-1/2}$ between $t_{k-1}$ and $t_k$.

It follows that any cycle value $v < 0$ of $C$ is a cycle value for $\Phi^{(a)}_C$, so if $\Phi^{(a)}_C$ has a period $p$, then we get $ucs(C) \cap \mathbf{R}^*_- \subseteq -p\mathbf{N}^*$ is non-empty because $C$ is positive recurrent, and as $ucs(C) + supp(C) \subseteq ucs(C)$, then $supp(C)$ (and also $ucs(C)$) must be a subset of $p\mathbf{Z} \cup \{\infty\}$, so $C$ will be periodic, which is contradictory.

2. Now, we deal with the case where $C$ is periodic, of fundamental period $p \in \mathbf{R}^+$, thus has a regular process $\tilde{C}$ of period 1. $\tilde{C}$'s convolution process is governed by integer increments, so it must be periodic with an integer period $e'$ at least 1. As all-time lows for $C$ are governed only by its "integer" part, then $C$'s descending process $\vec{C}$, thus $C$'s convolution process $\Phi_C^{(a)}$, is periodic of period $pe'$, so we want to prove that $e'$ is at most 1. In the rest of the proof, we shall note :
   — $V = supp(\tilde{C})$ is $\tilde{C}$'s cycle support ;
   — $U = ucs(\tilde{C})$ is $\tilde{C}$'s universal cycle support.
   As $\tilde{C}$ has fundamental period 1, we may find coprime integers $u_{k \leq q} \in -V$, denoting (negated) cycle values of $\tilde{C}$ for each cycle number $k$, for some $q \in \mathbf{N}^*$. After Bézout's identity, there are integers $m_{k \leq q} \in \mathbf{Z}$ such that
$$\sum_{k=1}^{q} u_k m_k = 1$$
   Hence, noting $m = 2 + \max_k (-m_k) \in \mathbf{N}^*$ for example, then for each $k \leq q$, $(m + m_k)u_k$ is the value of $(m + m_k) > 0$ concatenations of cycle number $k$, thus belongs to $-V$. Now, let us take any universal cycle for $\tilde{C}$ of value $x \in \mathbf{Z}$. As $U + V \subseteq U$, then we concatenate it to the following cycles to obtain new universal cycles :
   — Considering $m$ concatenations of each cycle, we get a cycle of negated value
$$y_1 = -x + \sum_{k=1}^{q} u_k m \in -U$$
   — Considering $m + m_k$ concatenations of cycle number $k$ and $m$ of the others, we get a cycle of negated value
$$y_2 = -x + \sum_{k=1}^{q} u_k (m + m_k) = y_1 + 1 \in -U$$
   Once again, we rotate the cycles so that their final values $y_1$ and $y_2$ are all-time lows, so belong to $\Phi_{\tilde{C}}^{(a)}$'s cycle support. We apply lemma 2.7.8 to $-y_1$ and $-y_2$, which proves that for some $X^- \in \mathbf{R}$, then for every $x \leq X^-$, $[x \pm 1/2] \cap U \neq \emptyset$. However, as $U \subseteq \mathbf{Z}$, then $[x \pm 1/2] \cap U \cap \mathbf{Z} \neq \emptyset$ : taking integer values for $x$, this is only possible if for any $x \in \mathbf{Z}$ and lower than $X^-$ we have $x \in U$. This implies that
   — There is a universal cycle of any value lower than $X^-$ (and this will be used later) ;
   — $U$ has period at most 1.
   So $U$ has 1 as a fundamental period.

This ends the proof.

### 2.10.3 Splitting between periodic and aperiodic cases

In this paragraph, we explain how to use the previous convolution process to prove the proposition 2.4.4 or the theorem 2.1. We remember that periodic C-processes were explicitly excluded from the theorem 2.1 because, as said above, "the references to previous values do not merge", and we are now going to explain this assertion in here.

#### Explanation

To clarify matters, we assume for now that $C$ is a Lévy process, so $\forall a, w^{(a)} = 1$. We consider a bounded stopping time $\tau$ for $C$'s convolution process $\Phi_C^{(a)}$ ; the martingale property from lemma 2.10.2 thus yields

$$\forall a \in \mathbf{R}^+, K^{(a)}(C(0)) = \mathbf{E}\left(K^{(a)}(C(\tau))\right)$$

We seek the local extrema for $K^{(a)}$ ; for example, let us imagine that we have sequences of reals $x_n^+$ (maxima) and $x_n^-$ (minima) such that

$$\lim_{n\to\infty}\left(K^{(a)}(x_n^+)\right) = l^+ \wedge \lim_{n\to\infty}\left(K^{(a)}(x_n^-)\right) = l^-$$

where $l^- \leq l^+$. To prove that $l^- = l^+$, our idea is to use $\Phi_C^{(a)}$ starting from some $x_n^+$, and halt it at a $\tau$ featuring with a positive probability some value $C(\tau) = x_m^-$ (for $n, m \in \mathbf{N}$). The line of thought is roughly :

1. $l^-$ keeps "pulling" values of $K^{(a)}$ at points $x_n^+$ to itself through a convex combination (given by the martingale property for $Y^{(a)}$) involving this positive coefficient ;
2. As a consequence, the discrepancy between $K^{(a)}(x_m^-)$ and $K^{(a)}(x_n^+)$ must vanish over time ;
3. So, as $K^{(a)}$ is bounded, it must converge. For now, we are not interested in the value of this limit $l^- = l^+$, which will be the subject of a further part of the study.

Unfortunately, this approach fails when $C(\tau)$ cannot hit any $x_m^-$. Indeed, the lemma 2.7.9 ensures that remote density allows "closing to" remote enough values $x_m^-$ only if $C$ is aperiodic, so the main theorem 2.1 will not hold for periodic C-processes : one gets the proposition 2.4.4 instead. The behaviour of a C-process may be described by either one of incoming lemmata, depending on whether $C$ is periodic or not.

— When $C$ is periodic, we recall that it suffices to prove the lemma 2.9.1 to get the theorem 2.4.4. To get it, we target a weaker form given below, the lemma 2.10.10.

— When it is not, we directly tackle the theorem 2.1, using the values of $\Phi_C^{(a)}$ hit by $\tau^{(a)}$, chosen to be "close" to the values $x_m^-$. We target the weaker lemma 2.10.11.

Before splitting the study, we start by creating $\tau$.

### Stopping time

As explained above, we aim at finding a suitable stopping time $\tau$, allowing to express the value

$$Y^{(a)}(0) = K^{(a)}_{M^{(a)}(0)}\left(\Phi_C^{(a)}(0)\right) = K^{(a)}_{M(0)}\left(C(0)\right)$$

as a convex combination of well-chosen values $K_i^{(a)}(x)$, for $x < C(0)$. A central idea is to consider the functions $K_i^{(a)}$ only over intervals whose forms are $[x-Q, x)$ for $x \in \mathbf{R}^+$. The reason behind this is that, as the increments $G^{(a)}_{i\to j}$ are deemed bounded, any path for $\Phi_C^{(a)}$'s successive values must hit some point in this region before going down : as a consequence, the global behaviour of the functions $K_i^{(a)}$ may be found looking at one such interval only. For this reason, for every $x \in \mathbf{R}^+$, we shall name $[x-Q, x)$ the $x$-interval, and look at the extrema of the functions $K_i^{(a)}$ over these.

**Definition 2.10.4** *Region-halting time*

*Let $\Phi_C^{(a)}$ be $C$'s convolution process at some point $a \in \boldsymbol{R}_+^*$, and $x \in \boldsymbol{R}^+$. We set the random process $\sigma^{(a)}$ as, for every $t \in \boldsymbol{N}$,*

— *If $\Phi_C^{(a)}(t) \geq x$, then $\sigma^{(a)}(t) = 1$ ;*

— *If $\Phi_C^{(a)}(t) \in [x-Q, x)$, then $\sigma^{(a)}(t) = 1/2$ ;*

— *Else $\sigma^{(a)}(t) = 0$.*

*We define random variables $Z(t)$ to be independent and identically distributed, of uniform distribution over $(0,1)$ and independent from anything else, and set $\rho^{(a)}$ to be the binary sequence defined by*

$$\forall t \in \boldsymbol{N}, \rho^{(a)}(t) = \boldsymbol{1}_{Z(t) > \sigma^{(a)}(t)}$$

*Finally, we set $\tau^{(a)}$ to be the stopping time defined by*

$$\tau^{(a)} = \min\left(\left\{t \in \boldsymbol{N}; \rho^{(a)}(t) = 1\right\}\right)$$

*We call $\tau^{(a)}$ the $x$-halting time of $\Phi_C^{(a)}$.*

Before studying the convolution process itself through the martingale property, we ensure that $\tau^{(a)}$ is eligible to this martingale property. As the functions $K_i^{(a)}$ are locally bounded and $\Phi_C^{(a)}$ is decreasing and bounded by $-Q$, this is tantamount to asking if $\tau^{(a)} < \infty$ almost surely.

**Lemma 2.10.7** *Finiteness of* $\tau^{(a)}$

*Let* $C$ *be a positive recurrent, bounded, not globally increasing C-process,* $a \in \boldsymbol{R}^+$*, and* $x \in \boldsymbol{R}^+$*. Let* $\tau^{(a)}$ *be the* $x$*-halting time of* $C$*'s convolution process at point* $a$ *:*

$$\boldsymbol{P}\left(\tau^{(a)} < \infty\right) = 1$$

Let us set $a \in \mathbf{R}^+$, and for every $\eta > 0$ the value

$$p(\eta) = \max_{i \leq A}\left(\sum_{j=1}^{A} P^{(a)}_{i\to j}\mathbf{P}\left(G^{(a)}_{i\to j} \leq \eta\right)\right)$$

We know that $p$ is a non-increasing function of $\eta$ and that

$$\lim_{\eta\to 0}\left(p(\eta)\right) = 0$$

so there is $\eta > 0$ such that $p(\eta) < 1/2$ (so $\eta \leq Q$), and we keep this value of $\eta$ in the sequel of the proof. By construction, if $\Phi^{(a)}_C$ goes below $x - Q$ at some time $t \in \mathbf{N}$, then $\tau^{(a)} \leq t < \infty$, so one only needs to prove that $\Phi^{(a)}_C$ will eventually lose $C(0) + Q - x$. Let us note

$$n = 1 + \left\lfloor \frac{C(0) + Q - x}{\eta} \right\rfloor$$

If $\Phi^{(a)}_C$ loses at least $\eta$ at least $n$ times, then $\Phi^{(a)}_C$ lost more than $C(0) + Q - x$ and thus stops $\tau^{(a)}$. However, we know at at each step $t \in \mathbf{N}$, no matter the present situation $\Phi^{(a)}_C(t)$ and $M^{(a)}(t)$,

$$\mathbf{P}\left(\Phi^{(a)}_C(t+1) < \Phi^{(a)}_C(t) - \eta\right) = \mathbf{P}\left(G^{(a)}_{M(t)\to M(t+1)} > \eta\right) > 1/2$$

This being for every $\Phi^{(a)}_C(t)$ and $M^{(a)}(t)$, Borel-Cantelli's lemma states that the event $G^{(a)}_{M(t)\to M(t+1)} > \eta$ will happen infinitely often almost surely, so this ends the proof.

### Case of periodic C-processes

In this paragraph, we assume that $C$ is periodic, and look at the hypotheses to the lemma 2.9.1 : hence, we may deem that $C$ lies in $\mathbf{Z}$ and has 1 as a fundamental period. Later on, this condition will be called "$C$ is regular". Hence, rather than controlling $K^{(a)}_i$'s values over whole $x$-intervals, we shall only look at their values over intervals like (when $x \in \mathbf{N}$)

$$[x - Q, x) \cap \mathbf{Z} = [|x - Q, x - 1|]$$

because the convolution equation will only involve negative integer increments of $\Phi_C^{(a)}$, so values in this latter set. As a consequence, the useful controls are local extrema given below, where we recall that $A'$ is $C$'s descending class.

**Definition 2.10.5** *Local discrete extrema*

*Let $K_i^{(a)}$ be the functions defined by the lemma 2.10.1. We define their local discrete extrema as follows.*

- *The local discrete maxima of positive recurrent functions $K_i^{(a)}$ over the x-interval is a function of $x$, given by*

$$K_+^{(a)} = \begin{pmatrix} \boldsymbol{N} & \rightarrow & \boldsymbol{R} \\ x & \rightarrow & \max_{y\in[|x-Q,x-1|],i\in A'}\left(K_i^{(a)}(x)\right) \end{pmatrix}$$

- *The local discrete maxima of all functions $K_i^{(a)}$ over the x-interval is a function of $x$ given by*

$$K_{++}^{(a)} = \begin{pmatrix} \boldsymbol{N} & \rightarrow & \boldsymbol{R} \\ x & \rightarrow & \max_{y\in[|x-Q,x-1|],i\leq A}\left(K_i^{(a)}(x)\right) \end{pmatrix}$$

- *The local discrete minima of positive recurrent, and all functions $K_i^{(a)}$ over the x-interval, are defined likewise and named repectively $K_-^{(a)}$ and $K_{--}^{(a)}$.*

To simplify the incoming work, we want to prove that they are monotone.

**Lemma 2.10.8** *Monotonicity of local extrema*

*The functions $K_+^{(a)}$, $K_{++}^{(a)}$, $K_-^{(a)}$, $K_{--}^{(a)}$ given in the definition 2.10.5 hold the following properties.*

1. *They are sorted by ascending order as in*

$$\forall x \in \boldsymbol{N}, K_{--}^{(a)}(x) \leq K_-^{(a)}(x) \leq K_+^{(a)} \leq K_{++}^{(a)}(x)$$

2. *$K_+^{(a)}$ and $K_{++}^{(a)}$ are non-increasing, $K_-^{(a)}$ and $K_{--}^{(a)}$ are non-decreasing.*
3. *All four of them converge to finite limits, respectively called $l_+^{(a)}$, $l_{++}^{(a)}$, $l_-^{(a)}$, $l_{--}^{(a)}$.*

Sorting these functions in ascending order comes from their definitions and $A' \subseteq [|1, A|]$, so we shall focus on the monotonicity property ; since the proofs are almost similar, we shall only present the case of $K_+^{(a)}$. Let $x \in \mathbf{N}$ : we know that, thanks to the convolution equation from lemma 2.10.2, every function $K_i^{(a)}$ rewrites so that

$$K_i^{(a)}(x) = \sum_{j=1}^{A}\sum_{d=1}^{Q} P_{i\to j}^{(a)}\mathbf{P}\left(G_{i\to j}^{(a)} = d\right) K_j^{(a)}(x-d)$$

If $i \in A'$, we know that the contributions of values $K_j^{(a)}(x-d)$ when $j \notin A'$ are non-existent because $A'$ is closed, thus $P_{i\to j}^{(a)} = 0$. However, the remaining terms $K_j^{(a)}(x-d)$ are involved in the local maximum $K_+^{(a)}(x)$, which means that

$$K_i^{(a)}(x) \leq \sum_{j\in A'} \sum_{d=1}^{Q} P_{i\to j}^{(a)} \mathbf{P}\left(G_{i\to j}^{(a)} = d\right) K_+^{(a)}(x) = K_+^{(a)}(x)$$

because the coefficients form a convex combination. This being for every $i \in A'$, we have that

- All values $K_j^{(a)}(x-d)$ for $j \in A', d \in [|1, Q-1|]$ are involved in the maximum $K_+^{(a)}(x)$ ;
- The values $K_j^{(a)}(x)$ for $j \in A'$ are no higher than $K_+^{(a)}(x)$.

It follows that

$$K_+^{(a)}(x+1) = \max\left(\max_{j\in A', d\in[|1,Q-1|]}\left(K_j^{(a)}(x-d)\right), \max_{j\in A'}\left(K_j^{(a)}(x)\right)\right) \leq K_+^{(a)}(x)$$

The case of $K_{++}^{(a)}$ being similar (removing the assumption on $A'$) and the cases of $K_-^{(a)}$ and $K_{--}^{(a)}$ being symmetrical, this ends the proof.

### Case of aperiodic C-processes

When $C$ is aperiodic, we need to control $K_i^{(a)}$'s values over whole $x$-intervals. The local extrema are thus defined as follows.

**Definition 2.10.6** *Local extrema*

*Let $K_i^{(a)}$ be the functions defined by the lemma 2.10.1. We define*

- *The local maxima of positive recurrent functions $K_i^{(a)}$ over the $x$-interval as a function of $x$, given by*

$$K_+^{(a)} = \begin{pmatrix} \boldsymbol{R}^+ & \rightarrow & \boldsymbol{R} \\ x & \rightarrow & \sup_{y\in[x-Q,x), i\in A'}\left(K_i^{(a)}(x)\right) \end{pmatrix}$$

- *The local maxima of all functions $K_i^{(a)}$ over the $x$-interval as a function of $x$ given by*

$$K_{++}^{(a)} = \begin{pmatrix} \boldsymbol{R}^+ & \rightarrow & \boldsymbol{R} \\ x & \rightarrow & \sup_{y\in[x-Q,x), i\leq A}\left(K_i^{(a)}(x)\right) \end{pmatrix}$$

- *The local minima of positive recurrent, and all functions $K_i^{(a)}$ over the $x$-interval, likewise and named repectively $K_-^{(a)}$ and $K_{--}^{(a)}$.*

*They are well-defined and locally bounded, since for every $a \in \boldsymbol{R}^+$, $i \leq A$ and $x \in \boldsymbol{R}^+$ we have*

$$K_i^{(a)}(x) \in \left[0, \frac{e^{\alpha(a)x}}{w_{[i]}^{(a)}}\right]$$

They are still monotone.

**Lemma 2.10.9** *Monotonicity of local extrema*

*The functions $K_+^{(a)}$, $K_{++}^{(a)}$, $K_-^{(a)}$, $K_{--}^{(a)}$ given in the definition 2.10.6 hold the following properties.*

1. *They are sorted by ascending order as in*

$$\forall x \in \boldsymbol{R}^+, K_{--}^{(a)}(x) \leq K_-^{(a)}(x) \leq K_+^{(a)} \leq K_{++}^{(a)}(x)$$

2. *$K_+^{(a)}$ and $K_{++}^{(a)}$ are non-increasing, $K_-^{(a)}$ and $K_{--}^{(a)}$ are non-decreasing.*
3. *All four of them converge to finite limits, respectively called $l_+^{(a)}$, $l_{++}^{(a)}$, $l_-^{(a)}$, $l_{--}^{(a)}$.*

This proof uses the $\eta$ and $p(\eta)$ from the proof of the lemma 2.10.7. Sorting these functions in ascending order comes from their definitions and $A' \subseteq [|1, A|]$, so we shall focus on the monotonicity property ; since the proofs are almost similar, we shall only present the case of $K_+^{(a)}$. We want to prove by induction on $n \in \mathbf{N}$ that for every $n \in \mathbf{N}$,

$$\sup_{i \leq A, z \in [x-\eta, x+n\eta)} (f_i(z)) \leq K_+^{(a)}(x)$$

For $n = 0$, this is true because $[x - \eta, x) \subseteq [x - Q, x)$ ; so we focus on the transmission of the property. Now, let us assume by contradiction that for some $x \in \mathbf{R}^+$,

$$\exists i_0 \in A', z_0 \in [x + n\eta, x + (n+1)\eta); (f_{i_0}(z_0)) - K_+^{(a)}(x) = y_0 > 0$$

We write $f_{i_0}(z_0)$ as given by the convolution equation. Since $p(\eta) < 1/2$, then there must be $i_1 \in A'$ and $z_1 \in [x + n\eta, z_0)$ such that

$$(f_{i_1}(z_1)) - K_+^{(a)}(x) = y_1 \geq 2y_0$$

else one would have

$$K_+^{(a)}(x) + y_0 = Y^{(a)}(t) = \mathbf{E}\left(Y^{(a)}(t+1)\right)$$

being bounded by hypothesis by

$$\begin{pmatrix} \mathbf{P}\left(\Phi_C^{(a)}(t+1) < x + n\eta\right) K_+^{(a)}(x) \\ +\mathbf{P}\left(\Phi_C^{(a)}(t+1) \geq x + n\eta\right)\left(K_+^{(a)}(x) + 2y_0\right) \end{pmatrix} \leq s(x) + 2p(\eta)y_0 < s(x) + y_0$$

because $M^{(a)}(t+1) \in A'$ since $A'$ is closed, which would be contradictory (N.B. : for $K_{++}^{(a)}$, one needs no condition on $M^{(a)}(t+1)$). Starting over with $x_1$ and $i_1$, one would progressively get sequences indexed by $u \in \mathbf{N}$, given by

— $x_u \in [x + n\eta, x + (n+1)\eta)$ ;
— $i_u \in A'$ ($i_n \leq A$ for $K_{++}^{(a)}$) ;
— $y_u \in \mathbf{R}_+^*$,

such that

$$(f_{i_u}(z_u)) - s(x) = y_u \geq 2^u y_0$$

but this is impossible if $y_0 > 0$ since all points $x_n$ belong to the finite interval $[x + n\eta, x + (n+1)\eta)$ and the functions $f_i$ are locally bounded. As the cases of $K_-^{(a)}$ and $K_{--}^{(a)}$ are symmetrical, we proved the monotonicity property, which in turn implies that the functions converge thanks to their natural inequalities.

**Separate lemmata**

We present the goal of the next step : intermediate lemmata, indicating that the twisted Laplace transforms converge (disregarding their limits). The rest of the work is now to :

1. Prove them ;
2. Find the value of the limits ;
3. Get back to the sought statements : proposition 2.4.4 and theorem 2.1, using them.

When $C$ is periodic, we aim at proving this lemma :

**Lemma 2.10.10** *Weak theorem for periodic C-processes*

*Let $C$ be a C-process such that*

— *$C$ is positive recurrent, bounded and not globally increasing ;*
— *For every $t \in \boldsymbol{N}$, $C(t) \in \boldsymbol{Z}$ almost surely ;*
— *$C$'s fundamental period is 1.*

*For every $a \in \boldsymbol{R}^+$, we set*

— *$C$'s martingale parameter at point $a$ as $\alpha(a)$ ;*
— *$C$'s dominant eigenvector at point $a$ as $w^{(a)}$.*

*We define the functions $K_i^{(a)}$ as given by the lemma 2.10.2. Then there are*

— *A function $K_\infty : \left(\boldsymbol{R}^+ \to \boldsymbol{R}_+^*\right)$ ;*
— *Exponential error functions*

$$Z : \left(\boldsymbol{R}^+ \to \boldsymbol{R}^+\right) \wedge \beta : \left(\boldsymbol{R}^+ \to \boldsymbol{R}_+^*\right)$$

*$Z$ and $\beta$ being continuous over $\boldsymbol{R}^+$,*

*such that*

$$\forall i \leq A, x \in \boldsymbol{N}, a \in \boldsymbol{R}^+, K_i^{(a)}(x) \in \left[K_\infty(a) \pm Z(a)e^{-\beta(a)x}\right]$$

When $C$ is aperiodic, we want this one :

**Lemma 2.10.11** *Weak theorem for aperiodic C-processes*

*Let $C$ be a positive recurrent, bounded and not globally increasing C-process, deemed aperiodic. For every $a \in \boldsymbol{R}^+$, we set*

— *C's martingale parameter at point a as $\alpha(a)$ ;*

— *C's dominant eigenvector at point a as $w^{(a)}$.*

*We define the functions $K_i^{(a)}$ as given by the lemma 2.10.2. Then there are*

— *A function $K_\infty : \left(\boldsymbol{R}^+ \to \boldsymbol{R}_+^*\right)$ ;*

— *A continuous error function*

$$e : \left(\left(\boldsymbol{R}^+ \times \boldsymbol{R}^+\right) \to \boldsymbol{R}^+\right)$$

*that converges uniformly over any compact set for its first variable, i.e.*

$$\forall a \in \boldsymbol{R}^+, \forall \epsilon > 0, \exists x_0(a,\epsilon); \forall b \in [0,a], \forall x \geq x_0(a,\epsilon), e(a,x) \leq \epsilon$$

*such that*

$$\forall i \leq A, x \in \boldsymbol{R}^+, a \in \boldsymbol{R}^+, K_i^{(a)}(x) \in [K_\infty(a) \pm e(a,x)]$$

In both cases, by definition of $K_i^{(a)}$, the remaining work will be to focus on the function $K_\infty$ to end the proofs.

### 2.10.4 Lemma 2.10.10

We want to prove the the weak theorem for periodic C-processes. To do this, we will set some $x \in \mathbf{R}^+$ defining the $x$-halting time $\tau^{(a)}$, and follow these steps :

1. Find a suitable lower bound $p^{(a)}$ to the probabilities of $\tau^{(a)}$ halting $\Phi_C^{(a)}$ on a specific situation

$$M^{(a)}\left(\tau^{(a)}\right) = A_i \wedge \Phi_C^{(a)}\left(\tau^{(a)}\right) = y$$

   for $i \in A'$ and $y \in \mathbf{N}$ ;
2. Prove that the discrepancy between $K_-^{(a)}(x)$ and $K_+^{(a)}(x)$ must decay exponentially thanks to this $p^{(a)}$ ;
3. Extend this control to $K_{--}^{(a)}(x)$ and $K_{++}^{(a)}(x)$ ;
4. Find out that the parameters controlling the latter discrepancy may be chosen as continuous functions of $a$, so they lead to the lemma 2.10.10.

### Lower bound

As $\Phi_C^{(a)}$ stays in $\mathbf{Z}$, we only need to prove that for some $x \in \mathbf{N}$, the $x$-halting time $\tau^{(a)}$ may halt $\Phi_C^{(a)}$ at any point

$$\Phi_C^{(a)}\left(\tau^{(a)}\right) = y \wedge M^{(a)}\left(\tau^{(a)}\right) = A_i$$

for $y \in [|x - Q, x - 1|]$ and $i \in A'$.

**Lemma 2.10.12** *Uniform references to previous values*

*Let $\Phi_C^{(a)}$ be C's convolution process at any point $a \in \boldsymbol{R}_+^*$, whose starting point is $C_0 \in \boldsymbol{N}$. We name $A'$ its descending class.*

1. *There is a minimal $x_0 \in \boldsymbol{N}^*$ such that, for every $a$, the $(C_0 - x_0)$-halting time $\tau^{(a)}$ of $\Phi_C^{(a)}$ holds for every $y \in [|C_0 - x_0 - Q, C_0 - x_0 - 1|]$ and $i \in A'$,*

$$\boldsymbol{P}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right) = y \wedge M^{(a)}\left(\tau^{(a)}\right) = A_i\right) > 0$$

   *We name :*
   - *This minimal $x_0$ is $\Phi_C^{(a)}$'s ergodic pace, noted $e(C)$ ;*
   - *Any value $h^{(a)}(C)$ no higher than both $1/2$ and all these probabilities is a reducing constant of $\Phi_C^{(a)}$.*
2. *When $a$ varies into $\boldsymbol{R}^+$,*
   - *$e(C)$ does not change ;*
   - *We may choose reducing constants so that $h^{(a)}(C)$ varies continuously.*

To prove the lemma 2.10.12, we take $\Phi_C^{(a)}$ to be $C$'s convolution process at any point $a \in \mathbf{R}_+^*$, whose
- Starting point is $C_0 \in \mathbf{N}$ ;
- Starting state is $s \leq A$ ;
- Descending class is $A'$.

We shall take an arbitrarily large $x_0$, then

1. Exhibit a path of $\Phi_C^{(a)}$ leading to any event afterwards noted

$$B(y,i) = \Phi_C^{(a)}\left(\tau^{(a)}\right) = y \wedge M^{(a)}\left(\tau^{(a)}\right) = A_i$$

2. Prove that $\Phi_C^{(a)}$ has a positive probability of following this path and then being stopped by $\tau^{(a)}$ at the end.

The lemma 2.10.12 will follow, when considering how $e(C)$ and $h^{(a)}(C)$ are built.

1. We begin by building a path from $A_s$ to $A_i$.

(a) First, as $A'$ is the only closed communicating class of $\Phi_C^{(a)}$, if $s \notin A'$, there is a way to $A'$ (as else $[|1, A|] \setminus A'$ would be closed, thus would have a closed communicating sub-class). We get (through the lemma 2.7.3) a path $P_1$ of minimal length from $A_s$ to some state $A_r$ with $r \in A'$, of finite value $v_1 \in -\mathbf{N}$.
   - This path $P_1$ works for all values of $a \in \mathbf{R}_+^*$ simultaneously thanks to the lemma 2.10.5.
   - As $\Phi_C^{(a)}$'s values are integers, following this path at precision e.g. $\epsilon = 1/3$ in the terms of lemma 2.7.2 means following it exactly, so $P_1$ has a positive probability $p_s(a) > 0$, that is continuous of $a \in \mathbf{R}^+$ (and depends on $A_s$).
   - As $P_1$'s length is minimal, it is at most $A$ ; since its payoffs are bounded by $Q$, then $-v_1 \leq AQ$.

(b) Since $A'$ is a closed communicating class of $\Phi_C^{(a)}$, there is a path $P_2$ of minimal length from $A_r$ to $A_i$. For the same reasons, its value is noted $v_2 \in [|-AQ, 0|]$, and its probability is noted $q_i(a) > 0$.

(c) We also know after lemma 2.7.9 that $\Phi_C^{(a)}$ has universal (in $A'$) cycles of any low enough integer value. In particular, there is a universal cycle starting from $A_i$ of every integer value $k \in [|-X-Q, -X+2AQ-1|]$ for some $X \in \mathbf{N}$. For each $k$, we note by $r_k(a) > 0$ the probability of such a cycle.

We concatenate the paths with every cycle. It follows that there are paths from $A_s$ to $A_i$ for every value in

$$[|-X-Q+v_1+v_2, -X+2AQ-1+v_1+v_2|]$$

As $v_1 + v_2 \in [|-2AQ, 0|]$, there are paths from $A_s$ to $A_i$ of every value in

$$\bigcap_{v=0}^{2AQ} [|-X-Q-v, -X+2AQ-1-v|] = [|-X-Q, -X-1|]$$

whose probabilities are at least

$$\min_{(s \leq A, i \in A', k \in [|-X-Q, -X+2AQ-1|])} (p_s(a) q_i(a) r_k(a)) = q(a) > 0$$

So, we proved that there are $q(a) > 0$ and $X \in \mathbf{N}$ such that for every $s \leq A$, $i \in A'$ and $v \in [|-X-Q, -X-1|]$, there is a path for from $A_s$ to $A_i$ whose value is $v$ and probability at least $q(a)$.

2. We shall prove that $x_0 = X$ works. Let us take
   - A "target" point

$$y \in [|C_0 - X - Q, C_0 - X - 1|]$$

— A "target" state $A_{i\in A'}$.
We start the convolution process $\Phi_C^{(a)}$ from $C_0 \in \mathbf{N}$ and $A_{s\leq A}$ : as proved above, it has a probability at least $q(a)$ of following the path leading to the target point $y$ and the target state $A_i$. If it follows this path, then the successive values $\rho(t)$ amount to
— 1 as long as $\Phi_C^{(a)}$ remains at least $C_0 - X$ ;
— 1/2 when $\Phi_C^{(a)}$ crosses $C_0 - X - 1$, staying there until the end of the path so for at most $Q$ time periods, since it decreases by at least 1 per step and becomes 0 when $\Phi_C^{(a)}(t)$ goes below $C_0 - X - Q$.
Hence, by construction of $\tau$, the probability of $\tau$ stopping precisely at the end of the path is at least $1/2^Q$ no matter the path taken. Since the path probability is at least $q(a)$, then

$$\forall y \in [|C_0 - X - Q, C_0 - X - 1|], i \in A', \mathbf{P}(B(y,i)) \geq \frac{q(a)}{2^Q}$$

3. We move on to the properties of $e(C)$ and $h^{(a)}(C)$ when $a$ changes.
— We remark that, no matter $a$, we have

$$\forall i \in A', y \in [|C_0 - x_0 - Q, C_0 - x_0 - 1|], \mathbf{P}(B(y,i)) > 0$$

iff there is a path for $\Phi_C^{(a)}$ from $A_s$ to $A_i$ whose value is $C_0 - y$. Since this property does not depend on $a$ thanks to the lemma 2.10.5, then the minimal $x_0$ does not either.
— As $e(C)$ is fixed, we investigate on $q(a)$ : as the probabilities of paths are non-zero and continuous of $a \in \mathbf{R}^+$ thanks to the lemma 2.10.5, then by construction $q$ is continuous, which implies that we may choose

$$h^{(a)}(C) = \min\left(\frac{1}{2}, \frac{q(a)}{2^Q}\right)$$

that is continuous of $a$.

This ends the proof.

#### Use of anterior values

We aim at proving that all (for $i \in A'$) twisted Laplace transforms $K_i^{(a)}$ converge exponentially, to a common limit named $l(a)$. Let us look at the value taken by $\Phi_C^{(a)}$ at its $(z - e(C))$-halting time $\tau^{(a)}$ as given by the lemma 2.10.12, when starting from $z = \Phi_C^{(a)}(0)$ and $M^{(a)}(0) = A_i$. As $Y_C^{(a)}$ is a martingale, the martingale property leads to

$$K_i^{(a)}(z) = Y_C^{(a)}(0) = \mathbf{E}\left(Y_C^{(a)}\left(\tau^{(a)}\right)\right)$$

which is

$$\sum_{x\in\mathbf{Z}}\sum_{j=1}^{A}\mathbf{P}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right)=x\wedge M^{(a)}\left(\tau^{(a)}\right)=A_j\right)K_j^{(a)}(x)$$

However, as $A'$ is closed, $i\in A'$, and by construction of $\tau^{(a)}$, the sum restricts to

$$K_i^{(a)}(z)=\sum_{x=z-e(C)-2Q}^{z-e(C)-1}\sum_{j\in A'}\mathbf{P}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right)=x\wedge M^{(a)}\left(\tau^{(a)}\right)=A_j\right)K_j^{(a)}(x)$$

We know that

- All terms $K_j^{(a)}(x)$ in this expression are bounded from above by
  - $K_+^{(a)}(z-e(C))$ if they belong to $[|z-e(C)-Q,z-e(C)-1|]$ ;
  - $K_+^{(a)}(z-e(C)-Q)$ if they belong to $[|z-e(C)-2Q,z-e(C)-Q-1|]$.

  As $J_+^{(a)}$ is non-increasing, we select $K_+^{(a)}(z-e(C)-Q)$.
- As $K_-^{(a)}$ is non-decreasing, by definition of the minimum we know that there are

  $$j\in A',x\in[|z-e(C)-Q,z-e(C)-1|]$$

  such that (noting $l_-(a)$ the limit of $K_-^{(a)}$ as in the definition 2.10.5)

  $$K_j^{(a)}(x)=K_-^{(a)}(z-e(C))\leq l_-(a)$$

  Moreover, we know thanks to the lemma 2.10.12 that its "weight" given by $P_{i\to j}^{(a)}$ is at least $h^{(a)}(C)$.

As a consequence,

$$K_i^{(a)}(z)\leq\left(1-h^{(a)}(C)\right)K_+^{(a)}(z-e(C)-Q)+h^{(a)}(C)l_-(a)$$

Doing this for every $i\in A'$ and $z\in[|C_0,C_0+Q-1|]$ yields

$$K_i^{(a)}(z)\leq\left(1-h^{(a)}(C)\right)\max_{z\in[|C_0,C_0+Q-1|]}\left(K_+^{(a)}(z-e(C)-Q)\right)+h^{(a)}(C)l_-(a)$$

However, the maximum appearing here is $K_+^{(a)}(C_0-e(C)-Q)$ because $K_+^{(a)}$ is non-increasing. This being for every $i$ and $z$, we get

$$\forall C_0\in\mathbf{N},K_+^{(a)}(C_0)\leq\left(1-h^{(a)}(C)\right)K_+^{(a)}(C_0-e(C)-Q)+h^{(a)}(C)l_-(a)$$

In particular, the sequence $u$ defined by

$$u=\begin{pmatrix}\mathbf{N} & \to & \mathbf{R}\\ n & \to & K_+^{(a)}\left(\left(e(C)+Q\right)n\right)-l_-(a)\end{pmatrix}$$

holds the geometric property

$$\forall n \in \mathbf{N}, u(n) \leq u(0) \left(1 - h^{(a)}(C)\right)^n$$

As $K_+^{(a)}$ is non-increasing, it follows that

$$\forall x \in \mathbf{N}, K_+^{(a)}(x) \leq K_+^{(a)}(0) \left(1 - h^{(a)}(C)\right)^{\left\lfloor \frac{x}{e(C)+Q} \right\rfloor} + l_-(a)$$

Finally, we simplify this expression :

— $K_+^{(a)}(0)$ is bounded by the highest value among $K_i^{(a)}(x)$ for $i \in A'$ and $x \in [|-Q, -1|]$, amounting to (at most)

$$\max_{i \in A'} \left( \frac{1}{w_{[i]}^{(a)}} \right) e^{-\alpha(a)}$$

— $h^{(a)}(C)$ was deemed no higher than 1/2, so we may replace the integer part of $x/(e(C)+Q)$ by $x/(e(C)+Q)-1$ and multiply by 2.

Hence, setting

$$\beta_1^{(a)} = \frac{-\ln\left(1 - h^{(a)}(C)\right)}{e(C)+Q} \wedge Z_1^{(a)} = \max_{i \in A'} \left( \frac{2}{w_{[i]}^{(a)}} \right) e^{-\alpha(a)}$$

then by definition of $J_+^{(a)}(x)$, one gets

$$\forall i \in A', x \in \mathbf{N}, K_i^{(a)}(x) \leq Z_1^{(a)} e^{-\beta_1^{(a)} x} + l_-(a)$$

This implies that $l_+(a) = l_-(a)$, and the convergence speed is driven by the exponential parameters $Z_1^{(a)}$ and $\beta_1^{(a)}$ (the case of $J_-^{(a)}(x)$ being symmetrical).

**Extension to outside the descending class**

When the starting state $A_i$ does not belong to $A'$, we use a similar trick, involving a reference to a value in $A'$. Indeed, when waiting for $A$ time periods, $M^{(a)}$ must contain a path from $A_i$ to some state $A_j$ with $j \in A'$ (as else $A_i$ would belong to a closed communicating class, however the lemma 2.10.3 forbids it). The proof for all states is quite similar to the one for states in $A'$, so we shall only explain its main steps. When waiting for $A$ time periods, we note by $g(a)$ a (continuous) lower bound for the probability of going from any state of $M^{(a)}$ to a state in $A'$ ; hence, we have

$$\forall x \in \mathbf{N}, K_{++}^{(a)}(x + AQ) \leq (1 - g(a)) K_{++}^{(a)}(x) + g(a) K_+^{(a)}(x)$$

Solving this inequation through the auxiliary sequence defined by

$$v^{(a)} = \left( \begin{array}{ccc} \mathbf{N} & \to & \mathbf{R} \\ n & \to & \left(\frac{1}{1-g(a)}\right)^n \left(K_{++}^{(a)}(nAQ) - l_-(a)\right) \end{array} \right)$$

yields eventually, thanks to the previous work,

$$\forall n \in \mathbf{N}, v^{(a)}(n+1) - v^{(a)}(n) = \frac{Z_1^{(a)}}{1-g(a)} e^{-n\left(\beta_1^{(a)} AQ + \ln(1-g(a))\right)}$$

Taking $g(a)$ to be positive and low enough so that

$$\forall a \in \mathbf{R}^+, \beta_1^{(a)} AQ + \ln\left(1 - g(a)\right) = \beta_2^{(a)} > 0$$

then one gets

$$\forall n \in \mathbf{N}, v^{(a)}(n) \leq v^{(a)}(0) + \frac{Z_1^{(a)}}{1-g(a)} \frac{1}{1-e^{-\beta_2^{(a)}}} = Z_2(a)$$

So, noting

$$\forall a \in \mathbf{R}^+, \beta_3^{(a)} = \min\left(\frac{-\ln\left(1-g(a)\right)}{AQ}, \frac{1}{2}\right) > 0$$

by hypothesis, one gets

$$\forall n \in \mathbf{N}, K_{++}^{(a)}(nAQ) \leq l_-(a) + Z_2(a) e^{-\beta_3^{(a)} nAQ}$$

which leads, as $K_{++}^{(a)}$ is non-increasing, and for the same reasons as before with $Z_3(a) = 2Z_2(a)$, to

$$\forall i \leq A, x \in \mathbf{N}, K_i^{(a)}(x) \leq l_-(a) + Z_3(a) e^{-\beta_3^{(a)} x}$$

The symmetrical inequality (for $K_{--}^{(a)}$) holds a similar way, eventually leading to functions $Z_4$ and $\beta_4$ such that

$$\forall i \leq A, x \in \mathbf{N}, K_i^{(a)}(x) \geq l_+(a) + Z_4(a) e^{-\beta_4^{(a)} x}$$

However, when $x$ goes to $\infty$, as we know that $l_-(a) \leq l_+(a)$, this is possible only if $l_-(a) = l_+(a)$ (which is the suitable $K_\infty(a)$), so setting

$$\forall a \in \mathbf{R}^+, Z(a) = \max\left(Z_3(a), Z_4(a)\right) \wedge \beta(a) = \min\left(\beta_3^{(a)}, \beta_4^{(a)}\right)$$

we have the solutions to the lemma 2.10.10, which ends the proof.

### 2.10.5 Lemma 2.10.11

When $C$ is aperiodic, one cannot force $\tau^{(a)}$ to stop $\Phi_C^{(a)}$ precisely on a given point $x \in \mathbf{R}^+$ instead of $\mathbf{N}$, e.g. when $\Phi_C^{(a)}$'s increments behave as a continuous distribution. However, the density alternative states that we may stop $\Phi_C^{(a)}$ arbitrarily close to $x$, so our idea is to

1. Control the functions $K_i^{(a)}$ locally around any $x \in \mathbf{R}^+$, to ensure that their values around $x$ may be viewed through $K_i^{(a)}(x)$ ;
2. Use the density alternative, so that for every $x \in \mathbf{R}^+$ remote enough from $C(0)$ one may ensure that

$$\mathbf{P}\left(M^{(a)}\left(\tau^{(a)}\right) \wedge \Phi^{(a)}\left(\tau^{(a)}\right) \in [x \pm \epsilon]\right)$$

   is bounded away from 0 ;
3. When expressing the martingale property, write $Y^{(a)}(0)$ as a convex combination involving average values of $K_i^{(a)}$ over some sets $[x, x+\epsilon]$, with controlled coefficients ;
4. When looking at the values of $i \in A'$ and $y \in \mathbf{R}$ where $K_i^{(a)}(y)$ is minimal, use the local control to bound the contribution of this set to $K_i^{(a)}$ from above ;
5. Rewrite the convex combination, finally yielding an arithmetico-geometric recursion scheme, whose "constant" part is driven by $\epsilon$. As $\epsilon$ may be chosen arbitrarily small, we shall get the result for $i \in A'$ ;
6. Extend to every $i \leq A$ a similar way as before.

Therefore, we will get the weak theorem for aperiodic C-processes.

#### Local control

The functions $K_i^{(a)}$ are differentiable over $[-Q, 0)$ by definition. As a consequence, there is a constant $\Xi \in \mathbf{R}_+^*$ such that

$$\forall x < y \in \mathbf{R}_-^*, \frac{K_i^{(a)}(y) - K_i^{(a)}(x)}{y - x} \leq \Xi$$

We want to prove that this property extends to $\mathbf{R}^+$, so that any value $K_i^{(a)}(y)$ is upper bounded by $K_i^{(a)}(x) + (y-x)\Xi$, locally controlling $K_i^{(a)}$ around an arbitrary point $x$.

**Lemma 2.10.13** *Half-Lipschitz functions*

*Let $a \in \boldsymbol{R}^+$, and $K^{(a)}_{i\leq A}$ be the functions coming from the convolution equation (lemma 2.10.1). Let us note by*

$$\Xi = \begin{pmatrix} \boldsymbol{R}^+ & \to & \boldsymbol{R}^+ \\ a & \to & \frac{\alpha(a)}{\min_{i\leq A}\left(w^{(a)}_{[i]}\right)} \end{pmatrix}$$

*Then $\Xi$ is a continuous function satisfying*

$$\forall i \leq A, \forall x < y \in \boldsymbol{R}^+, \frac{K^{(a)}_i(y) - K^{(a)}_i(x)}{y - x} \leq \Xi(a)$$

*We call it the half-Lipschitz function of $C$.*

To prove this lemma, we take $x < y \in [-Q, \infty)$, writing $y = x + z$ with $z > 0$, and solve the property depending on whether $x, y$ are negative or not. For every $i \leq A$, we write

$$\xi_i(x, z) = \frac{K^{(a)}_i(x+z) - K^{(a)}_i(x)}{z}$$

so we want $\forall i \leq A, \forall x \in \mathbf{R}^+, \forall z > 0, \xi_i(x, z) \leq \Xi(a)$.

— When $x + z < 0$, the statement is given by the mean value inequality, implying that

$$\frac{K^{(a)}_i(x+z) - K^{(a)}_i(x)}{z} \leq \sup_{u\in\mathbf{R}^*_-}\left(\frac{dK^{(a)}_i(u)}{du}\right)$$

The definition of $\Xi$ comes from the limit value of this derivative at point $u = 0$.

— When $x < 0$ and $x + z \geq 0$, we know thanks to the definition of $\Xi$ that

$$\forall i \leq A, u \in \mathbf{R}^*_-, \left(K^{(a)}_i(u)\right) \leq K^{(a)}_i(x) - x\Xi(a)$$

However, as $K^{(a)}_{++}$ is non-increasing, this implies that the same is true for every $z \in \mathbf{R}^+$, so

$$\xi_i(x, z) \leq \frac{\left(K^{(a)}_i(x) - x\Xi\right) - K^{(a)}_i(x)}{z} = \Xi\frac{-x}{z} \leq \Xi(a)$$

because $0 < -x \leq z$, which solves this case.

— When $x \geq 0$, we consider $\Phi_C^{(a)}$ starting from the state $A_i$ and the point $x$, and its random hitting time $\tau$ of $\mathbf{R}_-^*$ (it is eligible to the martingale property, thanks to the lemma 2.10.7). The martingale property thus yields

$$\begin{aligned} & K_i^{(a)}(x+z) - K_i^{(a)}(x) \\ = & \sum_{j=1}^{A} \begin{pmatrix} \mathbf{P}\left(M^{(a)}(\tau) = A_j\right) \\ \mathbf{E}\left(K_j^{(a)}\left(z + \Phi_C^{(a)}(\tau)\right) - K_j^{(a)}\left(\Phi_C^{(a)}(\tau)\right) | M^{(a)}(\tau) = A_j\right) \end{pmatrix} \end{aligned}$$

By definition of $\xi_i$, this comes to

$$\begin{aligned} & z\xi_i(x, x+z) \\ = & \sum_{j=1}^{A} \mathbf{P}\left(M^{(a)}(\tau) = A_j\right) \mathbf{E}\left(\xi_j\left(\Phi_C^{(a)}(\tau), z + \Phi_C^{(a)}(\tau)\right) | M^{(a)}(\tau) = A_j\right) \end{aligned}$$

As $\Phi_C^{(a)}(\tau) < 0$, we are driven back to one of the previous cases, however in both of them we know that

$$\forall u < 0, \xi_j\left(u, z+u\right) \leq z\Xi(a)$$

so what finally remains is

$$z\xi_i(x, x+z) \leq z\Xi(a)$$

which ends the proof.

**Density over intervals**

Let $\eta > 0$. We set $m = \lceil Q/\eta \rceil$ and split any $x$-interval on sub-intervals of length at most $\eta$, as in

$$[x - Q, x) \subseteq \bigcup_{k=1}^{m} [x - k\eta, x - (k-1)\eta)$$

We want to prove, thanks to the density alternative, that $\tau^{(a)}$ may halt $\Phi_C^{(a)}$ on every such interval. The following lemma "looks like" the lemma 2.10.12, changing only when considering that $\Phi_C^{(a)}$ hits an interval rather than a point : as a consequence, its length $\eta$ will be involved in the result.

**Lemma 2.10.14** *Uniform references to approximate previous values*

*Let $\Phi_C^{(a)}$ be $C$'s convolution process at any point $a \in \boldsymbol{R}^+$, whose starting point is $C_0 \in \boldsymbol{N}$. We name $A'$ its descending class. Let $\eta > 0$ be called a margin, supposed to be an integer fraction of $Q$, i.e. and $m = Q/\eta \in \boldsymbol{N}^*$.*

1. *There is some $x_0(\eta) \in \boldsymbol{R}_+^*$ such that, for every $a$, the $(C_0 - x_0(\eta))$-halting time $\tau^{(a)}$ of $\Phi_C^{(a)}$ holds, for every $\forall k \in [|1, m|]$ and $i \in A'$,*

$$\boldsymbol{P}\left(\begin{array}{c} \Phi_C^{(a)}\left(\tau^{(a)}\right) \in [C_0 - x_0(\eta) - k\eta, C_0 - x_0(\eta) - (k-1)\eta) \\ \wedge M^{(a)}\left(\tau^{(a)}\right) = A_i \end{array}\right) > 0$$

*We name :*
   - *Any value $x_0(\eta)$ is an ergodic pace of $\Phi_C^{(a)}$ at margin $\eta$, noted $e(C, \eta)$ ;*
   - *Any value $h^{(a)}(C, \eta)$ no higher than both $1/2$ and all these probabilities is a reducing constant of $\Phi_C^{(a)}$ at margin $\eta$.*
2. *When $a$ varies into $\boldsymbol{R}^+$ and $\eta$ is fixed,*
   - *We may choose $e(C, \eta)$ so that it does not change ;*
   - *We may choose reducing constants so that $h^{(a)}(C, \eta)$ varies continuously.*

The proof of this lemma follows the same steps as for the lemma 2.10.12.

