NMPC without terminal constraints ⋆

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Abstract: This paper provides a survey on recent results on NMPC without terminal constraints. We investigate stability, performance and feasibility issues, both for classical stabilizing NMPC and for economic NMPC. Besides explaning and comparing different approaches obtained during the last couple of years, the paper also contains previously unpublished results and proofs for exponential convergence of economic NMPC performance and for recursive feasibility of stabilizing NMPC without stabilizing terminal constraints. Several examples are presented to illustrate our findings.

Keywords: predictive control, nonlinear control, optimal control, stability analysis, performance analysis, constraints

1. INTRODUCTION

When looking at the NMPC literature, the vast majority of papers considers schemes with terminal constraints. This may lead to the impression that imposing terminal constraints is a necessary condition for obtaining rigorous proofs for, e.g., stability and feasibility of NMPC schemes. It is the purpose of this paper to demonstrate that this is not the case by summarizing and explaining recent advances in the analysis of NMPC schemes without terminal constraints.

Particularly, we will focus on properties like stability, performance and feasibility. We compare and explain different approaches which can be found in the literature, including some new results from our own research — both for stabilizing NMPC and for the relatively new area of economic NMPC — as well as alternative (and shorter) proofs for some known results. Moreover, we will discuss a motivating example which shows that it is not only possible but can even be advantageous to omit terminal constraints. Further examples are provided to illustrate certain results and phenomena described in the paper. While these examples often have linear dynamics (in order to keep them technically simple), we emphasize that all results presented in this paper hold for general nonlinear discrete time systems. For all results we provide proofs which are, however, often only sketched in order to highlight the main arguments with references to the appropriate literature for details.

The paper is organized as follows. After introducing the basic setting and notation in Section 2, we discuss stability and performance properties of stabilizing NMPC in Section 3. The same topics are investigated for economic NMPC in Section 4. Section 5 gives two results on feasibility of NMPC schemes without terminal constraints and Section 6 concludes the paper.

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2. SETTING AND PRELIMINARIES

We consider discrete time control systems with state $x \in X$ and control values $u \in U$, where X and U are normed spaces with norms denoted by $\|\cdot\|$. The control system under consideration is given by

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

with $f : X \times U \to X$. For any control sequence $u = (u(0), \dots, u(K-1)) \in U^K$ or $u = (u(0), u(1), \dots) \in U^\infty$, by $x_u(k, x)$ we denote the solution of (1) with initial value $x = x_u(0, x) \in X$. Note that the general setting with X and U being normed spaces particularly covers exact sampled data models of finite dimensional continuous time systems with sampling time $T > 0$ by setting $U = L_\infty([0, T], \mathbb{R}^m)$, i.e., by defining the discrete time control value $u(n)$ to be the piece of the continuous time control function acting on the $(n + 1)$st sampling interval. Of course, sampled data systems with zero order hold can be modelled as well by defining $u(n)$ to be the constant control input on each sampling interval. Likewise, sampled infinite dimensional systems governed by PDEs fit to our setting.

For given admissible sets of states $\mathbb{X} \subseteq X$ and control values $\mathbb{U} \subseteq U$ and an initial value $x \in \mathbb{X}$ we call the control sequences $u \in \mathbb{U}^K$ satisfying $x_u(k, x) \in \mathbb{X}$ for all $k = 0, \dots, K-1$ admissible. The set of all admissible control sequences is denoted by $\mathbb{U}^K(x)$. Similarly, we define the set $\mathbb{U}^\infty(x)$ of admissible control sequences of infinite length.

Given a state feedback map $\mu : \mathbb{X} \to \mathbb{U}$, we denote the solutions of the closed loop system $x(k+1) = f(x(k), \mu(x(k)))$ by $x_\mu(k)$ or by $x_\mu(k, x)$ if we want to emphasize the dependence on the initial value $x = x_\mu(0)$. We say that a feedback law μ is admissible if $f(x, \mu(x)) \in \mathbb{X}$ holds for all $x \in \mathbb{X}$.