1. We begin by building a path from any starting state $A_s$ to $A_i$ with $i \in A'$.
   (a) As $A'$ is the only closed communicating class of $\Phi_C^{(a)}$, if $s \notin A'$, there is a way to $A'$. We get (through the lemma 2.7.3) a path $P_1$ of minimal length from $A_s$ to some state $A_r$ with $r \in A'$, of finite value $v_1 \in -\mathbf{N}$.
      - This path $P_1$ works for all values of $a \in \mathbf{R}_+^*$ simultaneously thanks to the lemma 2.10.5.
      - We take $\eta/9$ as a precision for following this path at precision in the terms of lemma 2.7.2. $P_1$ has a positive probability to be followed at this precision, noted $p_s(a, \eta) > 0$, that is continuous of $a \in \mathbf{R}^+$ thanks to the lemma 2.10.5 (and depends on $A_s$).
      - As $P_1$'s length is minimal, it is at most $A$ ; since its payoffs are bounded by $Q$, then $v_1 \in [-AQ, 0]$.

   (b) Since $A'$ is a closed communicating class of $\Phi_C^{(a)}$, there is a path $P_2$ of minimal length from $A_r$ to $A_i$. For the same reasons, its value is noted $v_2 \in [-AQ, 0]$, and its probability to be followed at precision $\eta/9$ is noted $q_i(a, \eta) > 0$.

   (c) We also know after lemma 2.7.9 that setting $\epsilon = \eta/9$ and the associated $X^-$, then for every $y \leq X^-$, $\Phi_C^{(a)}$ has a universal (in $A'$) cycle of some value in $[y \pm \epsilon/2]$. In particular, let us take

$$b = \left\lceil \frac{(2A+1)Q}{\epsilon} \right\rceil$$

   Doing this for $y = X^- - k\epsilon$ for every $k \in [|0, b|]$, we get a universal cycle $Q_k$ starting from $A_i$ of some value in

$$\left[X^- - k\eta \pm \frac{\eta}{18}\right]$$

for every $k \in [|0, b|]$. For each $k$, we note by $r_k(a, \eta) > 0$ the probability of following such a cycle at precision $\eta/9$.

We know that $v_1 + v_2 \in [-2AQ, 0]$ ; hence, taking any $x$ in the interval

$$\left[X^- - (2A+1)Q, X^- - 2AQ\right]$$

we have

$$y = \frac{-x + X^- + v_1 + v_2}{\epsilon} \in \left[0, \frac{(2A+1)Q}{\epsilon}\right] \subseteq [0, b]$$

The closest integer $k$ to $y$ is thus at a distance at most $1/2$ of $y$ itself and belongs to $[|0, b|]$. Hence, there is a $k \in [|0, b|]$ such that

$$X^- - k\eta \in \left[x - v_1 - v_2 \pm \frac{\epsilon}{2}\right]$$

We concatenate the paths $P_1$ and $P_2$ with the cycle $Q_k$ for this $k$, whose value belongs to

$$\left[X^- - k\eta \pm \frac{\eta}{18}\right]$$

This leads to a path from $A_s$ to $A_i$ whose value is in

$$\left[X^- - k\eta + v_1 + v_2 \pm \frac{\eta}{18}\right] \subseteq \left[x \pm \frac{\eta}{9}\right]$$

Hence, we proved that for every $x \in [X^- - (2A+1)Q, X^- - 2AQ]$, there is a path from $A_s$ to $A_i$ whose value is at most $\eta/9$ apart from $x$. Besides, the probability of following it at a precision $\eta/9 + \eta/9 + \eta/9$ (each $\eta/9$ comes respectively from $P_1$, $P_2$ and $Q_k$) is at least

$$\min_{(s \leq A, i \in A', k \in [|0,b|])} (p_s(a, \eta) q_i(a, \eta) r_k(a, \eta)) = q(a, \eta) > 0$$

2. We shall prove that $x_0(\eta) = X^-$ works. Let us take
— A "target" interval, for $k \leq m$,

$$[C_0 - X^- - k\eta, C_0 - X^- - (k-1)\eta) = [y_k \pm \eta/2)$$

described by its central point

$$y_k = C_0 - X^- - (k - \frac{1}{2})\eta$$

— A "target" state $A_{i \in A'}$.

We shall note, for $k \leq m$ and $i \in A'$,

$$B(k,i) = \mathbf{P}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right) \in [y_k \pm \eta/2) \wedge M^{(a)}\left(\tau^{(a)}\right) = A_i\right)$$

We start the convolution process $\Phi_C^{(a)}$ from $C_0 \in \mathbf{R}^+$ and $A_{s \leq A}$ : as proved above, it has a probability at least $q(a,\eta)$ of following the path leading to
— At most $\eta/9$ apart of the target point $y$,
— And the target state $A_i$,
at precision $\eta/3$. If it follows this path at this precision, then the successive values $\rho(t)$ amount to
— 1 as long as $\Phi_C^{(a)}$ remains at least $C_0 - X^-$ ;
— 1/2 when $\Phi_C^{(a)}$ crosses $C_0 - X^-$ ;
— 0 if $\Phi_C^{(a)}$ crosses $C_0 - Q - X^-$.
However, the latter case is impossible, because $\Phi_C^{(a)}\left(\tau^{(a)}\right)$ is not further than $\eta/9 + \eta/3$ apart (approximation of $y$ + precision) from $y_k$, and

$$y_k = C_0 - X^- - (k - \frac{1}{2})\eta \geq C_0 - X^- - Q + \frac{\eta}{2}$$

thanks to $m = Q/\eta \in \mathbf{N}$, and $\eta/2 > 4\eta/9$. Now, the given paths are in a finite number, so we note by $N(\eta) \in \mathbf{N}$ the maximum of their lengths : it follows that the probability of $\tau$ stopping precisely at the end of the path is at least $1/2^{N(\eta)}$ no matter the path taken. Since the path probability is at least $q(a,\eta)$, then

$$\forall k \in [|1,m|], i \in A', \mathbf{P}\left(B(k,i)\right) \geq \frac{q(a,\eta)}{2^{N(\eta)}}$$

which ends this proof.

3. We move on to the properties of $e(C,\eta)$ and $h^{(a)}(C,\eta)$ when $a$ changes.
   — We remark that, no matter $a$, we have

   $$\forall i \in A', k \in [|1,m|], \mathbf{P}\left(B(k,i)\right) > 0$$

   as soon as there is a path for $\Phi_C^{(a)}$ from $A_s$ to $A_i$ whose value is at most $\eta/9$ apart from $C_0 - y_k$. Since this property does not depend on $a$ thanks to the lemma 2.10.5, then there is an $x_0$ that does not either.
   — As $e(C,\eta)$ is fixed, we investigate on $q(a,\eta)$ : as the probabilities of paths are non-zero and continuous of $a \in \mathbf{R}^+$ thanks to the lemma 2.10.5, then by construction $q$ is continuous, which implies that we may choose

   $$h^{(a)}(C,\eta) = \min\left(\frac{1}{2}, \frac{q(a,\eta)}{2^{N(\eta)}}\right)$$

   that is continuous of $a$.

This ends the proof.

**Use of the local control**

Let us take $x_0 \in \mathbf{R}^+$, yielding a value $y_0 = K_-^{(a)}(x_0)$. By definition, setting any $\epsilon > 0$ allows us to find $x_1 \in [x_0 - Q, x_0)$ and $i_1 \in A'$ such that

$$K_{i_1}^{(a)}(x_1) < y_0 + \epsilon$$

Now, let us take $\eta$ such that $Q/\eta = m \in \mathbf{N}^*$, $i_2 \in A'$, and

$$x_2 \in [x_1 + \eta + e(C,\eta), x_1 + \eta + e(C,\eta) + Q)$$

Thanks to the convolution equation, the value $K_{i_2}^{(a)}(x_2)$ may be expressed as a convex combination, involving the intervals from the lemma 2.10.14. Noting by $\Phi_C^{(a)}$ the convolution process of $C$ starting from $M^{(a)}(0) = A_{i_2}$ and $\Phi_C^{(a)}(0) = x_2$, and $\tau^{(a)}$ its $(x_2 - e(C,\eta))$-halting time, we have

$$\begin{aligned} & \forall k \leq m, j \in A', \exists c_{k,j} \geq h^{(a)}(C,\eta); K_{i_2}^{(a)}(x_2) \\ = & \sum_{j \in A'} \sum_{k=1}^{m} c_{k,j} \mathbf{E}\left(K_j^{(a)}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right)\right) | B(k,j)\right) \\ + & \mathbf{P}\left(\neg \bigcup_{j \in A'} \bigcup_{k=1}^{m} B(k,j)\right) \mathbf{E}\left(K_j^{(a)}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right)\right) | \neg \bigcup_{j \in A'} \bigcup_{k=1}^{m} B(k,j)\right) \end{aligned}$$

In particular, we consider the term with $j_0 = i_1$ and

$$k_0 = \left\lfloor \frac{x_2 - e(C,\eta) - x_1}{\eta} \right\rfloor \in [|1, m|]$$

by construction of $m$. Noting by $I$ the interval

$$I = [x_2 - e(C,\eta) - k_0\eta, x_2 - e(C,\eta) - (k_0 - 1)\eta)$$

the conditional expected value to this $B(k_0, j_0)$ rewrites as

$$\mathbf{E}\left(K_{j_0}^{(a)}\left(\Phi_C^{(a)}\left(\tau^{(a)}\right)\right) | \Phi_C^{(a)}\left(\tau^{(a)}\right) \in I \wedge M^{(a)}\left(\tau^{(a)}\right) = A_{j_0}\right)$$

However, by definition of $k_0$, $I$'s lower bound is at least $x_1$ and at most $x_1 + \eta$. Thanks to the half-Lipschitz property from lemma 2.10.13, the function $K_{j_0}^{(a)}$ is upper bounded over the involved set, as it is included in $[x_1, x_1 + 2\eta)$, by

$$K_{j_0}^{(a)}(x_1) + 2\eta\Xi(a) < y_0 + \epsilon + 2\eta\Xi(a)$$

Now, we know by definition of $\tau^{(a)}$ that almost surely,

$$\Phi_C^{(a)}\left(\tau^{(a)}\right) \geq x_1 - Q$$

When $B(k_0, j_0)$ does not happen, then

$$\forall j \in A', K_{j_0}^{(a)} \left(\Phi_C^{(a)} \left(\tau^{(a)}\right)\right) \leq K_+^{(a)}(x_1 - Q)$$

because $K_+^{(a)}$ is non-increasing (provided that $x_1 \geq Q$). As a consequence, we may bound $K_{i_2}^{(a)}(x_2)$ from above, by

$$K_{i_2}^{(a)}(x_2) \leq (1 - \mathbf{P}(B(k_0, j_0))) K_+^{(a)}(x_1 - Q) + \mathbf{P}(B(k_0, j_0)) (y_0 + \epsilon + 2\eta\Xi(a))$$

As $K_{i_2}^{(a)}(x_2) \leq K_+^{(a)}(x_1 - Q)$, we have the alternative :

— If $y_0 + \epsilon + 2\eta\Xi(a)$ is not higher than $K_+^{(a)}(x_1 - Q)$, then we may replace $\mathbf{P}(B(k_0, j_0))$ by its lower bound $h^{(a)}(C, \eta)$, and get

$$K_{i_2}^{(a)}(x_2) \leq \left(1 - h^{(a)}(C, \eta)\right) K_+^{(a)}(x_1 - Q) + h^{(a)}(C, \eta) (y_0 + \epsilon + 2\eta\Xi(a))$$

— If it is, the latter inequality still holds because

$$K_{i_2}^{(a)}(x_2) \leq K_+^{(a)}(x_1 - Q)$$

by definition of $K_+^{(a)}$.

So, we proved that for every $i_2 \in A'$ and $x_2 \in [x_1 + \eta + e(C, \eta), x_1 + \eta + e(C, \eta) + Q)$,

$$K_{i_2}^{(a)}(x_2) \leq \left(1 - h^{(a)}(C, \eta)\right) K_+^{(a)}(x_1 - Q) + h^{(a)}(C, \eta) (y_0 + \epsilon + 2\eta\Xi(a))$$

which leads by definition of $K_+^{(a)}$ and $y_0$ to

$$\begin{aligned} & K_+^{(a)} (x_1 + \eta + e(C, \eta) + Q) \\ \leq \; & \left(1 - h^{(a)}(C, \eta)\right) K_+^{(a)}(x_1 - Q) + h^{(a)}(C, \eta) \left(K_-^{(a)}(x_0) + \epsilon + 2\eta\Xi(a)\right) \end{aligned}$$

Finally, as

— $x_1 \in [x_0 - Q, x_0)$ by construction ;

— $K_+^{(a)}$ is non-increasing ;

— $K_-^{(a)}$ is non-decreasing and converges to $l_-^{(a)}$,

then we get, provided that $x_1 \geq Q$, so whenever $x_0 \geq 2Q$,

$$\begin{aligned} & K_+^{(a)} (x_0 + \eta + e(C, \eta) + Q) \\ \leq \; & \left(1 - h^{(a)}(C, \eta)\right) K_+^{(a)}(x_0 - 2Q) + h^{(a)}(C, \eta) l_-^{(a)} + h^{(a)}(C, \eta) (\epsilon + 2\eta\Xi(a)) \end{aligned}$$

We shall use this inequation to get an arithmetico-geometric inequality for $K_+^{(a)}$.

**Arithmetico-geometric convergence**

Let us take $\epsilon > 0$ and $\eta$ an integer fraction of $Q$. We define the sequence

$$u_{(\epsilon,\eta,a)} = \begin{pmatrix} \mathbf{N} & \to & \mathbf{R} \\ n & \to & K_+^{(a)}\left(2Q + \left(\eta + e(C,\eta) + 3Q\right)n\right) \end{pmatrix}$$

The previous inequality leads for every $n \in \mathbf{N}$ to the arithmetico-geometric inequality

$$u_{(\epsilon,\eta,a)}(n+1) \leq \left(1 - h^{(a)}(C,\eta)\right) u_{(\epsilon,\eta,a)}(n) + h^{(a)}(C,\eta) l_-^{(a)} + h^{(a)}(C,\eta)\left(\epsilon + 2\eta\Xi(a)\right)$$

Noting by $Z_1(\epsilon,\eta,a) = u_{(\epsilon,\eta,a)}(0)$, we eventually get

$$\forall n \in \mathbf{N}, u_{(\epsilon,\eta,a)}(n) \leq l_-^{(a)} + \epsilon + 2\eta\Xi(a) + Z_1(\epsilon,\eta,a)\left(1 - h^{(a)}(C,\eta)\right)^n$$

Since $K_+^{(a)}$ is non-increasing,

$$\forall x \geq 2Q, K_+^{(a)}(x) \leq l_-^{(a)} + \epsilon + 2\eta\Xi(a) + Z_1(\epsilon,\eta,a)\left(1 - h^{(a)}(C,\eta)\right)^{\left\lfloor \frac{x-2Q}{\eta+e(C,\eta)+3Q} \right\rfloor}$$

Hence, let us note by

$$\beta_1(\eta,a) = \frac{-\ln\left(1 - h^{(a)}(C,\eta)\right)}{\eta + e(C,\eta) + 3Q}$$

and as earlier, thanks to $h^{(a)}(C,\eta) \leq 1/2$,

$$Z_2(\epsilon,\eta,a) = 2Z_1(\epsilon,\eta,a)e^{2Q\beta_1(\eta,a)}$$

so that the inequation implies

$$\forall x \geq 2Q, K_+^{(a)}(x) \leq l_-^{(a)} + \epsilon + 2\eta\Xi(a) + Z_2(\epsilon,\eta,a)e^{-\beta_1(\eta,a)x}$$

Let us take $\zeta > 0$ and $a \in \mathbf{R}^+$. We set $\epsilon = \zeta/3$ ; as $\Xi$ is continuous, we may define its maximum $\bar{\Xi}(a)$ over $[0,a]$, then take $\eta(a,\zeta)$ to be any constant in

$$\left[0, \frac{\zeta}{6\bar{\Xi}(a)}\right] \cap \frac{Q}{\mathbf{N}^*}$$

This $\eta$ allows us to define in turn :

— Since $\beta_1$ is continuous of its second variable, a term

$$\bar{\beta_1}(\zeta,a) = \inf_{b\in[0,a]}\left(\beta_1(\eta(a,\zeta),b)\right) > 0$$

— Since $Z_2$ is continuous of its third variable, another term

$$\bar{Z}_2(\zeta, a) = \sup_{b\in[0,a]} Z_2(\zeta/3, \eta(a,\zeta), b) < \infty$$

Hence, provided that

$$x \geq \max\left(2Q, \frac{-\ln\left(\frac{\zeta}{3\bar{Z}_2(\zeta,a)}\right)}{\bar{\beta}_1(\zeta, a)}\right) = \bar{x}(\zeta, a)$$

then the inequation will lead to

$$K_+^{(a)}(x) \leq l_-^{(a)} + \zeta/3 + \zeta/3 + \zeta/3$$

Since $l_-^{(a)} \leq l_+^{(a)}$, we proved that for every $\zeta > 0$, $a \in \mathbf{R}^+$, there is a $\bar{x}(\zeta, a)$ such that for every $x \geq \bar{x}(\zeta, a)$ and every $b \in [0, a]$,

$$K_+^{(a)}(x) \in \left[l_-^{(a)}, l_-^{(a)} + \zeta\right]$$

To get the symmetrical property, the only changes in this proof are that

— To $x_0 \in \mathbf{R}^+$ is associated $y_0 = K_+^{(a)}(x_0)$, and $x_1, i_1$ such that

$$K_{i_1}^{(a)}(x_1) > y_0 - \epsilon$$

— One takes

$$x_2 \in [x_1 - \eta + e(C, \eta), x_1 - \eta + e(C, \eta) + Q)$$

— The set in which $B(k_0, j_0)$ falls lies in $[x_1 - 2\eta, x_1)$, and $K_{j_0}^{(a)}$ is bounded from below on this set by

$$K_{j_0}^{(a)}(x_1) - 2\eta\Xi$$

thanks to the lemma 2.10.13 ;

— So, we end up with every $K_{i_2}^{(a)}(x_2)$, and thus $K_-^{(a)}(x_1 - \eta + e(C, \eta) + Q)$, being lower bounded by a convex combination consisting of

— A term $y_0 - \epsilon$, of weight at least $h^{(a)}(\eta, C)$ ;

— Remaining terms $l_+^{(a)}$.

Solving this ultimately leads, the similar way, to the symmetrical property : for every $\zeta > 0$, $a \in \mathbf{R}^+$, there is a $\bar{x}(\zeta, a)$ such that for every $x \geq \bar{x}(\zeta, a)$ and every $b \in [0, a]$,

$$K_-^{(a)}(x) \in \left[l_+^{(a)} - \zeta, l_+^{(a)}\right]$$

As $l_-^{(a)} \leq l_+^{(a)}$, this is possible only if $l_-^{(a)} = l_+^{(a)}$, the common limit being $K_\infty(a)$. So, we proved the desired property for the states of $A'$.

**Extension to outside the descending class**

When considering $i \notin A'$, the same idea as before still works : when waiting for $A$ time periods, we have $g(a)$ as a lower bound for the probability of hitting $A'$. Let us take $a \in \mathbf{R}^+$ and $\epsilon > 0$ : we know that, thanks to the previous work, there is $x_0 \in \mathbf{R}^+$ such that

$$\forall b \leq a, \forall x \geq x_0, K_+^{(b)}(x) \leq K_\infty(b) + \epsilon/2$$

Hence, let us consider $n\left(\epsilon, a\right) \in \mathbf{N}$ such that

$$\forall b \leq a, K_{++}^{(b)}(0)\left(1 - g(b)\right)^{n(\epsilon,a)} < \epsilon/2$$

This is possible because $K_{++}^{(a)}$ and $g(a)$ are continuous expressions of $a$. When starting $\Phi_C^{(a)}$ from $x + n\left(\epsilon, a\right) AQ$ and any $A_i$, the convolution equation after a waiting time of $n\left(\epsilon, a\right) A$ time periods yields, for every $b \leq a$,

$$\begin{aligned} & K_i^{(b)}\left(x + n\left(\epsilon, a\right) AQ\right) \\ = \; & \mathbf{P}\left(M^{(b)}(n\left(\epsilon, a\right) A) \in A'\right) \mathbf{E}\left(K_i^{(b)}\left(\Phi_C^{(a)}(n\left(\epsilon, a\right) A)\right) | M^{(b)}(n\left(\epsilon, a\right) A) \in A'\right) \\ + \; & \mathbf{P}\left(M^{(b)}(n\left(\epsilon, a\right) A) \notin A'\right) \mathbf{E}\left(K_i^{(b)}\left(\Phi_C^{(a)}(n\left(\epsilon, a\right) A)\right) | M^{(b)}(n\left(\epsilon, a\right) A) \notin A'\right) \end{aligned}$$

We know that after $n\left(\epsilon, a\right) A$ time periods, $\Phi_C^{(a)}(n\left(\epsilon, a\right) A) \geq x_0$, so

- If $M^{(b)}(n\left(\epsilon, a\right) A) \in A'$, then the term in the conditional expectancy is upper bounded by $K_+^{(b)}(x_0)$ as we just proved ;
- If not, we use the universal bound

$$K_{++}^{(b)}(0) < \infty$$

  and the control of the corresponding probability (choice of $n\left(\epsilon, a\right)$).

Hence, one eventually gets that

$$K_i^{(b)}\left(x + n\left(\epsilon, a\right) AQ\right) \leq K_\infty(b) + \epsilon$$

Doing this for every $x \in [x_0, x_0 + Q)$ and $i \leq A$ yields

$$K_{++}^{(b)}\left(x_0 + Q + n\left(\epsilon, a\right) AQ\right) \leq K_\infty(b) + \epsilon$$

So, we proved that for every $a \in \mathbf{R}^+$, $\epsilon > 0$, there is

$$m = x_0 + Q + n\left(\epsilon, a\right) AQ$$

such that for every $x \geq m$ and $b \leq a$,

$$K_{++}^{(b)}(x) \leq K_\infty(b) + \epsilon$$

As the symmetrical inequality holds the same way, this ends the proof.

## 2.11 End of the main proofs

Now that the lemmata 2.10.12 and 2.10.14 are stated, we shall use them to find the term $K_\infty(a)$, that will lead to

— The lemma 2.9.1 for periodic C-processes ;

— The theorem 2.1 for aperiodic ones.

We shall also explain why the convergence may be arbitrarily slow (proposition 2.4.6).

### 2.11.1 Affine vector equation

Now that we are ensured of $K_i^{(a)}$'s convergence to $K_\infty(a)$, our idea is to

1. Apply the Laplace transformation to the convolution equation governing $K_i^{(a)}$, getting a "solved-form" result since $\Phi_C^{(a)}$ is decreasing ;
2. Get an equation for $K_i^{(a)}$'s Laplace transform around $0^+$, so that we shall use the final value theorem to find $K_\infty(a)$ later : this is possible because the functions $K_i^{(a)}$ converge.

We notice that, as the convergence is to be taken in a different sense when $C$ is periodic, we shall use the discrete Laplace transform (linked with the Z-transform) instead of the usual Laplace transform in this case.

#### Regular C-processes

When $C$ is regular, we shall use the discrete Laplace transform.

**Definition 2.11.1** *Discrete Laplace transform*

*Let $u : \left(\boldsymbol{N} \to \boldsymbol{R}^+\right)$ be a non-negative sequence. Its discrete Laplace transform (DLT) is defined, whenever possible, by*

$$\hat{u} = \begin{pmatrix} \boldsymbol{R} & \to & \boldsymbol{R}^+ \cup \{\infty\} \\ w & \to & \sum_{n=0}^{\infty} u(n) e^{-nw} \end{pmatrix}$$

The DLT obeys the usual properties.

**Lemma 2.11.1** *Properties of the DLT*

*Let $u : \left(\boldsymbol{Z} \to \boldsymbol{R}^+\right)$ be a non-negative sequence, and $\psi$ be a probability distribution over $\boldsymbol{N}^*$.*

1. *Final value theorem : if $u(\infty) \in \boldsymbol{R}^+$ exists, then*

$$\lim_{w \to 0^+} \left(w \hat{u}(w)\right) = u(\infty)$$

*2. Convolution : for every $w \in \boldsymbol{R}$, we have*

$$\begin{aligned}\widehat{u * \psi}(w) &= \sum_{x=0}^{\infty}\sum_{d=1}^{\infty} u(x-d)\psi(d)e^{-wx} \\ &= \widehat{u}(w)\widehat{\psi}(w) + \sum_{d=1}^{\infty}\psi(d)e^{-wd}\sum_{x=-d}^{-1}u(x)e^{-wx}\end{aligned}$$

*(possibly $+\infty$).*

As these properties are similar to those of the usual Laplace transform and come from computations, we shall admit them, so we can proceed with the study. Let us take the convolution equation from lemma 2.10.1 : we view the functions $K_i^{(a)}$ as sequences, and apply the DLT on them. We get that, for every $i \leq A$ and $w \in \mathbf{R}$,

$$\widehat{K_i^{(a)}}(w) = \sum_{j=1}^{A} P_{i\to j}^{(a)} \widehat{K_j^{(a)} * G_{i\to j}^{(a)}}(w)$$

so this rewrites as

$$\sum_{j=1}^{A} P_{i\to j}^{(a)} \widehat{K_j^{(a)}}(w)\widehat{G_{i\to j}^{(a)}}(w) + \sum_{j=1}^{A} P_{i\to j}^{(a)} \sum_{d=1}^{\infty} \mathbf{P}\left(G_{i\to j}^{(a)} = d\right) e^{-wd} \sum_{x=-d}^{-1} K_j^{(a)}(x)e^{-wx}$$

However, we know that

— Over $-\mathbf{N}^*$, $K_j^{(a)}$ is by definition of $L_j^{(a)}$

$$\forall x \in -\mathbf{N}^*, K_j^{(a)}(x) = \frac{e^{\alpha(a)x}}{w_{[j]}^{(a)}}$$

so the rightmost term ultimately simplifies (when $w \neq \alpha(a)$) to

$$\sum_{j=1}^{A} \frac{P_{i\to j}^{(a)}}{w_{[j]}^{(a)}} \sum_{d=1}^{\infty} \mathbf{P}\left(G_{i\to j}^{(a)} = d\right) \frac{e^{-wd} - e^{-\alpha(a)d}}{e^{\alpha(a)-w} - 1}$$

We introduce the vector function $V^{(a)}$, defined by its coordinates :

$$V^{(a)} = \left( \begin{array}{lll} \mathbf{R} & \to & \left(\mathbf{R}_+^*\right)^A \\ w & \to & \left(\sum_{j=1}^{A} \frac{P_{i\to j}^{(a)}}{w_{[j]}^{(a)}} \sum_{d=1}^{\infty} \mathbf{P}\left(G_{i\to j}^{(a)} = d\right) \frac{e^{-wd}-e^{-\alpha(a)d}}{e^{\alpha(a)-w}-1}\right)_{i\leq A} \end{array} \right)$$

We notice that, when $w = \alpha(a)$, the rightmost fraction is continuously prolongated to $de^{-\alpha(a)d}$, so that this point is not an issue. In particular,

$$V_{[i]}^{(a)}(\alpha(a)) = \sum_{j=1}^{A} \frac{P_{i\to j}^{(a)}}{w_{[j]}^{(a)}} \mathbf{E}\left(G_{i\to j}^{(a)} e^{-\alpha(a)G_{i\to j}^{(a)}}\right)$$

— We recognize

$$\forall w \in \mathbf{R}, P^{(a)}_{i\to j}\widehat{G^{(a)}_{i\to j}}(w) = \Big(L_{\Phi^{(a)}_C}(-w)\Big)_{i,j}$$

is the general term of a matrix product.

Hence, noting $\widehat{K^{(a)}}$ the vector of discrete Laplace transforms, we have

$$\widehat{K^{(a)}}(w) = L_{\Phi^{(a)}_C}(-w)\widehat{K^{(a)}}(w) + V^{(a)}(w)$$

which leads to

$$\widehat{K^{(a)}}(w) = \Big(Id - L_{\Phi^{(a)}_C}(-w)\Big)^{-1} V^{(a)}(w)$$

whenever 1 is not an eigenvalue of $L_{\Phi^{(a)}_C}(-w)$, which is automatic if $w > 0$.

**Aperiodic C-processes**

When $C$ is aperiodic, we use the usual Laplace transform, that obeys the usual properties.

**Lemma 2.11.2** *Properties of the Laplace transform*

*Let* $u : \Big(\boldsymbol{R} \to \boldsymbol{R}^+\Big)$ *be a non-negative measurable function, and* $\psi$ *be a probability distribution over* $\boldsymbol{R}^*_+$*.*

1. *Final value theorem : if* $u(\infty) \in \boldsymbol{R}^+$ *exists, then*

$$\lim_{w\to 0^+} (wL_u(w)) = u(\infty)$$

2. *Convolution : for every* $w \in \boldsymbol{R}$*, we have*

$$\begin{aligned} L_{u*\psi}(w) &= \int_{x=0}^{\infty}\int_{d=0}^{\infty} u(x-d)e^{-wx}d\psi(d)dx \\ &= L_u(w)L_\psi(w) + \int_{d=0}^{\infty} e^{-wd}\int_{x=-d}^{0} u(x)e^{-wx}dxd\psi(d) \end{aligned}$$

*(possibly* $+\infty$*).*

Once again, we admit these properties, so we proceed immediately with the study. Let us take the convolution equation from lemma 2.10.1 again, and apply the Laplace transform on it. We get that, for every $i \leq A$,

$$\forall w \in \mathbf{R}, L_{K^{(a)}_i}(w) = \sum_{j=1}^{A} P^{(a)}_{i\to j} L_{K^{(a)}_j * G^{(a)}_{i\to j}}(w)$$

Noting $\phi^{(a)}_{i\to j}$ the distribution of $G^{(a)}_{i\to j}$, we get

$$\begin{aligned} L_{K_i^{(a)}}(w) &= \sum_{j=1}^{A} P^{(a)}_{i\to j} L_{K_j^{(a)}}(w) L_{G^{(a)}_{i\to j}}(w) \\ &+ \sum_{j=1}^{A} P^{(a)}_{i\to j} \int_{d=0}^{\infty} e^{-wd} \int_{x=-d}^{0} K_j^{(a)}(x) e^{-wx} dx d\phi^{(a)}_{i\to j}(d) \end{aligned}$$

However, we know that

— Over $\mathbf{R}^*_-$, $K_j^{(a)}$ is still by definition of $L_j^{(a)}$

$$\forall x \in \mathbf{R}^*_-, K_j^{(a)}(x) = \frac{e^{\alpha(a)x}}{w^{(a)}_{[j]}}$$

so the rightmost term ultimately simplifies (when $w \neq \alpha(a)$) to

$$\sum_{j=1}^{A} \frac{P^{(a)}_{i\to j}}{w^{(a)}_{[j]}} \int_{d=0}^{\infty} \frac{e^{-wd} - e^{-\alpha(a)d}}{\alpha(a) - w} d\phi^{(a)}_{i\to j}(d)$$

We introduce another vector function $V$, defined by its coordinates :

$$V^{(a)} = \left( \begin{array}{ccc} \mathbf{R} & \to & \left(\mathbf{R}^*_+\right)^A \\ w & \to & \left(\sum_{j=1}^{A} \frac{P^{(a)}_{i\to j}}{w^{(a)}_{[j]}} \int_{d=0}^{\infty} \frac{e^{-wd} - e^{-\alpha(a)d}}{\alpha(a)-w} d\phi^{(a)}_{i\to j}(d)\right)_{i\leq A} \end{array} \right)$$

When $w = \alpha(a)$, $V$ is again continuously prolongated by

$$V^{(a)}_{[i]}(\alpha(a)) = \sum_{j=1}^{A} \frac{P^{(a)}_{i\to j}}{w^{(a)}_{[j]}} \mathbf{E}\left(G^{(a)}_{i\to j} e^{-\alpha(a) G^{(a)}_{i\to j}}\right)$$

— We still recognize

$$\forall w \in \mathbf{R}, P^{(a)}_{i\to j} L_{G^{(a)}_{i\to j}}(w) = \left(L_{\Phi_C^{(a)}}(-w)\right)_{i,j}$$

being the general term of a matrix product.

Hence, noting $L_{K^{(a)}}$ the vector of Laplace transforms, we have

$$L_{K^{(a)}}(w) = L_{\Phi_C^{(a)}}(-w) L_{K^{(a)}}(w) + V^{(a)}(w)$$

which leads to

$$L_{K^{(a)}}(w) = \left(Id - L_{\Phi_C^{(a)}}(-w)\right)^{-1} V^{(a)}(w)$$

whenever 1 is not an eigenvalue of $L_{\Phi_C^{(a)}}(-w)$, which is automatic if $w > 0$. This is the same equation as for regular C-processes, as only the definition of $V$ has changed.

### 2.11.2 Final value theorem

We want the value

$$\lim_{w\to 0^+}\left(wL_{K^{(a)}}(w)\right)=\lim_{w\to 0^+}\left(w\left(Id-L_{\Phi_C^{(a)}}(-w)\right)^{-1}V^{(a)}(w)\right)$$

(when $C$ is regular, it is $\widehat{K^{(a)}}$ that plays the part of $L_{K^{(a)}}$).

#### Preliminary formula

A scheme that we will encounter in this paragraph is the computation of

$$\lim_{w\to 0^+}\left((Id-L(-w))^{-1}w\right)$$

when $L$ is a differentiable matrix function $\left(\mathbf{R}\to\mathbf{M}_A\left(\mathbf{R}^+\right)\right)$, in particular when 1 is an eigenvalue for $L(0)$. We hereby evaluate this term.

**Lemma 2.11.3** *Computation of the local inverse*

*Let $C$ be a bounded, positive recurrent, not globally increasing C-process, and $L$ be the Laplace matrix function of its convolution process (at any point). We deem that 1 is the dominant eigenvalue of $L(0)$, whose characteristic eigenspace is 1-dimensional, spanned by a column vector $c\in\left(\boldsymbol{R}_+^*\right)^A$ or a row vector $r\in\left(\boldsymbol{R}_+^*\right)^A$, with $rc=1$. Then*

$$\lim_{w\to 0^+}\left((Id-L(-w))^{-1}w\right)=\frac{cr}{-r\frac{dL(w)}{dw}(0)c}$$

To prove this lemma, we first verify that $L(0)$ has a single dominant eigenvalue : thanks to Perron-Frobenius' theorem, it suffices to verify that it has a single closed communicating class, which is true thanks to the lemma 2.10.3. We use Jordan's reduction of the matrix $L(0)$, considering $P$ a change-of-basis matrix of $L(0)$ to $\Delta$ through the equation

$$L(0)=P\Delta P^{-1}$$

where

— $\Delta$ is an upper triangular matrix, with $\Delta_{1,1}=1$ and the rest of the first row is zero ;

— $P$'s first column is $c$ (which means that $P^{-1}$'s first row is $r$).

Now, let us note by

$$\forall w\in\mathbf{R}_+^*, f(w)=(Id-L(-w))^{-1}w$$

and rewrite it as

$$f(w) = \left(Id - P\Delta P^{-1} + L(0) - L(-w)\right)^{-1} w$$

As $L$ is continuously differentiable, we get a continuous function $H$ such that

$$f(w) = \left(P\left(Id - \Delta\right) P^{-1} - wH(w)\right)^{-1} w$$

that is also

$$f(w) = P\left(\frac{Id - \Delta}{w} - P^{-1}H(w)P\right)^{-1} P^{-1}$$

In particular, $-H(0)$ is the local derivative of $L$ at point 0, which means by construction that

$$H(0) = R_{\Phi_C}(0)$$

As $\Phi_C$ is decreasing almost surely by construction, $H(0)$'s entries are non-positive. Now, we look for the matrix $P^{-1}f(w)P$. We note by

- $T$ the upper triangular matrix of dimension $A-1$, being the restriction of $Id-\Delta$ to its rows and columns 2 to $A$ (we exclude the dominant eigenspace) ;
- $B(w), C(w), D(w), E(w)$ the block decomposition of $F(w) = -P^{-1}H(w)P$ relatively to the first or the other dimensions, written in the natural order (so that the block $C(w)$ is $1 \times (A-1)$).

We are going to find matrices $M_n(w)$, for $n \in \mathbf{N}$, decomposed in blocks $U_n(w)$, $V_n(w)$, $W_n(w)$, $X_n(w)$ the similar way, such that

$$\left(\frac{Id - \Delta}{w} + F(w)\right)\left(\sum_{n=0}^{\infty} M_n(w)w^n\right) = Id$$

where this sum converges to a matrix $M(w)$ for $w$ small enough.

- At order $-1$ in $w$, we find the equations $TW_0(w) = 0$ and $TX_0(w) = 0$, that translate to $W_0(w) = 0$ and $X_0(w) = 0$ because $T$ is triangular and its diagonal terms are not zero, since 1 was a single eigenvalue of $L(0)$, thus invertible.
- At order 0, we get four equations (one per block), yielding

$$\begin{aligned} B(w)U_0(w) &= 1 \\ B(w)V_0(w) &= 0 \\ TW_1(w) + D(w)U_0(w) &= 0 \\ TX_1(w) + D(w)V_0(w) &= Id \end{aligned}$$

  $B(w)$ is locally non-zero, because $B(w) = rH(w)c$ where $r$ and $c$ are positive, and $H(w)$ is non-zero and non-positive. As a consequence, we get $U_0(w) = 1/B(w)$, $V_0(w) = 0$, and then $W_1(w)$ and $X_1(w)$ because $T$ is invertible.

— At further orders $n \in \mathbf{N}^*$, we have

$$\begin{array}{rcl} B(w)U_n(w) + C(w)W_n(w) & = & 0 \\ B(w)V_n(w) + C(w)X_n(w) & = & 0 \\ TW_{n+1}(w) + D(w)U_n(w) + E(w)W_n(w) & = & 0 \\ TX_{n+1}(w) + D(w)V_n(w) + E(w)X_n(w) & = & 0 \end{array}$$

The similar way, we get successive values for $U_n(w)$ and $V_n(w)$, and then for $W_{n+1}(w)$ and $X_{n+1}(w)$ recursively.

As $B(w)$ is locally bounded away from 0, the values we find follow an arithmetico-geometric recursion scheme, and their growth rates are bounded by a geometric parameter $(\lambda(w))^n$. However, as all terms are continuous of $w$, $\lambda(w)$ is a continuous expression of $w$, thus we may take $w \leq w_0$ such that

— $B$ does not hit 0 over $[0, w_0]$ ;

— $\lambda(w)$ is then bounded by some $\lambda^+$ over $[0, w_0]$

Taking $w_1 < \min(1/\lambda^+, w_0)$ guarantees absolute and uniform convergence of the sum $M(w)$ over $[0, w_1]$. Now, we found by construction of $M(w)$ that around $w = 0$,

$$\forall i, j \leq A, M(w) = B(w)\mathbf{1}_{i=j=1} + O(w)$$

so that $P^{-1}f(w)P = M(w)$ leads to

$$f(w) = cr\frac{1}{B(w)} + O(w)$$

However, we also know that $-B(0) = rH(0)c$, so we finally get the value of $f(0)$ :

$$f(0) = \frac{cr}{r\frac{dL(w)}{dw}(0)c}$$

which ends the proof.

**Case** $\alpha(a) = 0$

When $\alpha(a) = 0$, which may happen only if $a = 0$, the above formula simplifies. Indeed, we get

$$\forall i, j \leq A, \left(L_{\Phi_C^{(0)}}(w)\right)_{i,j} = P^{(0)}_{i\to j}\mathbf{E}\left(e^{wG^{(0)}_{i\to j}}\right)$$

so $L_{\Phi_C^{(0)}}$ coincides with $L_{\vec{C}}$ (without a state $A_\infty$). We also have

1. In the regular case,

$$\forall i \leq A, V^{(0)}_{[i]}(w) = \sum_{j=1}^{A} \frac{P^{(0)}_{i\to j}}{w^{(0)}_{[j]}} \int_{d=0}^{\infty} \frac{e^{-wd} - 1}{e^{-w} - 1} d\phi^{(0)}_{i\to j}(d)$$

2. In the aperiodic case,

$$\forall i \leq A, V_{[i]}^{(0)}(w) = \sum_{j=1}^{A} \frac{P_{i\to j}^{(0)}}{w_{[j]}^{(0)}} \int_{d=0}^{\infty} \frac{e^{-wd}-1}{-w} d\phi_{i\to j}^{(0)}(d)$$

However, $\alpha(0) = 0$ means $w^{(0)} = \left(\vec{1}\right)$ thanks to the proposition 2.4.1 ; when $w$ goes to 0, we also have that

$$\lim_{w\to 0^+} \left(\frac{e^{-wd}-1}{e^{-w}-1}\right) = \lim_{w\to 0^+} \left(\frac{e^{-wd}-1}{-w}\right) = d$$

so all that remains is

$$V_{[i]}^{(0)}(0^+) = \sum_{j=1}^{A} P_{i\to j}^{(0)} \mathbf{E}\left(G_{i\to j}^{(0)}\right)$$

Hence, in the equation

$$\lim_{w\to 0^+} \left(w \left(Id - L_{\Phi_C^{(0)}}(-w)\right)^{-1} V^{(a)}(w)\right) = \left(\lim_{w\to 0^+} \left(w \left(Id - L_{\vec{C}}(-w)\right)^{-1}\right)\right) V_{[i]}^{(0)}(0^+)$$

thanks to the lemma 2.11.3, the limit matrix is (since $w^{(0)} = \left(\vec{1}\right)$)

$$\frac{\left(\vec{1}\right)\mu^{(0)}}{\mu^{(0)} \frac{dL_{\vec{C}}(w)}{dw}(0)\left(\vec{1}\right)}$$

However, as $V_{[i]}^{(0)}(0^+)$ is precisely $\frac{dL_{\vec{C}}(w)}{dw}(0)\left(\vec{1}\right)$ by definition, then we have

$$\begin{aligned} &\lim_{w\to 0^+} \left(w \left(Id - L_{\Phi_C^{(a)}}(-w)\right)^{-1} V^{(a)}(w)\right) \\ = \ & \frac{\left(\vec{1}\right)\mu^{(0)}}{\mu^{(0)} \frac{dL_{\vec{C}}(w)}{dw}(0)\left(\vec{1}\right)} \left(\frac{dL_{\vec{C}}(w)}{dw}(0)\left(\vec{1}\right)\right) = \left(\vec{1}\right) \end{aligned}$$

thus the functions $K_{[i]}^{(a)}$ all converge to 1, as well as the functions $L_{[i]}^{(a)}$. In this special case, we proved that if $\alpha(a) = 0$ (thus $a = 0$), then the default probability expressed by the functions $L_{[i]}^{(a)}$ converges to 1 when $C(0)$ increases ; as it is non-increasing of $C(0)$, this is possible only if default is almost certain as soon as $\alpha(0) = 0$. Recalling that this happens iff $E(C) \leq 0$, we recover a usual property of Lévy processes : a (not globally constant) down-drifted C-process eventually defaults almost surely.

**General case**

We assume now that $\alpha(a) \neq 0$. We shall only look at the aperiodic case, as the regular case is similar when changing
— The integrals over $d$ to sums ;
— The denominator $\alpha(a)$ for $e^{\alpha(a)-1}$.
We know that $V$ is continuous at point 0, so for every $i \leq A$,

$$\left(V^{(a)}(0)\right)_{[i]} = \sum_{j=1}^{A} \frac{P_{i\to j}^{(a)}}{w_{[j]}^{(a)}} \int_{d=0}^{\infty} \frac{1 - e^{-\alpha(a)d}}{\alpha(a)} d\phi_{i\to j}^{(a)}(d)$$

where the integral evaluates to

$$\int_{d=0}^{\infty} \frac{1 - e^{-\alpha(a)d}}{e^{\alpha(a)} - 1} d\phi_{i\to j}^{(a)}(d) = \frac{1 - \mathbf{E}\left(e^{-\alpha(a)G_{i\to j}^{(a)}}\right)}{\alpha(a)}$$

so we get

$$\left(V^{(a)}(0)\right)_{[i]} = \frac{1}{\alpha(a)} \left(\sum_{j=1}^{A} P_{i\to j}^{(a)} \frac{1}{w_{[j]}^{(a)}} - \sum_{j=1}^{A} P_{i\to j}^{(a)} \mathbf{E}\left(e^{-\alpha(a)G_{i\to j}^{(a)}}\right) \frac{1}{w_{[j]}^{(a)}}\right)$$

Noting by $v^{(a)} \in \left(\mathbf{R}_+^*\right)^A$ the vector whose coordinates are

$$\forall i \leq A, v_{[i]}^{(a)} = \frac{1}{w_{[j]}^{(a)}}$$

then this rewrites as

$$V^{(a)}(0) = \frac{1}{\alpha(a)} \left(P^{(a)} v^{(a)} - L_{\Phi_C^{(a)}}(-\alpha(a)) v^{(a)}\right)$$

Besides, the lemma 2.11.3 states that

$$\lim_{w\to 0^+} \left(\left(Id - L_{\Phi_C^{(a)}}(-w)\right)^{-1} w\right) = \frac{c^{(a)} r^{(a)}}{r^{(a)} \frac{dL(w)}{dw}(0) c^{(a)}}$$

where $r^{(a)}$ and $c^{(a)}$ are the dominant eigenvectors of $L_{\Phi_C^{(a)}}(0)$, i.e. of the matrix whose general entry $(i, j)$ is $P_{i\to j}^{(a)}$. However, this means that $c^{(a)} = \left(\vec{1}\right)$ and $r^{(a)}$ is $M^{(a)}$'s invariant distribution, so by the definition 2.3.11 of $\Phi_C^{(a)}$'s mean expectancy,

$$r^{(a)} \frac{dL(w)}{dw}(0) c^{(a)} = -E\left(\Phi_C^{(a)}\right)$$

so the value we look for holds

$$K_\infty^{(a)}\left(\vec{1}\right) = \frac{\left(\vec{1}\right) r^{(a)}}{-E\left(\Phi_C^{(a)}\right)} \frac{1}{\alpha(a)} \left(P^{(a)} v^{(a)} - L_{\Phi_C^{(a)}}(-\alpha(a)) v^{(a)}\right)$$

which rewrites as

$$K_\infty^{(a)} = \frac{r^{(a)}\left(P^{(a)} - L_{\Phi_C^{(a)}}(-\alpha(a))\right) v^{(a)}}{-E\left(\Phi_C^{(a)}\right)\alpha(a)}$$

**Properties of $K_\infty^{(a)}$**

To end the proof of lemma 2.9.1 and theorem 2.1, we will prove that $K_\infty^{(a)}$ holds the suitable properties. We shall only consider the case $\alpha(a) \neq 0$, as $\alpha(a) = 0$ is already solved.

1. It is positive, because by construction

$$K_\infty^{(a)} \geq K_{--}^{(a)}(0) > 0$$

since $K_{--}^{(a)}$ is non-decreasing.

2. It is continuous of $a$ over $\mathbf{R}_+^*$ by definition of its expression, as all terms are continuous and $-E\left(\Phi_C^{(a)}\right)$ and $\alpha(a)$ are positive. The only case we should look at is what happens at $a = 0$ when $\alpha(0) = 0$.
   — The vectors $c^{(a)} = \left(\vec{1}\right)$ and $r^{(a)}$ are the dominant eigenvectors of $L_{\Phi_C^{(a)}}(0)$, this matrix being $P_{i\to j}^{(a)}$, and we write $L_{\Phi_C^{(a)}}(-\alpha(a))$ as

$$\forall i, j \leq A, \left(L_{\Phi_C^{(a)}}(-\alpha(a))\right)_{i,j} = P_{i\to j}^{(a)}\mathbf{E}\left(e^{-\alpha(a)G_{i\to j}}\right)$$

So when $\alpha(a)$ goes to 0,

$$\forall i, j \leq A, \left(P^{(a)} - L_{\Phi_C^{(a)}}(-\alpha(a))\right)_{i,j} = P_{i\to j}^{(a)}\left(\alpha(a)\mathbf{E}\left(G_{i\to j}\right) + o\left(\alpha(a)\right)\right)$$

As $v^{(a)}$ converges to $\left(\vec{1}\right)$ by definition, we recover that

$$r^{(a)}\left(P^{(a)} - L_{\Phi_C^{(a)}}(-\alpha(a))\right) v^{(a)} = \alpha(a)\left(-E\left(\Phi_C^{(a)}\right) + o(1)\right)$$

so $K_\infty^{(a)}$ converges to $1 = K_\infty^{(0)}$ when $a$ goes to 0.