Our goal is now to find an admissible feedback controller which (approximately) solves the infinite horizon optimal control problem

$$\underset{u \in \mathbb{U}^\infty(x)}{\text{minimize}} \quad J_\infty(x,u) := \sum_{k=0}^{\infty} \ell(x_u(k,x), u(k)) \qquad (2)$$

where $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is called the *stage cost* or *running cost*. We define the optimal value function related to (2) by $V_\infty(x) := \inf_{u \in \mathbb{U}^\infty(x)} J_\infty(x,u)$. Note that the state constraints $x_u(k,x) \in \mathbb{X}$ are implicitly included in (2) since we minimize over $u \in \mathbb{U}^\infty(x)$. In order to measure the performance of a given feedback law $\mu : \mathbb{X} \to \mathbb{U}$ we define the closed loop cost

$$J_\infty^{cl}(x,\mu) := \sum_{k=0}^{\infty} \ell(x_\mu(k,x), \mu(x_\mu(k,x))).$$

Approximate infinite horizon optimality of μ then refers to the fact that $J_\infty^{cl}(x,\mu) \approx V_\infty(x)$ holds. A substantial portion of our results will focus on the case where the optimization objective in (2) is designed in order to solve a stabilization or tracking problem, see Section 3, below, for details. In this case, stability of the closed loop will be equally important as approximate optimality.

Since infinite horizon problems (2) are typically difficult to be solved directly, we use the NMPC receding horizon approach in order to compute a feedback law. To this end, we define the finite horizon counterpart of (2)

$$\underset{u \in \mathbb{U}^N(x)}{\text{minimize}} \quad J_N(x,u) := \sum_{k=0}^{N-1} \ell(x_u(k,x), u(k)) \qquad (3)$$

and the corresponding optimal value function $V_N(x) := \inf_{u \in \mathbb{U}^N(x)} J_N(x,u)$. We assume that for each $x \in \mathbb{X}$ a (not necessarily unique) optimal control sequence $u^\star \in \mathbb{U}^N(x)$ for (3) exists, i.e., satisfying $V_N(x) = J_N(x,u^\star)$. While most of the statements in this paper could alternatively be formulated via approximate mininizers, the existence of a minimizer considerably simplifies the presentation of the results.

The NMPC approach then consists of solving the open loop optimization problem (3) with initial value $x = x_\mu(k)$ at each sampling instant k for some given optimization horizon $N \in \mathbb{N}$ and then defining the feedback value $\mu(x) = \mu_N(x)$ to be the first element of the corresponding optimal control sequence, i.e.,

$$\mu_N(x) := u^\star(0).$$

Since nowadays efficient algorithms for the necessary online minimization of $J_N(x,u)$ are available (for instance, by converting the problem into a static nonlinear optimization problem followed by sequential quadratic programming (SQP) or an interior point method for solving this problem, see, e.g., [Grüne and Pannek, 2011, Chapter 10] and the references therein), this method is computationally feasible for large classes of systems. In this context we like to note that even when the system is too complex or the systems' dimension is too high for an online solution of (3) within one sampling period, NMPC can still be useful as an approximate numerical solution method for infinite horizon optimal control problems which would be computationally intractable otherwise.

We end this section by defining some notation and recalling some concepts which we will need in the sequel. The set $\mathbb{R}_0^+$ denotes the non negative real numbers. With $\mathcal{K}_\infty$ we denote the set of continuous functions $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which are strictly increasing and unbounded with $\alpha(0) = 0$. With $\mathcal{KL}$ we denote the set of continuous functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which are strictly increasing in the first argument, strictly decreasing to 0 in the second argument and satisfy $\beta(0,t) = 0$ for all $t \geq 0$. With $\lceil r \rceil$ and $\lfloor r \rfloor$ we denote the smallest integer $\geq r \in \mathbb{R}$ and the largest integer $\leq r \in \mathbb{R}$, respectively.

Although we do not use dynamic programming for actually solving our optimal control problems, in our analysis we will make extensive use of the dynamic programming principle, cf. Bertsekas [1995]. The form of this principle which applies here states that for the optimal control sequence $u^\star$ for the problem with finite horizon N and each $K \in \{1, \dots, N-1\}$ the equality

$$V_N(x) = \sum_{k=0}^{K-1} \ell(x_{u^\star}(k,x), u^\star(k)) + V_{N-K}(x_{u^\star}(K,x)) \qquad (4)$$

holds. As a consequence, since $\mu_N(x) = u^\star(0)$, for $K = 1$ we get

$$V_N(x) = \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x))). \qquad (5)$$

3. STABILIZING NMPC

This section will focus on stability and performance issues of stabilizing NMPC, i.e., of NMPC schemes which are designed to yield a controller stabilizing a given reference solution. In order to avoid problems with feasibility, we assume $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$, i.e., that for each initial value $x \in \mathbb{X}$ we can find a trajectory staying inside $\mathbb{X}$ for all future times. Note that this property immediately implies that for each $x \in \mathbb{X}$ there exists $u \in \mathbb{U}$ with $f(x,u) \in \mathbb{X}$, i.e., that $\mathbb{X}$ is *controlled forward invariant* or *viable*, cf. Aubin [1991]. Ways to relax this condition are discussed in Section 5, below.

In order to simplify the presentation, we restrict ourselves to the problem of asymptotically stabilizing the origin $x^e = 0$. To this end we assume that 0 is an equilibrium of f for some control value $u^e \in \mathbb{U}$, i.e., that $f(0, u^e) = 0$ holds. Asymptotic stability of the origin is then defined as follows.

Definition 3.1. Consider the system (1). Then we say that a feedback law $\mu : \mathbb{X} \to \mathbb{U}$ renders the origin *asymptotically stable* if there exists a function $\beta \in \mathcal{KL}$ such that the closed loop trajectory x_μ satisfies the inequality

$$\|x_\mu(k,x)\| \leq \beta(\|x\|, k)$$

for all $x \in \mathbb{X}$ and all $k \in \mathbb{N}$.

NMPC is easily adapted to more general asymptotic stability settings. For instance, we could use equilibria different from 0, time varying references (including periodic ones) or whole compact sets in place of a single point. All these extensions can be straightforwardly achieved by replacing the norms $\|x_\mu(k,x)\|$ and $\|x\|$ — in Definition 3.1 and in all subsequent statements — by the distances to the respective points or sets.

The well known idea of stabilizing NMPC is now to use a stage cost ℓ which penalizes the distance of the state x

to the origin. Moreover, we require that the corresponding infinite horizon problem is well defined in the sense that the optimal value function is bounded by a $\mathcal{K}_\infty$ function (if a stabilizing feedback law exists, then this can always be achieved by choosing ℓ appropriately, see, e.g., [Grüne and Pannek, 2011, Theorem 4.3]). Formally, these conditions are stated in the following assumptions.

Assumption 3.2. (i) There are functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\ell^\star(x) := \min_{u\in\mathbb{U}} \ell(x,u)$ satisfies

$$\alpha_1(\|x\|) \le \ell^\star(x) \le \alpha_2(\|x\|).$$

(ii) There exists a function $\alpha_3 \in \mathcal{K}_\infty$ such that

$$V_\infty(x) \le \alpha_3(\|x\|)$$

holds for all $x \in \mathbb{X}$.

Our goal is now to find conditions which ensure that the NMPC feedback law μ_N stabilizes the system (1) in the sense of Definition 3.1.

There exists an elaborate and elegant theory for ensuring stability of NMPC schemes by adding terminal constraints and terminal costs, see the seminal survey paper Mayne et al. [2000] or the monographs Rawlings and Mayne [2009] or Grüne and Pannek [2011]. This approach requires that the finite horizon problem (3) to be solved in each step is changed to

$$\underset{u\in\mathbb{U}^N(x)}{\text{minimize}} \;\; J_N(x,u) := \sum_{k=0}^{N-1} \ell(x_u(k,x),u(k)) + F(x_u(N,x)).$$

Here $F : \mathbb{X}_0 \to \mathbb{R}_0^+$ is called a *terminal cost* which is defined on a so called *terminal region* $\mathbb{X}_0$ and in order to be well defined we need to add the terminal constraint $x_u(N,x) \in \mathbb{X}_0$ as an additional constraint to (3). The terminal cost function is then assumed to be a control Lyapunov function on $\mathbb{X}_0$ which is compatible with ℓ. This means that for each $x \in \mathbb{X}_0$ there exists a control value $u \in \mathbb{U}$ satisfying $f(x,u) \in \mathbb{X}_0$ and

$$F(f(x,u)) \le F(x) - \ell(x,u).$$

In this paper, we are not going to apply this approach. In particular, we like to avoid using terminal constraints of the type $x_u(N,x) \in \mathbb{X}_0$. We illustrate our motivation for this by means of the following example.