3. When $C$ is an aperiodic Lévy process, the matrix $P^{(a)}$ is the unit and the vectors $v^{(a)}$ and $r^{(a)}$ are $\left(\vec{1}\right)$. Noting by $G$ the negated only random increment of $\Phi_C^{(a)}$, we have

$$K_\infty^{(a)} = \frac{1 - \mathbf{E}\left(e^{-\alpha(a)G}\right)}{\mathbf{E}(G)\alpha(a)}$$

We ensure it is less than 1 using that

$$\forall x \in \mathbf{R}^*, 1 - e^{-\alpha(a)x} < \alpha(a)x$$

so as $G > 0$ almost surely, one gets

$$K_\infty^{(a)} < \frac{\mathbf{E}\left(\alpha(a)G\right)}{\mathbf{E}(G)\alpha(a)} = 1$$

4. When $C$ is a regular Lévy process, the matrix $P^{(a)}$ is again the unit and the vectors $v^{(a)}$ and $r^{(a)}$ are $\left(\vec{1}\right)$. Noting by $G$ the negated only random increment of $\Phi_C^{(a)}$, we have this time

$$K_\infty^{(a)} = e^{-\alpha(a)} \frac{1 - \mathbf{E}\left(e^{-\alpha(a)G}\right)}{\mathbf{E}(G)\left(1 - e^{-\alpha(a)}\right)}$$

Let us set $f(x) = 1 - e^{-\alpha(a)x}$ for $x \in \mathbf{R}$. As $\alpha(a) > 0$, $f$ is strictly concave ; from $f(0) = 0$, it comes that

$$\forall x > 1, f(x) < xf(1)$$

Hence, unless $G = 1$ almost surely (and then we have an exact equality),

$$\mathbf{E}\left(1 - e^{-\alpha(a)G}\right) < \mathbf{E}\left(\left(1 - e^{-\alpha(a)}\right) G\right)$$

Thus, we proved that $K_\infty^{(a)} \leq e^{-\alpha(a)}$, with equality iff $G = 1$ almost surely. On a side note, the fact that $C$ is a regular Lévy process with $G = 1$ almost surely indicates that its default time $T_0$ may be expressed as

$$T_0 = \sum_{i=1}^{C(0)+1} T_i$$

where the random variables $T_i$ describe the successive descent times for $C$ (necessarily involving a loss of 1 per descent). These are independent and identically distributed (whose common distribution is named $T$'s one), so one has

$$\forall a \in \mathbf{R}^+, \mathbf{E}\left(e^{-aT_0}\right) = \left(\mathbf{E}\left(e^{-aT}\right)\right)^{1+C(0)}$$

However, we know that then

$$-\ln\left(\mathbf{E}\left(e^{-aT}\right)\right) = \alpha(a)$$

thanks to the martingale property

$$\mathbf{E}\left(e^{-\alpha(a)C(T)}e^{-aT}\right) = e^{-\alpha(a)C(0)}$$

and because $C(T)$ must be $C(0) - 1$ since $G = 1$ almost surely. So,

$$\forall a \in \mathbf{R}^+, \mathbf{E}\left(e^{-aT_0}\right) = e^{-\alpha(a)(1+C(0))} = K_\infty^{(a)}e^{-\alpha(a)C(0)}$$

which yields

$$\forall a \in \mathbf{R}^+, -\Lambda_{T_0}(a) = \alpha(a)C(0) + \alpha(a)$$

The additive term $\alpha(a)$ of the affine equation is related with, as said earlier in the study, the "way" $C$ defaults. As it may only default on $-1$ no matter the starting point, we recover that this term is

$$K(a) = \ln\left(e^{-\alpha(a)(-1)}\right) = \alpha(a)$$

which is the "minimum" term required by the lemma 2.9.1.

However, we remark that $K_\infty(a)$ need not be lower than $e^{-\alpha(a)}$ in general when $C$ is a regular C-process. Taking $C$ whose Laplace matrix function is given by

$$\forall w \in \mathbf{R}, L_C(w) = \begin{pmatrix} 0 & \frac{2}{3}e^{-w} + \frac{1}{3} \\ \frac{1}{2}e^{w} + \frac{1}{2} & 0 \end{pmatrix}$$

one gets for $a = 0$ that $\alpha(0) = \ln(2)$ and

$$K_\infty(0) = \frac{3}{5} > \frac{1}{2} = e^{-\alpha(0)}$$

This is because of the vector $w^{(a)}$, that makes the "way to default"

$$\mathbf{E}\left(w_{M(T_0)}^{(a)}e^{-\alpha(a)C(0)}\right)$$

possibly greater than $e^{-\alpha(a)}$ when $w^{(a)}$'s spread is too high.

**Recapitulation**

We review the statements we proved during this study to end with the proposition 2.4.4 and the theorem 2.1. Starting with $C$ a positive recurrent, bounded, not globally increasing C-process, we did the following :

1. We defined its martingale parameter $\alpha(a)$ and dominant eigenvectors $\mu^{(a)}$ and $w^{(a)}$ at any point $a \in \mathbf{R}^+$ (first $a \in \mathbf{R}_+^*$, then we extended to $a = 0$).
2. For $C$ periodic only :
   (a) We changed $C$ into a regular process $\tilde{C}$, with the idea of getting integer increments. As shown, $\alpha(a)$ is unchanged, and $\mu^{(a)}$ and $w^{(a)}$ are scaled by $C$'s offsets (the correction to apply so that it becomes regular).
   (b) We ensured that solving the case of regular processes allows to get the theorem 2.4.4 : this used the notions of breaking points, transforming the error parameters given by the lemma 2.9.1 into suitable terms to the theorem 2.4.4.

   Periodic processes are solved using regular processes, so we may deem that $C$ is either aperiodic or regular.
3. We transformed $C$ into a decreasing process satisfying the martingale property : this is the purpose of the descending process $\Phi_C^{(a)}$, whose main property is driving the convolution equation for the twisted Laplace transforms $K_i^{(a)}$.
4. In both cases, we proved that they converge to a common positive limit $K_\infty^{(a)}$, that is continuous of $a \in \mathbf{R}^+$. We also ensured that the convergence is of the desired form :
   — Uniformly exponential for regular C-processes ;
   — Uniform for aperiodic C-processes.

Computing $-\ln\left(K_i^{(a)}(x)\right)$ leads to the final results (respectively lemma 2.9.1 and theorem 2.1) ; since $K_\infty^{(a)}$ is bounded away from 0 on every compact set, this preserves the convergence properties, so this ends the proofs.

### 2.11.3 Proposition 2.4.6

To prove the slow convergence for Liouville processes, we will use the same idea as with periodic process : we exhibit intervals over which $L_i^{(a)}$ (herein renamed $L^{(a)}$, as since $C$ is a Lévy process, there is only one $i$) is constant. As a constant function is badly approximated by an affine non-constant function, this will lead us to the proposition 2.4.6.

### Defaulting paths

To get the value of $L^{(a)}(C_0)$, we analyze all paths $C$ may follow on its way to default.

**Definition 2.11.2** *Defaulting paths of a Liouville process*
*Let $C$ be a Liouville process.*

- *Its paths are determined by a length $T \in \boldsymbol{N}$ and a sequence $P : [|1, T|] \to \{1, L\}$ indicating their negated successive payoffs values.*
- *Such a path $P$ is said to be a defaulting path for the starting point $x \in \boldsymbol{R}^+$ iff following it from $C(0) = x$ leads to $T_0 = T$, i.e. both*
  1. *$C(T) < 0$, i.e.*
  $$\sum_{t=1}^{T} P(t) > C(0)$$
  2. *$C$ does not default before $T$, i.e. (as $L > 0$)*
  $$\sum_{t=1}^{T-1} P(t) \leq C(0)$$

In the sequel, we will assume that $L \notin \mathbf{Q}$. In particular, we have $L \neq 1$, so the probability of following a path of length $T$ exactly (at precision 0) is

$$\mathbf{P}\left(\forall t \leq T, D(t) = -P(t)\right) = 2^{-T}$$

Moreover, every non-trivial path may be described as follows :

- It consists in $n_1 \in \mathbf{N}$ transition payoffs of $-1$ and $n_2 \in \mathbf{N}$ transition payoffs of $-L$ ;
- Its length is $T = n_1 + n_2 \in \mathbf{N}^*$ ;
- Its value is $-v = n_1 + n_2 L \in \mathbf{R}_-^*$.

We shall use these notations in the next paragraph.

### Preservation of defaults

We want to prove the following lemma.

**Lemma 2.11.4** *Preservation of defaults*
*Let $C$ be any $f$-Liouville process. Let $k \in \boldsymbol{N}^*$, and $C_0 \in \boldsymbol{R}^+$ such that*

$$C_0 < \frac{10^{f(k+1)-f(k)}}{12} - 1 \wedge C_0 \in \left(\boldsymbol{N} + 2 * 10^{-f(k)-1}\right)$$

*Let $P$ be any path for $C$ : if it is a defaulting path for the starting point $C_0$, then it is also a defaulting path for all starting points in*

$$\left[C_0, C_0 + 10^{-f(k)-1}\right]$$

First, as $C_0$ is the lower starting point, one is not concerned about defaults before time $T$, so it suffices to prove that $C(T) = C_0 + v < 10^{-f(k)-1}$ when starting from $C_0$. Noting $L = L_f$ the $f$-Liouville number, the possible negated values for defaulting paths write as $-v = n_1 + n_2 L$ for $n_1, n_2 \in \mathbf{N}$. In particular, as $-v \leq C_0 + L$ by definition of the default time thanks to $L > 1$, and $n_1, n_2 \geq 0$, then

$$n_2 \leq \frac{C_0 + L}{L} < 1 + C_0$$

Let us look at these $-v$ modulo $10^{-f(k)}$ : as $n_1 \in \mathbf{N}$ and

$$L \in 10^{-f(k)}\mathbf{N} + \left[0, 1.2 * 10^{-f(k+1)}\right]$$

then thanks to the bound for $C_0$, we get that

$$-v = n_1 + n_2 L \in 10^{-f(k)}\mathbf{N} + \left[0, 10^{-f(k)-1}\right]$$

In particular, no defaulting path has a negated value in $\left(10^{-f(k)-1}, 10^{-f(k)}\right)$ modulo $10^{-f(k)}$. Now, as $P$ is a defaulting path for $C_0$, we have $-v > C_0$ ; however, as we just proved that

$$-v \notin 10^{-f(k)}\mathbf{N} + \left(10^{-f(k)-1}, 10^{-f(k)}\right)$$

and we recall that

$$C_0 \in \mathbf{N} + 2 * 10^{-f(k)-1} \subset 10^{-f(k)}\mathbf{N} + 2 * 10^{-f(k)-1}$$

then the minimal possible $-v$ is $C_0 + 8 * 10^{-f(k)-1}$, thus $C(T) \leq -(8)10^{-f(k)-1}$, which ends the proof.

**Bad approximation**

The same idea as for periodic C-processes applies now.

1. Let us take two starting points $C_1, C_2$ in the range

$$\left[C_0, C_0 + 10^{-f(k)-1}\right]$$

for some $k \in \mathbf{N}^*$ and $C_0$ that suits the requirement of the lemma 2.11.4. We know that the Laplace transform of $C$'s default time may be expressed as

$$\forall a \in \mathbf{R}^+, L_{T_0}(a) = \sum_P \mathbf{P}(P) e^{-aT(P)}$$

where the sum runs over all defaulting paths $P$, $\mathbf{P}(P)$ is the probability of following the defaulting path $P$ (exactly) and $T(P)$ is $P$'s length. However, as given, any defaulting path from $C_1$ is a defaulting path from $C_2$, so this formula yields the same values for both starting points, i.e.

$$L^{(a)}\left(C_1\right) = L^{(a)}\left(C_2\right)$$

This being over the whole range, it follows that $L^{(a)}$ is constant over the whole interval $\left[C_0, C_0 + 10^{-f(k)-1}\right]$.

2. Now, the martingale parameter $\alpha(a)$ has a linear (in $C_0$) effect on the main term in the theorem, so the approximation of a constant function over the previous range $\left[C_0, C_0 + 10^{-f(k)-1}\right]$ by an affine (multiplicative factor $\alpha(a)$) function of $C_0$ on this interval yields an error of at least

$$y(a,k) = 10^{-f(k)-1}\alpha(a)/2$$

It follows that the error function $e$ at $C_0$ and $a$ cannot decrease permanently below $y(a,k)$ before $C_0$ gets larger than

$$\max\left(\left(0, \frac{10^{f(k+1)-f(k)}}{12} - 1\right) \cap \left(\mathbf{N} + 2 * 10^{-f(k)-1}\right)\right) \geq x(k)$$

setting $x(k) = 10^{f(k+1)-f(k)}/12 - 2$. In particular, there is $x \geq x(k)$ such that $e(x,a) \geq y(a,k)$.

3. Now, let us take $g$ as in the proposition 2.4.6, $y \in \mathbf{R}^+$, and $a \in \mathbf{R}_+^*$. As $g$ converges to 0, then for every $k \in \mathbf{N}^*$ there is $z(k) \in \mathbf{R}^+$ after which $g$ is lower than $y(a,k)$ ; for future purposes, we may assume that the sequence $(z(k))_{k \in \mathbf{N}^*}$ goes to infinity. We choose recursively the terms of the function $f$ such that

$$\forall k \in \mathbf{N}^*, x(k) = \frac{10^{f(k+1)-f(k)}}{12} - 2 > z(k)$$

for example $f(1) = 1$ and then

$$\forall k \in \mathbf{N}^*, f(k+1) = k + f(k) + \left\lceil \frac{\ln\left(12z(k) + 24\right)}{\ln(10)} \right\rceil$$

We verify that $L_f$ defined as such is not a rational number, so that the $f$-Liouville process is aperiodic. If $L_f$ is a rational number, its decimal expansion is

— Either finite (but it is not, because there are infinitely many digits 1 given by the successive positions $f(k)$ since $f$ is increasing) ;

— Or ultimately periodic, so the successive number of 0 digits between digits 1 cannot go to infinity (but it is the case, because $f(k+1) - f(k) \geq k$)

Hence, the $f$-Liouville process holds the hypotheses of the theorem 2.1 :

— It is bounded (by $L_f$), positive recurrent (because it is a bounded Lévy), not globally increasing by construction ;
— We just proved that it is aperiodic.

For every $k \in \mathbf{N}^*$, there is there is $x \geq x(k) > z(k)$ such that $e(x, a) \geq y(a, k) > g(x)$ (since $x > z(k)$). Hence, as the sequence of successive terms $z(k)$ goes to infinity, there is $k \in \mathbf{N}^*$ such that $z(k) > y$, and taking this $k$ and an associated $x$ will end the proof.

4. Finally, if $\alpha(0) > 0$ (which is automatic if $E(C) > 0$), we recall that $\alpha$ is increasing so one may set

$$y(k) = 10^{-f(k)-1}\alpha(0)/2 \leq y(a, k)$$

and then work with $y(k)$ instead of $y(a, k)$.

This ends the proof.

### 2.11.4 Proposition 2.4.7

We aim at proving the exponential convergence for a suitable positive recurrent C-process $C$. Hence, let $T$, $i, j \leq A$, $u \in \mathbf{R}_+^*$ and $\eta > 0$ be as given.

#### Local weight

During this proof, we shall use the following definitions to state the property given by hypothesis.

**Definition 2.11.3** *Local weight*

*Let $X$ be a real (or $\infty$) random variable, $\eta > 0$ and $x \in \boldsymbol{R}$.*

— *$X$ is said to be $\eta$-locally heavy on $x$ iff for every $u \in (0, 1]$,*

$$\boldsymbol{P}(X \in [x \pm u]) \geq u\eta$$

— *$X$ is said to be $\eta$-locally heavy around $x$ iff there is $\gamma > 0$ such that $X$ is $\eta$-locally heavy on every $y \in (x \pm \gamma)$. We also say that it is $\eta$-heavy over the interval $(x \pm \gamma)$.*

*In both cases, it is said locally heavy iff $\eta$-locally heavy for some $\eta > 0$.*

We want to prove the following properties about local weight.

**Lemma 2.11.5** *Operations on local weight*

*Let $X$ and $Y$ be independent random variables over $\boldsymbol{R} \cup \{\infty\}$.*

1. *Let $U \subseteq \boldsymbol{R}$ be a compact interval such that $\boldsymbol{P}(X \in U) > 0$. There are $x \in U$ and $\eta > 0$ such that $X$ is $\eta$-locally heavy on $x$.*
2. *Let us deem that $X$ is a-locally heavy on $x$ and $Y$ is b-locally heavy around $y$ over $(y \pm \gamma)$. For every $\xi < \gamma$, $X+Y$ is c-locally heavy around $x+y$ over $(x+y \pm xi)$ with*
$$c = ab \min\left(\lambda, \lambda^2\right)$$
*where $\lambda = (\gamma - \xi)/2$.*
3. *If $X$ is a-heavy over the interval $(x \pm \eta)$ and $Y$ is b-heavy over the interval $(y \pm \epsilon)$, then for every $\xi < \eta+\epsilon$, $X+Y$ is heavy over the interval $(x+y \pm \xi)$.*

In particular, $X+Y$ is not necessarily heavy over the whole $(x+y \pm (\eta+\epsilon))$, take e.g. $X$ and $Y$ uniform distributions over $[0, 1]$.

1. Let us set $p_0 = \mathbf{P}(X \in U) > 0$. First, $U$ cannot be empty, and if it a singleton then $X$ is $p_0$-locally heavy on its only point. Without loss of generality, we may now deem that $U = [0, 1]$ after an affine transformation (that only modifies $\eta$ by a non-zero multiplicative factor). Let us set $z_0 = 0$ ; we define recusively the sequences $(z_n)_n$ and $(p_n)_n$ by
   (a) Compute the probabilities
   $$\begin{aligned} x_0 &= \mathbf{P}\left(X \in \left[z_n, z_n + 2^{-n-1}\right]\right) \\ x_1 &= \mathbf{P}\left(X \in \left[z_n + 2^{-n-1}, z_n + 2^{-n}\right]\right) \end{aligned}$$
   (b) If $x_0 \geq x_1$, then set $z_{n+1} = z_n$ and $p_{n+1} = x_0$ ; else set $z_{n+1} = z_n + 2^{-n-1}$ and $p_{n+1} = x_1$.
   (c) Start over to get the sequences.

   After an immediate recursion, $p_n \geq p_0/2^n$ by choice of each $z_n$. As the series of $2^{-n}$ converges, then $(z_n)_n$ converges to a limit called $x \in \mathbf{R}$. We want to prove that $x = z$ holds the desired property, so let $u$ be as described and $n = \lceil -\log_2(u) \rceil$. By construction of terms $z_n$ and $p_n$, we have
   $$\mathbf{P}\left(X \in \left[z_n, z_n + 2^{-n}\right]\right) \geq p_n \geq p_0/2^n$$
   However, one has $z_n \leq z$ because the sequence $(z_n)_n$ is non-decreasing, and $z_n + 2^{-n} \geq z$ because by construction
   $$z \leq z_n + \sum_{k=n+1}^{\infty} 2^{-k} = z_n + 2^{-n}$$
   It follows that $z_n \geq z - 2^{-n} \geq z - u$ and $z_n + 2^{-n} \leq z + u$, so
   $$\left[z_n, z_n + 2^{-n}\right] \subset [z \pm u]$$

which implies that

$$\mathbf{P}\left(X \in [z \pm u]\right) \geq \frac{p_0}{2^{\lceil -\log_2(u) \rceil}} \geq \frac{p_0 u}{2}$$

so $\eta = p_0/2$ works.

2. Let $\xi < \gamma$ and $\lambda = (\gamma - \xi)/2$ ; we are going to prove that $X + Y$ is locally heavy over $(x + y \pm \xi)$. Let us write that for $u \in (\pm\xi)$ and $\upsilon \leq \lambda$, we have

   $$\begin{aligned} & \mathbf{P}\left(Y \in [x + y + u - X \pm \upsilon]\right) \\ \geq\ & \mathbf{P}\left(Y \in [x + y + u - X \pm \upsilon] \,|X \in [x \pm \lambda]\right) \mathbf{P}\left(X \in [x \pm \lambda]\right) \end{aligned}$$

   As $X$ is $a$-locally heavy on $x$, we have

   $$\mathbf{P}\left(X \in [x \pm \lambda]\right) \geq a\lambda > 0$$

   As $X$ and $Y$ are independent, we bound this from below by

   $$\mathbf{P}\left(Y \in [x + y + u - X \pm \upsilon]\right) \geq \inf_{z \in [\pm\lambda]} \left(\mathbf{P}\left(Y \in [y + u - z \pm \upsilon]\right)\right) a\lambda$$

   Since $z$ and $\epsilon$ are bounded by $\lambda$ and $u$ strictly bounded by $\xi$, then

   $$[y + u - z \pm \upsilon] \subset (y \pm \gamma)$$

   and as $Y$ is $b$-locally heavy around $y$ for $\gamma$, this tells that

   $$\mathbf{P}\left(Y \in [x + y + u - X \pm \upsilon]\right) \geq \inf_{z \in [\pm\lambda]} (b\upsilon)\, a\lambda = \upsilon(ab\lambda)$$

   Finally, setting $q = ab\lambda$, we proved that for every $u \in (\pm\xi)$ and $\upsilon \leq \lambda$,

   $$\mathbf{P}\left(X + Y \in [x + y + u \pm \upsilon]\right) \geq \upsilon q$$

   which leads, for every $\upsilon \leq 1$, to

   $$\mathbf{P}\left(X + Y \in [x + y + u \pm \upsilon]\right) \geq \min\left(\upsilon, \lambda\right) q$$

   If $\lambda > 1$, this ends the proof ; if not, we take $q\lambda$ instead of $q$.

3. Applying the previous result at point $x + u$ (with $u \in (\pm\eta)$) for $X$, we get that for every $\xi < \epsilon$, $X + Y$ is $c$-heavy over $(x + y + u \pm \xi)$ with such a $c$. This being for every $u \in (\pm\eta)$, we get the result.

These properties will prove useful during the incoming analysis.

### Construction of heavy payoffs

First, let us prove that we have a descending transition payoff $F_{i \xrightarrow{n} j}$ such that

— $A_i$ and $A_j$ are in $C$'s descending class $A'$ ;

— It is locally heavy over some non-trivial interval $I_1 \subset \mathbf{R}$ (for the descending process $\vec{C}$).

By hypothesis, we have a transition payoff $D_{i \to j}$ that is locally heavy over some non-trivial interval $I_1 \subset \mathbf{R}$. When $C$ is positive recurrent and not globally increasing, it is possible to build a cycle of arbitrary low (negative) value, with a length $n \in \mathbf{N}^*$, starting from $A_i$ and going first through the transition $(i \to j)$. Conditionally to $M$ going through this cycle, let us express $C(n) - C(0)$ as the sum of its active increments :

— The first $D_{i \to j}$ is locally heavy over $I_1 = (-y, -x)$.

— Thanks to the lemma 2.11.5, the cycle has a finite value thus subsequent transition payoffs must be locally heavy on some points $x_k \in \mathbf{R}$.

Let us take a state $A_k \in A'$ and a path of finite value from $A_k$ to $A_i$ ; in this path, we select its last state in $A'$, called $A_y$. We look at the path between $A_y$ and $A_i$, whose length is $h$ and value is called $v_1 \in \mathbf{R}$. One must have $v_1 \geq 0$, else there would be a descending state between $A_y$ and $A_i$.

— If $v_1 - y < 0$, then we decompose the value $v_1$ over its successive transition payoffs $x_{1 \dots h}$. Choosing $\epsilon > 0$ small enough and $U = (-\infty, x_t + \epsilon)$ in the lemma 2.11.5 yields successive points $y_t \leq x_t + \epsilon$ such that the transition payoffs are locally heavy on $y_t$ ; thanks to the second part of the lemma 2.11.5, the concatenation of this path to the transition $D_{i \to j}$ yields a value $C(h) - C(0) \leq v_2 = v_1 - y + h\epsilon$, that may be chosen negative. Hence, this gets a descending payoff between $A_y$ and $A_j$ that is locally heavy over a non-trivial sub-interval of $(v_2, \min(v_2 + y - x, 0))$ (that run over a duration $h + 1$).

— If not, we know that from $A_j$ we may build a path of arbitrarily large negative value $v < 0$. It is decomposed in the same fashion and chosen such that $v_1 - y + v < 0$ (but $v_1 - y + v' \geq 0$ for all partial values of the path), and its concatenation eventually leads to a descending payoff between $A_y$ and its finishing state. Likewise, we control the lengh of the additional path, so we get in both cases a locally heavy descending transition payoff, finally called $F\ i \xrightarrow{t} j$.

Now we work on $\vec{C}$ only. As $\vec{C}$ admits a cycle going first through the transition $(i \to j)$, this allows for the existence of some descending transition payoffs $F\ i_k \xrightarrow{t_k} i_{k+1}$ with $k \leq n$. The use of the second part of the lemma 2.11.5 indicates that $\vec{C}(n) - \vec{C}(0)$ (conditionned to the cycle) is locally heavy on some non-trivial interval $I_2$ whose own measure is called $m > 0$. We have in particular

$I_2 \subset \mathbf{R}_-^*$ by definition of $\vec{C}$. We repeat this cycle $p = \lceil 2AQ/m \rceil + 1$ times : thanks to the last part of the lemma 2.11.5, $\vec{C}(np) - \vec{C}(0)$ (conditionally to the repeated cycle) is locally heavy on some interval $I_3 \subset \mathbf{R}_-^*$ whose length is greater than $2AQ$.

**End of the proof**

From this point, we transform the cycle for $\vec{C}$ into the corresponding cycle for $C$, so that it has now a total length of

$$t_0 = p \sum_{k=1}^{n} t_k$$

There are paths of controlled length $L_i \leq A - 1 \in \mathbf{N}$ going to every state of $M$, so for every state $A_i$ there is a path $Q_i$ of length at most $t = t_0 + A - 1$ such that conditionally to $Q_i$, $C(t_0 + L_i) - C(0)$ is locally heavy on an interval $J_i$ whose length is greater than $2AQ$ and at most $(A-1)Q$ apart from $I_3$ (because we add up to $A - 1$ times at most $Q$ to $I_3$). It follows that there is an interval $J$, e.g. $[(-A-1)Q, (-A+1)Q]$, belonging to the intersection of all intervals $J_i$,

— Of length at least $2Q$ (by substraction) ;

— Belonging to $\mathbf{R}_-^*$.

The final part is similar to the proof for aperiodic C-processes : the previous construction yields a "region-halting time" $\tau$ for $C$ itself, defined by stopping at every time period with 1/2 probability (and then systematically after $t$ time periods). Decomposition on the possible paths and stopping times indicates that for every $x$ high enough, $K_i^{(a)}(x)$ is a convex combination of other values $K_j^{(a)}(y)$.

The key point is that the chosen paths have a positive contribution to the value $K_i^{(a)}(C(0))$ and are associated with an expectancy

$$\mathbf{E}\left(K_j^{(a)}(C(t)) | \forall u \leq t, M(t) = P(t)\right)$$

while this conditioned $C(t) - C(0)$ was proven to be $c$-locally heavy over $J$ for some $c > 0$. Therefore, as $t$ is controlled, we have $C(t) - C(0) \geq -Qt$ ; let us consider the discrepancy $\delta$ between $K_j^{(a)}$'s local extrema $K^-$ and $K^+$ over $[C(0) - Qt, C(0)]$. Now, let us take an interval $I = [x, x + Q]$ and a $C(0)$ such that $I \subset C(0) + J$, i.e. $C(0) \in [x + AQ, x + (A+1)Q]$ ; let us compute the measures of $\left(K_j^{(a)}\right)^{-1}([K^-, K^- + \delta/2])$ and $(K^- + \delta/2, K^+]$ over $I$, named $m^-$ and $m^+$ respectively. As $C(t)$ is $c$-locally heavy over $C(0) + J$ (hence $I$), then the masses $m^-$ and $m^+$ have an effect of at least $m^{\pm}c/2$ on the convex combination evaluating $K_i^{(a)}(C(0))$. Thus

$$K_i^{(a)}(C(0)) \in \left[K^- + (\delta/2)m^+(c/2), K^+ - (\delta/2)m^-(c/2)\right]$$

This holds over $C(0) \in [x + AQ, x + (A+1)Q]$ for every $i$, which means that the discrepancy has become

$$\left(K^+ - (\delta/2)m^-(c/2)\right) - \left(K^- + (\delta/2)m^+(c/2)\right) = \delta - (\delta/2)(c/2)\left(m^- + m^+\right)$$

and as $m^- + m^+ = Q$ by construction, the new discrepancy is

$$(1 - cQ/4)\,\delta$$

so decreased by an exponential factor. Finally, as one may repeat this operation at an ergodic pace $Qt + (A+1)Q$, it follows that the convergence is exponential.

# Chapitre 3

# Décisions d'investissement et gestion coopérative dans une économie markovienne

Durant cette étude, nous nous intéressons au problème commun de la stratégie d'investissement idéale d'un acteur souhaitant échapper à la faillite. Nous avons expliqué dans l'introduction générale pourquoi travailler avec la contrainte d'illiquidité totale et dans un modèle markovien nommé C-processus.

## 3.1 Motivations de l'étude

Toutefois, comme avancé lors du travail précédent, la dépendance temporelle entre les fluctuations aléatoires des prix des actifs détruit de manière significative la qualité prédictive des modèles de la littérature quant à l'évaluation des risques de faillite. Nous avons donc décidé de revoir les questions sur la stratégie d'investissement dans le modèle de C-processus conçu précédemment.

### 3.1.1 Modélisation du problème

On commencera donc par créer un univers, nommé C-jeu, modélisant notre question : il s'agit de bâtir un jeu d'investissement dont le joueur est soumis à des C-processus différents suivant ses choix d'investir ou non au fil du temps. Le but de l'acteur dans ce modèle sera donc de minimiser la transformée de Laplace de son temps de faillite $T$ en un point $a \in \mathbf{R}^+$ (dépendant de ses préférences), soit donc

$$\min_S \left( \mathbf{E} \left( e^{-aT} \right) \mathbf{1}_{T<\infty} \right)$$

une quantité nommée espérance de faillite qui dépend de la stratégie d'investissement $S$ choisie par le joueur. On définira donc à la fois le jeu dans son ensemble et les stratégies comme $S$ à la disposition du joueur.

### 3.1.2 Différences entre Lévy et C-processus

Nous cherchons comment un acteur doit investir dans ce modèle pour combattre les risques inhérents à l'illiquidité totale des investissements, c'est-à-dire quelle est la meilleure stratégie $S$. Dans des cas particuliers de C-jeux, nous pourrons exhiber des concepts pour $S$ :

- Le plus souvent, $S$ est une stratégie dite "à seuil", qui ordonne l'achat quand le capital liquide disponible dépasse un certain seuil. De tels problèmes d'arrêt optimal en dynamique brownienne sont ainsi étudiés par [34].
- Le seuil en question $b$ s'exprime comme une fonction affine des coûts d'investissement $I$, couramment écrite selon la forme

  $$b = I/\gamma + H$$

  où $\gamma$ est une fonction nommée envie du ratio entre les paramètres martingalisants des C-processus impliqués dans le modèle, et $H$ une barrière nommée handicap indépendante de $I$ destinée à prévenir les risques de liquidité et tenir compte du caractère markovien du modèle.

L'expression exacte de $\gamma$ et $H$ donnera les interprétations de la stratégie optimale en termes économiques. Par exemple, on pourra mettre l'accent sur la valeur du "droit à l'attente" avant d'investir (comme [33]), concept inconnu dans le marché parfait de Modigliani et Miller, ou sur les particularités de chaque état du marché qui incitent ou non à investir, lesquelles n'existent pas dans les modèles de Lévy.

### 3.1.3 Investissement coopératif

Nous nous pencherons finalement sur les notions d'investissement coopératif. Ainsi, nous démontrerons que dans notre univers à base de C-processus, deux individus peuvent tirer parti du caractère markovien du marché afin de diminuer tous deux leurs risques de faillite sans apport exogène d'actifs, et nous indiquerons comment procéder pour atteindre le point d'équilibre. Pour ceci, nous mettons au point un "contrat de soutien" mutuel entre les acteurs, chacun s'engageant à verser une compensation à son homologue quand l'évolution du marché lui est plus favorable qu'à l'autre. Nous démontrerons que les contrats de soutien optimaux $S^*$ égalisent les transformées matricelles de Laplace des C-processus $C_1$ et $C_2$ des acteurs : en les calculant en les paramètres martingalisants $\alpha_1(a_1)$ et $\alpha_2(a_2)$ ainsi obtenus, on obtient

$$L_{C_1+S^*}(\alpha_1(a_1))\, e^{-a_1} = L_{C_2-S^*}(\alpha_2(a_2))\, e^{-a_2}$$

Nous retrouverons par ailleurs que la fusion totale entre les deux fonds, qui est optimale dans un modèle de Lévy, reste optimale dans le cadre plus général des C-processus. Ce concept de fusion expliqué dans [31, 44] contribue en effet toujours à réduire les risques de faillite de chacun, d'une manière que l'on exposera.

# Investment decisions and cooperative management in a Markovian economy

We are interested in common management issues when the market is governed by a Markovian process. Specifically, we want to solve investment decisions concerned with liquidity issues in a Markovian environment, and recover how the concept of asset securitization is still optimal between several agents to avoid bankruptcy even when the market's evolution has momentum.

## 3.2 Introduction

In this study, we revise the well-known solutions to the issues of common management decisions when the market is deemed to have a Markovian behaviour instead of being governed by a Lévy process. Specifically, we deem that the stochastic processes involved in the models follow the dynamics of C-processes described during the previous work : we use the main results about C-processes to compute default risks and management strategies as to avoid one's default.

### 3.2.1 Notion of opportunities

Throughout this study, we shall consider one or several agents whose cash flows are governed by C-processes. They are given management opportunities consisting in modifying their dynamics to other C-processes, and want to select the best choice of dynamics to escape from bankruptcy risks. Examples of management decisions include investment, typically described as the act of paying an immediate price in return for a hope of permanent long-term benefits, eventually overcompensating for the initial investment. Perpetuities are the simplest case of such a permanent rate of return, and one may refer to [15] when interested in the case of Brownian-driven cash flows. Investment decisions dealt with in this work include funding of companies, when a bank provides a required initial investment for a startup company to run, or an individual purchases stocks, then expects to raise profits in dividends. In a lesser extent, bank loans also fall in this category, except for that the payments are not permanent. Conversely, we may define a "liquidation decision" as the act of accepting an immediate amount of cash with a long-term expense, the counterpart of investment decisions. In particular, we are not intersted in speculative trading : the buyer seeks profits only from the dividends and not through a higher resale price of the stock or the assets, although we may see a speculative activity as the compound of an investment and a liquidation, because we will disregard price variations of investment opportunities in this model.

### 3.2.2 Single-player investment

First, we will deal with an investor (later named "he") who is subject to an exogenous market, in the sense that his exogenous profits and losses randomly depend on the market. The investor wants to avoid bankruptcy (liquidity reserves falling below 0), and is presented with an investment oppotunity as above. However, investment is risky in the sense that the expected incomes are not guaranteed to the investor whose incomes may be insufficient if "the market goes wrong", e.g. dividends are reduced for any reason, or even when a borrower gets insolvent and does not repay their debt at all. This is especially a concern for an investor with a low level of cash reserves ; because of his own dependency on the market, he may be confronted with more short-term liquidity issues because of quick "bad luck" on the market than he would if he had not invested (the quantity of money invested is not available to pay unexpected expenses). Typically, this happens when he must liquidate assets in an emergency, and the cumulated incomes earned before liquidation do not cover the liquidation costs.

Our first question throughout this study will be to solve management decisions of different kinds. When the market follows a common Brownian-shape behaviour, investment decisions have already been investigated ([24, 20]), so we shall especially focus on how modifying the nature of the market may change the choice of an investor for a same investment opportunity : for example, we investigate on the effects of market volatility and effects of "boom-bust"-like cycles on optimal investment decisions.

### 3.2.3 Questions about multi-player cooperation

We will also look at the issues of cooperative management under the same Markovian structure. As mentioned by [16], measures of risks other than the evaluation of volatility (variance) are not necessarily reduced by the well-known strategy of "pooling" (as explained in [31, 44]) ; however, we already stated previously that the variance statistic of a C-process is a misleading indicator of risk levels. This rose the question of whether risk mutualization by pooling of assets is still optimal to a group of agents in this Markovian model : we shall prove that this kind of securitization is still best, as well as quantify the benefits of partial or total pooling. Notice that we are still interested in partial (thus sub-optimal) risk mutualization, because of moral hazard phenomena happening when the risk is exceedingly shared : e.g. [26] states that large pools of risks reduce efforts and total incomes to the pools. We will also mention moral hazard in this study, but for this reason, we may sometimes limit ourselves to special cases of risk mutualization.

Our next question will be to find a bilateral contract aiming at reducing both players' default risks. We will use the structure of C-processes to our advantage, in order to quantify how each transition of its underlying Markovian process specifically affects each player, eventually finding out that both players' best interests are to support each other in moments of need.

## 3.3 Choice of the model

Our question calls for the use of a model based on a Markovian environment, namely C-processes. We build a universe accounting for multiple players $J_p$ subject to a market $M$, holding cash reserves following C-processes $C_p$ ; moreover, we want them to invest in order to enhance their payoffs, so we consider that they are able to change of C-processes along with investment. We detail the complete construction of the model in the next paragraphs.

### 3.3.1 Markovian process

To illustrate the effects of the market on the investor's cash flows, we chose to model them by a C-process, because unlike Lévy-like processes, C-processes allow for short-term dependency between successive incomes and an exogenous market, thus are more fit to our study. We have seen in the previous work that momentum effects of C-processes may drive the default probabilities far away from their "expected" values when neglecting time dependency ; as a consequence, we revise the solutions to common management problems when using the structure of a C-process instead of Lévy processes. We eventually aim at showing how the momentum behaviour changes investment and management decisions found in the cases where Lévy processes represent wealth.

The effect of an investment decision will be represented as changing of C-processes : depending on the players' decision about the investment opportunity, the modification of long-term incomes is described by a new C-process, with different transition payoffs. Hence, an investment decision amounts to the choice between several C-processes, with different

— Structures (distribution of random increments, states of the market), describing the future evolution of the investor's wealth ;
— Starting points (initial wealth), indicating the remaining liquidity e.g. after deciding to buy or not.

Later in the study, we will be interested in cooperative management between several investors (in a number $J \in \mathbf{N}^*$), as well as multiple successive investment opportunities. As a consequence, our model consists of $J$ players, each holding an amount $C_p$ of liquid assets, $C_p$ being a discrete-time random process.

### Temporary C-processes

We call C-games some games based upon C-processes, in which players may be confronted to strategic decisions modifiying the transitions : depending on past investment decisions, the increments of their assets follow different distributions. C-games are defined thanks to a "tree of C-processes" : edges starting from each node refer to outcomes of an investment opportunity, leading to a new C-process. By means of a recursion, the universe is defined thanks to nodes (C-processes) and branchings (investment opportunities) as follows.

**Definition 3.3.1** *Temporary C-processes*

*A temporary C-process is the determination of*

— *A Markovian time-homogeneous process* $(M(t))_{t\in \boldsymbol{N}}$ *with*
  — *A finite state space* $A$*, containing one starting state* $A_0$*, and* $f \in \boldsymbol{N}$ *finishing states* $A_i$ *for* $i \leq f$ *whose set is called* $F$ *;*
  — *Transition probabilities : for any* $t \in \boldsymbol{N}$*, we have*

$$\forall i, j \in A, P_{i\to j} = \boldsymbol{P}\left(M(t+1) = A_j | M(t) = A_i\right)$$

  *It is deemed that the starting state holds* $M(0) = A_0$ *almost surely, and the finishing states are absorbing :* $\forall i \in F, P_{i\to i} = 1$*. Moreover, we shall assume that* $A_0 \notin F$ *for the sake of simplicity.*

— *Random variables called transition payoffs, such that*
  — *For every* $i \in A\backslash F, j \in A$*, there is a probability distribution over* $\boldsymbol{R} \cup \{+\infty\}$ *defining a random variable* $D_{i\to j}$ *with respect to this distribution ;*
  — *For any* $t \in \boldsymbol{N}^*$*, we define the family* $\left(D_{i\to j}(t)\right)_{i,j}$ *as independent and identically distributed copies of the family* $\left(D_{i\to j}\right)_{i,j}$ *with respect to the time variable* $t \in \boldsymbol{N}^*$*.*

  *For every* $i \in A\backslash F, j \in A$ *and* $t \in \boldsymbol{N}$*,* $D_{i\to j}(t)$ *is called the transition payoff between states* $A_i$ *and* $A_j$ *at time* $t$*.*

— *A discrete-time process* $C$*, satisfying*
  — $C(0) = C_0 \in \boldsymbol{R}^+$ *is deterministic, called* $C$*'s starting point ;*

— *If $M(t) \notin F$, then C's next increment holds*

$$\forall t \in \boldsymbol{N}, C(t+1) = C(t) + D_{M(t)\to M(t+1)}(t+1)$$

— *If $M(t) \in F$, then $C$ is $+\infty$ after time $t$ (which is consistent with the other Markovian transition payoffs).*

*We say that $C$ is a temporary C-process whose :*

— *Underlying Markovian process is $M$ ;*
— *Starting state is $A_0$, finishing states are $A_i$ for $i \leq f$ ;*
— *Transition payoffs are (the distributions of) the random variables $D_{i\to j}$, for each state numbers $i \in A\backslash F$ and $j \in A$ ;*
— *Starting point is $C_0$.*

*We shall deem that $C$ is positive recurrent, i.e. for any $s \in \boldsymbol{N}$, for every $i, j \in A\backslash F$,*

$$\boldsymbol{P}(\exists t \in \boldsymbol{N}; M(t+s) = A_j \wedge C(t+s) < \infty | M(s) = A_i \wedge C(s) < \infty) > 0$$

According to this definition, finishing states will refer to accessible investment opportunies from $A_0$, so the temporary C-process indicates the moves of a player's wealth $C$ between investment opportunities.

### Temporary C-games

Taking a temporary C-process to model the wealth of each investor between successive investment times, we may define a temporary C-game as the description of everyone's assets.

**Definition 3.3.2** *Temporary C-game*

*Let $C_{1\ldots J}$ be $J$ temporary C-processes sharing the same underlying Markovian process $M$ (whose starting state is $A_0$, and finishing states are all the same ones, indicated by the subset $F$ common to all). We say that $C_{1\ldots J}$ form a temporary C-game, whose*

— *Markovian process is $M$ ;*
— *Starting state is $A_0$, finishing states are F's ones ;*
— *Transition payoffs are random variables, named $D_{i\to j}^{(p),[a_{1\ldots n}]}$ for every $i \notin F$, $j$ in M's state space, and $p \leq J$ indicating a player, whose distributions are defined through $C_{1\ldots J}$ : the family*

$$\left(D_{i\to j}^{(p),[a_{1\ldots n}]}\right)_{p\leq J}$$

*has the same distribution as*

$$\left(C_p(t+1) - C_p(t)\right)_{p\leq J} | M(t) = A_i \wedge M(t+1) = A_j$$

— *Starting point is* $(C_p(0))_{p\leq J} \in \boldsymbol{R}^J$.
*An elementary C-game is a temporary C-game with no finishing states.*

In particular, as temporary C-process with no finishing states form C-processes that will govern $C$'s increments permanently, players will play no part at all (no further strategical choices) ; as a consequence, these temporary C-processes are permanent and define pure-luck games, hereby named elementary C-games.

### 3.3.2 Investment opportunities

Contrarily to these purely random C-processes, players have some control over the dynamics of the processes $C_p$ : they may invest, trying to increase their future incomes, at the expense of an immediate price, and an increased risk of going bust earlier as described above.

#### Conditions of investment

To be allowed to invest, players must wait until the process $M^{[G(t)]}$ lands on a pre-determined state, named an "investment opportunity". The terms of the buying contract vary with the model, but in the main line, players are given the possibility of immediately changing of C-processes, for a more avantageous long-term behaviour. Sometimes this refers to an investment, costing some price $I \in \mathbf{R}^+$ (notice that when $I \in \mathbf{R}^*_-$, this is selling an asset, so it refers to a liquidation opportunity). When an investment opportunity is hit by $M$, involved players make a choice about it (on conditions depending on the model). Every possible outcome leads to another temporary C-process, maybe with different starting points indicating the price of this opportunity, paid by one (or more) of the players.

**Definition 3.3.3** *Investment opportunity*
*An investment opportunity is the determination of*
— *A triggering state, being a finishing state of a temporary C-process ;*
— *Possible outcomes, defining a list of temporary C-processes ;*
— *For each outcome, a method of price attributions to players, defined by a function of present cash values and investment opportunities.*

We will mainly focus on a model where investment is "take it or leave it" at some fixed price, depending on the present C-process : when opportunity knocks, one player is offered the possibility of spending the price in order to modify both players' C-processes. This kind of investment opportunities that we are going to study is called "single-player opportunities".

**Model 3.1** *Nature of investment opportunities*
*Investment opportunities are solved as follows upon realization.*

1. *When an investment opportunity happens, it indicates a player* $J_p$ *and a fixed price* $I \in \boldsymbol{R}$.
2. $J_p$ *decides alone whether or not to spend* $I$*, modifying the C-process accordingly.*
3. *If* $J_p$ *accepts, a cost* $-I$ *is attributed to* $J_p$ *and* 0 *to other players ; if not, a* 0 *cost is attributed to everyone.*

*The corresponding new C-process is then started.*

We will also encounter management opportunities, where two players agree on management in both players' interest. Management opportunities indicate a set of reachable C-processes to these players ; then they agree on one of them (the method of agreement will not be discussed, however we shall determine useful equilibria to get the possible outcomes of this agreement) and start these new C-processes subsequently.

**Structure of the game**

C-games may now be defined recursively using

— (Temporary) C-processes to govern asset values $C_{1\ldots J}$ ;

— Investment opportunities, indicating which C-process is active at some point during the game (after some specific investment opportunity).

As a consequence, we represent it as a tree-shaped oriented graph $G$, whose nodes are labelled as the list of previous branchings (outcomes of investment opportunities) in the form

$$G^{\left[(a_i, r_i)_{i \leq n}\right]}$$

where $n \in \mathbf{N}$ is the depth of the node in the graph, $a_i \in \mathbf{N}^*$ is the label of the $i^{th}$ investment opportunity to be hit (among accessible ones in this node), and $r_i \in \mathbf{N}^*$ its outcome number. When there is no possible confusion, we will often write the superscript as $[\ldots]$ to represent a fixed node, for the sake of simplicity. Thus, purchasing properties is represented by the graph $G$ : the node hit by time $t$, called $G(t)$, changes at any time when an investment opportunity arises. To each node is associated a C-process (with an underlying Markovian process), so that successive C-processes and prices of investment opportunities drive cash values $C_p(t)$.

**Definition 3.3.4** *Contents of a C-game*

*A C-game must contain the following elements in its universe :*

*—* $J \in \boldsymbol{N}^*$ *players* $J_{1\ldots J}$*, each of them holding liquid assets given by a process* $C_p : (\boldsymbol{N} \rightarrow \boldsymbol{R})$ *;*

— *A tree-shaped finite oriented graph $G$, whose nodes are labelled as the list of previous branchings as above. This graph is seen as a process, with $G(t)$ describing the node hit on the graph at time $t$.*

*Moreover, every state $G^{[\dots]}$ in $G$ consists in a temporary C-process, giving*

— *A discrete Markovian process $M^{[\dots]}$, with a finite state space $A^{[\dots]}$, with a starting state $A_0^{[\dots]}$ and finishing states forming the set $F^{[\dots]}$, whose transition probabilities are given by*

$$\forall i \notin F^{[\dots]}, j \in A^{[\dots]}, P_{i\to j}^{[\dots]} = \boldsymbol{P}\left(M^{[\dots]}(t+1) = j | M(t) = i\right)$$

*It is said active at time $t$ iff $G(t)$ corresponds to the node number $G^{[\dots]}$. At starting time, $G(0)$ is assumed to be the root of the graph, i.e. $G^{[]}$.*

— *Random transition payoffs for every player : for $i \notin F^{[\dots]}$, $j \in A^{[\dots]}$ and $p \leq J$, the transition payoff between states $i$ and $j$ for player $J_p$ is distributed like a random variable named $D_{i\to j}^{(p),[\dots]}$.*

Now that the contents for a C-game are defined, we may build C-games recursively using the tree structure.

**Definition 3.3.5** *Construction of a C-game*

*C-games are built recursively as follows.*

1. *Any elementary C-game is a C-game, whose*
   — *Graph $G$ is constituted of a single node called $G^{[]}$ ;*
   — *Markovian process at this node $G^{[]}$ is the elementary C-game's underlying Markovian process, called $M^{[]}$ ;*
   — *Transition payoffs at this node $G^{[]}$ are the elementary C-game's ones ;*
   — *Starting state is $M^{[]}$'s one ;*
   — *Price attributions do not exist, as there are none.*
2. *Let $K$ be a temporary C-game, whose finishing states are called $F_{1\dots n}$. For every $i \leq n$, let*
   — *$a_i$ be an investment opportunity triggered by $F_i$ ;*
   — *For every such $i$, some possible outcomes $u_r$ for some $r \in \boldsymbol{N}^*$ ;*
   — *For every outcome, indexed by a couple $(i, r)$, some method of price attributions yielding a price $P_{j,(i,r)}(C_{1\dots J})$ to every player $J_p$, in the configuration of liquid assets $C_{1\dots J}$ when confronted to outcome $(i, r)$.*
   *For every outcome $(i, r)$, we deem that $K_{(i,r)}$ is already a C-game, whose*
   — *Graph is $G_{(i,r)}$ ;*
   — *Starting state is $M_{(i,r)}$'s one, called $A_{(i,r)}$.*
   *Then we define a C-game $K$, whose*
   — *Graph $G$ is constituted of a root node called $G^{[]}$, and one edge from $G^{[]}$ to every root node of a subsequent graph $G_{(i,r)}$ ;*

— *Markovian process at the node* $G^{\square}$ *is the temporary C-game's underlying Markovian process, called* $M^{\square}$*, while the Markovian processes at other nodes are conserved from* $K_{(i,r)}$*'s ones ;*
— *Transition payoffs at the node* $G^{\square}$ *are the temporary C-game's ones ;*
— *Starting state is* $M^{\square}$*'s one ;*
— *Price attributions between* $G^{\square}$ *and the node* $G^{[(i,r)]}$ *are* $I_{j,(i,r)}(C_{1\ldots J})$*, while price attributions related to deeper nodes are unchanged.*

*Every item previously related to a node labelled by* $[\ldots]$ *in game* $K_{(i,r)}$ *is now labelled by* $[(i,r),\ldots]$ *in game* $K$*.*

This construction indicates the dynamics of player's assets while in each node, and how investment opportunities allow to change of nodes.

### Dynamics of a C-game

The dynamics of players' assets are thus governed by
— The node $G(t)$ on the graph $G$, describing the history of bought properties ;
— The present transition between $M^{G(t)}(t-1)$ and $M^{G(t)}(t)$ ;
— The prices paid during investment opportunities (changing of nodes).

For this reason, we are interested in the following processes.

**Definition 3.3.6** *Dynamics of a C-game*

*Additionally to the previous items, a C-game is given the discrete-time processes* $G$ *and* $M^G$*, working with the processes* $C_p$ *the following fashion.*
— *Initial values :* $G(0) = G^{\square}$ *is the root of graph* $G$*, a Markovian process* $M^{\square}$ *starts at* $t = 0$ *from its starting point and we set* $M^G(0) = M^{\square}(0)$*. The initial cash amounts* $C_p(0)$ *are called initial values of the C-game.*
— *If* $M^{[G(t)]}(t+1) \notin F^{[G(t)]}$*, i.e. the temporary* $M$ *does not hit a finishing state, then the game remains in the same node and the assets are randomly modified by the temporary C-process, so*
  — $G(t+1) = G(t)$ *(no node change) ;*
  — $M^G(t+1) = M^{[G(t)]}(t+1)$ *(according to* $M^{[G(t)]}$*'s transition probabilities) ;*
  — *For every player* $J_p$*,*

$$C_p(t+1) = C_p(t) + D^{(p),[G(t)]}_{M^{[G(t)]}(t)\to M^{[G(t)]}(t+1)}$$

  *accoring to the temporary transition payoffs.*
— *If* $M^{[G(t)]}(t+1) \in F^{[G(t)]}$*, i.e. the temporary* $M$ *hits a finishing state (investment opportunity number* $i$ *after node* $G(t)$*), then*
  1. *Transition payoffs are first observed ;*

*2. Players agree on the outcome $r$ of the investment opportunity, yielding prices $I_{p,(i,r)}$ for every player $J_p$ ;*

*3. Modifications of $G$, $M^G$ and $C_p$ are then realized.*

*The next step is thus given by*

— *The new node $G(t+1) = [G(t), (i,r)]$ ;*

— *A new Markovian process $M^{[G(t+1)]}$ starts at $t+1$ from its own starting state and we set $M^G(t+1) = M^{[G(t),(i,r)]}(t+1)$ ;*

— *Transition payoffs between $t$ and $t+1$ are cumulated to price attributions, thus for every $J_p$,*

$$C_p(t+1) = C_p(t) + I_{p,(i,r)} + D^{(p),[G(t)]}_{M^{[G(t)]}(t) \to M^{[G(t)]}(t+1)}$$

*The process $M^G$ is called the game's general Markovian process.*

We will say that

— A Markovian process $M^{[...]}$ is active at time $t$ iff $G(t)$ corresponds to the node number $G^{[...]}$.

— An increment $D^{(p),[...]}_{a \to b}$ is active at time $t$ iff $G(t)$ is the correct node, $M^{[...]}(t)$ indicates the state $a$, and $M^{[...]}(t)$ indicates the state $b$.

To clarify matters :

— Investment opportunities happen after the present transition payoff is paid and before the next one. In particular, if simultaneously $C_p(t) \geq 0$, the active increment at time $t+1$ for $J_p$ is $D^{(j)}(t+1) < -C_p(t)$, but an opportunity is reached at time $t+1$ with an outcome allowing $C_p(t+1)$ to remain non-negative, then there is no default at time $t+1$.

— By construction of the game, strategic choices only appear when $G$ moves.

— In particular, when $G^{[...]}$ is a leaf of the graph, the processes permanently behave as in an elementary C-game. If $G$ hits such a node, there will never be any future investment opportunity, and the liquid assets $C_p$ will be pure-luck random C-processes forever.

The "take it or leave it" behaviour of investment opportunities appears in the graph, because while accepting the offer means paying $I$ immediately and shifting $G$'s status, hereby changing future transition payoffs, rejecting it means that $G(t)$ also changes to a different node, therefore preventing players to reconsider their decisions.

### 3.3.3 Players' aims

In our study, we focus only on how players aim at avoiding the risk of short-term bankruptcy : decisions about an optimal consumption or dividend policy are outside the frame of this work, although we shall briefly introduce the concept of consumption later in order to explain the final results. The reader interested in managing dividends should refer to [38] or [3]. Default of player $J_p$ is defined by $C_p$ going down below 0. Each player is deemed to make use of their own investment decisions to minimize default risks, whose preferencies are quantified in this paragraph.

### Investor's dilemma

Most often, the decision will rely on a trade-off between

- Benefits from owning investment : as transition payoffs are modified with $G$, one wants to buy the best investment oppotunities to increase their own transition payoffs and lower the default risks.
- Risk of liquidity shortage : having to pay an immediate expense increases the risk of quick default in the case of short-term "bad luck", while the additional income from the property still does not cover its price.

Liquidity issues towards investment have already be investigated ([28, 33]). Intuitively, investment should happen when one's remaining cash after the expense is high enough to cover short-term default risk for long enough, so that the property eventually "pays for itself" (this will be referred to as the return on investment time, as seen below). This forms a concept of security level for liquid assets, that will be discussed further in this study. In particular, [33] investigates on the issue of "value of waiting", a concept that will be observed with our model.

Additionally, when multiple successive investment opportunities are at stake, further considerations come to mind when forecasting about later investment. Increasing one's income, at the expense of an immediate price, implies that the security level of cash for a future opportunity will be hit later if its effects on the fluctuations are "small" with respect to its price : this is like comparing affine functions $C(t) = E_1 t$ or $C(t) = E_2 t - I$, where $E_2 > E_1$ is the main drift of $C$ (better with investment) and $I \in \mathbf{R}^+$ is the investment cost. Hence, investment choices may help or prevent the player from buying a more valuable opportunity later on for these liquidity reasons.

This general concept of rentability vs. liquidity will be called the "investor's dilemma" hereafter.

### Strategies

When an investment opportunity arises, the players decide to buy upon past information about the game. However, as the game structure is Markovian, the only relevant variables are

- $G(t)$ the situation of bought properties ;
- $M^G(t)$ the present state of the market ;

— $C_p(t)$ the liquid assets of all players,

as well as perfect knowledge of the rules of the game, and of other players' preferencies, which are assumed throughout the study. This is why an investment choice is assumed to rely only on its price $I$ and these items at present time.

**Definition 3.3.7** *Strategies*

*For each player $J_p$, a strategy $S_p$ is is a function mapping $G(t)$, $M(t)$, every $C_k(t)$, the investment opportunity $g_k$, and its price $I$ to either 1 or 0 depending on whether or not $J_p$ decides to buy. As the game has complete and perfect information, we only look at strategies as deterministic functions of these relevant variables.*

Players solve their investment dilemmas by computation of a quantity named default expectancy. To penalize early default over long-term default, we chose it to be expressed as the Laplace transform of the default time, taken at a point $a_p \in \mathbf{R}^+$ expressing $J_p$'s discount factor, hereby called $J_p$'s Laplace parameter.

**Definition 3.3.8** *Default expectancy*

*For $J_p$ a player in the universe, we define*

— *The default time $T_p$ as*

$$T_p = \min\left(\{t; C_p(t) < 0\}\right)$$

*If default never happens, we set $T_p = \min(\emptyset) = +\infty$ ;*

— *The Laplace parameter $a_p \in \boldsymbol{R}^+$ as a constant expressing the exponential decay of $J_p$'s preferencies over time ;*

— *The default expectancy based upon strategies $S_k$ for all players $k \leq J$ as*

$$F_p\left(S_{1\ldots J}\right) = \boldsymbol{E}\left(e^{-a_p T_p}\right)$$

*provided that all players use their strategies $S_k$ during investment opportunities.*

*Players aim at minimizing $F_p$ with respect to their own $S_p$.*

As the game is perfect, it is natural to assume that every player has a unique optimal strategy $S_p^*$ (or at least, that every player has optimal strategies, with the probability of a conflict being zero, as is often the case with continuous random distributions of increments). We shall call

$$F_p^* = F_p\left(S_{1\ldots J}^*\right)$$

the result of all optimal strategies for player $J_p$. By extension, we will also name by

$$F_p^*(t) = F_p^*\left(G(t), M(t), C_{1\ldots J}(t)\right)$$

the value of $F_p^*$ when the game situation is as described by $G$, $M$ and $C_{1\ldots J}$.

## 3.4 Implications of the theory of C-processes

During this section, we shall focus on the case of a single player (hereby named "he"), i.e. $J = 1$, being able to choose immediately between several investment opportunities a single time, in order to avoid bankruptcy. Our main goal is to exhibit the differences between optimal investment strategies when comparing standard (Brownian) cases with different kinds of C-processes, quantifying positive feedback effects and indicating when they are detrimental to investment in the investor's dilemma.

### 3.4.1 Direct analysis

Let us deem that $n \in \mathbf{N}^*$ investment opportunities are available now, each one leading to an elementary C-process $K_i$ after paying a price $I_i \in \mathbf{R}^+$, whose starting state is named $i$ itself. In this paragraph, the investor wants to minimize his default expectancy through a single choice of some such $i$.

#### Approximative control of the C-processes

To simplify matters and prevent integrability issues, we deem that all C-processes involved in the study are bounded and aperiodic. In particular, we already know that they have a martingale parameter $\alpha_p$ (except when they are globally increasing, where the martingale parameter is assumed to be $+\infty$). When $G$ finally hits a leaf, an elementary C-game is played. Since it is purely random, one may compute directly the default expectancy for every player : assuming that the main theorem of C-processes applies as given during the previous work, it yields the expression

$$F_p^* \left(G(t), M(t), C_{1\ldots J}(t)\right) = w_{[M(t)]}^{(a_p)} e^{-\alpha_p(a_p)C_p(t)} \left(Z(a_p) + o(1)\right)$$

where the $o(1)$ refers to $C_p$ going to $+\infty$ while $a_p$ remains constant, $w^{(a_p)}$ is a vector indexed by $M^{[\ldots]}$'s state space called $C^{[\ldots]}$'s dominant eigenvector, and $Z(a_p)$ is a multiplicative term called $C$'s severity of default at point $a_p$.

**Definition 3.4.1** *Fundamental approximation*

*The fundamental approximation drops the $o(1)$ term. Hence, let $G_0$ be a leaf of $G$, $M_i$ be a state of $M^{[G_0]}$, and $C_{1\ldots J} \in \boldsymbol{R}^+$ be cash amounts. Under the fundamental approximation, the value to be minimized is assumed to be given through*

$$F_p^* \left(G_0, i, C_{1\ldots J}\right) \equiv F_p\left(i, C_p\right) = Z(a_p) w_{[i]}^{(a_p)} e^{-\alpha_p(a_p)C_p}$$

*where $w^{(a_p)}$ and $\alpha_p(a_p)$ depend only on $G_0$ and $a_p$.*

Henceforth, this symbol $\equiv$ will be used to indicate a property that derives from the fundamental approximation. When $C$ is actually a Lévy process, this approximation is commonly known as Cramér-Lundberg's approximation, see [7, 12] for additional information.

This simplification is necessary not only to simplify computations, but also to avoid boundary effects around $C \approx 0$. Mainly, we expect the player to invest iff his liquid assets at decision time are greater than some threshold like in [33], and we aim at computing it. However, as we work in discrete time, issues appear regarding the default severity term, and we can get an erratic behaviour of the buying decision, as shown in paragraph 3.8.1, not being subject to a threshold. Notice that the same constatation may rise with some continuous-time jump processes, so discrete time does not create a flaw in our study. To compute explicitly $\alpha_p$, $w^{(a_p)}$ and $Z(a_p)$, or to evaluate the errors done by the fundamental approximation, the reader may refer to the other study. In particular, we shall assume throughout this study that these approximation errors do not severely bias the optimal strategies : this seemingly arbitrary assumption is however justified when the convergence in Cramér-Lundberg's approximation ([7]) is exponential (e.g. when distributions have a smooth density, see the previous work).

### Reminders about the theory of C-processes

Naming $C_0 \in \mathbf{R}^+$ the player's initial assets and $a \in \mathbf{R}^+$ his own Laplace parameter, for every outcome $i$, we get a martingale parameter and a dominant eigenvector at point $a$ through the fundamental approximation for $K_i$. We call $\alpha_i \in \mathbf{R}^+$ the martingale parameter and $w_i \in \mathbf{R}^+$ the coordinate of the dominant eigenvector associated with the subsequent starting state $i$, so that

$$F(i, C_0) = Z_i w_i e^{-\alpha_i (C_0 - I_i)}$$

thus the player chooses the $i$ such that $F(i, C_0)$ is minimal. One recalls that the items appearing in $F$'s expression are interpreted as such :

— $\alpha_i$ is the martingale parameter, describing the exponential decay of the default expectancy, i.e. the marginal multiplicative effect of one additional unit of wealth on the default expectancy. When $a$ increases, $\alpha_i$ increases because late default is less severely penalized.

— $w_i$ is the multiplicative correction, related with the specificity of the starting state hit once in the game $K_i$. One has typically $w_i > 1$ if the state $i$ is among the "bad" states of $K_i$'s underlying Markovian process (incoming transition payoffs are bad to the player), because this bad state will cost some money to exit from, thus increasing the default risks.

— Similarly, one may define

$$\delta_i = \frac{-\ln(w_i)}{\alpha_i}$$

the corrective gain of switching to $K_i$ (related with the spread term of the vector $w$ from $K_i$) in monetary terms, so that owning $C$ in this state $i$ is "tantamount to" owning $C + \delta_i$ in an averaged state of $K_i$'s underlying Markovian process, in default expectancy terms.

We also remind that the Laplace parameter $a$ may be used to express the player's preferences over time : low values of $a$ (down to $a = 0$) account for the long-term default probability (the default expectancy becomes the exact default probability at $a = 0$), and high values of $a$ indicate willingness to avoid short-term default (no matter long-term risks).

### Comparison between investment opportunities

Let us take two investment opportunities, labelled $i = 1$ or $i = 2$. First, we shall dispose of the case $\alpha_1 = \alpha_2$, as if this holds, then the best investment opportunity is given by the minimal (exponential) value of $I_i - \delta_i$ for this common $\alpha$. Interestingly, this does not necessarily mean that for an identical market trend (same $\alpha$) one should go for the best price : indeed, the price must be corrected by the specificity of the next state $i$, because a "locally bad" state incurs an additional cost to the player, that may overcome the benefits of the lower price.

Henceforth, we assume that $\alpha_2 > \alpha_1$. Immediate computations on the default expectancy indicate that choice 2 (costing $I_2$) is better than choice 1 (costing $I_1$) iff

$$\alpha_1 (C_0 - I_1 + \delta_1) - \ln(Z_1) < \alpha_2 (C_0 - I_2 + \delta_2) - \ln(Z_2)$$

which eventually leads to

$$\begin{aligned} C_0 &> \frac{\alpha_1 (\delta_1 - I_1) - \alpha_2 (\delta_2 - I_2) + \ln\left(\frac{Z_2}{Z_1}\right)}{\alpha_2 - \alpha_1} \\ &= I_1 + \frac{\alpha_1 \delta_1 - \alpha_2 \delta_2 + \ln\left(\frac{Z_2}{Z_1}\right)}{\alpha_2 - \alpha_1} + (I_2 - I_1) \frac{\alpha_2}{\alpha_2 - \alpha_1} \end{aligned}$$

This condition expresses as a threshold for the present cash reserves : one buys the better invesment opportunity (better $\alpha_i$ for $i = 2$) as soon as $C_0$ is large enough. We are now investigating on this threshold, hereafter called $B$.

— Terms like $I_i - \delta_i$ appear rather than $I_i$ alone, because the investment costs also account for the corrective gains, like as above for $\alpha_1 = \alpha_2$.

— Not accounting for terms $Z_i$, if $I_1 - \delta_1 > I_2 - \delta_2$, then as $\alpha_1 < \alpha_2$ (with $\alpha_1 \geq 0$), we get $B < 0$ : this means that the investor should always buy

$i = 2$. This holds because under this condition, there is no upside to the choice of $K_1$ : the (corrected) expense is higher to go to $K_1$ and the game itself is less attractive (worse $\alpha_i$). However, we recall that the fundamental approximation is flawed when going to $C_0 \approx 0$, which accounts for boundary effects for a low $C_0$ (most notably, when $I_1 < C_0 < I_2$, one must choose $i = 1$ to avoid immediate bankruptcy).

— Neglecting the "constant" terms $\delta_i$ and $Z_i$, one gets a boundary of

$$I_1 + (I_2 - I_1) \frac{\alpha_2}{\alpha_2 - \alpha_1}$$

so if $I_2 > I_1$, setting aside the mandatory expense $I_1$ in both cases (hence the additive $I_1$), one needs cash reseves *directly proportional* to the additional expense $I_2 - I_1$.

This decomposition of $B$ leads us to considerations on what will be called the investor's incentive and handicap to buy an investment opportunity.

**Definition 3.4.2** *Incentive and handicap for an investment opportunity*

*Let $\alpha_1(a)$ and $\alpha_2(a)$ be the martingale parameters for the player's C-process after choosing the investment $I_1$ or $I_2$, where it is assumed that $\alpha_2(a) > \alpha_1(a)$. We call*

$$\gamma(a) = 1 - \frac{\alpha_1(a)}{\alpha_2(a)}$$

*the incentive of investment, and when $\gamma(a) \neq 0$,*

$$H(a) = \frac{\alpha_1(a)\delta_1(a) - \alpha_2(a)\delta_2(a) + \ln\left(\frac{Z_2(a)}{Z_1(a)}\right)}{\alpha_2(a) - \alpha_1(a)} = \frac{\ln\left(\frac{Z_2(a)w_2^{(a)}}{Z_1(a)w_1^{(a)}}\right)}{\alpha_2(a) - \alpha_1(a)}$$

*the handicap of investment, so that (under the fundamental approximation) the player buys iff*

$$C \geq I_1 + H(a) + \frac{I_2 - I_1}{\gamma(a)}$$

As $I_2$ is the most interesting investment, we shall say that the player "buys" or "invests" when he chooses it (as we shall often have $I_2 > I_1$ so that the problem is interesting). Conversely, we shall say that he "liquidates" when he chooses $I_1$.

### Interpretations of the incentive and the handicap

The previous equation allows us several observations.

— When $\alpha_1(a) = \alpha_2(a)$, $C$ plays no part as $B(a)$ goes to $\pm\infty$ (depending on $Z_i w_i^{(a)}$) and $\gamma(a)$ goes to 0. The decision only depends on the sign of

$$\ln\left(\frac{Z_2 w_2^{(a)}}{Z_1 w_1^{(a)}}\right) + (I_2 - I_1)\alpha_i(a)$$

which in turn simplifies to the sign of

$$(\delta_1(a) - I_1) - (\delta_2(a) - I_2) + \frac{\ln\left(\frac{Z_2}{Z_1}\right)}{\alpha_i(a)}$$

This means that both outcomes are roughly "equivalent" as far as long-term behaviour is concerned, and the question mainly refers to the corrected expense. The residual term in $Z_i$ is corrective and accounts for a discrepancy between severity of defaults in games $C_1$ and $C_2$.

— When $\alpha_2(a) > \alpha_1(a)$, the incentive $\gamma(a)$ is positive, and the C-process is better when accepting the offer. This means that (up to $H(a)$) investment should be accepted with cash reserves directly proportional to the incurred costs, the factor being $1/\gamma(a)$ :
  - — If $\gamma(a) \approx 0$, investment has weak effects on $C$'s martingale parameter, so its advantages fade out compared with liquidity shortage issues, and high cash reserves are required to cancel these risks ;
  - — If $\gamma(a) \approx 1$ because $\alpha_1(a)$ is low, investment is required to avoid probable default without acceptance of the offer, despite liquidity risks ; the ratio of proportionality closes to 1, incitating to buy whenever possible ($C \geq I_2 + H(a)$) ;
  - — If $\gamma(a) \approx 1$ because $\alpha_2(a)$ is high, investment allows to eliminate default risks even with low cash reserves, incitating to buy whenever possible once again.

  Finally, we notice that as $\alpha_1(a) \in \mathbf{R}^+$, then $\gamma(a) \leq 1$, which (up to the handicap, that deals with boundary effects) translates roughly to $C \geq I$ : one cannot buy without enough cash reserves.

— When $C$ is low, the investment is actually detrimental to the player for liquidity reasons ; for this reason, he chooses to liquidate (for a liquidation value of $I_2 - I_1$). Liquidation is optimal when $C$ is lower than the threshold, indicating liquidity distress : the player wants to escape from short-term default, even at the expense of increasing the risks of long-term default.
  - — If $\gamma(a) \approx 0$, liquidation has weak effects on $C$'s martingale parameter, so its drawbacks are minimal compared with liquidity shortage issues, and liquidation is optimal unless default risks are negligible (high $C$) ;

— If $\gamma(a) \approx 1$ because $\alpha_1(a)$ is high, liquidation means that the player has much to lose when liquidating, so should not unless forced (the inequation yields $C \leq 0$) ;
— If $\gamma(a) \approx 1$ because $\alpha_2(a)$ is low, liquidation means that the player is doomed if forced to liquidate, and yet again should not unless forced.

— The logarithmic term $H(a)$ indicates the spread between specificities of states $k_0$ and $k_1$. It is positive when the state $k_2$ hit after buying is more detrimental to survival than $k_1$ after liquidating ; in particular, it discourages investment, requiring $C$ to be higher to trigger the investment threshold.

In the case of Lévy processes, there are no spread terms $\delta_i(a)$. While the incentive does not directly depend on them, the investment threshold for a C-process is modified by the underlying Markovian process, because of direct corrections $\delta_i(a)$, but also because momentum effects may hamper the martingale parameters (like in the previous study). As a consequence, a modification of $\alpha_i(a)$ has a geometrical effect on the buying threshold through the expression of the incentive, so potentially yields higher estimation errors than misevaluating $H(a)$ through $\delta_i(a)$, in particular when high investment costs $I$ are at stake.

### 3.4.2 Expectancy of the default time

When the player is interested in minimizing long term default risks, e.g. $\mathbf{P}(T < \infty)$, he looks for the special case $a \approx 0$ of $T$'s Laplace transform. In particular, let us look at two outcomes with associated C-processes whose mean expectancies are respectively $E_1$ and $E_2$ :

— If $E_1 > 0$ and $E_2 > 0$, the above calculations work since we get $\alpha_i(0) > 0$.
— If $E_1 \leq 0$ and $E_2 > 0$, we have $\alpha_2(0) > \alpha_1(0) = 0$ and thus the player must choose $I_2$ to avoid almost sure eventual bankrpuptcy.
— However, if $E_1 < E_2 \leq 0$, we need more specific computations (this translates to “a higher order of Taylor series” for the Laplace transform).

It follows that $\mathbf{E}(T)$ is the concern because we have

$$\mathbf{E}\left(e^{-aT}\right) = \mathbf{P}\left(T < \infty\right) - a\mathbf{E}(T) + O\left(a^2\right)$$

As $E_i \leq 0$, the default probabilities are both 1, so the question amounts to maximizing $\mathbf{E}(T)$. Intuitively, if an investment decision is right to minimize $\mathbf{E}\left(e^{-aT}\right)$ for every small $a$, then it will also be correct to maximize $\mathbf{E}(T)$ by differentiation : for this reason, we are now interested in first-order properties of the martingale equation of a C-process. In this whole paragraph, we shall therefore assume that $E(C) < 0$ unless noted otherwise.

### Incentive and handicap for $a = 0$

Let us try and express the incentive and the handicap given by the definition 3.4.2 around $a \approx 0$. We shall name $\alpha'(a)$, $Z'(a)$ and $w'^{(a)}$ the derivatives of $\alpha$, $Z$ and $w$ at a point $a \in \mathbf{R}$, as they are ensured to exist after the other work, so that Taylor development around 0 of the terms of $\gamma$ and $H$ should yield them at $a = 0$.

**Proposition 3.4.1** *Incentive and handicap for $E < 0$*

*When $E_1 < E_2 < 0$ and $a$ converges to 0, the buying decison for $I_2$ against $I_1$ is determined as follows.*

— *The incentive is*

$$\gamma = 1 - \frac{E_2}{E_1}$$

— *We define a "differential severity of default parameter" $c_i$ for each C-process $C_i$ as*

$$c_i = \lim_{x\to\infty} \left( \frac{1}{E_i} \boldsymbol{E}(C_i(T)|C_i(0) = x) + \boldsymbol{E}\left( w'^{i,(0)}_{[M(T)]} | C_i(0) = x \right) \right)$$

*Provided that $C_i$ is aperiodic, $c_i$ is well-defined and amounts to $-Z_i'(0)$, where $Z_i(a)$ is the standard "severity of default" parameter of $C_i$ at point $a$.*

— *The handicap expresses as*

$$H = \frac{(c_1 - c_2) + (w_2' - w_1')}{\frac{-1}{E_2} - \frac{-1}{E_1}} = \left( \frac{E_1 E_2}{E_2 - E_1} \right) (c_1 - c_2 + w_2' - w_1')$$

*Under the fundamental approximation, the player still buys iff*

$$C \geq I_1 + H + \frac{I_2 - I_1}{\gamma}$$

In particular, one remarks that this $\gamma$ is the limit of $\gamma(a)$ when $a$ goes to 0. However, even if the computation of $H(a)$ for $a \approx 0$ by direct Taylor development yields the correct handicap (i.e. $H = H(0)$), this does not suffice to state the buying decision, because of the error term in the fundamental approximation : indeed, all we get for $a \approx 0$ is

$$C > I_1 + H(a) + \frac{I_2 - I_1}{\gamma(a)} + \frac{o(1)}{\alpha_2(a) - \alpha_1(a)}$$

where the error term is not necessarily controlled as $\alpha_i(0) = 0$. For the proof, we shall consequently refer to the paragraph 3.7.1. The terms $c_i$ and $w'^{(0)}_i$ will be interpreted in the next paragraph.

### Formulations of the handicap

We recall that for $E(C) > 0$, each state $A_i$ had an own contribution to $C$'s default probability, given by $w_{[i]}^{(0)}$, leading to the handicap of an investment opportunity. We are going to state a similar property for $E(C) < 0$ : in particular, we are interested in the vectors ${\mu'}^{(0)}$ and ${w'}^{(0)}$. By diffentiation of the eigenvector equations, one gets the following properties about them.

**Proposition 3.4.2** *Vector equations for ${\mu'}^{(0)}$ and ${w'}^{(0)}$*

*Let $C$ be as required for the proposition 3.4.1.*

1. *The vector ${\mu'}^{(0)}$ holds the following equation in vector $v \in \boldsymbol{R}^A$ :*

$$v\,(Id - P) = \frac{1}{E(C)}\mu R_C(0) - \mu$$

2. *The only solutions $v$ to this equation form a one-dimensional affine space, containing ${\mu'}^{(0)}$ and directed by the vector $\mu$.*
3. *The vector ${w'}^{(0)}$ holds the following equation in vector $v \in \boldsymbol{R}^A$ :*

$$\frac{1}{E(C)}R_C(0)\left(\vec{1}\right) - \left(\vec{1}\right) = (Id - P)\,v$$

4. *The only solutions $v$ to this equation form a one-dimensional affine space, containing ${w'}^{(0)}$ and directed by the vector $\left(\vec{1}\right)$.*

These equations come from differentiation of the eigenvector equation at point $a = 0$, since we know that $\alpha(0) = 0$. Recalling that $\forall a \in \mathbf{R}^+, \mu^{(a)}\left(\vec{1}\right) = 1$ and differentiating this equality yields in particular ${\mu'}^{(0)}\left(\vec{1}\right) = 0$, which completely determines ${\mu'}^{(0)}$ once given its vector equation. Likewise, ${w'}^{(0)}$ is detemined thanks to

$$0 = \frac{d\left(\mu^{(a)}w^{(a)}\right)}{da}(a=0) = {\mu'}^{(0)}\left(\vec{1}\right) + \mu {w'}^{(0)}$$

which yields $\mu {w'}^{(0)} = 0$.

### Asymptotical expectancies

Let us now take another look at $L_C$. Thanks to $C$'s mean expectancy, we know that the successive values $\mathbf{E}\left(C(t) - C(0)\right)$ roughly follow a line of slope $E(C)$ when $t$ grows. However, to find more accurate results about these expectancies, one should take into account the specificities of $M$'s states : as it happens, the corrective terms are additive and are each related to one single state. To get them, we already know that $L_C$'s powers yield periodically concatenated C-processes, and that differentiation leads to the expectancies of their transition payoffs. Thanks to Perron-Frobenius' theorem, we may look at the matrix

$$-\frac{d\left(L_C(\alpha)\right)^t}{d\alpha}(\alpha = 0)$$

for large values of $t$, ultimately leading to expectancies of large concatenated transition payoffs.

**Proposition 3.4.3** *Asymptotic linearity of cumulative transition payoffs*

*Let $C$ be a positive recurrent, bounded C-process, whose mean expectancy is $E(C)$. There are two vectors $E_{\to\infty}, E_{\infty\to} \in \boldsymbol{R}^A$, given by their coordinates*

$$E_{\to\infty} = \left(E_{i\to\infty}\right)_{i\leq A} \wedge E_{\infty\to j} = \left(E_{\infty\to j}\right)_{j\leq A}$$

*such that*

1. *They are centered relatively to the states :*

$$\sum_{i=1}^{A} \mu_{[i]} E_{i\to\infty} = 0 \wedge \sum_{j=1}^{A} E_{\infty\to j} = 0$$

2. *There is $\lambda < 1$ such that they express the expectancies of transition payoffs over several time periods with geometrical convergence, i.e. for every $i, j \leq A$ and $t \in \boldsymbol{N}$,*

$$\boldsymbol{E}\left(C(t) - C(0) | M(0) = A_i \wedge M(t) = A_j\right) = tE(C) + E_{i\to\infty} + E_{\infty\to j} + o\left(\lambda^t\right)$$

*whenever the condition $M(0) = A_i \wedge M(t) = A_j$ has non-zero probability.*

*The vectors $E_{\infty\to}$ and $E_{\to\infty}$ are unique. We will name*

— *$E_{\infty\to j}$ the asymptotical expectancy offset finishing on state $A_j$ ;*

— *$E_{i\to\infty}$ the asymptotical expectancy offset starting from state $A_i$.*

The vectors $E_{\infty\to}$ and $E_{\to\infty}$ may be computed thanks to $L_C$, and describe how a state $A_i$ is locally beneficial or detrimental to $C$.

— The asymptotical expectancy offset finishing on $A_j$ is the "bonus" of payoffs one is expected to get when landing on $A_j$, compared to the average payoffs driven by $E(C)$. Positive values mean that transitions leading to $A_j$ are somewhat higher than $E(C)$, while negative values mean that they are lower.

— The asymptotical expectancy offset starting from $A_i$ is the bonus that $C$ gets when leaving $A_i$ ; positive values mean that transitions exiting $A_i$ win a higher amount than the average given by $E(C)$, negative ones mean that they win less. The condition of scaling $\mu E_{\to\infty} = 0$ means that starting from a "random" state, with respect to $M$'s invariant measure, averages the earnings to zero.

We notice that $E_{i\to\infty} + E_{\infty\to i}$ is not zero in general. The sum of these expectancies measures how a state $A_i$ is globally beneficial (if positive) or detrimental (if negative) to $C$'s mean expectancy.

### Links between derivatives and expectancy offsets

Recalling the equation

$$\mathbf{E}\left(T|M(0) = i\right) \equiv \frac{C}{-E_i} + \left(c_i - w'_i + \frac{I_i}{E_i}\right)$$

we expect $C$'s default time to be roughly measured thanks to $C$'s mean expectancy ; as such, the terms appearing in the player's handicap refer to state specificities, which translate to expectancy offsets. Indeed, we have the following equalities.

**Proposition 3.4.4** *Expressions of $\mu'^{(0)}$ and $w'^{(0)}$*

*The expectancy offsets $E_{\infty\to}$ and $E_{\to\infty}$ from the proposition 3.4.3 hold the following equalities.*

1. *For every $j \leq A$,*

$$\frac{E_{\infty\to j}}{E(C)} = \frac{\mu'^{(0)}_{[j]}}{\mu_{[j]}} = \frac{d\ln\left(\mu^{(a)}_{[j]}\right)}{da}(a=0)$$

2. *For every $i \leq A$,*

$$\frac{E_{i\to\infty}}{E(C)} = w'^{(0)}_{[i]} = \frac{d\ln\left(w^{(a)}_{[j]}\right)}{da}(a=0)$$

In particular, this allows us to interpret the terms in the equation governing $\mathbf{E}(T)$.

— The main term $C(0)/(-E(C))$ codes for the effect of $C$'s drift, expecting $C$'s default time to behave linearly with respect to available initial cash reserves (as in "time equals length divided by speed").

— $w'^{(0)}_{[i]}$ indicates the specificity of the state after the investment decision : as mentioned, its value is $E_{i\to\infty}/E(C)$ for the new C-process. Recalling that $E(C) < 0$, a high (positive) value for $E_{i\to\infty}$ means that the state $A_i$ is beneficial to $C$, which has a positive incidence over $\mathbf{E}(T)$ : according to the "speed" interpretation, the additional net worth $E_{i\to\infty}$ of $A_i$ amounts to an additional survival time of $-E_{i\to\infty}/E(C)$.

— Finally, $c_i$ rewrites as the sum of two components :
  — Likewise, $w'^{(0)}_{[M(T)]}$ indicates the specificity of the state hit when default happens : as the player went bankrupt, the beneficial effect of $M(T)$ is wasted, hence the negative sign.
  — The term $C(T)/E(C)$ is once again a consequence of the "speed" interpretation. As the expected severity of default is $\mathbf{E}\left(C(T)\right)$, then this severity allows for an additional time before default is declared.

For example, the vector $E_{\to\infty}$ may be used to predict $C$'s future expectancies.

**Proposition 3.4.5** *Differentiated martingale*

*Let $C$ be a positive recurrent, bounded C-process, whose mean expectancy is $E(C)$ ; let $t \in \boldsymbol{N}^*$.*

1. *The process $X'_C$ defined by*

$$X'_C = \begin{pmatrix} \boldsymbol{N} & \to & \boldsymbol{R} \\ t & \to & C(t) - E_{M(t)\to\infty} - tE(C) \end{pmatrix}$$

   *is a martingale, named $C$'s differentiated martingale.*

2. *If $M$ is aperiodic, there is $\lambda < 1$ such that the expectancy of $C(t)$ (without any condition on $M(t)$) is*

$$\boldsymbol{E}(C(t)) = C(0) + tE(C) + E_{M(0)\to\infty} + o(\lambda^t)$$

*Additionally, when $M$ is periodic, a similar property holds when $t$ is restricted to $p\boldsymbol{N}$ where $p$ is $M$'s fundamental period.*

This statement may be obtained directly through differentiation of the martingale process $X^{(a)}$ with respect to $a$ at point $a = 0$ when $E < 0$ thanks to the identification of $w'^{(0)}$. We notice that the case $E > 0$ is treated likewise, making use of the process $-C$ instead of $C$, and observing that all expectancy-like terms are linear by construction, while the case $E = 0$ is obtained by continuity of said terms. These propositions may in turn lead to solving several investment decisions ; most often (e.g. if one wants to delay default, minimize expected losses, etc.), one aims only at maximizing $E$. However, as the proposition 3.4.1 indicates that this case is somehow a "limit" case of $\alpha > 0$, we will commonly discard it in future results.

### 3.4.3 Solution

We are now able to compute the player's default expectancy starting from any state $M(0)$ in the node $G(0)$ ; as it turns out, this only needs exponential parameters, as we shall see below.

**Early and late default**

The main idea when expressing the default expectancy starting from $M(0)$ is to distinguish between two types of default :

— Early default : if $C$ goes bust before $M$ hits an investment opportunity.

— Late default : if $M$ hits some investment opportunity, and the player busts after the buying decision.

The early default may be modelled through the current temporary C-process. Given a set $F \subset A$ of investment opportunities $k_{1\ldots n}$ (finishing states), the temporary C-process on $A \backslash F$ erases late defaults, as $C$ gets stuck at $+\infty$ after $M$ hits an investment opportunity, so $C'$ defaults iff $C$ defaults early : the "early default" part is taken care of thanks to $C'$. Conversely, the late default will be modelled only once the investment opportunity has already been hit, thanks to the Markovian property of the game.

**Induction scheme**

To get the default expectancy from the starting point, we use a backward induction scheme, computing default expectancies starting from $G$'s leaves and going back to $G(0)$. The main idea is to state that at every node $G(t)$, the default expectancy is roughly expressed by means of an exponential parameter $\alpha$ and a constant $v$, like in the asymptotical expression of the fundamental approximation, as

$$\mathbf{E}\left(e^{-aT}\right) = ve^{-\alpha C(t)}\left(1 + o(1)\right)$$

This is done by induction on future nodes for $G(t)$.

**Definition 3.4.3** *Characteristic items*

*Let us take a C-game whose temporary C-process at present node $G(t)$ is $C$ ; its underlying Markovian process runs over a state space $A$, with $A_0$ being its starting state and $F$ being the set of its finishing states. If there are both*

*— A constant $v \in \boldsymbol{R}^+$ ;*

*— And an exponential parameter $\alpha \in \boldsymbol{R}^+ \cup \{\infty\}$,*

*such that,*

$$\boldsymbol{E}\left(e^{-aT} | M^G(0) = A_0 \wedge C(0) = x\right) = \left(v + o(1)\right) e^{-\alpha x}$$

*then $v$ is called a characteristic constant and $\alpha$ a characteristic exponential parameter over $A \backslash F$.*

In particular, if $G$ has hit a leaf and $C$ is in a closed communicating class (which almost surely happens eventually), then we know that there $v$ and $\alpha$ exist, as $\alpha$ is $C$'s martingale parameter (on this class), and $v$ is the product of $C$'s dominant eigenvector taken at coordinate $M(0)$ by the multiplicative factor given by the default severity term. The sought induction property amounts to exhibit characteristic items : first for every communicating classes in a given node, then for every communicating classes leading to future nodes.

To clarify matters, let us start the game from the node $G(0)$, giving an underlying Markovan process $M$ starting from a state $A_i$, with a wealth $C(0) \in \mathbf{R}^+$. Defining $T$ to be $C$'s default time and $\tau$ to be $M$'s hitting time of $F$, the following events may happen to $C$ :

1. $C$ goes bankrupt before $M$ hits $F$, i.e. $T \leq \tau$. Thanks to the definition $C = +\infty$ once $F$ has been hit, we know that

$$\mathbf{E}\left(e^{-aT}\mathbf{1}_{T\leq\tau}\right) = \mathbf{E}\left(e^{-aT'}\right)$$

where $T'$ is the default time of the temporary C-process itself. As a C-process, it has $\zeta$ a martingale parameter, named the early exponential parameter for the node $G(t)$, and the fundamental approximation yields a vector $w$ such that

$$\mathbf{E}\left(e^{-aT}\mathbf{1}_{T\leq\tau}\right) \equiv w_{[i]}e^{-\zeta(a)C(0)}$$

2. $C$ first hits $F$ (either going bankrupt later, or not at all). Depending on the choices presented to the player, he will eventually go with a lower node (labelled $i$), already analyzed by induction hypothesis, with a constant $v_i$ and an exponential parameter $\theta_i$. Once in this node's starting state (after paying the investment costs $I_i$), his default expectancy is approximated as

$$\mathbf{E}\left(e^{-aT}\mathbf{1}_{T>\tau}\right) \equiv k_i\mathbf{E}\left(e^{-\theta_i C(\tau)}e^{-a\tau}\mathbf{1}_{T>\tau}|M^G(\tau) = A_i\right)$$

where the constant $k_i$ accounts for both $v_i$ at the next starting state and $I_i$. So, for $C(0)$ high enough (as explained above), the player chooses the highest $\theta_i$ (called $\theta$, associated with a constant $k_i$ called $k$). It follows that we are interested in the value

$$\mathbf{E}\left(e^{-\theta C(\tau)}e^{-a\tau}\mathbf{1}_{T>\tau}|M(\tau) = F_i\right)$$

where $F_i$ is the finishing state leading to $A_i$.

Comparison between $\zeta$ and $\theta$ yields $C$'s behaviour.

**Proposition 3.4.6** *Characteristic items between nodes*

*Let us take a C-game whose temporary C-process at present node $G^{[\dots]}$ is named $C$, and $F_{1\dots n}$ its accessible investment opportunities.*

— *For every $i \leq n$, the investment opportunity $F_i$ leads to possible nodes $G^{[\dots,(i,u)]}$ (for $u$ indexing the possible outcomes for the opportuniy $F_i$). We assume that these nodes allow for characteristic items $\alpha_{i,n}$ and $v_{i,n}$.*

— *As a temporary C-process, C has $\zeta$ a martingale parameter, named the early exponential parameter for the node $G^{[\dots]}$.*

*We call*

$$\theta = \min_i \left( \max_n (\alpha_{i,n}) \right)$$

*the late exponential parameter for the node $G(t)$. Then*

1. *The game has characteristic items in the node $G^{[\dots]}$ ;*
2. *The exponential parameter is* $\min(\zeta, \theta)$.

The exponential parameter thus acts as the counterpart of a martingale parameter for C-games prior to final nodes.

**Solution and interpretation**

As the nodes are well-ordered, we are now able to find the default expectancy starting from the initial node of the graph by means of recursion, as given through a final exponential parameter $\theta$.

— The maximum $\alpha_i$ among investment possibilities $\alpha_{i,n}$ denotes that short-term expense has little effect when $C(t)$ goes to infinity, and one should only look for the best martingale parameter (the incentive being positive for better martingale parameters after investment).

— The minimum over exponential parameters $\alpha_i$ indicates that as martingale parameters behave exponentially, the "worst" future becomes the dominant term among late defaults.

— The minimum between $\zeta$ and $\theta$ refers to both possibilities of early and late defaults.

The expression of the final $\theta$ looks pessimistic (worst outcome among the random accessible nodes). In particular, let us look at the node $G^{[g]}$ where it appears along with a characteristic constant $v^{[g]}$ :

— As lower nodes have higher exponential parameters, it must mean that this node itself is harmful to $C$. When transposing the study to an investor's wealth, this node denotes local instability (low $\zeta$), which means that the player hopes that "the situation changes" (i.e. an investment opportunity arises) ; indeed, $\zeta$ stands for the early default expectancy, which means that danger is immediate to the player as long as $G$ stands on $G^{[g]}$.

— Higher nodes recursively keep $\zeta$ as the exponential parameter. However, their characteristic constants are expressed as convex combinations of future nodes, so they should decay with the height to $G^{[g]}$, thanks to the probability of $G$ exiting the path to $G^{[g]}$ at each node change ; therefore, the final characteristic constant should be "small" (roughly, the probability of $G$

eventually going to $G^{[g]}$, weighted by the exponential losses through this path).

Hence, this asymptotical result conveys little information on the default probability : indeed, it is assumed that the investor always chooses the maximum exponential parameter when possible. This poor conclusion is the result of the asymptotical behaviour of default probabilities, as exponential functions are ordered by the relation of asymptotic comparison $o(\cdot)$ : the worst-case scenario dominates all other outcomes for huge initial assets $C(0)$. This may not happen when cash reserves are limited (when the incentive is insufficient) and an investment opportunity is too expensive for the present $C(t)$ ; the best choice is to reject the investment, sacrificing the exponential parameter in exchange for a decrease in the multiplicative constant. Therefore, the asymptotical expression may be really inaccurate for low values of $C$.

Unfortunately, we did not find any simpler way to determine the best investment strategy than computing numerical values for each C-game. Therefore, we shall look at several examples to get the main ideas.

## 3.5 Examples of single-player behaviour

We are interested in typical investment decisions for a single player ($J = 1$), whose assets are governed by the theory of C-processes in the case of a stationary market. Most often, we will seek an "investment ratio" as the optimal allocation of resources between investment (improving $C$'s transition payoffs) and liquidity (as to avoid short-term default). The optimal strategy of investment will depend on the player's own Laplace parameter $a$, leading to different investment policies : more specifically, we consider the total investment so far, defining a non-decreasing process $B : \left(\mathbf{R}^+ \to \mathbf{R}^+\right)$, and investment opportunities, increasing the game's characteristic exponential parameter, taken as a non-decreasing function $\beta$ of $B$'s value at present time. The player controls investment $B$ and aims at using it to minimize his default expectancy : our goal is to look at the "shape" of the investment decision, i.e. when the player chooses to increase $B$ with respect to $C$. We are also interested in how the Laplace parameter $a$ modifies the investment decision, eventually comparing the behaviour of the buying threshold obtained here with the thresholds obtained with the more classical Lévy-driven processes. The reader may compare this analysis with [8], also investigating exponential utility and default probabilities to find the best investment policies.

### 3.5.1 High binary tree model

During this section, we revise the issue of optimal buying time for an investment opportunity in a Markovian model : the investor is deemed to be subject to some stationary C-process between each buying time, modifying the C-process depending on the investment. We are going to write this model as a C-game, with a graph $G$ whose nodes contain said C-processes ; to account for the hypothesis of a stationary market, we shall deem that

— The tree defining $G$ has an arbitrarily high height $h \in \mathbf{N}$ and is constituted of binary nodes, each one containing at most one investment opportunity, each one leading to either one of the outcomes : the player "declines" (outcome number 1) or "accepts" (outcome number 2) the investment opportunity. For the sake of simplicity, the tree is assumed to be complete (all leaves have a height of $h$).

— When an investment opportunity is declined (outcome number 1), the C-process in the node thus hit is identical to the one just exited, i.e. nothing changes if investment is declined. In particular, all three of $M^{[\dots]}$'s starting state, $M^{[\dots]}$'s finishing state, and $M^{[\dots,1]}$'s starting state are identified.

— All investment opportunities are "identical", i.e. they have the same effect on $C$'s increments ; moreover, they are interchangeable, which means that only the total $B$ is relevant to $\beta$. In other words, all non-leaf nodes $G^{[\dots]}$ such that the sequence of nodes $[\dots]$ yield the same cumulative $B$ in investment costs define the same temporary C-process $C^{[\dots]}$, while corresponding leaves have C-processes defined by $C^{[\dots]}$ looped back on its starting state after hitting its finishing state. As a consequence, we shall commonly call it "the C-process before buying".

We shall use these hypotheses as to simplify further computations and the expressions of optimal strategies, leading to acceptable approximations during the computations, to be detailed in this paragraph. Namely, we are going to look for an approximately optimal strategy, whose losses compared with optimality are "small" thanks to the fundamental approximation ; however, we will not evaluate these losses explicitly.

#### Threshold effect

As this will often happen, the investment decision will be beneficial when $C$ is greater than a "threshold" defined by the strategy, so we shall call this threshold $S(B, n)$ where $B$ is the present total investment and $n$ the number of remaining branchings in $G$ before hitting leaves (with the idea to work with high values of $n$). The model may now be defined by the nature of its transition payoffs, as we will focus on how the lightness of the distribution tails affects $S(B, n)$, but also the nature of the opportunity itself, and its effects on the terms appearing in the C-game's default expectancies on each state. We emphasize on the fact that we work *under the fundamental approximation*, to avoid erratic behaviour of the buying decision, as it roughly allows the equations of incentive and handicap to govern the buying choice ; once again, look at the paragraph 3.8.1 to get an example of non-threshold buying decision when the fundamental approximation is removed. Hence, our first hypothesis is to admit that the fundamental approximation yields a threshold-like strategy to be optimal.