3.1 A motivating example

Example 3.3. We consider a swarm of P "agents" moving in $\mathbb{R}^2$ given by

$$\dot{x}_i = f(x_i,u_i)$$

for $i = 1,\dots,P$ with $x_i = (x_{i1},x_{i2},x_{i3},x_{i4})^T \in X_i = \mathbb{R}^4$, $u_i = (u_{i1},u_{i2})^T \in U_i = \mathbb{R}^2$ and $f : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$f(x_i,u_i) = (x_{i2},u_{i1},x_{i4},u_{i2})^T.$$

The system and the following simulations are taken from Jahn [2010], to which we also refer for all details of the parallel implementation of the NMPC algorithm on a graphics processor (GPU).

The overall state space of the system is $X = \mathbb{R}^{4P}$ and the control inputs lie in the space $U = \mathbb{R}^{2P}$. Each agent i can be considered as a point moving in the plane with position $(x_{i1},x_{i3})^T$ and velocity (x_{i2},x_{i4}) whose acceleration can be controlled by the control input $u_i = (u_{i1},u_{i2})^T$. While the system dynamics is linear, the constraints render the overall problem nonlinear: We impose control input constraints $\mathbb{U} = [-12,12]^{2P}$ and state constraints given by

$$\mathbb{X} := \left\{ x \in X \;\middle|\; \begin{array}{l} \|(x_{i1},x_{i3})^T - (x_{j1},x_{j3})^T\| \ge 0.1 \\ \text{for all } i,j = 1,\dots,P \text{ with } i \ne j \\ \text{and } (x_{i1},x_{i3})^T \notin \overline{B}_{0.3}(y_p),\ p = 1,2,3 \\ \text{and } \|(x_{i2},x_{i4})^T\| \le 1 \\ \text{for all } i = 1,\dots,P \end{array} \right\}$$

with $y_1 = (1.4,0.4)^T$, $y_2 = (1.4,-0.4)^T$, $y_3 = (2.1,0)^T$ and $\overline{B}_r(y)$ denoting the closed ball with radius r around y in $\mathbb{R}^2$. The first constraints are non-collision constraints for the agents, the second constraints define three disc-shaped obstacles which cannot be crossed by the subsystems and the third constraints limit the speed of the agents.

The discrete time system (1) is obtained from the associated zero order hold sampled data system for sampling period $T = 0.02s$. The goal of the optimal control problem is to first move all agents to the origin $x_1^e = 0 \in \mathbb{R}^4$, i.e., to the position $x_1^p = (0,0)^T$. After $t = 20s$, i.e., after $k = 1000$ sampling times, the control task is changed and the agents are supposed to move to $x_2^e = (3,0,0,0)^T \in \mathbb{R}^4$, i.e., to the position $x_2^p = (3,0)^T$. To this end, the stage cost

$$\ell(x,u) = \sum_{i=1}^{P} \Big(\|(x_{i1},x_{i3})^T - x^p\| + \|(x_{i2},x_{i4})^T\|/50 \Big)$$

is used with $x^p = x_1^p$ for the sampling instants $k \in \{0,\dots,999\}$ and $x^p = x_2^p$ for $k \in \{1000,\dots,1999\}$. Observe that u is not penalized in the cost. The initial state for each agent is $x_i^0 = ((i-1)0.12,0,1,0)^T$ and the problem was solved by NMPC without terminal constraints with horizon $N = 6$. Figure 1 shows the positions of a swarm of $P = 64$ agents (depicted as small blue discs) under the NMPC feedback law at different times of the simulation. The system shows exactly the desired behavior: the agents first move to a position as close as possible to the origin which is reached at about $k = 700$. After $k = 1000$, i.e., after changing the functional, the swarm moves through the obstacles (depicted as large red discs) to the new desired position $x_2^p = (3,0)^T$.

In fact, this example uses a slight variation of the basic NMPC scheme outlined in Section 2 which is explained in Section 3.3(d), below. However, we emphasize that neither terminal constraints nor Lyapunov function terminal costs were used in this implementation. In fact — given the constraints of the velocity on the system — the design of a terminal constraint set $\mathbb{X}_0$ rendering the initial configuration in our simulation feasible (i.e., ensuring that there exists a control sequence u satisfying $x_u(N,x_0) \in \mathbb{X}_0$) would either require the terminal cost F to be defined on a very large terminal region $\mathbb{X}_0$ (including the obstacles which make the design of a control Lyapunov functions a hard task) or a considerable enlargement of the optimization horizon N leading to a very difficult optimization problem to be solved in each step of the NMPC scheme.