Now, let us assume that in the node $G^{[\ldots]}$, $M^{[\ldots]}$ has just hit an investment opportunity, leaving the player with two options :

— Declining the investment opportunity, leading to the node $G^{[\ldots,1]}$ associated with a default expectancy characterized by its parameters as

$$\mathbf{E}\left(e^{-aT}|M^G(0) = M^{[\ldots,1]}(0) \wedge C(0) = x\right) = F^{[\ldots,1]}(x)$$

defined by

$$F^{[\ldots,1]}(x) = Z^{[\ldots,1]}(a)e^{-\alpha^{[\ldots,1]}(a)x}$$

— Accepting the investment opportunity after paying a price $I$, leading to the node $G^{[\ldots,2]}$ associated with a default expectancy characterized by its parameters as

$$\mathbf{E}\left(e^{-aT}|M^G(0) = M^{[\ldots,2]}(0) \wedge C(0) = x - I\right) = F^{[\ldots,2]}(x - I)$$

defined by

$$F^{[\ldots,2]}(x - I) = \left(Z^{[\ldots,2]}(a)e^{\alpha^{[\ldots,2]}(a)I}\right) e^{-\alpha^{[\ldots,2]}(a)x}$$

The player thus chooses the lower default expectancy

$$\min\left(F^{[\ldots,1]}(x), F^{[\ldots,2]}(x - I)\right)$$

leading in turn to the default expectancy starting from $M^{[\ldots]}(0)$ : noting $M^{[\ldots]}(0)$'s hitting time of the investment opportunity by $\tau$, and $D = C(\tau) - C(0)$ is the random variable of $C$'s cumulated increments up to $\tau$, one gets

$$F^{[\ldots]}(x) = \mathbf{E}\left(e^{-aT}\mathbf{1}_{T<\tau}\right) + \mathbf{E}\left(e^{-a\tau}\min\left(F^{[\ldots,1]}(x + D), F^{[\ldots,2]}(x + D - I)\right)\mathbf{1}_{T\geq\tau}\right)$$

If infinitely many successive investment opportunities are available, the situation

$$C(0) = x \wedge M^G(0) = M^{[\ldots]}(0)$$

is "equivalent" to the same situation after declining one investment opportunity

$$C(0) = x \wedge M^G(0) = M^{[\ldots,1]}(0)$$

Intuitively, we want to assume that when the tree's height increases, horizon effects related with the "final node" tend to vanish (for an identical amount of investment) ; that is to say, $F^{[\ldots]}(x) = F^{[\ldots,1]}(x)$. This leads us to our second hypothesis in this single-player analysis : the optimal strategy $S(B,n)$ is roughly constant of (high values of) $n$, thus we shall seek a strategy $S(B)$ with a single threshold called $b(B)$.

### Reverse fundamental approximation

Using this model, we seek the best threshold $b$ in the case of infinitely many investment opportunities, so we introduce the default expectancy

$$F^{(a,b)} = \begin{pmatrix} \mathbf{R}^+ & \to & [0,1] \\ x & \to & \mathbf{E}\left(e^{-aT_0}|C(0)=x\right) \end{pmatrix}$$

under the strategy $S(B)$ yielding a threshold $b$ for $G$'s current position, assumed to be constant of $n$ as in the previous paragraph. Our idea is to start from some $C(0) < b$ and decompose over the possible outcomes of the game.

— $C$ never buys and never defaults, however this event has a zero contribution to the default expectancy so shall be discarded.

— $C$ goes bust before hitting the investment opportunity $F$ with $b$ of cash reserves (event called $\neg A$) : this event is analyzed with $C$ a purely random C-process, as there are no investment opportunities in the meantime.

— $C$ first buys before going bankrupt (event called $A$, since the first one had zero probability) : this part is tantamount to considering the hitting time $\tau$ of a C-process from below (using its negative martingale parameter), and then the subsequent default time $T_1$ is computed.

The event $A$ may thus be written as

$$\exists t \in \mathbf{N}; \left(M^{[}\ldots](t) = F \wedge C(t) \geq b \wedge \forall s < t, C(s) \geq 0\right)$$

In order to compute the contribution of the latter case to the default expectancy, we shall need to "cut" the time counter at buying time $\tau$, expressing the default time in $F^{(a,b)}$ as the sum of the buying time $\tau$ and the additional default time $T_1$, starting once the purchase has been completed, giving the decomposition

$$\mathbf{E}\left(e^{-aT_0}\mathbf{1}_A\right) = \mathbf{E}\left(e^{-a\tau}\mathbf{1}_A e^{-aT_1}\right)$$

In the latter case, this decomposition will involve both $\tau$ and $T_1$, so we must assume that they are roughly independent to compute their Laplace transforms. Actually, as the C-process is Markovian, the random couple $(\tau, \mathbf{1}_A)$ is independent of $T_1$ given $C(\tau)$ (and $M^{[}\ldots](\tau)$, but by definition of $\tau$ one must have $M^{[}\ldots](\tau) = F$) ; so our idea is to dismiss the dependency between $\tau$ and $C(\tau)$. This is actually the same idea as the fundamental approximation, where this time $C$ starts *below* the threshold $b$, and we seek a hitting time for $b - C$.

**Definition 3.5.1** *Reverse fundamental approximation*

*Let $C$ be a positive recurrent, not globally decreasing C-process, associated with*

*— A negative martingale parameter $\omega(a)$ ;*

*— A multiplicative constant $Y(a)$.*

*The reverse fundamental approximation approximates the hitting time $T$ of the region $[b, \infty)$ given by a threshold $b \in \boldsymbol{R}_+^*$, when starting from $C(0) = 0$, as*

$$\boldsymbol{E}\left(e^{-aT}\right) \equiv Y(a)e^{-\omega(a)b}$$

*when $b$ goes to $\infty$.*

In particular, not only it comes from this approximation that $\tau$ holds the same property, but it allows to evaluate the expectancy of "overshooting" the threshold (i.e. the value of $C(\tau)-b$) at buying time, as the counterpart of the default severity parameter.

**Proposition 3.5.1** *Laplace transforms at time $\tau$*

*Let $\omega(a)$ be $C$'s negative martingale parameter at point $a \in \boldsymbol{R}^+$. Let $\tau$ be a stopping time whose form is*

$$\tau = \inf\left(\left\{t \in \boldsymbol{N}; M^{[\ldots]}(t) = F \wedge C(t) \geq b\right\}\right)$$

*for $F$ the investment opportunity and $b \in \boldsymbol{R}^+$.*

1. *There is $Y_0(a) \in \boldsymbol{R}^+$ such that*

$$\boldsymbol{E}\left(e^{-a\tau}\right) = Y_0(a)e^{-\omega(a)(b-C(0))}\left(1 + o(1)\right)$$

*when $b - C(0)$ goes to $\infty$. This $Y_0(a)$ is called $C$'s multiplicative negative parameter (for the finishing state $F$).*

2. *For every $\alpha_i(a) \in \boldsymbol{R}^+$, there is $Y_i(a) \in \boldsymbol{R}^+$ such that*

$$\boldsymbol{E}\left(e^{-\alpha_i(a)(C(\tau)-b)}\right) = Y_i(a) + o(1)$$

*when $b - C(0)$ goes to $\infty$, this being for $i \in \{1, 2\}$. This $Y_i$ is called $C$'s multiplicative corrective term for $\alpha_i(a)$.*

It will come handy to extend the reverse fundamental approximation to these asymptotic expressions, dropping the $o(1)$ terms on further computations.

**Best threshold**

Using the hypotheses of the model justified above :

— The best strategy is deemed to be a threshold-like buying decision ;

— The direct and reverse fundamental approximations are enforced,

we may express the best investment strategy for the player as to minimize his default expectancy.

**Proposition 3.5.2** *Approximative default expectancy*

*Let us deem that the C-game at present time $t$ involves*

- *A C-process $C^{[G(t)]}$ before buying, characterized by*
  - *A martingale parameter $\alpha_1(a)$ associated with a multiplicative parameter $Z_1(a)$, and*
  - *A negative martingale parameter $\omega(a)$ ;*
- *A default expectancy after buying (on the node $C^{[G(t),2]}$), characterized by an exponential parameter $\alpha_2(a)$ associated with a multiplicative parameter $Z_2(a)$ ;*
- *A threshold $b \in \boldsymbol{R}^+$ such that the player's strategy on node $G(t)$ is to buy whenever an investment opportunity is hit at some time $t$ when $C(t) \geq b$, yielding a stopping time $\tau$.*

*We introduce $Y_0(a)$, $Y_1(a)$ and $Y_2(a)$, as computed thanks to the proposition 3.5.1, so that the following properties hold.*

1. *The player's default expectancy starting from $C(0) < b$ is expressed by*

$$\begin{aligned}\boldsymbol{E}\left(e^{-aT_0}|C(0)=x\right) &\equiv Z_1(a)e^{-\alpha_1(a)x} \\ &+ Y_0(a)e^{-\omega(a)(b-x)}Y_2(a)Z_2(a)e^{-\alpha_2(a)(b-I)} \\ &- Y_0(a)e^{-\omega(a)(b-x)}Y_1(a)Z_1(a)e^{-\alpha_1(a)b}\end{aligned}$$

   *where this approximation holds when both $b$ and $b-x$ go to $\infty$ ;*

2. *Recalling the incentive $\gamma(a)$ and the handicap $H(a)$ from the definition 3.4.2, the approximative best threshold $b$ is expressed by*

$$b = \frac{I}{\gamma^*(a)} + H^*(a)$$

   *where the new incentive is still $\gamma^*(a) = \gamma(a)$, and the new handicap is $H^*(a) = H(a) + H^+(a) - H^-(a)$ with the additional handicaps are given by*

$$H^+(a) = \frac{\ln\left(\frac{(\alpha_2(a)+\omega(a))}{(\alpha_1(a)+\omega(a))}\right)}{\alpha_2(a)-\alpha_1(a)} = \frac{\ln\left(1+\frac{\alpha_2(a)-\alpha_1(a)}{\alpha_1(a)+\omega(a)}\right)}{\alpha_2(a)-\alpha_1(a)}$$

   *and*

$$H^-(a) = \frac{\ln\left(\frac{Y_1(a)}{Y_2(a)}\right)}{\alpha_2(a)-\alpha_1(a)}$$

*In particular, $H^+(a)$ and $H^-(a)$ are always positive.*

For the proof, we refer to the paragraph 3.8.1. We recognize the notions of incentive and handicap to buy an investment opportunity. In particular, we find that

- The incentive to buy is similar to the "take it or leave it" case, indicating that the price of the investment opportunity itself governs the gist of the buying decision independently of the model. On a side note, let us look at the investment dilemma as a simplified optimal stopping time problem, involving two drifted Brownian motions $C_1$ before buying and $C_2$ after buying. Using the fundamental approximation on $C_2$'s default probability means that we roughly look for $\tau$ that minimizes

  $$\mathbf{P}\left(T_0 < \infty\right) \equiv \mathbf{E}\left(e^{-\alpha_2(0)C_1(\tau)}\right)$$

  (the case of ruin before buying is dismissed for the sake of explanations), and by Jensen's inequality we get

  $$\mathbf{E}\left(e^{-\alpha_2(0)C_1(\tau)}\right) \geq \left(\mathbf{E}\left(e^{-\alpha_1(0)C_1(\tau)}\right)\right)^{\left(\frac{\alpha_2(0)}{\alpha_1(0)}\right)}$$

  However as $\alpha_1$ is $C_1$'s martingale parameter, the inner expectancy is given by the martingale property, so we know that

  $$\mathbf{E}\left(e^{-\alpha_1(0)C_1(\tau)}\right) = e^{-\alpha_1(0)C_1(0)}$$

  and we finally get that

  $$\mathbf{E}\left(e^{-\alpha_2(0)C_1(\tau)}\right) \geq e^{-\alpha_2(0)C_1(0)}$$

  which means that the default expectancy is roughly increased by waiting. In other words, immediate purchase is preferable, which sides with the idea that an investment opportunity should be declined only because of insufficient cash reserves, rather than the player's tactical decisions. Actually, the only term to measure the discrepancy between investment decisions in the cases "repeated node" and "take it or leave it" is the small error term $H^+(a)$, as explained later : neglecting it means that any investment opportunity may thus be taken as being "take it or leave it".
- The old handicap $H(a)$ keeps the same interpretation as before : if the multiplicative parameters discourage investment, one needs more cash to purchase.
- The additional $H^+(a)$ indicates the benefits of waiting when presented with a "repeated node" investment opportunity rather than a "take it or leave it" one. To understand this, we shall again focus on the Brownian motions $C_1$ and $C_2$ for $a = 0$.

— In the "take it or leave it" model, the threshold is exactly obtained by $b = I/\gamma(0)$, as continuous trajectories indicate that there is no handicap.
— In the "repeated node" model, let us assume that an investment opportunity is available at the starting time with $C(0) = b$. Now declining the investment opportunity yields the additional advantage of observing $C$'s short-term future before commiting to a decision when the next opportunity strikes at time $\tau$ : if the conjoncture turns good to the player, he can revert his decision and buy at time $\tau$, whereas if it turns bad, he shall keep postponing the purchase.

As a consequence, the "repeated node" model allows the player to postpone investment when the situation is too unclear (when $C(t)$ is too close to $b$), which translates to a *positive* additional handicap : this ability to postpone investment in order to get more information on the immediate future increases the expected benefits of waiting, thus discourages investment. This "value of waiting" concept is explored deeper in [33].

— The additional $-H^-(a)$ comes from the effects of overshooting the threshold $b$ when deciding to buy : even if it is tactically optimal to buy a "take it or leave it" opportunity as given by exactly $b$, this choice is skewed by the effects of upward jumps for $C$. To clarify this, let us assume that an investment opportunity strikes at time 0 when $C(0) = b - \epsilon$ with a positive $\epsilon << 1$. If it is declined, chances are that a jump will carry $C$ to a higher value than $b$, e.g. $C(\tau) \geq b + 1$ where the opportunity will be accepted ; as a consequence, one effectively buys at the suboptimal threshold $b + 1$ with non-negligible probability, with a loss in default expectancy as indicated by construction of $H^-(a)$. It follows that buying at $C(0) = b - \epsilon$ rather than $b$ yields a small enough loss to compensate this, provided that $\epsilon$ itself is small enough. The optimal value of $\epsilon$ constitutes $H^-(a)$.

**Relative scale of $H^+(a)$ and $H^-(a)$**

Finally, we want to assume that $H^+(a)$ and $H^-(a)$ are "small" relatively to the main term $I/\gamma(a) + H(a)$, so that in the comments we shall neglect the effect of $H^+(a) - H^-(a)$ on the optimal threshold. To justify these assumptions, let us start with

$$H^+(a) = \frac{\ln\left(\frac{\alpha_2(a)+\omega(a)}{\alpha_1(a)+\omega(a)}\right)}{\alpha_2(a) - \alpha_1(a)}$$

As $\omega(a) \geq 0$, one gets

$$H^+(a) \leq \frac{\ln\left(\frac{\alpha_2(a)}{\alpha_1(a)}\right)}{\alpha_2(a) - \alpha_1(a)} \leq \frac{1}{\alpha_1(a)}$$

Hence, provided that $\alpha_1(a)$ is not too small, $H^+(a)$ is well-controlled. On a side note, we notice that $H^+(a)$ mainly increases when $\alpha_1(a) + \omega(a)$ decreases to 0, which only happens for $a = 0$ when the C-process $C_1$ admits a minimum at point 0, indicating that $C$'s mean expectancy is zero. To understand why this foils the control, let us try and compute the optimal threshold for $C_1$ the standard Brownian. The probability of hitting $b$ before 0 is now $C(0)/b$, a linear function instead of an exponential-shaped function, because of first-order phenomena happening with double solutions for the martingale parameter (see the other study). As a consequence, one eventually wants to solve

$$e^{\alpha_2(a)(b-I)} = 1 + \alpha_2(a)b$$

so in particular

$$b \in I + \left[\frac{\ln\left(1+\alpha_2(a)I\right)}{\alpha_2(a)}, \sqrt{\frac{2I}{\alpha_2}}\right]$$

cannot be expressed as an affine function of $I$. It follows that the handicap must eventually "blow up" to keep up with $b$ as $\gamma(a) = 1$.

Likewise, we aim at controlling $H^-(a)$ : rather than looking for an exact expression, we shall give ideas as why $H^-(a)$ may be considered small. For the purposes of this explanation, we shall assume that the distribution of $C(\tau) - b$ may be represented as a single random variable $C_f$ and considered independent of $\tau$, as deemed within the fundamental approximation. Starting with

$$H^-(a) = \frac{\ln\left(\frac{\mathbf{E}\left(e^{-\alpha_1(a)C_f}\right)}{\mathbf{E}\left(e^{-\alpha_2(a)C_f}\right)}\right)}{\alpha_2(a) - \alpha_1(a)}$$

one recognizes a difference quotient form for the log-Laplace transform of $C_f$ applied at points $\alpha_1(a)$ and $\alpha_2(a)$, so one has by differentiation

$$H^-(a) \leq \max_{\alpha\in[\alpha_1(a),\alpha_2(a)]}\left(\frac{\mathbf{E}\left(C_f e^{-\alpha C_f}\right)}{\mathbf{E}\left(e^{-\alpha C_f}\right)}\right)$$

As a log-Laplace is convex, the maximum must be hit on $\alpha = \alpha_1(a)$. Now, we know that $C_f$ is the random value of $C_1(\tau) - b$, with $\tau$ indicating when $M$ first hits $F$ while $C_1(\tau) \geq b$ ; as $C_1$'s transition payoffs are independent of $\tau$, one should assume that $\tau$ has a roughly exponential tail, as well as $C_1(\tau)$. Replacing $C_f$ by a random variable $X$ with exponential distribution (of parameter $\lambda \in \mathbf{R}_+^*$), one gets after computations that $H^-(a)$ is approximatively dominated by

$$\frac{1}{\alpha_1(a) + \lambda} \leq \frac{1}{\alpha_1(a)}$$

for the same control as earlier.

#### Insights on the best strategy

The proposition 3.5.2 indicates when to buy as far as a "next" default expectancy is concerned, so we may finally express the approximative best strategy along with its default expectancy ; however, as the exact form of the optimal strategy is not computable explicitly, we shall actually only give a canonical strategy "close to" being optimal. For this purpose, we shall qualify by $n$-node a node of $G$ where the player has bought $n \in \mathbf{N}$ opportunities so far.

Let us start our analysis from the bottom of $G$. Comparing the default expectancy of an $n$-node with an $(n+1)$-node (of depths called $d \in \mathbf{N}^*$), one gets a threshold given thanks to the definition 3.4.2 though the characteristic items $Z_n(a)$ and $\alpha_n(a)$. The recursion is now started for the previous node of depth $d-1$ leading to them, approximating the long-term default expectancy starting from its starting state as

$$\mathbf{E}\left(\min\left(Z_n(a)e^{-\alpha_n(a)(C(0)+D_n)}, Z_{n+1}(a)e^{-\alpha_{n+1}(a)(C(0)+D_n-I)}\right)\right)$$

where $D_n$ is the random variable of $C_n$'s total increments up to the hitting time of state $F$, the disjunction indicating the place of the threshold $b_n^*$. However, as said during the previous paragraph, $b_n^*$ is "close" to the "take it or leave it" threshold $b_n = I/\gamma_n(a) + H_n(a)$, so we shall work with the strategy using $b_n$ instead of $b_n^*$. Now, the above expectancy may be computed : we know that

$$\mathbf{E}\left(e^{-\alpha_n(a)D_n}\right) = e^a$$

because $\alpha_n$ is $C_n$'s martingale parameter, and we shall call

$$X_{n+1}(a) = \mathbf{E}\left(e^{-\alpha_{n+1}(a)D_n}\right)$$

which is greater than $e^a$ since $\alpha_{n+1}(a) > \alpha_n(a) \geq 0$, even possibly $\infty$. Using the same disjunction between short-term and long-term default as earlier in the proposition 3.4.6, short-term default is dominant when $X_{n+1} = \infty$ and long-term default is dominant otherwise. It follows that the shape of the default expectancy function when entering this node is

- Exponential, governed by the function $f$ defined by

  $$f(x) = Z_n(a)e^{-\alpha_n(a)x}$$

  for a starting point $x < b_n$ ;
- Another exponential, either

  $$f(x) = Z_{n+1}(a)X_{n+1}(a)e^{-\alpha_{n+1}(a)(x-I)}$$

  if $\alpha_{n+1}(a) < \zeta_n(a)$ the quick exponential default parameter, or

  $$f(x) = Z'_n(a)e^{-\zeta_n(a)x}$$

  otherwise, this being when $x > b_n$.

Notice that the default expectancy for $x < b_n$ is not affected by quick default, as $\zeta_n > \alpha_n$ by construction (which is recovered in terms of $D_n$'s finite Laplace transform at point $\alpha_n(a)$). We may now proceed with nodes at height $d-2$, and so on until the starting node is reached ; ideally, doing this for $d$ high enough should lead to the default expectancy function. Successive iterations of this induction scheme lead to a function $f$ describing the default expectancy, whose form is piecewise exponential. Actually, when the sequence $(b_k)_k$ is increasing, we find out that the approximatively best strategy buys accordingly to the thresholds $b_n$.

**Definition 3.5.2** *Ascending canonical strategy*

*For every $n \in \boldsymbol{N}$, let $C_n$ be the C-process governing nodes of G where the player has bought n investment oppotunities yet. Each C-process yields*

- *A martingale parameter $\alpha_n(a)$, associated with a multiplicative parameter $Z_n(a)$ ;*
- *A negative martingale parameter $\omega_n(a)$, associated with its multiplicative negative parameter $Y_n(a)$ ;*
- *Multiplicative corrective terms, respectively called $Y_n^{=}(a)$ and $Y_n^{+}(a)$ when taken for $\alpha_n(a)$ and $\alpha_{n+1}(a)$.*

*We define successive thresholds $b_n$ as*

$$\forall n \in \boldsymbol{N}, b_n = \frac{I}{\gamma_n(a)} + H_n(a)$$

*where for every $n \in \boldsymbol{N}^*$, the incentive number n is given by*

$$\gamma_n(a) = 1 - \frac{\alpha_{n-1}(a)}{\alpha_n(a)}$$

*and the handicap number n is given by*

$$H_n(a) = \frac{\ln\left(\frac{Z_n(a)}{Z_{n-1}(a)}\right) + \ln\left(1 + \frac{\alpha_n(a)-\alpha_{n-1}(a)}{\alpha_{n-1}(a)+\omega_{n-1}(a)}\right) - \ln\left(\frac{Y_n^{=}(a)}{Y_n^{+}(a)}\right)}{\alpha_n(a) - \alpha_{n-1}(a)}$$

*The canonical strategy accepts the investment opportunity number n (i.e. when the player bought $n-1$ so far) iff his wealth at present time is at least $b_n$.*

Reasons why the discrepancy to optimality is small enough to be neglected afterwards are presented in the proof, in paragraph 3.8.2.

### 3.5.2 Purchasing of a drift

We are going to present some examples of this study when investment allows for a long-term permanent income, reason why we shall refer to a model of "drift purchasing". Investment opportunities are thus each deemed to increase old transition payoffs by a built-in constant $\delta \in \mathbf{R}_+^*$, and cost $I \in \mathbf{R}^+$.

### Brownian model

We shall look at a canonical case, where $C$ is a Brownian motion with some natural drift $E$ and variance $\sigma^2$ per time unit. We shall work like with continuous-time Markov modulated Brownian models, with non-jump trajectories, indicating that the values $Z_n$, $Y_n^+$ and $Y_n^=$ describing "severity of default" are actually 1. In particular, we shall assume that the hitting times of investment opportunities (that had a geometrical distribution in discrete time) now have an exponential distribution.

**Model 3.2** *Lévy Brownian drift purchasing*

*The C-game conststs in a binary gragh $G$, whose non-leaf nodes consist in C-processes given by*

— *2 states, one being the starting state $A_0$ and the other one the finishing state $A_1$ ;*

— *At each time step, the probability of being given an investment opportunity (i.e. $M$ going to $A_1$) is $1 - e^{-\rho}$ for some $\rho \in (0, 1)$ ;*

— *All transition payoffs are Gaussian-distributed with drift $E$ and volatility $\sigma^2$, where $\sigma^2$ is fixed throughout the model and $E$ amounts to $E_0 + k\delta$, where $E_0 > 0$ is the initial drift and $k$ is the quantity of purchased investment opportunities so far.*

*The leaves are Brownian motions whose increments are distributed likewise.*

Therefore, the total additional drift purchased by $B$ amounts to $B\delta/I = Bp$ where we hereby set $p = \delta/I$. We know that for a drift $E \in \mathbf{R}$ and a volatility $\sigma^2 \in \mathbf{R}_+^*$,

— The martingale parameter on a leaf writes as

$$\alpha(a) = \frac{E + \sqrt{E^2 + 2a\sigma^2}}{\sigma^2}$$

which translates into buying terms as

$$\beta(B) = \frac{E_0 + Bp + \sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}{\sigma^2}$$

— Likewise, the negative exponential parameter is

$$\omega(a) = \frac{-E + \sqrt{E^2 + 2a\sigma^2}}{\sigma^2}$$

which translates as

$$\psi(B) = \frac{-E_0 - Bp + \sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}{\sigma^2}$$

— The characteristic short-term exponential parameter on a non-leaf node is computed likewise to

$$\zeta(B) = \frac{E_0 + Bp + \sqrt{(E_0 + Bp)^2 + 2(a+\rho)\sigma^2}}{\sigma^2}$$

In particular, $\zeta(B)$ gets outweighted by $\beta(B+I)$ when

$$Ip + \sqrt{(E_0 + (B+I)p)^2 + 2a\sigma^2} > \sqrt{(E_0 + Bp)^2 + 2(a+\rho)\sigma^2}$$

These expressions lead to the threshold to the investment decision.

**Proposition 3.5.3** *Thresholds for Brownian drift purchasing*

*In the model 3.2, there is an upper threshold $B^+$ such that for $B \geq B^+$, the quick default outweighs long-term default risks. It is given by $E_0 + B^+p = U$ where*

$$U = \frac{\rho\sigma^2}{2Ip} - \frac{a+\rho}{\rho} Ip$$

*The threshold decision given by the canonical strategy from definition 3.5.2 when having purchased $B$ worth in investment opportunities is thus given by*

1. *If $B < B^+$, comparison of terms $\beta$, yielding*

$$S(B) \approx \frac{\sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}{p} + \frac{\sigma^2}{2\sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}$$

2. *If $B > B^+$, comparison of terms $\zeta$, yielding*

$$S(B) \approx \frac{\sqrt{(E_0 + Bp)^2 + 2(a+\rho)\sigma^2}}{p} + \frac{\sigma^2}{2\sqrt{(E_0 + Bp)^2 + 2(a+\rho)\sigma^2}}$$

*these approximations being valid when $I$ is small. Dropping the handicap term yields an expression given only by the incentive, defined as*

$$\tilde{S}(B) = \frac{\sqrt{(E_0 + Bp)^2 + 2u\sigma^2}}{p}$$

*for $u = a$ or $u = a + \rho$ depending on $B^+$.*

We may observe the following facts on this $S(B)$.

— The threshold $S(B)$, indicating optimal cash reserves, never falls below the value of illiquid assets. This is tantamount to saying that the so-called optimal "liquidity-to-investment" ratio in this model is always at least 50-50 in favour of liquidity. Notice that there is no contradiction with common banking investment policies, banks typically having much less liquidity, because of the possibilities of issuing debt, artificially increasing the amount of available liquidity. One may look e.g. to [22] for a broader explanation of leverage effects.
— As $E_0 > 0$, $S(B)$ is decreasing of $p$. Indeed, for a high $p$, indicating a high change in drifts with respect to investment costs (i.e. opporunities are cheaper), investment is more beneficial to the player.
— $S(B)$ is increasing of $\sigma$. This means that high uncertainty (volatility) deters investment because of enhanced short-term default risks after an immediate purchase.

We also know that the handicap term is bounded from below by $1/\alpha(a)$ and decreasing of $B$ ; when allowed to drop it in view of the other approximations, we may proceed the analysis with $\tilde{S}(B)$.

— $\tilde{S}(B)$ is increasing of $B$. When the drift is increased by investment, the remaining liquidity risks will be rather short-sighted as the law of large numbers protects $C$ from a long-term default ; as a consequence, one should emphasize on avoiding short-term risks, leading to being more cautious with future investment opportunities. As a matter of fact, $\tilde{S}(B)$ increasing is the reason why the canonical strategy works well, as suggested after the definition 3.5.2.
— For the same reason, $\tilde{S}(B)$ is increasing of $E_0$, as $E_0$ may be viewed as an "initial $B$".
— $\tilde{S}(B)$ is increasing of $a$. This is not a surprise either, since a higher value of $a$ indicates that the player is more interested in avoiding short-term default rather than limiting the overall default probability.
— When $B$ hits $B^+$, short-term default risks (before an investment opportunity is available) become predominant in default computations ; as a consequence, an additional $\rho$ comes with $a$, discouraging investment. More specifically, $B^+$ is increasing of $\rho$, which was expected as allowing more frequent investment opportunities decreases the impact of quick exponential parameters.

**Exponential tails**

Let us look at what happens when one allows the negative tail of the distribution to get fatter, e.g. governed by an exponential parameter $\lambda \in \mathbf{R}_+^*$. For now, we shall assume that $C_n$ is a discrete-time Lévy process, whose increments are distributed with a density given by

$$\forall x \in \mathbf{R}^-, \frac{\mathbf{P}\left(D \in [x + Bp, x + Bp + dx]\right)}{dx} = \lambda e^{\lambda x}$$

for $B \in \mathbf{R}^+$ such that $\mathbf{E}(D) > 0$. Therefore, investment opportunities still contibute to $C$'s drift through an additive constant $Bp$ given to $D$. The martingale parameter now expesses as the positive solution $\alpha$ to

$$e^{-\alpha Bp} \frac{\lambda}{\lambda - \alpha} = e^a$$

We find out that the cash reserves $S$ form an exponential-shaped function of $B$, and the liquidity-to-investment ratio goes to 100 % liquidity when the cash amounts at stake increase (see the proof for detailed computations, at paragraph 3.8.3). Required cash reserves are thus really huge when compared with the Brownian case, principally because further investment does not increase $C$'s martingale parameter more than $\lambda$, so not only the effects of additional incomes are limited, but the player should remain cautious of a sudden negative cash flow, as described by the fatter tails. We also remark that $S$ is decreasing of $\lambda$, which is no surprise as $\lambda$ measures the decay of $D$'s negative tail : higher values of $\lambda$ indicate that the future is safer, so they encourage investment.

### 3.5.3 Interpretations

These examples on Brownian-like analysis encourage us to focus on general lines of thought governing investment strategies to avoid bankruptcy of an individual. In the next examples, investment may effect on $E$ or $\sigma$, like in the theory of diversification ; the results found there are somewhat linked with Markowitz's efficient frontier ([30]).

#### Links with diversification theory

Let us assume that the player has a constant consumption rate $c > 0$ and is presented with several investment opportunities, whose effects on his wealth are modelled by drifted Brownian motions. Such a player may buy any amount of each investment opportunity and aims at adjusting his portfolio as to avoid going bust. As we work with exponential-shaped functions (the default expectancy is an exponentially decaying function thanks to the fundamental approximation), we expect Markowitz's efficient frontier to indicate the guidelines of the best investment strategy, defining a portfolio of zero cost with unit drift $E > 0$ and unit variance $\sigma^2$. Hence, for a purchased quantity $b$ of such a portfolio, the player's wealth $C$ will follow a drift $E(C) = -c + bE$ and a volatility $V(C) = b^2\sigma^2$, which means that its martingale parameter is

$$\alpha(b) = \frac{-c + bE + \sqrt{(-c+bE)^2 + 2ab^2\sigma^2}}{b^2\sigma^2}$$

Maximization of $\alpha(b)$ yields after computations

$$b = \frac{2cE}{2a\sigma^2 + E^2}$$

It so turns out that, to avoid default, the quantity $b$ to purchase is

— Increasing of $c$, because if the player has a higher consumption, he will logically need higher incomes to keep up with his expenses.

— Decreasing of $a$, for the same reason as exposed earlier : being more afraid of short-term default discourages from being subject to a high volatility. Interestingly, we find that when the default probability should be minimized (doing $a = 0$) then $b = 2c/E$, which means that one should invest for *twice* the necessary consumption amount to sustainability.

— Decreasing of $\sigma$, because investment is less interesting when riskier.

**Final thoughts**

After looking at these examples of fixed-prices investment opportunities, we remark several general lines of thought common to all cases.

— When the incentive $\gamma$ is expressed as a function of invested capital $B$, the most notable idea comes from the approximations

$$\gamma(B) \approx \frac{I\beta'(B)}{\beta(B)}$$

(the approximation working when the width of investment costs are negligible) thus when neglecting handicap terms, the cash reserves should be inversely proportional to the player's incentive to buy, as

$$S(B) \approx \frac{\beta(B)}{\beta'(B)}$$

Therefore, some considerations may be drawn from the shape of the function $\beta$.

— When the distribution tails (on the negative side) come fatter, the Laplace transform of $C$'s transition payoffs soars faster, thus $\beta$ will have a "concave" shape with respect to $B$, and $S$ shall rise faster than $B$. As said above, the player should be more cautious than in the Brownian model.

— $S$ is increasing of $a$. This means that players with a higher Laplace parameter are more restrictive on investment opportunities : indeed, as they mainly focus on avoiding quick default, they are not willing to take the risk of an immediate liquidity shortage in exchange for long-term benefits.
— The parameter $p$ measures an additional drift obtained by a unit price, which means that its unit of measurement is $(\$/s)/\$ = 1/s$, i.e. $p$ is a frequency. Actually, $1/p$ may be seen as the "return on investment" time : having a return on investment time of $1/p$ means that the opportunity's additional drift pays for its own price after a time $1/p$. Therefore, the investor should only be concerned about the risk of a default before time $1/p$, as a later default cannot be avoided by means of refusing the investment opportunity. For additional information on how the return on investment time allow to evaluate assets, as well as other market indicators relative to an investment opportunity, the reader may refer to [36].

One may extrapolate to general C-processes, where investment opportunities are tantamount to purchasing a drift at a fixed price. Indeed, for an additional drift $Bp$, the inherent eigenvector equation to $C$ yields

$$L_C(\beta)w = e^{a+\beta Bp}w$$

where $\beta$ is $C$'s present martingale parameter (with drift $Bp$) and $w$ its dominant eigenvector. Hence, noting by $\alpha$ the original martingale parameter of $C$, one gets the equation

$$\beta = \alpha(a + \beta Bp)$$

where $\alpha$ is still considered as a function of the Laplace parameter (originally $a$).

— When $\alpha$ roughly behaves asymptotically as the square root function, $\beta$ comes as a linear function of $B$, thus $S \approx \beta/\beta'$ is also linear. This is an exact case for the Brownian motion (of variance $V$), where one has

$$\forall a \in \mathbf{R}^+, \alpha(a) = \sqrt{\frac{2a}{V}}$$

so one gets for $a = 0$ that

$$\forall B, \beta(B) = \frac{2B}{pV}$$

and finally $S(B) = \beta(B)/\beta'(B) = B$.
— When $\alpha$ increases faster (e.g. for bounded C-processes, it is asymptotically linear), $\beta$ has a super-linear behaviour and $S = \beta/\beta'$ is sub-linear, indicating that cash security levels are lower (investment is encouraged).

— Contrariwise, when $\alpha$ increases slower than the square root function (when the tails of $C$'s increments are fatter), $S$ turns out to be super-linear, discouraging investment.

We notice that, for $a = 0$, the equation $\beta(B) = \alpha(\beta(B)Bp)$ yields

$$S(0) = \frac{1}{p\alpha'(0)}$$

When $C$ is a C-process with non-positive mean expectancy $E(C) \leq 0$, this translates to $S(0) = -E(C)/p$. In other words, one should buy iff not expecting $C$ to default during the return on investment time, which is not surprising.

## 3.6 Multiplayer models

In this new paragraph, we deal with the concept of player cooperation in a Markovian environment. We discuss about two-player models, where players are subject to a game with investment opportunities, and we want to investigate how the structure of a C-game affects the relationship between players. Specifically, we tackle the issue of cooperative risk management for two players. When the market is governed by Brownian motions, it is well-known ([31, 44]) that pooling assets is an efficient way to reduce default risks of both parties. We want to find a similar method of risk management for C-processes, indicating when player cooperation is possible and beneficial to both players so that they both limit bankruptcy risks.

For instance, let us assume that $M^G$ hits an investment opportunity, whose outcomes are either acceptance or rejection by $J_1$ (hereby named "he") at some price $I \in \mathbf{R}^+$. When investment has a "positive" effect on economy (be it positive externality, reduction of risks, etc.), it may be desirable to $J_1$ except for the investment costs $I$ that prevent him to invest because of high short-term liquidity risks : it may happen that by himself, $J_1$ solves his own investment dilemma by refusing to buy. Eventually, the opportunity is declined and the players squandered a possiblity to benefit from it for want of coordination. In our model, $J_1$ is allowed to call for $J_2$'s help to invest in the opportunity. $J_2$ (hereby named "she") must be willing to help $J_1$ to provide fundings : her incentive to contribute are translated by a permanent increase of her own future transition payoffs, either directly (under the natural effects of investment in the next node), or through a compensation ($J_1$ repays $J_2$ with a permanent "rent", in addition to the standard transition payoffs). Thus, we aim at finding the terms of some "contract", indicating $J_2$'s contribution to the investment and $J_1$'s payback to her, such that both

— $J_2$ is willing to invest (her rent will exceed her default risks) ;

— $J_1$ is willing to call for $J_2$'s help ($J_1$ must not be better off rejecting investment, nor bearing all default risks by himself without help).

In particular, we want to know whether or not there is such a contract, and when there is one, to find a "best" contract to both players.

We will assume that the martingale parameters involved in the computations are all non-zero, as $\alpha_p(0) = 0$ means that the player $J_p$ is bound to eventually default and constitutes a trivial case.

### 3.6.1 Cooperative management

Before investigating how players can cooperate in order to purchase an investment opportunity, we shall focus on how they can support each other to minimize their default expectancies. Let us work on a final node of the C-game : both players are subject to the random risks governed by C-processes $C_1$ and $C_2$. We assume that players agree on a risk reduction method whose concept is to help the "player in need", e.g. if the active transition $(i \to j)$ favours $J_2$ more than $J_1$, then $J_1$ should percieve a compensatory payment $s_{i\to j}$ from $J_2$ to cancel this effect (and reciprocally).

To agree on this long-run contract, both players must be satisfied, in the sense that both default expectancies must be reduced by agreeing. As under the use of the fundamental approximation, they are expressed by

$$\mathbf{E}\left(e^{-a_p T_p}\right) \equiv Z_p(a_p) w_{p,[M(0)]}^{(a_p)} e^{-\alpha_p(a_p) C_p(0)}$$

our main concerns are to

1. Enhance the martingale parameters $\alpha_p(a_p)$ ;
2. When optimal martingale parameters are found, decrease the starting term of the dominant eigenvector.

We will not be concerned by $Z_p(a_p)$ throughout this paragraph. Actually, as it measures the severity of default, it make sense that $J_p$ does not want to reduce the default expectancy at the price of a greater severity if default happens nonetheless.

#### Support contracts

In this paragraph, we indicate when it is possible to find a family of compensatory payoffs suitable to both martingale parameters. We call it a support contract between players.

**Definition 3.6.1** *Support contract*

*Let $J_1, J_2$ be two players in a C-game, subject to C-processes $C_1$ and $C_2$ in a given node of $G$, that are deemed independent conditionally to the underlying Markovian process $M$. A support contract is the determination of a C-process $S$*

*over $M$'s state space, such that the processes giving $J_1$'s and $J_2$'s assets under $S$ are given by*

$$C_1'(t) = C_1(t) + S(t) \wedge C_2'(t) = C_2(t) - S(t)$$

*Moreover, the transition payoffs of $S$ are deemed to depend only on $M$'s transition at present time, each transition payoff given $M(t) = A_i$ and $M(t+1) = A_j$ is a deterministic value $s_{i\to j} \in \boldsymbol{R}$.*

The concept of a support transition $s_{i\to j} > 0$ is to help $J_1$ in financial distress because of a locally "bad" state of a market for him (if $s_{i\to j} < 0$, then $J_1$ actually helps $J_2$). We aim at finding such a support contract that decreases both players' default expectancies. If we succeed, it will be profitable to both players to contract to support each other through payments given by $S$, so that they both avoid default.

**Proposition 3.6.1** *Best martingale parameters*

*Let us assume that $C_1$ and $C_2$ are C-processes describing the players' assets. We call respectively*

- *$a_p \in \boldsymbol{R}$ their Laplace parameters ;*
- *$\alpha_p(a_p) \in \boldsymbol{R}$ their martingale parameters ;*
- *$w_p^{(a)}$ and $\mu_p^{(a)}$ their associated column and row eigenvectors,*

*for $p \in \{1, 2\}$. For every states $A_i, A_j$ in $M$'s state space called $A$, we note by $D_{i\to j}^{(p)}$ the transition payoff attributed to $J_p$ over the transition $(i \to j)$. Let us set*

$$z_{p,i\to j} = \mu_{p,[i]}^{(a)} w_{p,[j]}^{(a)} P_{i\to j} \boldsymbol{E}\left(e^{-\alpha_p(a_p) D_{i\to j}^{(p)}}\right)$$

*Unless either $\alpha_p(a_p) = 0$, or for every $i, j \leq A$,*

$$e^{-a_1} z_{1,i\to j} = e^{-a_2} z_{2,i\to j}$$

*the latter being called the identity condition, there is a support contract $S$ between $J_1$ and $J_2$ such that*

- *$C_1 + S$ and $C_2 - S$ hold the identity condition ;*
- *The martingale parameter of $C_1 + S$ is higher than $C_1$'s one ;*
- *The martingale parameter of $C_2 - S$ is also higher than $C_2$'s one.*

The proof of this statement also provides a construction of such a support contract :

1. First, we built a compact subset $K$ of the set of support contracts $\mathbf{R}^{A^2}$ such that for every support contract $S$ outside of $K$, there is $S' \in K$ such that the martingale parameters $\alpha_p'(a_p)$ of $S'$ are both no lesser than $\alpha_p(a_p)$ (we say that "$S'$ weakly beats $S$"). As a consequence, we are interested only in support contracts in $K$.

2. We work recursively, starting from $S_0 = 0$ and $C_{p,0} = C_p$. From any sub-optimal support contract $C_{p,n}$, a direction $s$ is given to modify the present C-processes $C_{p,n}$ by a support contract $s$, yielding $C_{p,n} \pm s$ for both players (we say that the new support contract "strictly beats" the old one). Thanks to the previous step, there is $S_{n+1} \in K$ such that $C_{p,n+1} = C_p \pm S_{n+1}$ weakly beats $C_p + S_n + s$, thus strictly beats $C_{p,n}$.
3. As the sequence $(S_n)_{n\in\mathbf{N}}$ is in $K$, it has a limit point $S^*$. Provided that the width of increments $s$ is well-chosen, this gradient-ascent method must lead to an $S^*$ that cannot be beaten further, which means that it holds the identity contition.

On a side note, calling $\Delta(v)$ the diagonal matrix whose elements are given by the vector $v$, the previous identity condition rewrites as

$$e^{-a_1}\Delta\left(\mu_1^{(a_1)}\right)L_{C_1}\left(\alpha_1(a_1)\right)\Delta\left(w_1^{(a_1)}\right) = e^{-a_2}\Delta\left(\mu_2^{(a_2)}\right)L_{C_2}\left(\alpha_2(a_2)\right)\Delta\left(w_2^{(a_2)}\right)$$

However, we know that positive recurrent C-processes have positive eigenvectors, so the diagonal matrices are invertible. We also know that right multiplication of this identity by $\left(\vec{1}\right)$ leads to

$$e^{-a_1}\left(\mu_{1,[i]}^{(a_1)}w_{1,[i]}^{(a_1)}\right)_{i\leq A} = e^{-a_2}\left(\mu_{2,[i]}^{(a_2)}w_{2,[i]}^{(a_2)}\right)_{i\leq A}$$

by definition of eigenvectors. Therefore, noting by

$$\delta = \Delta\left(\left(\left(\frac{w_{2,[i]}^{(a_2)}}{w_{1,[i]}^{(a_1)}}\right)^{\left(\frac{1}{\alpha_1(a_1)+\alpha_2(a_2)}\right)}\right)_{i\leq A}\right)$$

the identity condition yields

$$e^{-a_1}\delta^{\alpha_1(a_1)}L_{C_1}\left(\alpha_1(a_1)\right)\delta^{-\alpha_1(a_1)} = e^{-a_2}\delta^{-\alpha_2(a_2)}L_{C_2}\left(\alpha_2(a_2)\right)\delta^{\alpha_2(a_2)}$$

which means that the Laplace matrix functions are similar up to a multiplicative constant, a fact to be used afterwards in this study.

**Interpretation**

Support contracts help both players without creating money : they mainly help the player in temporary distress (because of an adverse state of $M$), in exchange for the promise of a contribution to save other players from their own liquidity shortages later. It follows that support contracts are most effective when states of $M$ that are detrimental to one player are beneficial to the other one. A common example of support contract is an insurance policy, issued between a holder $J_1$ and an insurer $J_2$ about some risk modelled e.g. by entrance of $M$ in state $A_2$, incurring a drift $-R < 0$ and a variance $\rho^2$ : modelling $J_2$'s wealth by e.g. a Brownian motion of drift $E_2$ and volatility $\sigma_2^2$,

- If the risk is roughly independent from $J_2$'s wealth, additivity of quadratic risks means that $J_2$ is willing to insure the risk in exchange for a permanent premium $P$ such that
$$\frac{E_2 - R + P}{\sigma_2^2 + \rho^2} > \frac{E_2}{\sigma_2^2}$$
that simplifies to
$$P = R + \frac{\rho^2}{\sigma_2^2} E_2$$
while $J_1$ may be willing to pay such a $P$ to eliminate this risk from his own C-process, even if the insurance policy has an overall negative effect on his wealth ($P > R$).
- However, if risk is correlated with the insurer's wealth, e.g. if she has already insured lots of individuals against a general risk, her volatility can increase up to $\sigma_2^2 + \rho^2 + 2\sigma_2\rho$, where the previous computation leads to
$$P = R + \frac{\rho^2 + 2\sigma_2\rho}{\sigma_2^2} E_2$$
The impact is particularly significant if (as is the case with many insurance companies) $\sigma_2 >> \rho$ and $E_2$ is large, where the premium obtained in the former case is largely underestimated.

The condition of identity on Laplace matrix functions indicates that players with similar transition payoffs cannot efficiently help each other, which amounts roughly to saying that diversification cannot be achieved using only identical assets : if two players are too identical, there will be no "win-win" contract between them, as a support contract does not create any assets, being only a redistribution of cash flows. Once diversification has been done, the martingale parameters are said to hit a "local optimum", since we cannot modify them in a profitable manner to both players.

### Offsets

Up to now, players only wanted to maximize their martingale parameter in order to reduce default expectancy by means of support payoffs. From now on, we grant them the opportunity to influate on the dominant eigenvectors (in a way that does not modify the martingale parameters, that already have been optimized). For this purpose, we shall only look at a special case of support contracts, chosen such that they do not alter martingale parameters.

**Definition 3.6.2** *Offsets support contract*

*A support contract $S$ is qualified as an offsets support contract iff for each state $A_i$, there is a constant $r_i \in \boldsymbol{R}$ named $A_i$'s offset such that the transition payoffs are all*

$$\forall i, j \leq A, S_{i \to j} = r_j - r_i$$

*For every C-process $C$ with a martingale parameter $\alpha$ and a dominant eigenvector $w$, under such a support contract :*

1. *The martingale parameter of $C + S$ is still $\alpha$ ;*
2. *The dominant eigenvector at point $a \in \boldsymbol{R}^+$ becomes*

$$\left( \sum_{i=1}^{A} \mu_{[i]}^{(a)} e^{-\alpha(a) r_i} \right) \Delta \left( \left( e^{\alpha(a) r_i} \right)_{i \leq A} \right) w^{(a)}$$

If we assume that tractations between $J_1$ and $J_2$ have led so far to some support contract holding the previous identity condition, the next step towards optimization lies in the choice of optimal offsets to both players.