The attentive reader may moreover have noticed that in this example the desired positions x_1^e and x_2^e are not even admissible equilibria for the overall system due to the non-collision constraints. Hence, in this example the goal is not to stabilize the system at the point x^e but rather at the set of admissible states $x \in \mathbb{X}$ at which $\ell(x)$ attains its minumum. Using terminal constraints, this set would

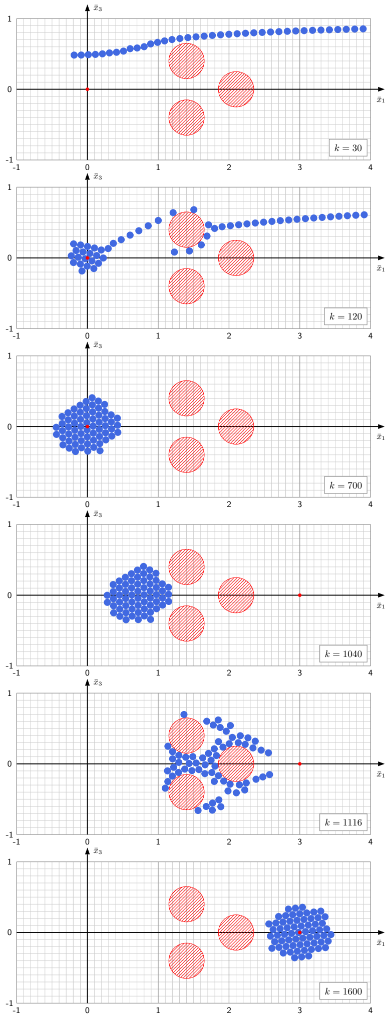

Fig. 1. Position of the agents at times $k = 30$, 120, 700, 1040, 1116 and 1600 (top to bottom)

have to be computed beforehand and used in the design of the terminal cost. However, as our simulation shows, for NMPC without terminal constraints this is not needed, at all, since the NMPC scheme is able to find the "best" state as part of the optimization taking place in each step of the algorithm.

3.2 Stability results

There are several papers in which stability results for NMPC without stabilizing terminal constraints have been developed, e.g., Alamir and Bornard [1994], Primbs and Nevistić [2000], Jadbabaie and Hauser [2005], Grimm et al. [2005], Tuna et al. [2006], Grüne and Rantzer [2008], Grüne [2009], Grüne et al. [2010a]. Here, we explain the main ideas behind the last five of these references since they use similar conditions and arguments. Before discussing these arguments, we state the following sufficient condition for both stability and approximately optimal performance.

Proposition 3.4. Let[1] $N \geq 2$ and assume that the optimal value function V_N and the NMPC feedback law μ_N satisfy the inequality

$$V_N(f(x, \mu_N(x))) \leq V_N(x) - \alpha \ell(x, \mu_N(x)) \qquad (6)$$

for some $\alpha \in (0, 1]$ and all $x \in \mathbb{X}$. Then the inequality

$$J_\infty^{cl}(x, \mu_N) \leq V_\infty(x)/\alpha$$

holds for all $x \in \mathbb{X}$. If, moreover, Assumption 3.2 holds, then μ_N stabilizes (1) in the sense of Definition 3.1.

[1] Note that our definition of J_N implies that $N = 2$ is the shortest meaningful horizon, since for $N = 1$ the stage cost is evaluated only at the initial value.

The proof of the first assertion, which uses quite straightforward dynamic programming arguments, can be found in [Grüne and Rantzer, 2008, Proposition 2.2]. Note that for $\alpha = 1$ the inequality is in fact an equality, since $J_\infty^{cl}(x, \mu) \geq V_\infty(x)$ holds for any admissible controller μ. The second assertion follows since Assumption 3.2 in conjunction with the obvious inequalities $\ell^\star(x) \leq V_N(x) \leq V_\infty(x)$ implies

$$\alpha_1(\|x\|) \leq V_N(x) \leq \alpha_3(\|x\|)$$

for all $x \in \mathbb{X}$. Together with (6) this implies that V_N is a Lyapunov function for the closed loop from which asymptotic stability can be concluded, cf. [Grüne and Pannek, 2011, Theorem 4.11].

One could now try to compute α by computing the optimal value functions V_N, which is essentially the approach from Primbs and Nevistić [2000]. Since the computation of V_N, however, is only feasible in exceptional cases, we will instead use suitable bounds on the value functions. More precisely, our main assumption is the following.