**Proposition 3.6.2** *Best offsets*

*Let $C_1$ and $C_2$ be C-processes holding the identity condition from the proposition 3.6.1. We define the optimum offsets support contract $S^*$ through its offsets*

$$\forall i \leq A, r_i = \frac{\ln \left( \frac{w_{2,[i]}^{(a_2)}}{w_{1,[i]}^{(a_1)}} \right)}{\alpha_1(a_1) + \alpha_2(a_2)}$$

1. *Under $S^*$, the players have the same Laplace matrix functions at points $a_p$ up to the discount factors, i.e*

$$L_{C_1 + S^*} \left( \alpha_1(a_1) \right) e^{-a_1} = L_{C_2 - S^*} \left( \alpha_2(a_2) \right) e^{-a_2}$$

   *It follows that they have the same dominant eigenvectors $x_p$ at points $a_p$, whose coordinate number $M(0)$ will be called $x$.*

2. *$S^*$ is optimum to $w$ among acceptable offsets support contracts, i.e. for every offsets support contract $S \neq S^*$ defining dominant eigenvectors $y_p$ such that*

$$y_{1,[M(0)]}^{(a_1)} \leq x_{1,[M(0)]}^{(a_1)} \wedge y_{2,[M(0)]}^{(a_2)} \leq x_{1,[M(0)]}^{(a_1)}$$

   *(both players are keen to accept the contract), we have both*

$$y_{1,[M(0)]}^{(a_1)} > x \wedge y_{2,[M(0)]}^{(a_2)} > x$$

This proof is similar to the proof of the proposition 3.6.1, where this time starting from a suboptimal contract, we build corrective offsets to enhance both dominant eigenvectors.

### 3.6.2 Mutualization of risk

The previous work has an interesting interpretation when we “detail” the transition payoffs of a C-process $C$. For example, if $a_1 = a_2 = a$ and $C_p$'s transition payoffs are deterministic, equal Laplace matrix functions mean that $C_1$'s and $C_2$'s transition payoffs hold

$$\alpha_1(a)D^{(1)}_{i\to j} = \alpha_2(a)D^{(2)}_{i\to j}$$

and therefore $C_1$'s and $C_2$'s variations are proportional. This means that the total cash amounts are split in a constant fashion : $J_1$ and $J_2$ actually do securitization on $C_1 + C_2$. In this paragraph, we shall enlight why securitization is the “best” way to avoid both players' default when they are subject to a C-process, in a sense to be specified below.

**Total decomposition of transition payoffs**

We begin by transforming the C-processes $C_p$ into C-processes $C'_p$ whose transition payoffs are deterministic. To do this, let us select a large integer $N \in \mathbf{N}^*$ and consider each transition payoff $D^{(p)}_{i\to j}$. We build C-processes $C'_p$, whose underlying state space is $A \times [|1, N|]^2$, defined by the transition probabilities

$$\forall (i,(n_1,n_2)),(j,(m_1,m_2)) \in A \times [|1,N|]^2, P_{(i,(n_1,n_2))\to(j,(m_1,m_2))} = \frac{1}{N^2} P_{i\to j}$$

and where the transition payoffs named

$$D^{(p)}_{(i,(n_1,n_2))\to(j,(m_1,m_2))}$$

are chosen such that their distributions are the corresponding $D^{(p)}_{i\to j}$'s ones once conditioned to being in the $(m_p)^{th}$ quantile out of $N$. Taking $N$ large enough, we may approximate the resulting transition payoffs to be roughly deterministic. Therefore, the two-player game with a larger state space and deterministic transitions has Laplace matrix transforms whose entries at point $\alpha$ may each be written as

$$P_{x\to y} e^{-\alpha d^{(p)}_{x\to y}}$$

where $x$ and $y$ are state numbers in $A \times [|1, N|]^2$ and $d^{(p)}_{x\to y}$ is the transition payoff in $\mathbf{R}$. Scaling the condition of identity by use of the best offsets as above, we find an optimal support contract such that the players' Laplace matrix functions are proportional at the chosen points, which means that for every $(x, y)$ (such that $P_{x\to y} \neq 0$),

$$\alpha_1(a_1)d^{(1)}_{x\to y} + a_1 = \alpha_2(a_2)d^{(2)}_{x\to y} + a_2$$

Considering the total income

$$d_{x\to y} = d^{(1)}_{x\to y} + d^{(2)}_{x\to y}$$

of both players, this solves to

$$d^{(1)}_{x\to y} = \frac{\alpha_2(a_2)}{\alpha_1(a_1)+\alpha_2(a_2)} d_{x\to y} + \frac{a_2 - a_1}{\alpha_1(a_1)+\alpha_2(a_2)}$$

which means that the best way they could share the total income is in an *affine function fashion*. Most notably, it features

— A "slope" term in $(0, 1)$, indicating how to share the total income ;

— An additive term, expressing how the discrepancy in player's Laplace parameters indicates that the longer-sighted player (with the lower $a_p$) should be granted with a higher cash allocation.

We thus find out that a "best" support contract between $J_1$ and $J_2$ may be found by considering the total process $C_1 + C_2$ and choosing two parameters :

— A ratio $r \in (0, 1)$, splitting its total income ;

— A constant $R \in \mathbf{R}$, indicating a permanent payoff to the longer-sighted player,

such that for an income of $D$, $J_1$ gets a $rD + R$ and $J_2$ gets $(1 - r)D - R$ of the income, with $r$ and $R$ chosen such that both players are satisfied.

### Choice of the parameters

We admit now that $J_1$ and $J_2$ have agreed on a choosing some support contract defining a distribution of $C = C_1 + C_2$ by an affine function, and they still can agree on the choice of $r$ and $R$. Using the same idea as for optimization of support contracts, we find that $R$ may be expressed as a function of $r$ to be optimal.

**Proposition 3.6.3** *Sharing the martingale parameter*

*Let $C$ be a C-process, whose martingale parameter at point $a$ is $\alpha(a) > 0$, deemed not to be a straight line, i.e. there is no $k \in \boldsymbol{R}$ such that $C(t) = C(0) + kt$ for every $t \in \boldsymbol{N}$ almost surely. Let $r \in \boldsymbol{R}^*_+$ and $Q \in \boldsymbol{R}$ such that the players' C-processes are defined by*

$$C_1(t) = rC(t) + Qt \wedge C_2(t) = (1 - r)C(t) - Qt$$

*We define the function*

$$R = \begin{pmatrix} [0,1] & \to & \boldsymbol{R} \\ r & \to & \frac{(a_2 - a_1) r (1-r)}{\alpha(r a_1 + (1-r) a_2)} \end{pmatrix}$$

*Unless $Q = R(r)$, there are $r' \in [0, 1]$ and $Q' \in \boldsymbol{R}$ such that*

1. *$Q' = R(r')$ holds ;*
2. *The martingale parameter of $C'_1$ defined by $C'_1(t) = r'C(t) + Q't$ is greater than $C_1$'s one ;*

*3. The martingale parameter of $C'_2$ defined by $C'_2(t) = (1-r')C(t) - Q't$ is greater than $C_2$'s one.*

Notice that when $C$ is a straight line, the controls $Q$ and $r$ are redundant, so we may set $R(r)$ arbitrarily, e.g. still as given. Moreover, this optimal $R(r)$ simplifies the computations for the martingale parameters.

**Proposition 3.6.4** *Martingale parameters for the optimal $R(r)$*
*Let $C$ be a C-process, whose martingale parameter at point $a$ is $\alpha(a) > 0$. Let $r \in (0,1)$ such that the players' C-processes are defined by*

$$C_1(t) = rC(t) + R(r)t \wedge C_2(t) = (1-r)C(t) - R(r)t$$

*with $R(r)$ as in the proposition 3.6.3. Then $C_1$'s martingale parameter at point $a_1 \in \boldsymbol{R}^+$ exists and is*

$$\alpha_1(a_1) = \frac{\alpha\left(ra_1 + (1-r)a_2\right)}{r}$$

*and $C_2$'s one at point $a_2$ is*

$$\alpha_2(a_2) = \frac{\alpha\left(ra_1 + (1-r)a_2\right)}{1-r}$$

*Moreover, the players' dominant eigenvectors are identical and constant of $r$, i.e. there are $\mu, w \in \boldsymbol{R}^A$ holding the scaling equations such that for every $r \in (0,1)$, the dominant eigenvectors $w_p^{(a_p)}$ hold*

$$\mu_1^{(a_1)} = \mu_2^{(a_2)} = \mu \wedge w_1^{(a_1)} = w_2^{(a_2)} = w$$

*no matter $r$.*

It follows from this proposition that the default expectancies may be written as

$$\mathbf{E}\left(e^{-a_p T_p}\right) \equiv Z_p(a_p) w_{[M(0)]} e^{-\alpha(ra_1+(1-r)a_2)\frac{C_p(0)}{r_p}}$$

with $r_1 = r$ and $r_2 = 1-r$, where $Z_p(a_p)$ depends on $r$ but not $w$.

### Initial compensation

To end with the minimization of the default expectancies, we grant the opportunity for the players to agree on $r$ when $C_p(0)$ is an adjustment variable. Specifically, it may happen that $J_2$ is unhappy enough of $r$ to be willing to offer $J_1$ some amount of cash $Q$ immediately, in exchange for a variation of $r$ (the effect on $Z_p(a_p)$ will be neglected).

**Proposition 3.6.5** *Minimal default expectancies*

*Let $C$ be a C-process, shared with $r \in (0,1)$ and $R(r)$ as in the proposition 3.6.4. We set $\alpha_p^*(a_p)$ the martingale parameters obtained with these, and*

$$k(r) = \frac{\alpha'\left(ra_1 + (1-r)a_2\right)}{\alpha\left(ra_1 + (1-r)a_2\right)}(a_1 - a_2)$$

*Unless we have the equality*

$$\frac{C_1(0)}{r} - \frac{C_2(0)}{1-r} = k(r)\left(C_1(0) + C_2(0)\right)$$

*named the critical equality for $C_1(0)$, $C_2(0)$ and $r$, there are $C_p^*(0)$ such that $C_1^*(0) + C_2^*(0) = C(0)$ and $r^* \in [0,1]$ such that*

1. *Either $r^* \in (0,1)$, $C_1^*(0)$, and $C_2^*(0)$ hold the critical equality, or $r = 0$ and $C_1(0) = 0$, or $r = 1$ and $C_2(0) = 0$ ;*
2. *The default expectancies are reduced by virtue of*

$$\alpha\left(ra_1 + (1-r)a_2\right)\frac{C_p(0)}{r_p} < \alpha\left(r^*a_1 + (1-r^*)a_2\right)\frac{C_p^*(0)}{r_p^*}$$

   *where $r_1^* = r^*$ and $r_2^* = 1 - r^*$.*

*The cases where an extremal $r \in \{0,1\}$ is hit are called "business proposal" configurations. In these cases, any expression of the form $x/0$ for $x \in \boldsymbol{R}_+^*$ must be understood as $\infty$.*

The function $k$ is non-increasing over $(0,1)$ (see a proof in the appendix). The case $r = 0$ means that $J_2$ buys the whole $J_1$ (and reciprocally for $r = 1$), hence the name "business proposal". As in some sense "$J_2$ sustains for $J_1$'s assets", it is normal to recover that $J_1$ cannot default. In particular, recalling that the default expectancies at the equilibrium points are

$$\mathbf{E}\left(e^{-a_pT_p}\right) \equiv Z_p(a_p)w_{[M(0)]}e^{-\alpha_p(a_p)C_p(0)}$$

and rewriting the critical equality as

$$\alpha_1(a_1)C_1(0) - \alpha_2(a_2)C_2(0) = \alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2)C(0)$$

we get that the default expectancies of players $J_p$ are in a ratio whose logarithm is this latter term (up to additive terms $\ln(Z_p)$). Therefore :

— The value $r$ plays as an adjustment factor, depending on players' bargaining power, but its effect is limited to the convex combination $ra_1 + (1-r)a_2$. As $r \in (0,1)$, it will be convenient to neglect this and approximate

$$\alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2) \approx \alpha(a_1) - \alpha(a_2)$$

eventually leading to

$$\alpha_1(a_1)C_1(0) - \alpha_2(a_2)C_2(0) \approx \left(\alpha(a_1) - \alpha(a_2)\right)C(0)$$

so the discrepancy between default expectancies is roughly recovered thanks to the total process $C$, as the difference between the "natural" terms appearing in $C$'s default expectancy $\alpha(a_p)C(0)$.

— $a_1 > a_2$ implies that this term is positive (favorable to $J_1$). This is not surprising, since higher Laplace parameters lead to lower default expectancies.

— Buisness proposals may happen only if $\alpha(a_p) \geq \alpha_p(a_p)$ for either player $J_p$. Notice that this means that the *worse*-shaped player is more likely to buy the other one than the converse : this may look like a paradox, as it comes in contradiction with common sense and observations of reality.

An explanation of the paradox may rise if we take consumption into account along with C-processes. Let us take players $J_1$ and $J_2$ with consumption rates $c_1, c_2 \in \mathbf{R}_+^*$ : $J_2$ has the better-shaped C-process, i.e. $\alpha_2(a_2) > \alpha_1(a_1)$. If $J_1$ wants to purchase $C_2$, when substracting players' consumption (like wages or dividends), $J_1$ must be prepared to actually get $C_2 - c_2$ as he must pay $c_2$ to $J_2$ to sustain her consumption ; for want of this, $J_2$ will refuse to sell her C-process. Since owners of huge firms typically percieve higher dividends than owners of tiny firms, and dividends are commonly roughly equal to the benefits, $J_1$ will not earn much by taking charge of $C_2$. He will be a simple "manager" of $C_2$, while $J_2$ still percieves her dividends.

**Case** $a_1 = a_2$

When the players have identical preferencies regarding their discount factors, i.e. $a_1 = a_2 = a \in \mathbf{R}^+$, the expressions found throughout these propositions simplify :

— We get $R(r) = 0$, so $C$ is shared in a proportional fashion ;

— The martingale parameters become

$$\alpha_1(a) = \frac{\alpha(a)}{r} \wedge \alpha_2(a) = \frac{\alpha(a)}{1-r}$$

— In particular, as $J_p$ is subject to the C-process $C_p = r_pC$, the default time $T_p$ may be seen as the default time of the process $C$ starting from $C_p/r_p$ by multiplicative scaling. It follows that $C_p$'s default expectancy is written as

$$\mathbf{E}\left(e^{-aT_p}\right) \equiv Z(a)w^{(a)}_{[M(0)]}e^{-\alpha(a)C_p(0)/r_p}$$

where $Z$ is $C$'s multiplicative term, $w$ is its dominant eigenvector, and $\alpha$ is its martingale parameter.

— After the critical equality, this also means that at any equilibrium point (choice of $r$), both players' default expectancies are *equal*. As the trajectories of $C_1$ and $C_2$ are proportional, they indeed default simultaneously, at the time when $C_1 + C_2$ goes bust.

On the other hand, this is the best we can do for both players simultaneously : intuitively, as we found the expression of $C$'s default expectancy, and we know that $C$'s default implies either $C_1$'s or $C_2$'s earlier default, we expect $C_p$'s default expectancy to be no lesser than $C$'s one. This intuitive idea is rigourously expressed thanks to the next proposition.

**Proposition 3.6.6** *Sub-additivity for* $\alpha^{-1}$

*Let $C_1$ and $C_2$ be C-processes with martingale parameters $\alpha_1$ and $\alpha_2$. We deem that $C = C_1 + C_2$ has $\alpha$ as a martingale parameter.*

1. *For every $a \in \boldsymbol{R}^+$,*
$$\frac{1}{\alpha(a)} \leq \frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}$$
2. *This inequality is an equality iff there are a constant $u \in \boldsymbol{R}_+^*$ and a globally constant C-process $C^=$ such that almost surely,*
$$\forall t \in \boldsymbol{N}, C_2(t) = uC_1(t) + C^=(t)$$

Let us assume now that two players $J_1$ and $J_2$ have the same Laplace parameter $a$, with martingale parameters $\alpha_p(a)$. If they merge their incomes and share them with a ratio $\alpha_2(a)/(\alpha_1(a) + \alpha_2(a))$ to $J_1$, setting

$$x(a) = \alpha(a) \left( \frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)} \right) \geq 1$$

then $J_1$'s new martingale parameter is $\alpha_1(a)x(a)$ and $J_2$'s one is $\alpha_2(a)x(a)$. Hence, unless players have already shared incomes as prescribed, they enhance both martingale parameters when doing so. Offsets may now be optimized thanks to the proposition 3.6.2, eventually indicating that $C^= = 0$, and so all that remains to do is to find the optimal initial compensation through the proposition 3.6.5, given by the critical equality. If we have e.g. the inequality $C_1(0)/r < C_2(0)/(1-r)$, let us set $c_0 > 0$ and $r_0 > 0$ such that

$$\frac{C_1(0)}{r} < \frac{c_0}{r_0} < \frac{C_2(0)}{1-r}$$

Increasing $C_1$ by $c_0$ along with $r$ by $r_0$ leads to exponential terms in the default expectancies, being

$$\frac{C_1(0)+c_0}{r+r_0} > \frac{C_1(0)}{r}$$

for $J_1$, and

$$\frac{C_2(0)-c_0}{1-r-r_0} > \frac{C_2(0)}{1-r}$$

for $J_2$. However, we still have

$$\frac{C_1(0)+c_0}{r+r_0} < \frac{c_0}{r_0} < \frac{C_2(0)-c_0}{1-r-r_0}$$

so we may start again, never hitting an equilibrium and stopping only when the boundary is hit with $r = 1$ : $J_1$ eventually buys $C_2$. Likewise, the converse inequality leads to $r = 0$ as only optimal solution : when $a_1 = a_2$, some buisness proposal is the only equilibrium (except if the critical equality already holds : in this case, the players have a set of equilibria, defined by allocation $r$ of incomes equal to the allocation of initial wealth, including $r = 0$ and $r = 1$). Therefore, $J_1$ is willing to buy $C_2$ provided that

$$\frac{C_1(0)}{r} < C_1(0) + C_2(0)$$

thanks to $\alpha_1 = \alpha/r$, thus when $r > C_1(0)/C(0)$, while $J_2$ is willing to buy $C_1$ when $r < C_1(0)/C(0)$. This leads us back to the critical equality, indicating equilibrium when $r = C_1(0)/C(0)$, which means that the players $J_p$ take under their own responsibilities a fraction of $C$ proportional to their own initial cash amounts.

### 3.6.3 Discussion

In this paragraph, we observe the therories of support contracts and diversification through an example ; this will lead to insights on cooperation between players, indicating investment strategies in two-player games. Let us define the C-processes $C_1$ and $C_2$ as follows.

— $M$ is given by the transition matrix

$$\frac{1}{10}\begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix}$$

with $A_1$ as the starting state ;

— $C_1$'s transition payoffs are given by

  — $D_{1\to1}$, $D_{1\to2}$ and $D_{2\to1}$ amount to 1 almost surely ;

— $D_{2\to 2}$ is Gaussian, with mean $-1$ and variance 1, starting from $C_1(0) = 10$ ;
— $C_2$'s transition payoffs are given by
  — $D_{1\to 2}$, $D_{2\to 1}$ and $D_{2\to 2}$ amount to 1 almost surely ;
  — $D_{2\to 2}$ is Gaussian, with mean $-1$ and variance 1, starting from $C_2(0) = 10$.

Both players aim at minimizing their default probabilities, i.e. $a_1 = a_2 = 0$.

**Computations**

Computations on the Laplace matrix functions yield the martingale parameters

$$\alpha_1(0) = \alpha_2(0) \approx .02053$$

as well as the dominant eigenvectors

$$\mu_1^{(0)} \approx \begin{pmatrix} .453 & .547 \end{pmatrix} \wedge w_1^{(0)} \approx \begin{pmatrix} .898 \\ 1.084 \end{pmatrix}$$

with coordinates reversed for $J_2$. It comes to no one's surprise that the state $A_1$ is "good" to $J_1$ and "bad" to $J_2$, as indicated by $w_{1,[1]}^{(0)} < w_{1,[2]}^{(0)}$ and $w_{2,[1]}^{(0)} > w_{2,[2]}^{(0)}$. Therefore, we will compute a support contract between both players as given by the proposition 3.6.1.

— The condition of identity does not hold, as may be verified through numerical computations ;
— A suitable infinitesimal support contract is given by the matrix $s$, computed as in the proof, whose entries $s_{i\to j}$ are

$$s = \begin{pmatrix} -x & 0 \\ 0 & x \end{pmatrix}$$

for $x \approx .122$.
— For example, using the support contract of $S = s/x$ leads to two C-processes sharing the new martingale parameter $\alpha_p(0) \approx .320$. In particular, the terms

$$w_{p,[M(0)]}^{(0)} e^{-\alpha_p(0) C_p(0)}$$

(coming from the fundamental approximation of the default probability) were respectively approximated by .731 for $J_1$ and .883 for $J_2$, and now become .033 and .046. The constants $Z_p(0)$ will not be computed exactly, but we shall admit that their effects do not significatively impede the blatant discrepancy in default probabilities.

Computations may now be restarted from the resulting C-process, using a gradient ascent method until a support contract holding the identity condition is found. Doing this for this choice of gradient variations $s$ leads to

$$s \approx \begin{pmatrix} -1.0915 & 0 \\ 0 & 1.0915 \end{pmatrix}$$

with martingale parameters

$$\alpha_1(0) = \alpha_2(0) \approx .366$$

As a matter of fact, as we find that $w_1^{(0)} = w_2^{(0)}$ under this support contract $S$, the optimal offsets are 0 and $S$ is already optimal with respect to offsets. Neglecting the multiplicative terms $Z_p(a_p)$ in front of the fundamental approximation, we conclude that both players' default expectancies are optimally reduced thanks to $S$ : the terms

$$w_{p,[M(0)]}^{(0)} e^{-\alpha_p(0)C_p(0)}$$

are now both approximated by .026. However, there is *no* globally optimum martingale parameters for both players, because the players have several ways of enhancing both martingale parameters : we may modify $S$ towards any direction $s$ such that both gradients ($z_1$ and $z_2$) are correctly altered, i.e.

$$\langle s, z_1 \rangle > 0 \wedge \langle s, z_2 \rangle < 0$$

maybe finally hitting a different support contract $S$ holding the identity condition because the martingale parameters are different from another case. In our previous example, setting a support contract defined by

$$s \approx \begin{pmatrix} -1.075 & -.634 \\ .366 & 1.108 \end{pmatrix}$$

also yields C-processes verifying the identity, but this time with martingale parameters

$$\alpha_1(0) \approx .387 \wedge \alpha_2(0) \approx .345$$

The values $(\alpha_1(0), \alpha_2(0))$ effectively hit once player agree on a support contract will not be investigated in this study, as they depend mainly on the "bargaining power" of $J_1$ against $J_2$ : a more powerful $J_1$ will "pull" the support contract by some $s$ closer to his own gradient $z_1$, while a powerful $J_2$ pulls towards her own $z_2$.

### Outcomes of securitization

To illustrate the power of securitization for these C-processes $C_1$ and $C_2$, we compute the martingale parameters under the hypotheses given by the proposition 3.6.4. In our case $a_1 = a_2 = 0$, starting from $C_1(0) = C_2(0)$, equilibriums points are found for $r \in (0, 1)$ when the initial compensation makes $C_1(0) = rC(0)$. For example, $r = 2/3$ leads to

$$\alpha(0) \approx .320 \wedge \alpha_1(0) \approx .480 \wedge \alpha_2(0) \approx .960$$

The total income from each transition $(x \to y)$ being split 2/3 to $J_1$ and 1/3 to $J_2$, one gets $\alpha_1(a_1) = \alpha_2(a_2)/2$ for this $2 : 1$ allocation. As $w$ is the constant vector of ones, the terms

$$w^{(0)}_{p,[M(0)]} e^{-\alpha_p(0)C_p(0)}$$

are now both approximated by .017. Incidentally, we notice that the reason why these proportions are found comes from the definition of the default for a C-process : multiplying all increments of a C-process by a constant $x$ has a similar effect to dividing $C(0)$ by $x$ and then scaling the obtained trajectory, which explains why the martingale parameter is inversely proportional to the width of increments. As a consequence, the default probabilities would still have been the same .017 no matter $r$, provided that the initial compensation is adjusted accordingly.

### Thoughts on the principal-agent model

When $M$ denotes the exogenous market and players have no influence on it whatsoever, these support contracts may also be designed to limit the consequences of moral hazard in principal-agent models (see [26] for explanations about moral hazard). During this explanation, we will call $J_p$ the principal and $J_a$ the agent, call $G^{[1]}$ the node hit without investment and $G^{[2]}$ the node hit with it, and assume that the agent's work produces a quantity $C_w$ of assets, deemed to be a C-process. As is well-known (e.g. [32]), we look for a debt-shaped distribution of assets, which encourages us to write $J_p$'s transition payoffs after investment as

$$C_p^{[2]}(t+1) - C_p^{[2]}(t) = C_p^{[1]}(t+1) - C_p^{[1]}(t) + d(t+1)$$

where $d(t+1)$ is the amount of debt repayment for the time period between $t$ and $t+1$, while the agent gets the equity as

$$C_a^{[2]}(t+1) - C_a^{[2]}(t) = C_a^{[1]}(t+1) - C_a^{[1]}(t) + C_w(t+1) - C_w(t) - d(t+1)$$

Therefore, the principal should calibrate $d$, which is tantamount to finding a support contract such that under the expected amount of effort by the agent, both players are satified by the contract. As we found out, taking $d$ as a support contract may yield different transition payoffs depending on $M$'s situation : in particular, if the conjoncture is "bad" to the incomes issuing from work, one expects $d$ to be lower than for "good" cases. In other words, the principal should not punish the agent for bad luck with $M$.

On the other hand, laziness from the agent will lead to lower increments of $C_w$, while $d$ has been calibrated with respect to proper work. It follows that $J_a$ will be the one to suffer from her own moral hazard, as sought by the choice of a debt contract. This also marks the limit to the decomposition of transition payoffs because $M$ should be observed by the principal for the model to work, so as long as his information is incomplete, the transition payoffs will not be totally decomposed. Interestingly, we notice that the model fails when $J_a$ modifies transition probabilities with her work decisions, because this makes her able to pretend that poor incomes are a consequence of bad luck rather than laziness, so the principal accepts to decrease his pledgeable income as explained above and eventually suffers indirectly from $J_a$'s lack of work. For this reason, the principal should not proceed with total decomposition of $C_w$ once all "exogenous information" has been considered with $M$, $d$ acting as a partial compensation for luck.

### Thoughts on investment strategies

Our model finally recovers the concept of risk mutualization : the best choice for both players is to share their incomes and risks into the total process $C = C_1 + C_2$, in order to support each other against bankruptcy risks. When $a_1 = a_2$, equilibria are found when one player is in charge of the whole $C$ ; we remark that there must be such a player, e.g. $J_1$, willing to be in charge of $C$ thanks to the proposition 3.6.6, while $J_2$ is "sustained" by $J_1$ and is assured to never default. This however enlights a shortcoming of our model : when taking $J_2$'s consumption into account like previously, it is not natural to assume that $J_2$ keeps percieving her consuption eternally once $J_1$ failed to sustain her, e.g. the holder of a perpetuity stops getting paid once the issuer goes bust (although such a case may be investigated, e.g. when a CEO retires and sells their buisness for a large enough price to maintain their future consumption). When $J_2$ must keep playing once $J_1$ defaulted, she should instead compute her default expectancy at $J_1$'s default time, going on with $C$ once $J_1$ retires from the market : it turns out that she should accept cooperation iff

$$(C_1(0) + C_2(0))\, \alpha(a) > C_1(0)\alpha_1(a)$$

i.e. when sharing the incomes lowers her own default risks. A special case is $a_1 = a_2 \approx 0$ when the total mean expectancy is negative. As default is almost certain under this hypothesis, optimization refers to the first-order properties from the proposition 3.4.1, requiring mean expectancies instead of martingale parameters.

However, as support contracts have zero sums, there is no way to improve both mean expectancies simultaneously, and the only adjustment may be made on the ratios $C_p/(-E_p)$, eventually leading to one player taking the whole $C_1 + C_2$ as in the general case $a_1 = a_2$. On the other hand, if $a_1 \neq a_2$, the term $k(a)$ may or may not allow for an inner equilibrium, depending on the situation. As a main idea, when $a_1$ comes close to $a_2$, $k(a)$'s magnitude decreases and the likelihood of an inner equilibrium decreases.

Investment decisions in a two-player C-game rely on the default expectancies, and are obtained recursively like in the single-player models. Starting from a leaf in the graph $G$, players compute an equilibrium support contract to get the optimal characteristic items of this node, then decide to buy (or not) depending on the results. We also notice that even when the default probabilities are equal at equilibrium points (when $a_1 = a_2$), the players need not have the same interests in the game, because of multiple equilibria. For example, let us assume that all C-processes involved in here are Brownian motions, calling by $W(E, \sigma^2)$ the Brownian motion of drift $E$ and volatility $\sigma^2$. The last investment opportunity is hit at time 0, available to $J_1$, where

— Acceptance would lead to $C_1$ following a $W(3,1)$ and $C_2$ following a perfectly positively correlated $W(1,1)$ ;
— Rejection would lead to both $C_1$ and $C_2$ following the same $W(2,1)$.

At present time, we have $C_1(0) = 9$ and $C_2(0) = 11$ while the investment cost amounts to 1, and both players have $a_p = 0$ : they aim at maximizing their negative logarithmic default probabilities $-\ln(\mathbf{P}(T_p < \infty))$, hereby named $J_p$'s "score". In both cases, the total income $C$ follows a $W(4,4)$.

— If $J_1$ buys, his martingale parameter is $\alpha_1(0) = 3$, and he scores $-\Lambda_1(0) = 48$, while $J_2$'s score drops to 22. However, as $C$'s score after buying is $-\Lambda_C(0) = 38$, the only useful equilibrium to the players is that $J_2$ takes charge of the whole $C_1$, getting 38 while $J_1$ never defaults.
— If $J_1$ rejects, his martingale parameter becomes $\alpha_1(0) = 2$ for $-\Lambda_1(0) = 36$ while $-\Lambda_2(0) = 44$. This time $J_1$ is going to buy $C_2$ to score 40, while $J_2$ never defaults.

As a consequence, even is the investment opportunity has a negative externality (positive price and no effect on $C$), $J_1$ is tempted to buy it in order to "force" $J_2$ to sustain him : coordination is not perfect. We may however object that at the step before the investment choice, $J_2$ could also buy $C_1$ immediately in exchange for the promise from $J_1$ that *he will reject the investment* ; if this kind of contract is allowed, $J_1$ is still protected from default, while $J_2$ gets 40 instead of 38 because mo money has been wasted in the investment opportunity. Such promises contribute to maintaining the choice of players towards the "greater good", i.e. the best martingale parameter for $C$. Finally, this phenomenon does not appear when under the same consideration as before after one player defaults : as the players share the same default expectancy, the negative effects of egoistic actions downgrade the default expectancy of their instigators too. This is seen through the same example, where buying yields 38 for $J_2$ (that translates here to 38 for $J_1$ because $J_1$ fails now with $J_2$), while not buying leads to 40 for $J_1$.

The prospect of a future advantageous support contract may also help players to reach the "greater good". Let us now assume that the C-processes are :

— After acceptation, identical $W(2,1)$ for $J_1$ and $W(2,1)$ for $J_2$ ;

— After rejection, identical $W(1,1)$ for $J_1$ and $W(1,1)$ for $J_2$.

At present time, we have $C_1(0) = 3$ and $C_2(0) = 5$ while the investment cost amounts to 2, and both players have $a_p = 0$. Let us compute the possible outcomes of this game.

— If $J_1$ rejects, his martingale parameter is 2 so he scores 6, while $J_2$'s martingale parameter is 2 so she scores 10.

— If $J_1$ accepts, his martingale parameter becomes 4 but he would score only 4 because of too high investment costs, while $J_2$ scores 20.

The investment opportunity with positive externality will be squandered because the investor has insufficient cash reserves ; however, the total C-process $W(4,4)$ after acceptation scores 12, which makes it interesting to both players. It follows that $J_1$ and $J_2$ will find an interesting support contract to make the opportunity advantageous. However, had $C$ scored lower than 10, $J_2$ would not have accepted to share it with $J_1$, because the equilibrium point after buying is not satisfying to her. The only possibility is for $J_1$ to buy $J_2$'s C-process (the case $r = 1$ of the proposition 3.6.5), and works only in the model where $J_2$'s consumption is unaffected by $J_1$'s bankruptcy.

# Proofs

This part is devoted to the proofs of the proposition given in the study. Each paragraph title referring to a proposition number contains the proof of this proposition.

## 3.7 Single-player analysis

This paragraph is devoted to the construction of the player's best strategy in the general case.

### 3.7.1 Proposition 3.4.1

In this paragraph, we prove the expressions for the incentive and handicap at $a = 0$. We begin by introducing the terms coming from Taylor development, and the proof of the buying decision will follow next.

#### Taylor development

Let us recall the expressions of the incentive and the handicap as given by the definition 3.4.2. As we know that $\alpha_i$ and $w_i^{(a)}$ are smooth around 0 as soon as $E_i \neq 0$, Taylor development yields

$$\gamma(a) = 1 - \frac{\alpha_1(0) + a\alpha_1'(0) + O\left(a^2\right)}{\alpha_2(0) + a\alpha_2'(0) + O\left(a^2\right)}$$

We know that $\alpha_i(0) = 0$ and $\alpha_i'(0) = -1/E_i$ when $E_i < 0$, so this leads to

$$\gamma(a) = 1 - \frac{E_2}{E_1} + O(a)$$

Likewise,

$$H(a) = \frac{\ln\left(\frac{Z_2(0)w_2^{(0)} + aZ_2(0){w'}_2^{(0)} + aZ_2'(0)w_2^{(0)} + O\left(a^2\right)}{Z_1(0)w_1^{(0)} + aZ_1(0){w'}_1^{(0)} + aZ_1'(0)w_1^{(0)} + O\left(a^2\right)}\right)}{\alpha_2(0) + a\alpha_2'(0) - \alpha_1(0) - a\alpha_1'(0) + O\left(a^2\right)}$$

We know that $w_i^{(0)} = 1$ and $Z_i(0) = 1$ so this yields

$$H(a) = \frac{\ln\left(1 + a{w'}_2^{(0)} + aZ'_2(0) - a{w'}_1^{(0)} - aZ'_1(0) + O\left(a^2\right)\right)}{a(-1/E_2) - a(-1/E_1) + O\left(a^2\right)}$$

and after computations we get

$$H(a) = \frac{\left(Z'_2(0) + {w'}_2^{(0)}\right) - \left(Z'_1(0) + {w'}_1^{(0)}\right)}{\frac{1}{E_1} - \frac{1}{E_2}} + O(a)$$

since $E_1 \neq E_2$.

**Buying decision for $a = 0$**

Let us consider two investment outcomes, defining $C_1$ and $C_2$ to be positive recurrent, bounded and not globally increasing C-processes, with $E_i = E(C_i)$ as required, whose underlying Markovian process respectively

— Are called $M_1$ and $M_2$ ;

— Have transition matrices $P_1$ and $P_2$ (necessarily positive recurrent, as $C_1$ and $C_2$ are positive recurrent) ;

— Have thus invariant measures called $\mu_1$ and $\mu_2$.

For $i = 1$ or $i = 2$, we introduce

— $L_i$ is $C_i$'s Laplace matrix function, and $R_i$ is $C_i$'s diff-Laplace matrix function : they are both defined over $\mathbf{R}$, since $C_i$ is bounded ;

— For every Laplace parameter $a \in \mathbf{R}$, $\alpha_i(a)$ is $C_i$'s martingale parameter, and $w^{i,(a)}$ is $L_i(\alpha_i(a))$'s dominant eigenvector ;

— We shall name $\alpha'_i(a)$ and ${w'}_i^{(a)}$ their derivatives at point $a \in \mathbf{R}$, as ensured to exist.

In particular, one has thanks to these definitions

— The eigenvector equation for $C_i$'s martingale parameter :

$$\forall t \in \mathbf{N}, \forall a \in \mathbf{R}^+, \left(L_i(\alpha_i(a))\right)^t w^{i,(a)} = e^{at} w^{i,(a)}$$

— The definition of $C_i$'s mean expectancy :

$$E_i = \mu_i R_i(0)\left(\vec{1}\right)$$

— The martingale equation for $T$ the default time

$$\forall a \in \mathbf{R}^+, w_{[0]}^{i,(a)} e^{-\alpha_i(a) C_i(0)} = \mathbf{E}\left(w_{[M(T)]}^{i,(a)} e^{-\alpha_i(a) C_i(T)} e^{-aT}\right)$$

First-order properties for $a = 0$ and $E(C) < 0$ are given by differentiation of $C$'s martingale equation for the default time $T$, which yields

$$\begin{aligned} & {w'}^{i,(a)}_{[0]} e^{-\alpha_i(a)C_i(0)} - \alpha_i'(a)C_i(0)w^{i,(a)}_{[0]} e^{-\alpha_i(a)C_i(0)} \\ = \; & \mathbf{E}\left({w'}^{i,(a)}_{[M(T)]} e^{-\alpha_i(a)C_i(T)} e^{-aT}\right) \\ - \; & \alpha_i'(a)\mathbf{E}\left(C_i(T) w^{i,(a)}_{[M(T)]} e^{-\alpha_i(a)C_i(T)} e^{-aT}\right) \\ - \; & \mathbf{E}\left(T w^{i,(a)}_{[M(T)]} e^{-\alpha_i(a)C_i(T)} e^{-aT}\right) \end{aligned}$$

As we know that $\alpha_i(0) = 0$, $\alpha_i'(0) = -1/E_i$, and $w^{i,(0)} = \left(\vec{1}\right)$ because $E(C) < 0$, then doing $a = 0$ yields

$$w'^{i,(0)}_{[0]} + \frac{1}{E_i}C_i(0) = \mathbf{E}\left(w'^{i,(0)}_{[M(T)]}\right) + \frac{1}{E_i}\mathbf{E}\left(C_i(T)\right) - \mathbf{E}(T)$$

which translates to

$$\mathbf{E}(T) = \frac{-1}{E_i}C_i(0) + \left(\frac{1}{E_i}\mathbf{E}\left(C_i(T)\right) + \mathbf{E}\left(w'^{i,(0)}_{[M(T)]}\right)\right) - w'^{i,(0)}_{[0]}$$

In particular, the final situation of $C_i(T)$ and $M_i(T)$ yields a distribution of a random couple $(C_i, M_i)(T)$. When $C_i$ is aperiodic, the point is that $\mathbf{E}\left(C_i(T)\right)$ and all probabilities $\mathbf{P}\left(M(T) = A_j\right)$ (for $A_j$ a state among $M_i$'s state space) converge to a limit when $C_i(0)$ goes to $\infty$, as described in the proof of the theory of C-processes. This means that there is a constant $c_i$ defined by

$$c_i = \lim_{x\to\infty}\left(\frac{1}{E_i}\mathbf{E}\left(C_i(T)|C(0) = x\right) + \mathbf{E}\left(w'^{i,(0)}_{[M(T)]}|C(0) = x\right)\right)$$

whose existence will be discussed in the next paragraph, that depends only on $C_i$'s structure, holding

$$\mathbf{E}(T) = \frac{-1}{E_i}C_i(0) + c_i - w'^{i,(0)}_{[0]} + o(1)$$

This leads to the investment decision when $a$ goes to $0$ : once again dropping the $o(1)$ like in the fundamental approximation, and setting $w_i' = w'^{i,(0)}_{[0]}$ the value associated with $C_i$'s starting point, one chooses the investment that maximizes

$$\frac{C}{-E_i} + \left(c_i - w_i' + \frac{I_i}{E_i}\right)$$

Comparing these terms for $i = 2$ with $i = 1$ yields the buying condition

$$\frac{C}{-E_2} - \frac{C}{-E_1} > \left(c_1 - w_1' + \frac{I_1}{E_1}\right) - \left(c_2 - w_2' + \frac{I_2}{E_2}\right)$$

Similarly to the case $\alpha(a) > 0$, we rewrite this as

$$\left(\frac{1}{-E_2} - \frac{1}{-E_1}\right) C > (c_1 - c_2 - w'_1 + w'_2) + \left(\frac{1}{E_1} - \frac{1}{E_2}\right) I_1 + \frac{I_1}{E_2} - \frac{I_2}{E_2}$$

so that as $E_1 < E_2 < 0$, it simplifies to

$$C > I_1 + \left(\frac{(c_1 - c_2) + (w'_2 - w'_1)}{\frac{-1}{E_2} - \frac{-1}{E_1}}\right) + (I_2 - I_1)\frac{1}{1 - \frac{E_2}{E_1}}$$

which ends the proof.

**Identification of $-c_i$ and $Z'_i(0)$**

We want to build and ensure the existence of $c_i$. In particular, we want to state that the term given by Taylor development of $H(a)$ around 0 coincides with the handicap from the proposition 3.4.1. We shall use the same trick as in the study on C-processes, considering $\vec{C}$ to be $C$'s descending process (we have $C < \infty$ almost surely because $E(C) < 0$). For every state $A_j$, let $Q$ be the boundary of $C$'s increments, and let us call $f_j$ the function defined by

$$f_j = \begin{pmatrix} [-Q, 0) & \to & \mathbf{R} \\ x & \to & \frac{-x}{E_i} - {w'}^{i,(0)}_{[j]} \end{pmatrix}$$

By construction of $\vec{C}$, let us deem $T'$ to be $\vec{C}$'s default time. $-c_i$ is the limit for high starting points $C(0)$ of the expectancy

$$\mathbf{E}\left(\frac{1}{E_i} C_i(T) + {w'}^{i,(0)}_{[M(T)]}\right) = \mathbf{E}\left(\frac{1}{E_i} \vec{C}_i(T') + {w'}^{i,(0)}_{[\vec{M}_i(T')]}\right) = \mathbf{E}\left(f_{[\vec{M}(T')]}\left(\vec{C}_i(T')\right)\right)$$

The functions $f_j$ being non-decreasing and half-Lipschitz, we conclude to the existence of $c_i$ like in the aforementioned work.

To identify $-c_i$, we introduce some basis functions $h_u$ defined for every $u \in [0, Q]$ by

$$h_u = \begin{pmatrix} [-Q, 0) & \to & \mathbf{R} \\ x & \to & (x+u)\mathbf{1}_{x+u>0} \end{pmatrix}$$

First, we are going to express the functions named $K_j^{(a)}$ over $[-Q, 0)$ (provided that $j$ belongs to $C$'s descending class $A'$) as a linear comination of

- A convex combination of functions $h_u$ ;
- The unit constant function ;
- The identity function.

We recall that for every $x \in [-Q,0)$,

$$K_j^{(a)}(x) = \frac{e^{\alpha_i(a)x}}{w_{[j]}^{i,(a)}}$$

and we use the calculus equality (for $f$ twice continuously differentiable over $[-Q,0)$, may be verified using integration by parts)

$$f(x) = \int_{u=-Q}^{0} f''(-u)h_u(x)du + xf'(-Q) + f(-Q) + Qf'(-Q)$$

In particular for $f = K_j^{(a)}$, this yields

$$\begin{aligned} K_j^{(a)}(x) &= \int_{u=-Q}^{0} \frac{\alpha_i^2(a)e^{-\alpha_i(a)u}}{w_{[j]}^{i,(a)}} h_u(x)du \\ &+ x\frac{\alpha_i(a)e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} \\ &+ \left(Q\frac{\alpha_i(a)e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} + \frac{e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}}\right) \end{aligned}$$

The limit term $Z(a)$ may be seen as a linear and continuous application $F$ of the functions $K_j^{(a)}$ for $j \in A'$ over $[-Q,0)$ (with respect e.g. to uniform convergence). Let us name

- $Y_j(u)$ the image by $F$ of the functions : $h_u$ on entry number $j$, 0 on other entries (well-defined because $h_u$ is non-decreasing and half-Lipschitz) ;
- $X_j^{(0)}$ the image of the unit function and $X_j^{(1)}$ the image of the identity function on entry number $j$.

As $F$ is continuous and multi-linear, and as $K_j^{(a)}(x)$ is defined thanks to a convex combination, the contribution $Z_{i,j}(a)$ of the entry number $j$ to $Z_i(a)$ is

$$\begin{aligned} Z_{i,j}(a) &= \int_{u=-Q}^{0} \frac{\alpha_i^2(a)e^{-\alpha_i(a)u}}{w_{[j]}^{i,(a)}} Y_j(u)du \\ &+ X_j^{(1)}\frac{\alpha_i(a)e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} + X_j^{(0)}Q\frac{\alpha_i(a)e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} + X_j^{(0)}\frac{e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} \end{aligned}$$

indeed the function $u \to h_u$ is continuous in the sense of uniform convergence, so

$Y$ is continuous. Now, differentiating yields

$$\begin{aligned} Z_i'(a) &= \int_{u=-Q}^{0} \frac{2\alpha_i(a)\alpha_i'(a)e^{-\alpha_i(a)u}}{w_{[j]}^{i,(a)}} Y_j(u)du + \int_{u=-Q}^{0} \alpha_i^2(a) \frac{d\left(\frac{e^{-\alpha_i(a)u}}{w_{[j]}^{i,(a)}}\right)}{da} Y_j(u)du \\ &+ X_j^{(1)} \frac{\alpha_i'(a)e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} + X_j^{(1)}\alpha_i(a) \frac{d\left(\frac{e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}}\right)}{da} \\ &+ X_j^{(0)} Q \frac{\alpha_i'(a)e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} + X_j^{(0)} Q\alpha_i(a) \frac{d\left(\frac{e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}}\right)}{da} \\ &+ X_j^{(0)} \left(-\alpha_i'(a)Q\right) \frac{e^{-\alpha_i(a)Q}}{w_{[j]}^{i,(a)}} + X_j^{(0)} \frac{-e^{-\alpha_i(a)Q} w'^{i,(a)}_{[j]}}{\left(w_{[j]}^{i,(a)}\right)^2} \end{aligned}$$

so for $a = 0$, one recalls that $\alpha_i(0) = 0$, $\alpha_i'(0) = -1/E_i$ and $w^{i,(0)} = \left(\vec{1}\right)$ to get

$$Z_{i,j}'(0) = X_j^{(1)} \frac{-1}{E_i} + X_j^{(0)} Q \frac{-1}{E_i} + X_j^{(0)} \left(\frac{1}{E_i} Q\right) - X_j^{(0)} w'^{i,(0)}_{[j]}$$

The sum over all entries yields

$$Z_i'(0) = \sum_{j\in A'} \left(\frac{X_j^{(1)}}{-E_i} - X_j^{(0)} w'^{i,(0)}_{[j]}\right)$$

However, we also have

$$-c_i = \lim_{C(0)\to\infty} \left(\mathbf{E}\left(f_{[\vec{M}(T')]}\left(\vec{C}_i(T')\right)\right)\right)$$

where each $f_j$ is the linear combination of $-1/E_i$ times the identity function and $-w'^{i,(0)}_{[j]}$ times the unit function by definition. It follows that

$$-c_i = F\left((f_j)_{j\in A'}\right) = \sum_{j\in A'} X_j^{(1)} \frac{-1}{E_i} + X_j^{(0)} \left(-w'^{i,(0)}_{[j]}\right)$$

so we ultimately get $-c_i = Z_i'(0)$, which ends the proof.

### 3.7.2 Linearity properties

In this paragraph, we prove the properties of asymptotic linearity for cumulated transition payoffs. We shall use some of the notations from the first study, particularly about concatenated transition payoffs like $D_{i\xrightarrow{n}j}$.

**Proposition 3.4.3**

We present a proof for the existence of asymptotical expectancy offsets, that works only if $C$'s Markovian process $M$ is aperiodic. However, if $M$ has a period $q > 1$, then as $C$ positive recurrent, there is $n_0 \in \mathbf{N}$ such that $P_{i \xrightarrow{n} j}$ is zero whenever $n \notin n_0 + q\mathbf{N}$. It follows that we may consider only the values of $n$ in $n_0 + q\mathbf{N}$, i.e. the process whose values are $C(n_0 + qt)$ for $t \in \mathbf{N}$, whose underlying Markovian process $M'$ at time $t$ is on $M'(t) = M(n_0 + qt)$ :

— $M'$ is an aperiodic Markovian process ;

— $M'$'s starting state is positive recurrent by hypothesis, and any non-trivial cycle of $M$ going from $M'(0)$ to $M'(0)$ must have a length $T \in q\mathbf{N}^*$, so defines a cycle for $M'$. Hence, $M'$ is still positive recurrent once restricted to the states that it may hit.

So, one might assume that $M$ is aperiodic after all : as it is positive recurrent, we shall say that it is ergodic. We shall use the characterization of $C$'s $n$-periodically concatenated process $C_{T_n}$ given during the other study.

Let us start with $n \in \mathbf{N}^*$ and $C_{T_n}$'s Laplace matrix function around zero : for every $i, j \leq A$, for $\alpha \in I$ an opened interval around 0,

$$P_{i \xrightarrow{n} j} \mathbf{E}\left(e^{-\alpha D_{i \xrightarrow{n} j}}\right) = \left(L_C(\alpha)^n\right)_{i,j}$$

Local differentiation at $\alpha = 0$ yields

$$P_{i \xrightarrow{n} j} \mathbf{E}\left(-D_{i \xrightarrow{n} j}\right) = \left(\frac{d\left(L_C(\alpha)^n\right)(w)}{d\alpha}(\alpha = 0)\right)_{i,j}$$

Using the formula for the derivative of an integer power of a matrix, this forms the matrix

$$\sum_{u=1}^{n} L_C(0)^{u-1} \left(\frac{d\left(L_C(\alpha)\right)(w)}{d\alpha}(\alpha = 0)\right) L_C(0)^{n-u}$$

At this point, as periodic time concatenation of an ergodic $M$ does not modify $M$'s invariant measure $\mu$, $C_{T_n}$'s mean expectancy is

$$E(C_{T_n}) = \mu \sum_{u=1}^{n} L_C(0)^{u-1} R_C(0) L_C(0)^{n-u} \left(\vec{1}\right)$$

However, $L_C(0)$ is $M$'s transition matrix $P$ by construction. Since $\mu$ is a row eigenvector for $P$, and $\left(\vec{1}\right)$ is a column eigenvector for $P$, then

$$E(C_{T_n}) = \mu \sum_{u=1}^{n} R_C(0) \left(\vec{1}\right) = n\mu R_C(0) \left(\vec{1}\right)$$

One recognizes the expression of $E(C)$, so $E(C_{T_n}) = nE(C)$. As $P$ is a stochastic matrix deemed positive recurrent and aperiodic, then Perron-Froebenius' theorem states that there are :

- A change-of-basis matrix $Q$, whose first column is the eigenvector $\left(\vec{1}\right)$, such that the first row of $Q^{-1}$ is the row eigenvector $\mu$ ;
- A matrix $\Delta$, with
  - $\Delta_{1,1} = 1$ ;
  - For every $i \neq 1$, $\Delta_{i,1} = \Delta_{1,i} = 0$ ;
  - For $\Delta'$ the sub-matrix of rows and columns 2 to $A$ of $\Delta$, the greatest absolute eigenvalue of $\Delta'$, i.e. the second greatest absolute eigenvalue of $\Delta$ is less than $\lambda < 1$,

such that $P = Q\Delta Q^{-1}$. Hence,

$$\left(P_{i \xrightarrow{n} j} \mathbf{E}\left(-D_{i \xrightarrow{n} j}\right)\right)_{i,j} = \sum_{u=1}^{n} Q\Delta^{u-1}Q^{-1}R_C(0)Q\Delta^{n-u}Q^{-1}$$

Noting $S = Q^{-1}\left(P_{i\to j}\mathbf{E}\left(D_{i\to j}\right)\right)_{i,j} Q$, this rewrites as

$$\left(P_{i \xrightarrow{n} j} \mathbf{E}\left(D_{i \xrightarrow{n} j}\right)\right)_{i,j} = Q\left(\sum_{u=1}^{n} \Delta^{u-1}S\Delta^{n-u}\right)Q^{-1}$$

However, as $\Delta$ is a diagonal block matrix constituted of a 1 and $\Delta'$ whose eigenvalues are no wider than $\lambda < 1$, then the sum develops as the sum of

1. The term $S_{1,1}$ at place $(1,1)$, $n$ times ;
2. The first row of $S$, multiplied by the diagonal block matrix constituted of a 0 and $\sum_{u=1}^{n} \Delta'^{n-u}$ ;
3. The diagonal block matrix constituted of a 0 and $\sum_{u=1}^{n} \Delta'^{u-1}$, multiplied by the first column of $S$ ;
4. A term whose order of magnitude is no larger than

$$\sum_{u=1}^{n} \lambda^{n-u}\lambda^{u-1} = n\lambda^{n-1}$$

After multiplication by $Q$ and $Q^{-1}$, these terms form matrices, respectively :

1. Whose entry number $(i,j)$ is $n\mu_{[j]}S_{1,1}$ ;
2. Whose rows are identical ;
3. Whose rows are proportional to $\mu$ ;
4. Whose order of magnitude is $O\left(n\lambda^{n-1}\right)$.

However, we recall that $M$ is ergodic, so the terms $P_{i \xrightarrow{n} j}$ converge exponentially, being $\mu_{[j]}\,(1 + o(\lambda^n))$ by definition of $\lambda$ with $\mu_{[j]} > 0$. So finally, division of the entry number $(i, j)$ by $P_{i \xrightarrow{n} j}$ yields a matrix expressing the expectancies of concatenated transition payoffs, sum of matrices :

1. Whose entries are all $nS_{1,1} + o(n\lambda^n)$ ;
2. Whose rows are identical with an approximation $o(\lambda^n)$ ;
3. Whose columns are identical with an approximation $o(\lambda^n)$ ;
4. Whose order of magnitude is $O\left(n\lambda^{n-1}\right)$.

Replacing the expressions like $\sum_{u=1}^{n} \Delta'^{u-1}$ (obtained by multiplying the dominant rows and columns of $S$ above) by the constant term $\sum_{u=0}^{\infty} \Delta'^{u}$ yields an error of magnitude $o(\lambda^n)$ because $\lambda < 1$, so finally we get

— A constant term $X$ ;
— A constant row $H$ ;
— A constant column $V$ ;
— A residual error matrix $Z(n)$ in $O\left(n\lambda^{n-1}\right)$ ;

such that, for every $i, j \leq A$, $n \in \mathbf{N}$,

$$\mathbf{E}\left(D_{i \xrightarrow{n} j}\right) = nX + H_{[j]} + V_{[i]} + Z_{i,j}(n)$$

As the only requirement for $\lambda$ was to be greater than $P$'s second eigenvalue, then one may take any intermediate $\lambda'$ between it and $\lambda$ to express $Z_{i,j}(n)$ as $o\left(\lambda'^n\right)$. We may now express the vectors $E_{\infty\rightarrow}$ and $E_{\rightarrow\infty}$ using $X$, $H$ and $V$.

— We verify that $X$ is $C$'s mean expectancy. As $C_{T_n}$'s mean expectancy is $nE(C)$, then we have

$$nE(C) = nX\mu\left(\vec{1}\right) + \mu\left(\vec{1}\right)H\left(\vec{1}\right) + \mu V\left(\vec{1}\right)^*\left(\vec{1}\right) + \mu Z(n)\left(\vec{1}\right)$$

where we noted $\left(\vec{1}\right)$ the vertical column of ones and $\left(\vec{1}\right)^*$ its transposition, the horizontal row of ones, as to avoid confusion. As $\mu\left(\vec{1}\right) = 1$, then all that remains is

$$n(E(C) - X) = H\left(\vec{1}\right) + A\mu V + \mu Z(n)\left(\vec{1}\right)$$

Since $H\left(\vec{1}\right) + A\mu V$ is a constant of $n$ and $\mu Z(n)\left(\vec{1}\right) = o\left(\lambda^n\right)$, this is possible only if $n(E(C) - X)$ converges, so $E(C) = X$.

— This means that $H\left(\vec{1}\right) + A\mu V = 0$, so

$$\frac{1}{A}\sum_{j=1}^{A} H_{[j]} + \sum_{i=1}^{A} \mu_{[i]} V_{[i]} = 0$$

Hence, setting $x = \sum_{i=1}^{A} \mu_{[i]} V_{[i]}$, the vectors given by

$$\forall i, j \leq A, E_{i \to \infty} = V_{[i]} - x \wedge E_{\infty \to j} = H_{[j]} + x$$

still satisfy the equality, and hold the additional constraints

$$\frac{1}{A} \sum_{j=1}^{A} E_{\infty \to j} = \sum_{i=1}^{A} \mu_{[i]} E_{i \to \infty} = 0$$

Finally, we need to prove that the vectors $E_{\infty \to}$ and $E_{\to \infty}$ are unique. Considering only the condition on expectancies of $D_{i \xrightarrow{n} j}$, if one sets arbitrarily the value of e.g. $E_{\infty \to 1}$, then we get successively all other values following this line of thought :

1. Since $M$ is ergodic, there is $n_0 \in \mathbf{N}$ such that for every $i, j \leq A$ and $n \geq n_0$, one has $P_{i \xrightarrow{n} j} > 0$. Let us name

   $$d_{i \xrightarrow{n} j} = \mathbf{E}\left(D_{i \xrightarrow{n} j}\right) - nE$$

   The sequence $\left(d_{i \xrightarrow{n} j}\right)_n$ converges to some value $d_{i,j}$ when $n$ goes to infinity, because we proved that the sequence of error matrices $(Z(n))_{n \in \mathbf{N}}$ converges to 0. By construction of $E_{\infty \to}$ and $E_{\to \infty}$, they must hold

   $$\forall i, j \leq A, d_{i,j} = E_{\infty \to j} + E_{i \to \infty}$$

2. Setting $E_{\infty \to 1} = x \in \mathbf{R}$ leads to the values $E_{i \to \infty}$ through $d_{i,1}$ by

   $$E_{i \to \infty} = d_{i,1} - E_{\infty \to 1} = d_{i,1} - x$$

3. This leads to all other values $E_{\infty \to j}$ using

   $$E_{\infty \to j} = d_{i,j} - E_{i \to \infty} = d_{i,j} - d_{i,1} + x$$

Hence, the only possible degree of freedom for $E_{\infty \to}$ and $E_{\to \infty}$ is an additive uniform and simultaneous shift of $x$ for all of their coordinates. However, the condition imposing these vectors to be centered yields a unique possible value for $x$, which ends the proof.

### Proposition 3.4.4

To characterize of the asymptotical expectancy offsets, we are going to prove that they hold the vector equations in the proposition 3.4.2, after what uniqueness properties will ensue from the equations of scaling. Let us start with the proposition 3.4.3, applied for a number $n+1 \in \mathbf{N}^*$ sufficiently large of concatenated transitions : for every $i, j \leq A$,

$$\mathbf{E}\left(D_{i \xrightarrow{n+1} j}\right) = (n+1)E(C) + E_{\infty \to j} + E_{i \to \infty} + o(\lambda^n)$$

By construction of the concatenated transition payoffs, and disjunction on the first step out of $n+1$, this expectancy rewrites as

$$\mathbf{E}\left(D_{i \xrightarrow{n+1} j}\right) = \sum_{k=1}^{A} \left(\frac{P_{i \to k} P_{k \xrightarrow{n} j}}{P_{i \xrightarrow{n+1} j}}\right) \left(\mathbf{E}\left(D_{i \to k} + D_{k \xrightarrow{n} j}\right)\right)$$

We know that $P_{i \xrightarrow{n} j} = \mu_{[j]} + o(\lambda^n)$, and that no $\mu_{[j]}$ is zero, so this comes to

$$\begin{aligned} & (n+1)E(C) + E_{\infty \to j} + E_{i \to \infty} + o(\lambda^n) \\ = & \sum_{k=1}^{A} \left(P_{i \to k} + o(\lambda^n)\right) \left(\mathbf{E}\left(D_{i \to k} + nE(C) + E_{\infty \to j} + E_{k \to \infty} + o(\lambda^n)\right)\right) \end{aligned}$$

which simplifies to

$$E(C) + E_{i \to \infty} = \sum_{k=1}^{A} P_{i \to k} \left(\mathbf{E}\left(D_{i \to k} + E_{k \to \infty}\right)\right) + o(n\lambda^n)$$

This is possible only if the term $o(n\lambda^n)$ is exactly zero, so we get

$$E(C) + E_{i \to \infty} = \sum_{k=1}^{A} \left(R_C(0)\right)_{i,k} + \sum_{k=1}^{A} P_{i \to k} E_{k \to \infty}$$

In the right-hand side, we recognize the terms number $i$ of vectors $R_C(0)\left(\vec{1}\right)$ and $PE_{\to \infty}$. We may rewrite the above equation simultaneously for all values $i \leq A$ as

$$\left(\vec{1}\right) E(C) + E_{\to \infty} = R_C(0)\left(\vec{1}\right) + PE_{\to \infty}$$

It follows that the vector $E_{\to \infty}/E(C)$ holds the column equation given in the proposition 3.4.2 ; thanks to the scaling constraint $\mu E_{\to \infty} = 0$, we eventually get that it must be $w'^{(0)}$. Likewise, we get a similar result for $E_{\infty \to}$, using decomposition over the last step rather than the first and the identity $\mu P = \mu$, eventually leading to

$$\forall j \leq A, \sum_{k=1}^{A} \left(\frac{\mu_{[k]} P_{k \to j}}{\mu_{[j]}} E_{\infty \to k} + \frac{\mu_{[k]}}{\mu_{[j]}} \left(R_C(0)\right)_{k,j}\right) = E + E_{\infty \to j}$$

Noting by $\Delta$ the diagonal matrix containing $\mu$, this eventually yields

$$E_{\infty \to} \Delta \left(Id - P\right) = \mu R_C(0) - E\mu$$

after computations.

### 3.7.3 Proposition 3.4.6

We aim at proving the building a C-game's characteristic items. We shall split the study, depending on whether the short-term martingale parameter $\zeta(a)$ indicating the dominant risks of default is lower or higher than the long-term risks $\theta_i(a)$ given by induction hypothesis on the next nodes of the game. Hence, we shall set some $a_0 \in \mathbf{R}^+$ and define

$$\theta_0 = \left( \begin{array}{ccc} \mathbf{R}^+ & \to & \mathbf{R}^+ \\ a & \to & \min_i \left(\theta_i(a)\right) \end{array} \right)$$

We shall use the decomposition of the game's default expectancy depending on whether default happens before or after buying (at time $\tau$), so

$$\mathbf{E}\left(e^{-aT}\right) = \mathbf{E}\left(e^{-aT}\mathbf{1}_{T\leq\tau}\right) + \mathbf{E}\left(e^{-aT}\mathbf{1}_{T>\tau}\right)$$

The rightmost term may be expressed conditionally to $C(\tau)$ and $M(\tau)$ thanks to the induction hypothesis : decomposing on all finishing states $F$, there are some values $Z_i(a) \in \mathbf{R}^+$ such that it amounts to

$$\sum_{i\in F} Z_i(a)\mathbf{E}\left(e^{-\theta_i(a)C(\tau)}e^{-a\tau}\mathbf{1}_{T>\tau}\mathbf{1}_{M(\tau)=F_i}\right)$$

We seek the dominant term in $\mathbf{E}\left(e^{-aT}\right)$.

**Case** $\zeta(a_0) < \theta_0(a_0)$

We deal with the case $\zeta(a_0) < \theta_0(a_0)$. For some finishing state $F_i$, let us look at the function $f$ defined by

$$f_i = \left( \begin{array}{ccc} \left(\mathbf{R}^+ \times \mathbf{R}^+\right) & \to & [0,1] \\ (a,\theta) & \to & \mathbf{E}\left(e^{-\theta C(\tau)}e^{-a\tau}\mathbf{1}_{T>\tau}\mathbf{1}_{M(\tau)=F_i}\right) \end{array} \right)$$

Since $C(\tau) \geq 0$ by hypothesis on $T$, it follows that $f_i$ is well-defined and infinitely differentiable over $\mathbf{R}^+ \times \mathbf{R}^+$, so that $\theta_0(a_0)$ and $a_0$ lead to a value $y = f_i\left(a_0, \theta_0(a_0)\right)$. Differentiation of $f_i$ indicates that it is non-increasing of $\theta$ and decreasing of $a$ (except when $\tau \geq T$ almost surely, but then $y = 0$ and this case is not interesting). Hence, the global implicit function theorem yields an infinitely differentiable, non-increasing function $g$ such that

$$\forall\theta \in I, f_i\left(g(\theta), \theta\right) = y$$

where $g(\theta_0(a_0)) = a_0$ and $I$ is a maximal non-trivial interval. Moreover, as $f_i\left(a, \theta\right)$ may be made arbitrarily small for $\theta = 0$ and large values of $a$, it follows that there must be $a_0^* \in (a_0, \infty)$ such that $f_i\left(a_0^*, 0\right) = y$. In particular, one must have $0 \in I$. Now we consider $\zeta$ the martingale parameter. As we know,

— $\theta_0(a_0) > \zeta(a_0)$, so $a_0 = g(\theta_0(a_0)) \leq g(\zeta(a_0))$ ;
— $\zeta(a_0^*) > 0$, so either $\zeta(a_0^*) \leq \theta_0(a_0)$, so it belongs to $g$'s domain and $g(\zeta(a_0^*)) \leq g(0) = a_0^*$, or $\zeta(a_0^*) > \theta_0(a_0)$ and thanks to the intermediate value theorem there is $a_1^* \in (a_0, a_0^*)$ such that $\zeta(a_1^*) = \theta_0(a_0)$ and so $g(\zeta(a_1^*)) = g(\theta_0(a_0)) = a_0 < a_1^*$. In both cases there is $a^* > a_0$ such that $g(\zeta(a^*)) \leq a^*$.

The intermediate value theorem indicates that there must be $x \in (a_0, a_0^*)$ such that $g(\zeta(x)) = x$. Using this $x$, we get (since $f_i$ is constant when following $g$)

$$f_i\left(a_0, \theta_0(a_0)\right) = f_i\left(g(\zeta(x)), \zeta(x)\right)$$

which is by definition

$$\mathbf{E}\left(e^{-\zeta(x)C(\tau)}e^{-x\tau}\mathbf{1}_{T>\tau}\mathbf{1}_{M(\tau)=F_i}\right)$$

Thanks to the martingale equation, and provided that $\mathbf{P}\left(M(\tau) = F_i\right) > 0$, there is a constant $k_i$ such that this is bounded by

$$k_i e^{-\zeta(x)C(0)} = o\left(e^{-\zeta(a_0)C(0)}\right)$$

because $x > a_0$. Adding this for all finishing states $F_i$ indicates that

$$\mathbf{E}\left(e^{-\theta_0(a_0)C(\tau)}e^{-a\tau}\mathbf{1}_{T>\tau}\right) = o\left(e^{-\zeta(a_0)C(0)}\right)$$

It follows that the dominant term in $\mathbf{E}\left(e^{-aT}\right)$ is the short-term default

$$\mathbf{E}\left(e^{-aT}\mathbf{1}_{T\leq\tau}\right) \equiv w^{(a)}_{[M(0)]}e^{-\zeta(a)C(0)}$$

where $w^{(a)}$ is the temporary C-process' dominant eigenvector.

**Case** $\zeta(a_0) \geq \theta_0(a_0)$

Let us name by $i$ an index of some investment opportunity leading to a value $\theta_i(a_0) \leq \zeta(a)$. As above, we compute its contribution to the default expectancy as

$$\mathbf{E}\left(e^{-\theta_i(a_0)C(\tau)}e^{-a_0\tau}\mathbf{1}_{T>\tau}\mathbf{1}_{M(\tau)=F_i}\right)$$

Actually, all we want to prove is that there is a boundary $Q \in \mathbf{R}^+$ such that all these terms $Q_i$ are well-defined :

$$Q_i = \lim_{C(0)\to\infty}\left(\mathbf{E}\left(e^{-\theta_i(a_0)(C(\tau)-C(0))}e^{-a_0\tau}\mathbf{1}_{T>\tau}\mathbf{1}_{M(\tau)=F_i}\right)\right)$$

as if we succeed, then the sum of long-term default expectancies will be controlled as

$$\sum_{i\in F} e^{-\theta_i(a_0)C(0)}Q_i = (Q + o(1))e^{-\theta_0(a_0)C(0)}$$

for some $Q \in \mathbf{R}^+$. We know that

— $\tau$'s distribution does not change with the starting point $C(0)$, so we may call by $D$ a random variable whose distribution is $C(\tau) - C(0)$'s one for every $C(0)$ ;
— The random variable $\mathbf{1}_{T>\tau}$ is non-decreasing of $C(0)$ ;
— $\tau$ is finite almost surely ;
— All probabilities $\mathbf{P}(T > n)$ for every $n \in \mathbf{N}$ go to 1 as $C(0)$ increases,

then by monotone convergence of $\mathbf{1}_{T>\tau}$ to 1 we get

$$Q_i = \left(\mathbf{E}\left(e^{-\theta_i(a_0)D} e^{-a_0\tau} \mathbf{1}_{M(\tau)=F_i}\right)\right)$$

To prove that $Q_i < \infty$, we make use of Jensen's inequality stating that for $A$ and $B$ non-negative random variables and $f$ a concave non-negative function,

$$\frac{\mathbf{E}\left(f(A)B\right)}{\mathbf{E}(B)} \leq f\left(\frac{\mathbf{E}(AB)}{\mathbf{E}(B)}\right)$$

where here we set the random variables

$$A = e^{-\zeta(a_0)D} \wedge B = e^{-a_0\tau} \mathbf{1}_{M(\tau)=F_i}$$

and $f$ the function defined by $\forall x \in \mathbf{R}^+, f(x) = x^{\theta_i(a_0)/\zeta(a_0)}$ ; $f$ is concave by hypothesis on $\theta_0(a_0)$. It follows that $\mathbf{E}\left(f(A)B\right)$ is the sought term, so it is no higher than

$$\mathbf{E}\left(e^{-a_0\tau} \mathbf{1}_{M(\tau)=F_i}\right) \left(\frac{\mathbf{E}\left(e^{-\zeta_0(a_0)D} e^{-a_0\tau} \mathbf{1}_{M(\tau)=F_i}\right)}{\mathbf{E}\left(e^{-a_0\tau} \mathbf{1}_{M(\tau)=F_i}\right)}\right)^{\theta_0(a_0)/\zeta(a_0)}$$

Thanks to the martingale equation, the exponential expectancy of $D$ is bounded by a constant $R_i$(whose value is incidentally $w^{(a_0)}_{[M(0)]}/w^{(a_0)}_{[F_i]}$, controlled by $C$'s spread), so the above simplifies to

$$\mathbf{E}\left(e^{-a\tau}\right)^{\left(1-\frac{\theta_0(a_0)}{\zeta(a_0)}\right)} R_i^{\theta_0(a_0)/\zeta(a_0)}$$

so is bounded, which indicates that $Q_i < \infty$. It follows that for $\zeta(a_0) \geq \theta_0(a_0)$, the contributions to the default expectancy are

— Short-term default : an exponential term in $\zeta(a_0)$, to be counted towards the dominant term only if $\zeta(a_0) = \theta_0(a_0)$ ;
— Long-term defaults for $\theta_i(a_0)$ holding $\theta_i(a_0) > \zeta(a_0)$, having a negligeable effect with respect to $\zeta(a_0)$ as in the previous paragraph ;
— Long-term defaults for $\theta_i(a_0)$ holding $\zeta(a_0) \geq \theta_i(a_0) > \theta_0(a_0)$, having a negligeable effect with respect to $\theta_0(a_0)$ as said here ;
— Long-term defaults for $\theta_i(a_0) = \theta_0(a_0)$, whose multiplicative constants $Q_i$ add up.

The sum of relevant constants yield the desired result.

## 3.8 Repeated node model

In this paragraph, we apply the generic construction above to the case of the repeated node model.

### 3.8.1 Proposition 3.5.2

We aim at finding the best threshold decision. In particular, we will need the proposition 3.5.1 to do it.

#### Non-threshold buying decision

Before we start, we indicate why the fundamental approximation is necessary to avoid boundary effects around $C \approx 0$. Let us build the C-game defined by a high enough tree, each non-leaf node containing a temporary C-process consisting in two states $A_0$ (starting) and $A_1$ (finishing). The underlying Markovian process is trivial with $P_{A_0 \to A_1} = 1$, this transition $(0 \to 1)$ yielding a random payoff amounting to

— $-1.002$ with probability $1/3$ ;

— $+1$ with probability $2/3$.

A rejected buying decision leads to a similar node (up to the final leaf after an arbitrarily high number od nodes, where a Lévy process is hit with this same distribution permanently) ; an accepted buying decision costs $I = 1$ and leads to a leaf, governed by a Lévy process whose increments are

— $-1$ with probability $1/3$ ;

— $+1$ with probability $2/3$.

The player wants to minimize the default probability. To help the computations, we describe $C$'s trajectories by independent and identically distributed sequences of signs $+$ (with probability $2/3$) and $-$ (with probability $1/3$) ; $C$ goes up or down according to the sign drawn and whether or not investment was purchased. For every $t \in \mathbf{N}$, let us name $x(t)$ the number of signs $-$ up to time $t$, and $y(t)$ the number of signs $+$ up to time $t$.

— For $C(t) < 2$ before choosing, one should wait. Indeed if the next increment is $-$ the buyer defaults (while never buying may survive if lucky enough, as the mean expectancy is positive), and if it is $+$ it has made no difference to delay the purchase.

— For $C(t) = 2.001$, the strategy of buying immediately fails iff there is a time $t \in \mathbf{N}$ such that $x(t) \geq y(t) + 2$, the first one being called $T$. We shall also call $U$ the time of the first $-$ in the sequence. Looking at such a trajectory for $C$, we observe that the player was doomed regardless of the buying time $\tau$ :

    - If $\tau < U$ : $C$ cannot default before $\tau$ and the successive values of $C(t)$ for $t \geq \tau$ are identical to the values after an immediate purchase, so default occurs at time $T$.
    - If $U \leq \tau$ : $C$ cannot default before $U$, the successive values of $C(t)$ for $t \geq \tau$ are still dominated by the values after an immediate purchase, and the value of $C(t)$ for $U \leq t < \tau$ is upper bounded by $1.999 - x(t) + y(t)$ since $x(t) \geq 1$. As a consequence, if $T$ holds $U \leq T < \tau$ then the equation $x(T) \geq y(T) + 2$ indicates that $C(T) < 0$.

  It follows that immediate purchase is optimal, while strict optimality comes from the repetition of the sequence $(-+)$ for 1001 times before going to $\infty$ with positive probability. Indeed, as it is not optimal to buy for $C(t) < 2$, the player undergoing this sequence after refusing the opportunity the first time will still reject it until finally going bankrupt at time $t = 2001$.
- For $C(t) = 2.999$, let us compare the strategies $S_0$ "buy immediately" vs. $S_3$ "buy once $C$ crosses 3. To help us, let us introduce a $\pm 1$ random walk $W$ on integers (with 2/3 probability of going +1) to express the default probabilities
    - $S_0$ defaults iff there is a time $t \in \mathbf{N}$ such that $x(t) \geq y(t) + 2$ : we say that "$W$ hits 0 starting from 2".
    - For $S_3$ to default, it is necessary that either :
        - $x$ hits 500 before $C$ crosses either 0 or 3 (so that $C$'s integer part is damaged by $x$), which is tantamount to "$W$ stays between 0 and 2 starting from 2 for 499 time periods",
        - Or $C$ crosses 3 before $x$ hits 500 and "$W$ hits 0 starting from 3"

  The probabilities $p(n)$ of $W$ hitting 0 starting from $n \in \mathbf{N}$ are given by the recursion equation

$$p(n) = \frac{1}{3}p(n-1) + \frac{2}{3}p(n+1)$$

  leading to $p(n) = 2^{-n}$, so in particular $p(2) = .25$ and $p(3) = .125$. Moreover, we know that $W$ has a probability at least 1/3 of exiting the interval between 0 and 2 each time it hits either 0 or 2, so every second time period : the probability of $W$ staying between 0 and 2 starting from 2 for 499 time periods is thus bounded by

$$\left(\frac{2}{3}\right)^{\left\lfloor \frac{499}{2} \right\rfloor}$$

  and by addition to $p(3)$, it follows that $S_3$'s the default probability is strictly lower than $S_0$'s, so waiting is optimal.
- Finally, as the fundamental approximation holds for $C(0)$ large enough, there is $c > 3$ such that buying is optimal when starting from $c$.

Hence, we proved that buying immediately is

— Not optimal for $C(0) = .999$ ;
— Optimal for $C(0) = 2.001$ ;
— Not optimal for $C(0) = 2.999$ ;
— Optimal for $C(0) = c > 3$,

so the "buying zone" is not an interval. Notice that we took a periodic C-process in this example, but as there are arbitrarily small irrational numbers in $\mathbf{R}_+^*$ we can slightly modify the transition payoffs so that it is not a concern. We also deemed that the tree was arbitrarily high, but for the purposes of this proof taking a height of 9999 is sufficient.

**Proposition 3.5.1**

We aim at obtaining the shapes of the Laplace transforms in the repeated node model. The core of this proof lies in the fact that as only times $t$ when $M^{[\ldots]}(t) = F$ are investigated, we are actually driven back to the case of a restricted Lévy process (on the state $F$) in the terms of the study on C-processes. As a consequence, the "classical" Cramér-Lundberg approximation indicates the default expectancy of the process $b - C$ thanks to the negative martingale parameter, yielding $Y_0(a)$.

As for $Y_i(a)$, we shall consider the C-process $B = b - C$ and use the trick of time concatenations described in the theory of C-processes, defining the binary determination sequence $\rho$ by $\rho(t) = 1$ iff

$$M^{[\ldots]}(t) = F \wedge \forall s < t, \left(M^{[\ldots]}(s) \neq F \vee B(s) > B(t)\right)$$

i.e. following the names of the other work, we eventually got the "descending Lévy restricted process" called $\vec{C}$, whose default time is $\sigma$. We introduce the function $f$ defined by

$$f = \left( \begin{array}{ccc} \mathbf{R} & \rightarrow & \mathbf{R}^+ \\ x & \rightarrow & \mathbf{E}\left(e^{\alpha_i(a)\vec{C}(\sigma)} | \vec{C}(0) = x \wedge M^{[\ldots]}(0) = F\right) \end{array} \right)$$

The function $f$ is exponential and non-decreasing over $\mathbf{R}_-^*$ by definition, so is half-Lipschitz over $\mathbf{R}^-$ ; as $C$ was deemed aperiodic, we know that it means that $f$ is bounded and converges to a constant towards $\infty$. Now, let us call $D$ the random variable indicating the cumulated transition payoff between $t = 0$ and the first hitting time of $F$, so that by definition of $f$ we have

$$Y_i(a) = \mathbf{E}\left(f\left(b - C(0) - D\right)\right)$$

As $f$ is bounded and converges, this must imply that $Y_i(a)$ converges when $b - C(0)$ goes to $\infty$, which ends the proof.

**Default expectancy in the repeated node model**

We aim at computing the player's default expectancy provided that the buying time is hit. To do this, we shall decompose $T_0$ thanks to the buying time $\tau$ as $T_0 = \tau + T_1$ as explained.

**Lemma 3.8.1** *Default expectancy after buying*

*We define the multiplicative parameters $Y_0(a)$, $Y_i(a)$ and $Z_i(a)$ as given by the proposition 3.5.1 for $i \in \{1,2\}$. Under the natural and reverse fundamental approximations,*

1. *When the player does not buy,*

$$\boldsymbol{E}\left(e^{-aT_0}\boldsymbol{1}_A|C(0)=x\right) \equiv Y_0(a)Y_1(a)Z_1(a)e^{-\omega(a)(b-x)}e^{-\alpha_1(a)b}$$

2. *When the player buys accordingly to the strategy,*

$$\boldsymbol{E}\left(e^{-aT_0}\boldsymbol{1}_A|C(0)=x\right) \equiv Y_0(a)Y_2(a)Z_2(a)e^{-\omega(a)(b-x)}e^{-\alpha_2(a)(b-I)}$$

*when both $x$ and $b-x$ go to $\infty$.*

First, the game is Markovian and homogenous, and fully described by $M^{[\ldots]}$ and $C$ at present time, since the effect of $G$ is neglected as said before. It follows that decomposition with respect to the situation at time $\tau$ yields conditional independence of the past (random couple $(\tau, \mathbf{1}_A)$) and the future ($T_1$) given the present (random couple $(C(\tau), M(\tau))$) ; as $M(\tau) = F$ by definition, we get

$$\mathbf{E}\left(e^{-aT_0}\mathbf{1}_A\right) = \mathbf{E}\left(\mathbf{E}\left(e^{-a\tau}\mathbf{1}_A|C(\tau)\right)\mathbf{E}\left(e^{-aT_1}|C(\tau)\right)\right)$$

The latter term is given through the characteristic parameters :

— Directly for $i = 1$ ;

— After purchasing and paying a price $I$ for $i = 2$ :

$$\mathbf{E}\left(e^{-aT_1}|C(\tau)\right) \equiv Z_i e^{-\alpha_i(a)C(\tau)}e^{\mathbf{1}_{i=2}\alpha_2(a)I}$$

As by construction $C(\tau) \geq b$, we get by integration by parts

$$\mathbf{E}\left(e^{-aT_0}\mathbf{1}_A\right) \equiv Z_i\alpha_i(a)\int_{u=b}^{\infty}\mathbf{E}\left(e^{-a\tau}\mathbf{1}_A\mathbf{1}_{C(\tau)\in[b,u]}\right)e^{-\alpha_i u}due^{\mathbf{1}_{i=2}\alpha_2(a)I}$$

Now, let us set the function $g$ to be

$$g = \begin{pmatrix} [b,\infty) & \rightarrow & \mathbf{R}^+ \\ x & \rightarrow & \mathbf{E}\left(e^{-a\tau}\mathbf{1}_A\left(x-C(\tau)\right)\mathbf{1}_{(x-C(\tau))\geq 0}\right) \end{pmatrix}$$

so that in particular, for every $x \geq b$,

$$g(x) = \int_{u=b}^{x} \mathbf{E}\left(e^{-a\tau}\mathbf{1}_A\mathbf{1}_{C(\tau)\in[b,u]}\right) du$$

which means that after a second integration by parts of the sought expression, one finally gets

$$\mathbf{E}\left(e^{-aT_0}\mathbf{1}_A\right) \equiv Z_i\left(\alpha_i(a)\right)^2 \int_{u=b}^{\infty} g(u)e^{-\alpha_i u}du e^{\mathbf{1}_{i=2}\alpha_2(a)I}$$

To simplify $g(u)$, we recall the mechanism of the fundamental approximation as above.

**Uses of the fundamental approximations**

We work with $g(u) = g_+(u) - g_-(u)$, where we set

$$g_+(u) = \mathbf{E}\left(e^{-a\tau}\left(u - C(\tau)\right)\mathbf{1}_{(u-C(\tau))\geq 0}\right)$$

Dealing with $g_-(u)$, the condition $\neg A$ indicates that $C$ must pass below 0 before hitting $b$, thus one gets thanks to the reverse fundamental approximation

$$g_-(u) = O\left(e^{-\omega(a)b}\right)$$

We also know that, using a similar scheme as in the theory of C-processes with $\vec{C}$'s convolution process $\Phi_C$ at point $a$, we eventually look for

$$g_+(u) = \mathbf{E}\left(f\left(\Phi(\tau)\right)\right)$$

where $f$ is the function defined by $f(x) = x + b$ for $x \in (-b, 0)$ and 0 elsewhere. This encourages us to "drop" the dependency in $C(\tau)$, stating that the couple of events

$$\left(C(\tau) \in [b, u], A\right)$$

is roughly independent of the random time $\tau$ provided that $C(0)$ is far enough from $b$ (this time on the lower side). $f$ being half-Lipschitz, the reverse fundamental approximation allows to replace

$$g_+(u) \equiv Y_0(a)e^{-\omega(a)(b-C(0))}\mathbf{E}\left(\left(u - C(\tau)\right)\mathbf{1}_{(u-C(\tau))\geq 0}\right)$$

It follows that when $b - C(0)$ and $C(0)$ go to $\infty$, $g_+(u)$ comes negligible with respect to $g_-(u)$ ; for this reason, we approximate

$$g(u) \equiv Y_0(a)e^{-\omega(a)(b-C(0))}\mathbf{E}\left(\left(u - C(\tau)\right)\mathbf{1}_{(u-C(\tau))\geq 0}\right)$$

which after cancellation of the integration by parts reverts to

$$\mathbf{E}\left(e^{-aT_0}\mathbf{1}_A\right) \equiv Z_i(a)Y_0(a)e^{-\omega(a)(b-C(0))}\mathbf{E}\left(e^{-\alpha_i(a)C(\tau)}\right)e^{\mathbf{1}_{i=2}\alpha_2(a)I}$$

Finally, the latter expectancy is expressed using $C$'s negative martingale parameter $\omega(a)$ alongside its multiplicative parameter $Y_i(a)$, which ends the proof of the lemma 3.8.1.

**Evaluation**

This lemma 3.8.1 now allows us to conclude the proof for the proposition 3.5.2. As we want to express $F^{(a,b)}(x)$ for $x < b$, let us rewrite it as

$$\mathbf{E}\left(e^{-aT_0}\mathbf{1}_A\right) + \mathbf{E}\left(e^{-aT_0}\left(1-\mathbf{1}_A\right)\right)$$

The first term is given by the lemma 3.8.1. To get the other one, we know that under the event $\neg A$, $T_0$ is the default time $T_0'$ of a simple C-process called $C'$, since there are no investment opportunities, whose characteristic items are $\alpha_1(a)$ and $Z_1(a)$. Thus we get

$$\mathbf{E}\left(e^{-aT_0}\left(1-\mathbf{1}_A\right)|C(0)\right) \equiv Z(a)e^{-\alpha_1(a)C(0)} - \mathbf{E}\left(e^{-aT_0'}\mathbf{1}_A\right)$$

Likewise, the default expectancy when buying is also computed thanks to the lemma 3.8.1, where this time the term that was obtained from after the purchase now still follows the C-process $C'$. Thus, it works with the exponential parameter $\alpha_1(a)$ alongside with the multiplicative parameters $Y_0(a)$, $Y_1(a)$, and $Z_1(a)$ :

$$\mathbf{E}\left(e^{-aT_0'}\mathbf{1}_A|C(0)=x\right) \equiv Y_0(a)Y_1(a)Z_1(a)e^{-\omega(a)(b-x)}e^{-\alpha_2(a)b}$$

This finally yields the approximative default expectancy

$$\begin{aligned}\mathbf{E}\left(e^{-aT_0}|C(0)=x\right) &\equiv Z_1(a)e^{-\alpha_1(a)x}\\ &+ Y_0(a)Y_2(a)Z_2(a)e^{-\omega(a)(b-x)}e^{-\alpha_2(a)(b-I)}\\ &- Y_0(a)Y_1(a)Z_1(a)e^{-\omega(a)(b-x)}e^{-\alpha_1(a)b}\end{aligned}$$

Now we find the best threshold $b$ according to this approximation. Differentiation with respect to $b$ yields the condition

$$\begin{aligned}&Y_2(a)Z_2(a)\left(\alpha_2(a)+\omega(a)\right)e^{\alpha_2(a)I}e^{\omega(a)x}e^{-(\alpha_2(a)+\omega(a))b}\\ =\ &Y_1(a)Z_1(a)\left(\alpha_1(a)+\omega(a)\right)e^{\omega(a)x}e^{-(\alpha_1(a)+\omega(a))b}\end{aligned}$$

that simplifies to

$$b = \frac{I}{\gamma(a)} + \frac{\ln\left(\frac{Y_2(a)Z_2(a)(\alpha_2(a)+\omega(a))}{Y_1(a)Z_1(a)(\alpha_1(a)+\omega(a))}\right)}{\alpha_2(a)-\alpha_1(a)}$$

Computing the corresponding default expectancy ensures that this $b$ actually yields a minimum. Finally, recalling the definitions of the incentive and handicap from the definition 3.4.2 lead to the result.

### 3.8.2 Low discrepancy

We are going to clarify why the canonical strategy given by the definition 3.5.2 has low losses with respect to optimality. For the sake of clarity, the nodes $G^{[\dots]}$ will be renamed regarding

— $n \in \mathbf{N}$, the number of accepted opportunities so far, and

— $r \in \mathbf{N}$, the number of remaining opportunities before hitting the bottom of the graph,

as the (identical) node $G^{[n,r]}$. In particular, $r = 0$ indicates that $G$ hit a leaf.

#### Bottommost default expectancies

With each node $G^{[n,r]}$, we associate a default expectancy function called $f^{[n,r]}$ indicating the player's default expectancy when using the canonical strategy, starting from this node and its starting state, as a function of his assets $C$ at starting time ; namely,

$$\forall x \in \mathbf{R}^+, f^{[n,r]}(x) = \mathbf{E}\left(e^{-aT_0} | M^G(0) = M^{[n,r]}(0) \wedge C(0) = x\right)$$

We know from the fundamental approximation that at the bottom of the tree $G$,

$$\forall n \in \mathbf{N}, f^{[n,0]}(x) \equiv Z_n(a) e^{-\alpha_n(a)x}$$

Therefore, the investment choice at the last investment time yields a default expectancy given by

$$g^{[n,1]}(x) \equiv Z_n(a) e^{-\alpha_n(a)x} \mathbf{1}_{x<b_{n+1}} + Z_{n+1}(a) e^{-\alpha_{n+1}(a)(x-I)} \mathbf{1}_{x \geq b_{n+1}}$$

The discrepancy between $g^{[n,1]}(x)$ and the actual minimum of $f^{[n,0]}(x)$ and $f^{[n,1]}(x-I)$ appears only when $x$ falls between $b_{n+1}$ and the threshold coming from the definition 3.4.2. As we justified in the study, this interval may be considered small enough so that $g^{[n,1]}(x)$ is an acceptable approximation of the optimal default expectancy before the decision. Finally, as the buying decision is taken after $M^{[n,1]}$ hits the investment opportunity, one gets a random variable $D$ indicating the cumulated transition payoffs between entering time into $G^{[n,1]}$ and exiting time out of it. In particular, we will be interested in the value

$$X_{n,k}(a) = \mathbf{E}\left(e^{-a\tau} e^{-\alpha_{n+k}(a)D}\right)$$

when $D$ follows the increments of $C_n$, for $n, k \in \mathbf{N}$. Using Jensen's inequality, we know that it is higher than

$$\mathbf{E}\left(e^{-a\tau}\right) \left(\frac{\mathbf{E}\left(e^{-a\tau} e^{-\alpha_{n+k-1}(a)(C(\tau)-I)}\right)}{\mathbf{E}\left(e^{-a\tau}\right)}\right)^{\left(\frac{\alpha_{n+k}(a)}{\alpha_{n+k-1}(a)}\right)}$$

so that we get

$$X_{n,k}(a) \geq \mathbf{E}\left(e^{-a\tau}\right)^{1-\left(\frac{\alpha_{n+k}(a)}{\alpha_{n+k-1}(a)}\right)} X_{n,k-1}(a)^{\left(\frac{\alpha_{n+k}(a)}{\alpha_{n+k-1}(a)}\right)}$$

As $\alpha_{n+k}(a) \geq \alpha_{n+k-1}(a)$, $\tau \geq 0$, and $X_{n,0}(a) = 1$ thanks to the martingale property, it follows that the sequence $(X_{n,k}(a))_{k\in\mathbf{N}}$ is non-decreasing.

### Shape of the default expectancy

The default expectancy may thus be approximated as follows :

— If $x < b_{n+1}$, the local shape of $g^{[n,1]}$ is the lower exponential function. We neglect that the curve breaks at point $b_{n+1}$ : actually, we assume that $D$ is small enough so that the event $x + D \geq b_{n+1}$ is rare enough for low values of $x$ ; and besides, the discrepancy between $g^{[n,1]}$'s exponential shapes is minimal close to $b_{n+1}$ by definition. We are thus encouraged to find

$$\mathbf{E}\left(e^{-a\tau}e^{-\alpha_n(a)C(\tau)}\right)$$

for $\tau$ the hitting time of the investment opportunity. The martingale property thus indicates that this is

$$\frac{w^{(a)}_{n,[M(0)]}}{w^{(a)}_{n,[F]}}e^{-\alpha_n(a)x}$$

where $w^{(a)}_n$ is $C_n$'s dominant eigenvector at point $a$. However, as $M(0) = F$ by construction, the piece of $f^{[n,1]}$ below $b_{n+1}$ simplifies to

$$f^{[n,1]}(x) \equiv Z_n(a)e^{-\alpha_n(a)x}$$

— Likewise, if $x \geq b_{n+1}$ and $\alpha_{n+1}(a) \leq \zeta_n(a)$ (the early exponential default parameter), we want to find

$$\mathbf{E}\left(e^{-a\tau}e^{-\alpha_{n+1}(a)(C(\tau)-I)}\right)$$

As stated above, this term is actually $X_{n,1}(a)$, so the piece of $f^{[n,1]}$ above $b_{n+1}$ simplifies to

$$f^{[n,1]}(x) \equiv Z_{n+1}(a)X_{n+1}(a)e^{-\alpha_{n+1}(a)(x-I)}$$

— Finally, if $x \geq b_{n+1}$ and $\alpha_{n+1}(a) > \zeta_n(a)$, the piece of $f^{[n,1]}$ above $b_{n+1}$ is given by the early default parameter as

$$f^{[n,1]}(x) \equiv Y_n(a)e^{-\zeta_n(a)}$$

for some $Y_n(a) \in \mathbf{R}^+$.

Therefore, we approximate the default expectancy when entering $G^{[n,1]}$ by either

$$f^{[n,1]}(x) \equiv \mathbf{1}_{x<b_{n+1}} Z_n(a) e^{-\alpha_n(a)x} + \mathbf{1}_{x \geq b_{n+1}} Z_{n+1}(a) X_{n+1}(a) e^{-\alpha_{n+1}(a)(x-I)}$$

if $\alpha_{n+1}(a) \leq \zeta_n(a)$, and

$$f^{[n,1]}(x) \equiv \mathbf{1}_{x<b_{n+1}} Z_n(a) e^{-\alpha_n(a)x} + \mathbf{1}_{x \geq b_{n+1}} Y_n(a) e^{-\zeta_n(a)}$$

otherwise.

### Comparison

To understand what happens to the default expectancy when we proceed with the recursion, we shall investigate its next step $r = 2$. If the player has a wealth $x$ when he is about to exit $G^{[n,2]}$, he chooses between $f^{[n,1]}(x)$ and $f^{[n+1,1]}(x - I)$ accordingly to $b_{n+1}$. Therefore, we have the following cases depending on $x$ and the thresholds $b_{n+1}$ and $b_{n+2}$.

— If $x < b_{n+1}$, the choice is between

$$f^{[n,1]}(x) = Z_n(a) e^{-\alpha_n(a)x}$$

if declining, and

$$f^{[n+1,1]}(x - I) = Z_{n+1}(a) e^{-\alpha_{n+1}(a)(x-I)}$$

if accepting. However, this comparison has already been solved by definition of $b_{n+1}$ : one should not buy. Thus

$$\forall x < b_{n+1}, g^{[n,2]}(x) = Z_n(a) e^{-\alpha_n(a)x}$$

— If $b_{n+1} \leq x < b_{n+2}$ and $\alpha_{n+1}(a) \leq \zeta_n(a)$, the choice is between

$$f^{[n,1]}(x) = Z_{n+1}(a) X_{n,1}(a) e^{-\alpha_{n+1}(a)(x-I)}$$

if declining, and

$$f^{[n+1,1]}(x - I) = Z_{n+1}(a) e^{-\alpha_{n+1}(a)(x-I)}$$

if accepting. As $X_{n,1}(a) \geq 1$ after Jensen's inequality, buying as early as possible is encouraged : one should buy and get

$$\forall x \in [b_{n+1}, b_{n+2}), g^{[n,2]}(x) = Z_{n+1}(a) e^{-\alpha_{n+1}(a)(x-I)}$$

— If $x \geq b_{n+2}$ and $\alpha_{n+1}(a) \leq \zeta_n(a)$, as investment was already encouraged for $x \geq b_{n+1}$ and $b_{n+2} > b_{n+1}$, it is still encouraged in this case. More specifically, even if the player decides arbitrarily not to buy while exiting $G^{[n+1,1]}$, his default expectancy should still be lower when buying now.

— Finally, in cases where $x \geq b_{n+1}$ and $\alpha_{n+1}(a) > \zeta_n(a)$ or $\alpha_n(a) > \zeta_n(a)$, the choice is between
$$f^{[n,1]}(x) = Y_n(a)e^{-\zeta_n(a)x}$$
if declining, and
$$f^{[n+1,1]}(x-I) \leq Z_{n+1}(a)e^{-\alpha_{n+1}(a)x}$$
(this is the value obtained if the player decides arbitrarily not to buy while exiting $G^{[n+1,1]}$) if accepting, so one should buy.

It so follows from the fact that the Jensen terms $X_{n,k}(a)$ are non-decreasing of $k$ that one has no incentive to postpone the purchase when granted with enough cash : indeed, buying immediately will increase immediately the martingale parameter, thus avoid an increase in $k$ and thus a term $X_{n,k}(a) > 1$ to appear in the default expectancy.

### 3.8.3 Examples

In this paragraph, we focus on the thresholds given by the ascending strategy from definition 3.5.2. Throughout this paragraph, it will be useful to assume that $\beta$ varies "smoothly" with time, so terms like $\alpha_n(a) - \alpha_{n-1}(a)$ may be viewed as
$$\beta(nI) - \beta((n-1)I) \approx I\beta'(nI).$$

**Proposition 3.5.3**

To find the threshold when the player buys drifts in the model 3.2, we shall work with the equation
$$S(B) = \frac{I}{\gamma} + H$$
where $\gamma$ indicates the incentive, and $H$ the handicap, after a total purchase of $B$. Recalling that this approximation is correct only when the martingale parameters involved in the study are bounded away from 0, we shall assume that $E_0 > 0$, so that we avoid one of the future C-processes to have a mean expectancy of $E_0 + nI$ close to 0. Let us start with the incentive
$$\gamma = \frac{\alpha_n(a) - \alpha_{n-1}(a)}{\alpha_n(a)}$$
by definition, which rewrites as
$$\gamma(B) = \frac{Ip + \sqrt{(E_0 + Bp)^2 + 2a\sigma^2} - \sqrt{(E_0 + (B-I)p)^2 + 2a\sigma^2}}{E_0 + Bp + \sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}$$

Like assumed above, first-order development with respect to $I$ small yields

$$\gamma(B) \approx \frac{Ip + Ip\frac{E_0+Bp}{\sqrt{(E_0+Bp)^2+2a\sigma^2}}}{E_0 + Bp + \sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}$$

which simplifies to

$$\gamma(B) \approx \frac{Ip}{\sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}$$

Similarly, starting from

$$H_n(a) = \frac{\ln\left(1 + \frac{\alpha_n(a)-\alpha_{n-1}(a)}{\alpha_{n-1}(a)+\omega_{n-1}(a)}\right)}{\alpha_n(a) - \alpha_{n-1}(a)}$$

as the martingale parameter is continuously differantiable of $B$ indicates that

$$H_n(a) \approx \frac{1}{\alpha_{n-1}(a) + \omega_{n-1}(a)}$$

which translates to

$$H(B) \approx \frac{1}{\beta(B) + \psi(B)} = \frac{\sigma^2}{2\sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}$$

So, the threshold is given by

$$S(B) \approx \frac{\sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}{p} + \frac{\sigma^2}{2\sqrt{(E_0 + Bp)^2 + 2a\sigma^2}}$$

As suggested in the study, we finally look for the value of $B$ such that $\beta(B+I) > \zeta(B)$. To simplify the notations, let us call

$$A = (E_0 + Bp)^2 + 2a\sigma^2$$

so that the inequation $\beta(B + I) > \zeta(B)$ translates to

$$\sqrt{I^2p^2 + 2Ip\,(E_0 + Bp) + A} > \sqrt{A + 2\rho\sigma^2} - Ip$$

All values involved in here are non-negative, so this implies

$$\sqrt{A + 2\rho\sigma^2} > \frac{\rho}{Ip}\sigma^2 - (E_0 + Bp)$$

which implies in turn

$$E_0 + Bp > \frac{\rho}{2Ip}\sigma^2 - \frac{a + \rho}{\rho}Ip$$

As the reciprocal implication holds after a similar computation, this ends the proof.

#### Exponential drift purchasing

When the player buys exponential drifts, the martingale parameter now expesses as the positive solution $\alpha$ to

$$e^{-\alpha Bp}\frac{\lambda}{\lambda-\alpha}=e^{a}$$

Hence, we may define the implicit function $\beta$ of a variable $B$ through the function $f$ defined by

$$f=\begin{pmatrix} \mathbf{R}\times(-\infty,\lambda) & \to & \mathbf{R}_{+}^{*} \\ (B,\beta) & \to & e^{-\beta Bp}\frac{\lambda}{\lambda-\beta} \end{pmatrix}$$

and the equation $f(B,\beta(B))=e^{a}$. Differentiation with respect to $B$ indicates that

$$\frac{\partial f}{\partial 1}(B,\beta(B))+\beta'(B)\frac{\partial f}{\partial 2}(B,\beta(B))=0$$

so computations yield

$$-\beta(B)p+\beta'(B)\left(-Bp+\frac{1}{\lambda-\beta(B)}\right)=0$$

which leads to

$$\frac{\beta(B)}{\beta'(B)}=\frac{1}{\left(\lambda-\beta(B)\right)p}-B$$

indicating that the incentive $\gamma(B)$ may be approximated thanks to

$$\frac{1}{\gamma(B)}\approx\frac{\beta(B)}{I\beta'(B)}=\frac{1}{\left(\lambda-\beta(B)\right)Ip}-\frac{B}{I}$$

so that by definition of $\beta(B)$,

$$\frac{1}{\gamma(B)}\approx\frac{e^{a}}{Ip}e^{\beta(B)Bp}-\frac{B}{I}$$

As $\beta(B)$ goes to $\lambda$ when $B$ increases, it follows that the incentive $\gamma(B)$ decays roughly as an exponential function of parameter $\lambda p$. Therefore, the threshold $S(B)$ is expected to behave exponentially with $B$.

## 3.9 Models of cooperative investment

This paragraph investigates on the support contracts found in the study, indicating how players may coordinate to avoid default.

### 3.9.1 Proposition 3.6.1

In this paragraph, we prove the existence of a support contract on the conditions of optimal martingale parameters. To do this, we set

- Two C-processes $C_1$ and $C_2$, as given by the proposition ;
- A family of support payoffs $s_{i\to j} \in \mathbf{R}$ ;
- Some $\epsilon << 1$, defining the support contract $S$ whose transition payoffs are $\epsilon s_{i\to j}$.

We are going to express the effect of a variation of $D_{i\to j}$ by $\epsilon s_{i\to j}$ on $\alpha_p(a_p)$ through the Laplace matrix functions $L_{C_p}$ of $C_1$ and $C_2$. This will yield a differentiated row-vector of $L_{C_p}$'s dominant eigenvalue (indexed by the couple $(i, j)$, called $z_p \in \mathbf{R}^{A^2}$, where $A$ is the cardinal of the underlying state space), so that once applied to the transition payoffs $\epsilon s_{i\to j}$, we find the marginal effect of $S$ of $\alpha_p(a_p)$. In particular, we can find a vector $v$ such that both $z_1 \cdot v > 0$ and $z_2 \cdot v > 0$ as soon as $z_1$ and $z_2$ are not colinear and of opposite signs (i.e. as they are non-zero, there is no $k < 0$ such that $z_1 = kz_2$). The latter condition eventually translates to the hypothesis given in the proof, while $v$ indicates the direction of a profitable support contract for both players.

#### Offsets support contract

Before starting the proof, we require the correctness of the statements given under the definition 3.6.2 for offsets support contracts. Let us start from the definition of $(C + S)$'s Laplace matrix function, indicating that

$$L_{C+S}(\alpha) = \Delta\left(\left(e^{\alpha(a)r_i}\right)_{i\le A}\right) L_C(\alpha)\, \Delta\left(\left(e^{\alpha(a)r_i}\right)_{i\le A}\right)^{-1}$$

As $L_{C+S}(\alpha)$ and $L_C(\alpha)$ are similar matrices, they have the same eigenvalues (thus, martingale parameters). The eigenspace is twisted by the appropriate diagonal matrix, while the multiplicative terms come from the equations of scaling : the new row eigenvector $\mu_1^{(a)}$ is now

$$\mu_1^{(a)} = \frac{\mu^{(a)}\Delta\left(\left(e^{\alpha(a)r_i}\right)_{i\le A}\right)^{-1}}{\sum_{i=1}^{A}\mu_{[i]}^{(a)}e^{-\alpha(a)r_i}}$$

and thus the new column eigenvector $w_1^{(a)}$ is now

$$w_1^{(a)} = \frac{\Delta\left(\left(e^{\alpha(a)r_i}\right)_{i\le A}\right) w^{(a)}}{\mu_1^{(a)}\Delta\left(\left(e^{\alpha(a)r_i}\right)_{i\le A}\right)} = \Delta\left(\left(e^{\alpha(a)r_i}\right)_{i\le A}\right) w^{(a)}\left(\sum_{i=1}^{A}\mu_{[i]}^{(a)}e^{-\alpha(a)r_i}\right)$$

thanks to the equations of scaling.

**Existence of an optimum**

We begin by proving that there really is an optimal support contract to both players, e.g. the martingale parameters do not grow asymptotically while the support contract escapes in the direction $s_{i\to j} \to \infty$. We are actually going to

1. Control the support contract's maximum drifts $\delta^{\pm}(S)$ ;
2. Use this control to build a support contract of identical martingale parameters, whose transition payoffs are controlled ;
3. Find out that there is compact set $K$ containing the best support contracts.

Let us take $S$ any support contract, and one of its cycles that never hits the same state twice (except for the finishing step, that must get back to the starting state), so that its length is at most $A$. Its occupied state numbers are called $a_i \leq A$ for $i \leq T$ ; $S$ has a value $v \in \mathbf{R}$, and a probability $p > 0$ of being followed by $M$. Let us now take the same cycle for $C_1$ : as $C_1$ is bounded (e.g. by $Q \in \mathbf{R}^+$), it has a probability $p$ of following it and getting a cumulated increment at most $AQ$ ; therefore, $C_1 + S$ has a probability $e^{-Au} > 0$ of cumulating an increment at most $AQ + v$ over a cycle of length $T$ where $e^{-u}$ is the lowest non-zero probability of transitions (positive as they are in a finite number). If $v < -AQ$, this cycle alone creates a downwards-shaped trajectory for $C_1 + S$ by repetitions, eventually driving it bankrupt after a number of repetitions of at most

$$n = \left\lfloor \frac{C_1(0)}{-AQ - v} \right\rfloor + 1$$

It follows that the default expectancy of $C_1 + S$ is at least

$$e^{-(u+a_1)An} \geq e^{-(u+a_1)A} e^{-C_1(0)\frac{(u+a_1)A}{-AQ-v}}$$

Therefore, its martingale parameter is at most $-(u+a_1)A/(AQ+v)$, which means that for $\alpha_1$ to be increased by the support contract, one must have

$$-v \leq x^- = (u + a_1)\, A\alpha_1 + AQ$$

It follows that cycles of $S$ that never hits the same state twice have a bounded (from below) value ; as every cycle is a combination of such cycles, we get $-\delta^-(S) \leq x^-$. Likewise, studying $C_2 - S$ yields $x^+$ such that $\delta^+(S) \leq x^+$. Hence, every support contract whose cycle support is not inside $[-x^-, x^+]$ is worse than the zero support contract either for $J_1$ or for $J_2$.

Now, let us take such a support contract $S$ holding these inequalities. We are going to modify it by means of offsets, such that all of its transition payoffs are upper bounded by $\delta^+(S)$ without altering the martingale parameters. Let us consider the C-process $S'$ defined by $S'(t) = S(t) - t\delta^+(S)$, drifting all of the increments of $S$ by $-\delta^+(S)$. By definition, $S'$ is globally decreasing, so it rewrites as $S^-(t) + S^=(t)$ with $S^-$ non-increasing and $S^=$ globally constant. We know that $S^=(t)$ may be written with offsets as in $\forall t, S^=(t) = r_{[M(t)]}$. It follows that the support contract $S^* = S - S^=$ has the same effect on the martingale parameters as $S$, and that it rewrites as

$$S^*(t) = t\delta^+(S) + S^-(t)$$

and as $S^-$ is non-increasing, its increments are upper bounded by $\delta^+(S) \leq x^+$. It also follows from this that $S^*$'s increments are lower bounded by

$$-A(x^- + 1)(A - 1)x^+$$

because if not, positive recurrence of $S^*$ yields a cycle going through the faulty increment and of value at most $-A(x^- + 1)$, thus $S^*$ would have a value lower than $-x^- - 1$ in its cycle support, and so does $S$, which is impossible. Hence, every support contract whose cycle support is inside $[-x^-, x^+]$ yields the same martingale parameters as some support contract $S^*$ whose transition payoffs are well-bounded.

Noting by $f$ the function that associates a support contract (viewed as an element of $\mathbf{R}^{A^2}$) with both players' martingale parameters (in $\mathbf{R}^2$), this means that there is a compact subset $K \subset \mathbf{R}^{A^2}$ such that for every support contract $S$, there is another $S^* \in K$ such that $f(S^*) \geq f(S)$. This means that $f$ must hit its maximal values somewhere on $K$, thus an optimal support contract exists.

**Dominant eigenvalue**

During the next paragraph, we will need to quantify the variation of the dominant eigenvalue of a matrix $M$ with respect to the variations of $M$. Let us start from the set $U \subset M_A\left(\mathbf{R}^+\right)$ of positive matrices satisfying Perron-Frobenius' theorem : both matrices $L_{C_p}(\alpha_p(a_p))$ belong to $U$, because they are positive recurrent (see the theory of C-processes). We define the function

$$F = \begin{pmatrix} (M_A(\mathbf{R}) \times \mathbf{R}) & \rightarrow & \mathbf{R} \\ (M, \lambda) & \rightarrow & \det(M - \lambda Id) \end{pmatrix}$$

Taking $M \in U$ and $\lambda$ its dominant eigenvalue, we know that

$$F(M, \lambda) = 0 \wedge \frac{\partial F}{\partial 2}(M, \lambda) \neq 0$$

as matrices in $U$ have a one-dimensional dominant eigenspace by Perron-Frobenius' theorem. It follows that $F$ defines an implicit function $\lambda$ that associates with any such matrix $M$ its dominant eigenvalue $\lambda(M)$ over a local opened set $W$ around $M$. Now, we know that $M_A\left(\mathbf{R}_+^*\right)$ is $\sigma$-compact, thus it is possible to define $\lambda$ globally over it ; and as $W$ is opened and $U \subset M_A\left(\mathbf{R}^+\right)$, $W$ must encounter it, thus we may define $\lambda$ over some opened set $V \supset U$, so that $\lambda$

— Is $C^\infty$ over $V$ ;

— Associates every $M \in U$ to its dominant eigenvalue.

Likewise, associated dominant eigenvectors $\mu$ and $w$ may be associated with $M$, defining $C^\infty$ functions over such a $V$. We will not explain the proof of the previous assertion, as it is similar to the theory of C-processes. We may thus compute $\lambda$'s derivative application at point $M \in U$. To do this, let us set $i, j \leq A$ and look for

$$\frac{\partial \lambda}{\partial(i,j)}(M) = \lim_{\epsilon \to 0}\left(\frac{\lambda(M + \epsilon e_{i,j}) - \lambda(M)}{\epsilon}\right)$$

where $e_{i,j}$ is the matrix whose only non-zero entry is 1 at position $(i,j)$. Let us write the eigenvector equation $Mw(M) = \lambda(M)w(M)$ and differentiate it in the direction $e_{i,j}$, getting

$$e_{i,j}w(M) + M\frac{\partial w}{\partial(i,j)}(M) = \frac{\partial \lambda}{\partial(i,j)}(M)w(M) + \lambda(M)\frac{\partial w}{\partial(i,j)}(M)$$

Since $\mu(M)M = \lambda(M)\mu(M)$, left multiplication by $\mu(M)$ yields

$$\mu(M)e_{i,j}w(M) = \mu(M)\frac{\partial \lambda}{\partial(i,j)}(M)w(M)$$

and as $\mu(M)w(M) = 1$ by definition, we get eventually

$$\frac{\partial \lambda}{\partial(i,j)}(M) = (\mu(M))_{[i]}\,(w(M))_{[j]}$$

We will use this equality in the next part of the proof.

### Effects of the support contract

Let us take the Laplace matrix functions $L_{C_p}$ at points $\alpha_p(a_p)$, called $M_p$, so that by definition

$$\lambda\left(M_p\right) = e^{a_p}$$

An increase of $\epsilon s_{i\to j}$ of the random variable $D_{i\to j}$ has an additive effect on $M_p$'s entry number $(i,j)$ of

$$-\epsilon\alpha_p(a_p)s_{i\to j}P_{i\to j}\mathbf{E}\left(e^{-\alpha_p(a_p)D^{(p)}_{i\to j}}\right) + o\left(\epsilon\right)$$

which means an overall effect on $\lambda$ of

$$-\epsilon\left(\mu(M_p)\right)_{[i]}\left(w(M_p)\right)_{[j]}\alpha_p(a_p)s_{i\to j}P_{i\to j}\mathbf{E}\left(e^{-\alpha_p(a_p)D^{(p)}_{i\to j}}\right)+o\left(\epsilon\right)$$

Considering that an increase of $\alpha_p(a_p)$ by $d\alpha$ would lead likewise to an increase of $\lambda$ by $x_p d\alpha + o\left(d\alpha\right)$ (with $x_p > 0$ since the martingale parameter is increasing of $a_p$), this means that the increase by $\epsilon s_{i\to j}$ of the random variable $D^{(p)}_{i\to j}$ has an additive effect on the martingale parameter of

$$\frac{\epsilon}{x_p}\left(\mu(M_p)\right)_{[i]}\left(w(M_p)\right)_{[j]}\alpha_p(a_p)s_{i\to j}P_{i\to j}\mathbf{E}\left(e^{-\alpha_p(a_p)D^{(p)}_{i\to j}}\right)+o\left(\epsilon\right)$$

As $\left(\mu(M_p)\right)_{[i]}=\mu^{(a_p)}_{p,[i]}$ and $\left(w(M_p)\right)_{[j]}=w^{(a_p)}_{p,[j]}$, let us set for $i,j\leq A$

$$z_{p,i\to j}=\mu^{(a_p)}_{p,[i]}w^{(a_p)}_{p,[j]}P_{i\to j}\mathbf{E}\left(e^{-\alpha_p(a_p)D^{(p)}_{i\to j}}\right)$$

The family $s_{i\to j}$ defines (an infinitesimal) favorable support contract to both players iff both $\alpha_p(a_p)>0$ and the value

$$\sum_{i=1}^{A}\sum_{j=1}^{A}s_{i\to j}z_{p,i\to j}$$

is positive for $p=1$ and negative for $p=2$. To find a condition of existence, we look at the families $z_p\in\mathbf{R}^{A^2}$ as vectors containing all values of $z_{p,i\to j}$ for $i,j\leq A$. The families $z_p$ are non-zero by construction of the Laplace matrix functions, so let us set

$$s=\left(s_{i\to j}\right)_{i,j\leq A}=\left\|z_2\right\|z_1-\left\|z_1\right\|z_2$$

where $\left\|\cdot\right\|$ denotes the Euclidean norm on $\mathbf{R}^{A^2}$. Therefore, the sought value for $z_1$ becomes

$$\langle s,z_1\rangle=\left\|z_1\right\|\left(\left\|z_1\right\|\left\|z_2\right\|-\langle z_1,z_2\rangle\right)$$

which is positive unless $z_1$ and $z_2$ are colinear and have the same sign. Likewise,

$$\langle s,z_2\rangle=\left\|z_2\right\|\left(\langle z_1,z_2\rangle-\left\|z_1\right\|\left\|z_2\right\|\right)$$

is negative unless the same condition happens. Finally, this condition $\exists k\in\mathbf{R}^*_+; z_2=kz_1$ rewrites as for every $i,j\leq A$,

$$k\mu^{(a_1)}_{1,[i]}w^{(a_1)}_{1,[j]}P_{i\to j}\mathbf{E}\left(e^{-\alpha_1(a_1)D^{(1)}_{i\to j}}\right)=\mu^{(a_2)}_{2,[i]}w^{(a_2)}_{2,[j]}P_{i\to j}\mathbf{E}\left(e^{-\alpha_2(a_2)D^{(2)}_{i\to j}}\right)$$

so noting by $\Delta(v)$ the diagonal matrix whose elements are given by the vector $v$, we get the condition

$$k\Delta\left(\mu^{(a_1)}_1\right)L_{C_1}\left(\alpha_1(a_1)\right)\Delta\left(w^{(a_1)}_1\right)=\Delta\left(\mu^{(a_2)}_2\right)L_{C_2}\left(\alpha_2(a_2)\right)\Delta\left(w^{(a_2)}_2\right)$$

Left multiplication by the row vector of ones alongside right multiplication by the column vector of ones thus yields

$$k\mu_1^{(a_1)} L_{C_1}\left(\alpha_1(a_1)\right) w_1^{(a_1)} = \mu_2^{(a_2)} L_{C_2}\left(\alpha_2(a_2)\right) w_2^{(a_2)}$$

and by definition of eigenvectors, this simplifies to $k = e^{a_2 - a_1}$. As optimal martingale parameters must be hit, this ends the proof.

### 3.9.2 Proposition 3.6.2

In this paragraph, we optimize the offsets $r_i$ of a support contract $S$ for C-processes $C_p$ under the identity condition. For reasons of simplicity, we will always assume that $r_{M(0)} = 0$, since a translation of all offsets by the same additive constant yields the same C-process by definition of offsets ; therefore, the set of offsets will be deemed $\mathbf{R}^{A-1}$.

#### Bounds for the offsets

Let us set, for each player $J_p$,

$$\mu_p^* = \max_{i \leq A} \left( \frac{1}{\mu_{p,[i]}^{(a_p)}} \right)$$

It is finite because a Laplace matrix function has positive eigenvectors. As thanks to the proposition 3.6.2, the coordinate number $M(0)$ of the resulting dominant eigenvector after use of $S$ for player $J_1$ is

$$\left( \sum_{i=1}^{A} \mu_{1,[i]}^{(a)} e^{-\alpha_1(a_1) r_i} \right) w_{p,[M(0)]}^{(a_1)}$$

Therefore, if any $r_i$ is lower than

$$-r_- = \frac{-\ln\left(\mu_1^*\right)}{\alpha_1(a_1)}$$

then the multiplicative term $w_{1,[M(0)]}^{(a_1)}$ is increased by the transition payoffs, thus $S$ is unfavorable to $J_1$. Likewise, if any $r_i$ is higher than than

$$r^+ = \frac{\ln\left(\mu_2^*\right)}{\alpha_2(a_2)}$$

then $w_{2,[M(0)]}^{(a_1)}$ is increased and $S$ is unfavorable to $J_2$. As a consequence, acceptable support contracts have offsets in $[-x^-, x^+]$, i.e. belong to a compact subset $K \subset \mathbf{R}^{A-1}$.

**Variation of offsets**

Let us modify $C_p$'s transition payoffs by a support contract $S$ consisting of infinitesimal transition payoffs defined by offsets $dr_i$. Thanks to the proposition 3.6.2, $J_1$'s new dominant eigenvector is at first order

$$w_1^{(a_1),r} = \left(\sum_{i=1}^{A} \mu_{1,[i]}^{(a_1)} \left(1 - \alpha_1(a_1) dr_i\right)\right) \Delta \left((1 + \alpha_1(a_1) dr_i)_{i \leq A}\right) w_1^{(a_1)}$$

so the equations of scaling lead to

$$w_1^{(a_1),r} = w_1^{(a_1)} + \alpha_1(a_1) \left(dr_i w_{1,[i]}^{(a_1)}\right)_{i \leq A} - \alpha_1(a_1) \left(\sum_{i=1}^{A} \mu_{1,[i]}^{(a_1)} dr_i\right) w_1^{(a_1)}$$

Therefore, $J_1$ is satisfied provided that

$$dr_{[M(0)]} - \left(\sum_{i=1}^{A} \mu_{1,[i]}^{(a_1)} dr_i\right) > 0$$

Likewise, $J_2$ is satisfied provided that

$$dr_{[M(0)]} - \left(\sum_{i=1}^{A} \mu_{2,[i]}^{(a_2)} dr_i\right) < 0$$

Hence, unless the vectors $\mu_1^{(a_1)}$ and $\mu_2^{(a_2)}$ are colinear (thus identical, thanks to the equations of scaling), there will be a favorable support contract to both players (this is the same idea as when boulding support contracts). Restarting recursively, like previously, finally yields an offsets support contract that holds $\mu_1^{(a_1)} = \mu_2^{(a_2)}$ ; however, as it still holds the identity condition

$$e^{-a_1} \delta^{\alpha_1(a_1)} L_{C_1}\left(\alpha_1(a_1)\right) \delta^{-\alpha_1(a_1)} = e^{-a_2} \delta^{-\alpha_2(a_2)} L_{C_2}\left(\alpha_2(a_2)\right) \delta^{\alpha_2(a_2)}$$

by construction, this implies $w_1^{(a_1)} = w_2^{(a_2)}$, which in turn means that the Laplace matrix functions at points $\alpha_p(a_p)$ are identical up to the multiplicative constants $e^{-a_p}$.

**uniqueness of optimal offsets**

Finally, to ensure uniqueness of these optimal offsets (over $K$, so up to the aforementioned additive translation, without any effect on $C_p$), let us take a support contract such that $w_1^{(a_1)} = w_2^{(a_2)}$, the vector being renamed $w$. Adding some $S$ with offsets $r_i$ (with $r_{[M(0)]} = 0$) yields new vectors $w_p^{(a_p)}$ ; if $S$ is acceptable to both players and optimal, they must also hold the identity condition, so after the proposition 3.6.2, the vectors

$$v_1 = \left(e^{\alpha_1(a_1)r_i} w_{[i]}\right); v_2 = \left(e^{-\alpha_2(a_2)r_2} w_{[i]}\right)$$

are colinear, thus identical since $r_{[M(0)]} = 0$. As martingale parameters are positive, this may happen only if every $r_i$ is zero, which means that there is no other optimal offsets support contract.

Hence, it suffices to use the offsets

$$\forall i \leq A, r_i = \frac{\ln\left(\frac{w_{2,[i]}}{w_{1,[i]}}\right)}{\alpha_1(a_1) + \alpha_2(a_2)}$$

to get for $J_1$

$$L_{C_1+S}\left(\alpha_1(a_1)\right) = \Delta\left(\left(e^{\alpha_1(a_1)r_i}\right)_i\right) L_C\left(\alpha_1(a_1)\right) \Delta\left(\left(e^{-\alpha_1(a_1)r_i}\right)_i\right)^{-1}$$

which is the left-hand side of the identity condition ; likewise, taking $L_{C_2-S}$ at point $\alpha_2(a_2)$ yields the right-hand side. This means that if the identity condition holds, the support contract $S$ defined by these offsets is optimum to both players.

### 3.9.3 Properties of total decomposition

In this paragraph, we prove the properties obtained when using the trick of total decomposition for the increments of a C-process.

#### Proposition 3.6.3

We aim at proving the proposition 3.6.3 relative to the affine transformation of payoffs. Before we start, we mention that the optimal $Q'$ and $r'$ to be found must lead to C-processes holding the identity condition : indeed, if they do not, we may find C-processes with strictly better martingale parameters, then use use the decomposition trick to recover some $Q''$ and $r''$ beating $Q'$ and $r'$. Moreover, $Q$ must remain bounded, because if $Q$ is too high, $J_2$'s martingale parameter collapses below her starting $\alpha_2(a_2)$ like is previous proofs, and if $Q$ is too low, $J_1$'s martingale parameter collapses below his starting $\alpha_1(a_1)$. It follows that $(Q, r)$ belongs to a compact subset of $\mathbf{R}^2$, and thus any $(Q, r)$ not holding the identity condition are beaten by some $(Q^*, r^*)$ holding it. For this reason, we shall deem that the initial $Q$ and $r$ hold the identity condition.

We start by the definition of $C_p$'s martingale parameters : there is some $\alpha_1(a) \in \mathbf{R}^+$ such that

$$\lambda\left(\left(P_{i\to j}\mathbf{E}\left(e^{-\alpha_1(a)(rD_{i\to j}+Q)}\right)\right)_{i,j}\right) = e^a$$

provided that $C_1$ is not globally increasing, and some $\alpha_2(a) \in \mathbf{R}^+$ such that

$$\lambda\left(\left(P_{i\to j}\mathbf{E}\left(e^{-\alpha_2(a)((1-r)D_{i\to j}-Q)}\right)\right)_{i,j}\right) = e^a$$

provided that $C_2$ is not globally increasing. Therefore, we get that

$$\lambda\left(\left(P_{i\to j}\mathbf{E}\left(e^{-(r\alpha_1(a))D_{i\to j}}\right)\right)_{i,j}\right) = e^{a+\alpha_1(a)Q}$$

so if $Q \in \mathbf{R}^+$, injectivity of the martingale parameter over $\mathbf{R}^+$ leads to the identification

$$r\alpha_1(a) = \alpha\left(a + \alpha_1(a)Q\right)$$

Likewise, if $Q \in \mathbf{R}^-$, we get

$$(1-r)\alpha_2(a) = \alpha\left(a - \alpha_2(a)Q\right)$$

We also recall that $C_1$ and $C_2$ must hold the identity condition to be optimal, and that it eventually led to transition payoffs holding

$$d^{(1)}_{x\to y} = \frac{\alpha_2(a_2)}{\alpha_1(a_1)+\alpha_2(a_2)}d_{x\to y} + \frac{a_2-a_1}{\alpha_1(a_1)+\alpha_2(a_2)}$$

so we get (except perhaps if all transition payoffs $d_{x\to y}$ are identical, therefore $C$ is an affine function of time almost surely)

$$\begin{aligned} r &= \frac{\alpha_2(a_2)}{\alpha_1(a_1)+\alpha_2(a_2)} \\ Q &= \frac{a_2-a_1}{\alpha_1(a_1)+\alpha_2(a_2)} \end{aligned}$$

Considering this as an equation system in unknowns $\alpha_p(a_p)$ yields (except if $Q = 0$)

$$\begin{aligned} \alpha_1(a_1) &= \frac{a_2-a_1}{Q}(1-r) \\ \alpha_2(a_2) &= \frac{a_2-a_1}{Q}r \end{aligned}$$

We may now replace the values $\alpha_p(a_p)$ in the identification equation for $a = a_p$ to get

$$r(1-r)\frac{a_2-a_1}{Q} = \alpha\left(ra_1 + (1-r)a_2\right)$$

if $Q \geq 0$, and the same equation if $Q \leq 0$, so finally

$$Q = \frac{(a_2-a_1)r(1-r)}{\alpha\left(ra_1+(1-r)a_2\right)}$$

We notice that in the special case $a_1 = a_2$ leading to $Q = 0$, we still recover $Q = 0$ with this formula, which ends the proof.

**Proposition 3.6.4**

To optimally share the total C-process, we are going to state that both $a_1 + \alpha_1(a_1)R(r)$ and $a_2 - \alpha_2(a_2)R(r)$ are non-negative, so that the identifications of the above proof into

$$\alpha_p(a_p) = \frac{\alpha\left(a_p \pm \alpha_p(a_p)R(r)\right)}{r_p}$$

(with $r_1 = r$ and $r_2 = 1-r$) are valid. Recalling that

$$\alpha_1(a_1) = \frac{a_2 - a_1}{R(r)}(1-r) \wedge \alpha_2(a_2) = \frac{a_2 - a_1}{R(r)}r$$

under optimality conditions (identity condition and optimal $R(r)$), we get

$$a_1 + \alpha_1(a_1)R(r) = a_1 + (a_2 - a_1)(1-r) = ra_1 + (1-r)a_2 > 0$$

as well as

$$a_2 - \alpha_2(a_2)R(r) = a_2 - (a_2 - a_1)r = ra_1 + (1-r)a_2 > 0$$

Hence, identification of martingale parameters leads to the desired expressions. To get the eigenvectors, we use the fact that for $J_1$,

$$L_{rC+R(r)}\left(\alpha_1(a_1)\right) = e^{-\alpha(ra_1+(1-r)a_2)R(r)/r}\left(P_{i\to j}\mathbf{E}\left(e^{-\alpha(ra_1+(1-r)a_2)D_{i\to j}}\right)\right)_{i,j\leq A}$$

so is proportional to $L_C\left(\alpha\left(ra_1 + (1-r)a_2\right)\right)$, as well as for $J_2$.

**Proposition 3.6.5**

We aim at maximizing the terms appearing in the exponential function in the default expectancies, leading to the optimal initial compensation. Actually, thanks to the proposition 3.6.4, we look for a couple $(r, x)$ that maximizes in the functions

$$F_1 = \begin{pmatrix} [0,1]\times[0,C(0)] & \to & \mathbf{R}^+ \cup \{\infty\} \\ (r,x) & \to & x^{\frac{\alpha(ra_1+(1-r)a_2)}{r}} \end{pmatrix}$$

for $J_1$, and

$$F_2 = \begin{pmatrix} [0,1]\times[0,C(0)] & \to & \mathbf{R}^+ \cup \{\infty\} \\ (r,x) & \to & (C(0)-x)^{\frac{\alpha(ra_1+(1-r)a_2)}{1-r}} \end{pmatrix}$$

where we set $F_1(0,x) = \infty$ and $F_2(1,x) = \infty$. These boundary values actually make sense, since $r \in \{0,1\}$ leads to $R(r) = 0$, thus the corresponding $C_p$ will be a constant almost surely and never defaults : for this reason, the exponential term in $J_p$'s default expectancy may be considered $\infty$. Since $r$ and $x$ belong to a compact set, and the functions $F_p$ are upper semi-continuous, optimal points must exist like in the previous cases. Moreover, for $r = 0$, as only $F_2$ is relevant, we should only look at the case $x = 0$ to minimize the default expectancies (and $x = C(0)$ for $r = 1$).

A variation of $r$ has an effect on $F_1$ given through

$$\frac{\partial F_1}{\partial 1}(r,x) = \frac{x}{r^2}\left(\alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2)r - \alpha\left(ra_1 + (1-r)a_2\right)\right)$$

while a variation of $x$ yields

$$\frac{\partial F_1}{\partial 2}(r,x) = \alpha\left(ra_1 + (1-r)a_2\right)\frac{1}{r}$$

$F_2$'s differential application is computed likewise, and once again if $(r,x)$ is optimal then either these differential applications must be colinear, or $(r,x)$ has hit a boundary and thus is $(0,0)$ or $(1, C(0))$ by virtue of the the previous remark. We shall now only look at the former case : colinearity means that

$$\begin{aligned} & \frac{\frac{x}{r^2}\left(\alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2)r - \alpha\left(ra_1 + (1-r)a_2\right)\right)}{\alpha\left(ra_1 + (1-r)a_2\right)\frac{1}{r}} \\ = \; & \frac{\frac{C(0)-x}{(1-r)^2}\left(\alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2)(1-r) + \alpha\left(ra_1 + (1-r)a_2\right)\right)}{\alpha\left(ra_1 + (1-r)a_2\right)\frac{-1}{1-r}} \end{aligned}$$

which simplifies to

$$\begin{aligned} & x\left(-\alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2) + \alpha\left(ra_1 + (1-r)a_2\right)\frac{1}{r}\right) \\ = \; & (C(0) - x)\left(\alpha'\left(ra_1 + (1-r)a_2\right)(a_1 - a_2) + \alpha\left(ra_1 + (1-r)a_2\right)\frac{1}{1-r}\right) \end{aligned}$$

Division by $\alpha\left(ra_1 + (1-r)a_2\right)$ leads to the result.

Finally, to prove that $k$ is non-increasing, we introduce the function

$$f = \begin{pmatrix} (0,1) & \to & \mathbf{R} \\ r & \to & \ln\left(\alpha\left(ra_1 + (1-r)a_2\right)\right) \end{pmatrix}$$

We verify that $f'(r) = k(r)$. As ln is increasing and concave, and $\alpha$ is known to be concave, it follows that $f$ is concave, therefore $k$ is non-increasing.

### Proposition 3.6.6

To prove the sub-additivity of $\alpha^{-1}$, we take $C_1$ and $C_2$ as described, and set $T$ the stopping time defined as follows :

— We set $\tau_1$ the first return time of $M$ at it starting state $M(0)$, and $\tau_2$ its second return time ;
— We define a random Bernouilli variable $B$ with 1/2 probabilities, and independent of the rest of the model, such that

$$T = \mathbf{1}_{B=1}\tau_1 + \mathbf{1}_{B=0}\tau_2$$

We want to use the martingale property on $C$'s martingale process $X_C^{(a)}$ for every Laplace parameter $a \in \mathbf{R}^+$, defined by

$$X_C^{(a)} = \left( \begin{array}{ccc} \mathbf{N} & \to & \mathbf{R}^+ \\ t & \to & w_{[M(t)]}^{(a)} e^{-at} e^{-\alpha(a)C(t)} \end{array} \right)$$

with stopping time $T$. To verify integrability properties, we want to create an integrable random variable $Y$ that dominates $X_C^{(a)}(T)$. Decomposing $X_C^{(a)}$ by its successive increments, we set

$$Y = X_C^{(a)}(0) + \sum_{t=0}^{\infty} \left| X_C^{(a)}(t+1) - X_C^{(a)}(t) \right| \mathbf{1}_{t \leq T-1}$$

so by definition, we get

$$\begin{array}{rl} & \mathbf{E}(Y) - \mathbf{E}\left(X_C^{(a)}(0)\right) \\ = & \displaystyle\sum_{t=0}^{\infty} e^{-at} \mathbf{E} \left( \begin{array}{c} e^{-\alpha(a)C(t)} \mathbf{1}_{t \leq T-1} \\ \mathbf{E}\left( \left| w_{[M(t+1)]}^{(a)} e^{-a} e^{-\alpha(a) D_{M(t) \to M(t+1)}} - w_{[M(t)]}^{(a)} \right| |\mathbf{F}(t) \right) \end{array} \right) \end{array}$$

The innermost expectancy is bounded by a constant $K(a)$ by hypothesis on transition payoffs. As $X_C^{(a)}(0)$ is a constant value, we shall have $\mathbf{E}(Y) < \infty$ as soon as

$$y = \sum_{t=0}^{\infty} e^{-at} \mathbf{E}\left(e^{-\alpha(a)C(t)} \mathbf{1}_{t \leq T-1}\right) < \infty$$

Let us consider the C-process $C'$ obtained by increasing all of $C$'s transition payoffs going to $M(0)$ by 1. By definition of $T$, the condition $t \leq T-1$ means that $M$ returns to $M(0)$ at most once during the time interval $[|1, T-1|]$, so $C'(t) \leq C(t) + 1$. $C'$'s Laplace matrix function at point $\alpha(a)$ has a dominant eigenvalue $e^b < e^a$ because $C$ is positive recurrent. It follows that we rewrite

$$y \leq e^{\alpha(a)} \sum_{t=0}^{\infty} e^{-(a-b)t} \mathbf{E}\left(e^{-bt} e^{-\alpha(a)C'(t)}\right)$$

Using the martingale property for $C'$ yields a dominant eigenvector, and thus a new constant $k'(a)$ such that

$$y \leq k'(a) \sum_{t=0}^{\infty} e^{-(a-b)t} < \infty$$

since $a > b$, so $\mathbf{E}(Y) < \infty$ and thus we have the martingale property

$$\mathbf{E}\left(X_C^{(a)}(T)\right) = X_C^{(a)}(0)$$

that translates, since $M(T) = M(0)$, to

$$\mathbf{E}\left(e^{-aT}e^{-\alpha(a)C(T)}\right) = e^{-\alpha(a)C(0)}$$

Now, let us set $a \in \mathbf{R}^+$, and for $p \in \{1, 2\}$, the random variables

$$\begin{aligned} D &= C(T) - C(0) + \frac{aT}{\alpha(a)} \\ D_p &= C_p(T) - C_p(0) + \frac{aT}{\alpha_p(a)} \end{aligned}$$

therefore we have by construction

$$\mathbf{E}\left(e^{-\alpha(a)D}\right) = 1$$

thanks to the martingale property, and

$$D = D_1 + D_2 + aT\left(\frac{1}{\alpha(a)} - \left(\frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}\right)\right)$$

Now, let us deem that

$$\frac{1}{\alpha(a)} - \left(\frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}\right) \geq 0$$

We set $r = \alpha_2(a)/\left(\alpha_1(a) + \alpha_2(a)\right)$ so that

$$\frac{\alpha(a)}{r} = \alpha(a)\alpha_1(a)\left(\frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}\right) \wedge \frac{\alpha(a)}{1-r} = \alpha(a)\alpha_2(a)\left(\frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}\right)$$

which leads by hypothesis to

$$\frac{\alpha(a)}{r} \leq \alpha_1(a) \wedge \frac{\alpha(a)}{1-r} \leq \alpha_2(a)$$

It also comes from $D \geq D_1 + D_2$ that

$$\mathbf{E}\left(e^{-\alpha(a)D}\right) \leq \mathbf{E}\left(e^{-\alpha(a)D_1}e^{-\alpha(a)D_2}\right)$$

Hölder's inequality then leads to

$$1 \leq \mathbf{E}\left(e^{-\frac{\alpha(a)}{r}D_1}\right)^r \mathbf{E}\left(e^{-\frac{\alpha(a)}{1-r}D_2}\right)^{1-r}$$

but Jensen's inequality indicates that

$$\mathbf{E}\left(\left(e^{-\alpha_1(a)D_1}\right)^{\left(\frac{\alpha(a)/r}{\alpha_1(a)}\right)}\right) \leq \left(\mathbf{E}\left(e^{-\alpha_1(a)D_1}\right)\right)^{\left(\frac{\alpha(a)/r}{\alpha_1(a)}\right)}$$

which is 1 by construction of $D_1$. Likewise for $D_2$, we eventually end up with the inequality $1 \leq 1$. The only equality case for Hölder's inequality being when the random variables

$$X_1 = e^{-\frac{\alpha(a)}{r}D_1} \wedge X_2 = e^{-\frac{\alpha(a)}{1-r}D_2}$$

are proportional almost surely, we proved that the inequation

$$\frac{1}{\alpha(a)} \geq \frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}$$

implies that there is $k \in \mathbf{R}_+^*$ such that $X_1 = kX_2$ almost surely, which rewrites as

$$\frac{1-r}{r}D_1 + \frac{1-r}{\alpha(a)}\ln(k) = D_2$$

so $rD_2-(1-r)D_1$ is some constant $x \in \mathbf{R}$ almost surely. However, this means that conditioning to $B = 1$ indicates that all cycles of the C-process $rC_2 - (1-r)C_1$ that return only once to $M(0)$ have a value $x$ ; it follows that all of its cycles that return twice to $M(0)$ have a value $2x$, while conditioning to $B = 0$ indicates that these have value $x$. From $2x = x$ comes $x = 0$, and so $rC_2 - (1-r)C_1$ has a cycle support of $\{0\}$ and is globally constant. Noting it by $C_0^=$, we get

$$C_2 = \frac{1-r}{r}C_1 + \frac{1}{r}C_0^=$$

which is the desired form under the equality condition. Finally, this means that after $C = C_1 + C_2$, we have

$$C_1 = rC - C_0^=$$

so $C_1$ has the same martingale parameter as $rC$ since globally constant C-processes do not modify martingale parameters, which is $\alpha(a)/r$ by homogeneity of martingale parameters (of exponent $-1$). Likewise, we get $\alpha_2(a) = \alpha(a)/(1-r)$, and this leads to equality in the inequality condition. To sum things up, we proved that the inequation

$$\frac{1}{\alpha(a)} \geq \frac{1}{\alpha_1(a)} + \frac{1}{\alpha_2(a)}$$

implied the corresponding equation, as well as the desired property, which ends the proof.

# Chapitre 4

# Discussion

Pour terminer le travail, nous indiquons des suites naturelles aux études précédentes, et des possibilités de développement dans notre modèle de C-jeu ; nous nous intéressons à des modèles plus précis, et donnons des intuitions sur le comportement prévu de nos résultats quand la structure du modèle est modifiée. Pour la simplicité des explications, nous les présenterons en termes du Monopoly.

To end the work, we mention natural continuations of the previous studies, investigating on sharper models and giving insights on the expected behaviour of our results when modifying the model structure. We look for possibilities of further development in our C-game model. For the sake of explanations, we will present them in Monopoly terms.

## 4.1 Maximal consumption

We give a short insight on what should happen when replacing the utility functions of default expectancies by descriptions of consumption and dividends. First, let us try and maximize the discounted firm value, obtained through a Laplace parameter at time $t$, accounting for the value $C(t)$ provided that $C$ has not defaulted yet (condition of illiquidity) by

$$V(a) = \mathbf{E}\left(\sum_{t=0}^{\infty} C(t)\mathbf{1}_{\forall s \leq t, C(s) \geq 0} e^{-at}\right)$$

After a short investigation, we conjectured that for $C$ a Lévy process,

$$\left(1 - e^{-a}\right) V(a) \approx C(0) + E(C)\frac{1 - e^{-\alpha(a)C(0)}}{1 - e^{-a}} e^{-a}$$

where $E(C)$ is $C$'s mean expectancy and $\alpha(a)$ is its martingale parameter, and the approximation is to be understood for high values of $C(0)$ like in the fundamental approximation. For C-processes, we expect $w^{(a)}$ and its derivative with respect to $a$ to show up somewhere in the expression, but the main feature remains the martingale parameter, indicating that the strategies looking for maximization of value or minimization of default risks should behave similarly. However, for high values of $C(0)$, the effect of the martingale parameter fades out with respect to the mean expectancy : investors with “deep pockets” should focus on enhancing their drifts with little regards towards default probability.

Another issue involves distributing dividends in a sustainable way, as done for Brownian motions e.g. by [39], without impeding investment (like [11]). This time, we expect the dividend policy to behave roughly in a common fashion (distributing only when cash reserves pass some threshold $b$), but we believe $b$ to depend on $M$'s state and thus on $w^{(a)}$. Similar behaviour is discussed thanks to $C$'s asymptotical expectancies to determine the “value” of $M$'s states to the investor.

## 4.2 Discussion on Monopoly rules

Let us look at the changes in our main results when we modify the rules of Monopoly to fit better to real-life investment decisions.

The rules of Monopoly imply that prices are not subject to inflation ; actually, they remain constant over time. However, as the total cash flow to players $C = C_1 + C_2$ is mainly positive, constant prices are an unrealistic model of reality. A more accurate model would account for inflation when setting rent prices, or at least prevent $C_p$ to go to $+\infty$ ; this happens when levelling out the “Go” salary so that $C$'s mean expectancy is cancelled. We presume that this nullifies the advantages of heavy investment (green properties) by preventing long-run games, although we are unsure whether this leaves enough liquid assets to players to strategically develop properties, so the winner will be mostly determined by luck.

Another idea to keep $C_p$ to realistic values would be to remove the investment limits, allowing players to build several hotels per property. This succeeds because each time $C_p$ exceeds an investment threshold, $J_p$ is willing to invest as told by our study. With this assumption, we expect the wealthiest player to invest more than other players, eventually enhancing his revenues and allowing to invest more, launching a positive feedback loop and eventually busting all opponents. Indeed, assuming that previous revenues are invested and next revenues are proportional to investment indicates that $J_p$'s wealth should diverge exponentially with time as long as he has opponents paying for his rents ; as $C$ grows only in a linear fashion, the game must end after a finite time.

The concept of trading assets aims at improving both traders' conditions, thus may be linked with the ideas of support contracts. We proved under our hypotheses that when both players aim at avoiding default, they should agree on a support contract that removes state specificities ; as an extreme case, they should share revenues to support each other. Translating this in Monopoly terms, we get the complete opposite of developping properties. Actually, once all properties of Monopoly are purchased, the best trades to do between players as allowed by the rules are to split each color group between $J_1$ and $J_2$, so that no development is possible and average rents are as even as possible. However, this behaviour will not happen with real investments for several reasons.

- When there are more than 2 players, two-way monopoly trades are advantageous to players allowed to develop, because the other players become additional sources of income to them. Therefore, investors undergo increases in both drift and volatility of their C-processes, which is particularly justified if (at least) one suffers from a negative mean expectancy for the time being.
- When there is inflation, assuming that the average purchasing power of players remains constant (as said above, we recover this when offsetting the total mean expectancy to 0), we recall that the total C-process $C$ is sure to default, indicating that there will be only one outstanding player (at most, but we neglect the case of a simultaneous default of $J_1$ and $J_2$, specifically impossible in standard Monopoly). This means that the game is actually zero-sum, so there are no profitable trades to both players.
- In a lesser extent, allowing infinite investment may lead to a zero-sum game for the same reason ; indeed, we stated above that positive return on investment rates enhance disequilibrium between players and should lead to a single winner. However, we do not know whether this perspective is encouraged from the viewpoint of game theory : identifying selfish development with defection in the prisoner's dilemma, many players may coordinate well enough to prevent an individual defection.

On a side note, when players are motivated by permanent consumption rather than survival, the relevant variable to them becomes the mean expectancy ; since investment in Monopoly has near zero total externality, cooperation is impossible in a zero-sum game.

## 4.3 Ending remarks

The computation of cooperative decisions in risk management requires full knowledge of the relevant parameters to the model. Bankruptcies may happen either when these parameters are not accurately estimated (e.g. replacing a C-process by a Lévy process erases momentum and overestimates the martingale parameter, and underestimation of risk leads to the crisis of 2008), or when players are motivated by greed rather than own survival. The game of Monopoly allows to recover several typical thoughts on investment strategies. We believe that the underlying structure of C-processes accurately accounts for the high volatility and momentum behaviour of several indicators of economy like stock prices, although a statistical test has not been done (we suspect that the underlying Markovian process $M$ calls for a Baum-Welch algorithm like in [5], with inherent complexity of computations). We finally investigated on the consequences of using such a process to model investment decisions rather a Lévy process, explaining e.g. “seasonal” behaviour of investors and dependency on the conjoncture. The remarks are consistent with common sense, so we will conclude that C-processes allow for a better than Lévy processes, and acceptable enough, model of real cash flows.

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# Ruin and investment in a Markovian environment

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This thesis aims at modelling and optimize an agent's (called "he") investment strategies when subjected to a Markovian environement, and to a liquidity risk happening when he runs out of liquid assets during an expense. Thoughout this work, we deem that he aims at avoiding default ; for this purpose, investment opportunities are available to him, allowing to increase his future expected incomes at the price of an immediate expense, therefore risking premature bankruptcy since investment is deemes illiquid : our goal is to find conditions under which incurring such liquidity risks is more advisable than declining a permanent income.

Keywords : ruin theory, Markovian environement, stochastic processes, risky investissement.

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L'objet de cette thèse est de modéliser et optimiser les stratégies d'investissement d'un agent soumis à un environnement markovien, et à un risque de liquidité se déclarant quand il ne peut plus faire face à une sortie d'argent faute d'actifs liquides. Durant cette étude, nous supposerons que son objectif est d'éviter la faillite ; il dispose pour cela d'opportunités d'investissement, lui permettant d'accroître ses gains futurs en échange d'une dépense immédiate, risquant ainsi une ruine prématurée puisque l'investissement est supposé illiquide : le but du travail est de déterminer les conditions sous lesquelles il est plus judicieux de courir un tel risque de liquidité que de renoncer à un revenu permanent.

Mots-clés : théorie de la ruine, environnement markovien, processus stochastiques, investissement risqué.
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