Assumption 3.5. There exists $\gamma > 0$ such that the inequality

$$V_N(x) \leq \gamma \ell^\star(x)$$

holds for all $N \geq 2$, all $x \in \mathbb{X}$ and $\ell^\star$ from Assumption 3.2.

One way to ensure Assumption 3.5 is by assuming an exponential controllability condition w.r.t. the stage cost ℓ of the following type:

There exist constants $C > 0$ and $\sigma \in (0, 1)$ such that for each $x \in \mathbb{X}$ and each $N \in \mathbb{N}$ there is $u \in \mathbb{U}^N(x)$ such that

$$\ell(x_u(k, x), u(x)) \leq C\sigma^k \ell^\star(x) \qquad (7)$$

holds for $k = 0, \ldots, N - 1$.

This assumption immediately implies

$$J_N(x, u) \leq \sum_{k=0}^{N-1} C\sigma^k \ell^\star(x) \leq \frac{C}{1-\sigma} \ell^\star(x)$$

and thus Assumption 3.5 with $\gamma = C/(1 - \sigma)$.

Note that it is not necessary that the system itself is exponentially controllable to the origin for (7) to hold. As an example, consider the 1d system

$$x(k + 1) = x(k) + u(k)x(k)^3$$

with $\mathbb{X} = [-1, 1]$ and $\mathbb{U} = [-1, 1]$. While the system itself is not exponentially controllable to 0 (due to the nonlinearity $x(k)^3$ multiplied with $u(k)$), (7) can still be satisfied by choosing, e.g.,

$$\ell(x, u) = e^{-\frac{1}{2x^2}},$$

for details see [Grüne and Pannek, 2011, Example 6.5]

We now claim that Assumption 3.5 implies (6) for sufficiently large N and describe three different ways to prove this claim leading to three different estimates for α in (6).

Variant 1: Consider the optimal trajectory $x_{u^\star}(k, x)$ of (3) for some $N \geq 2$ and abbreviate $\ell_k = \ell(x_{u^\star}(k, x), u^\star(k))$.

This implies $V_N(x) = \sum_{k=0}^{N-1} \ell_k$ and $\ell_k \ge \ell^\star(x_{u^\star}(k,x))$. Now Assumption 3.5 implies $\sum_{k=0}^{N-1} \ell_k \le \gamma\ell_0$ and thus $\ell_p \le \gamma\ell_0/N$ for at least one $p \in \{0,\dots,N-1\}$. Supposing $N > \gamma$ moreover yields $p \ge 1$. Hence, using Assumption 3.5 for $x = x_{u^\star}(p,x)$ yields

$$V_{N-p+1}(x_{u^\star}(p,x)) \le \gamma\ell_p \quad \text{and} \quad \ell_p \le \gamma\ell_0/N \tag{8}$$

for some $p \in \{1,\dots,N-1\}$. We denote the optimal control sequence corresponding to $V_{N-p+1}(x_{u^\star}(p,x))$ by $\tilde u^\star$. Now, if we use the control sequence $\tilde u$ consisting of the control values $(u^\star(1),\dots,u^\star(p-1),\tilde u^\star(0),\dots,\tilde u^\star(N-p))$ we obtain

$$\begin{aligned} V_N(x_{u^\star}(1,x)) &\le J_N(x_{u^\star}(1,x),\tilde u) \\ &= \sum_{k=1}^{p-1} \ell_k + V_{N-p+1}(x_{u^\star}(p,x)) \qquad (9)\\ &= \sum_{k=1}^{p} \ell_k - \ell_p + V_{N-p+1}(x_{u^\star}(p,x)) \\ &\le \sum_{k=1}^{p} \ell_k + (\gamma-1)\ell_p \\ &\le \sum_{k=1}^{p} \ell_k + \gamma(\gamma-1)\ell_0/N. \end{aligned}$$

Using that by definition of μ_N we have $f(x,\mu_N(x)) = x_{u^\star}(1,x)$, we thus end up with

$$\begin{aligned} V_N(f(x,\mu_N(x))) &\le \sum_{k=1}^{p-1} \ell_k + \gamma(\gamma-1)\ell_0/N \\ &\le V_N(x) - \ell_0 + \gamma(\gamma-1)\ell_0/N \\ &\le V_N(x) - (1-\gamma(\gamma-1)/N)\ell_0. \end{aligned}$$

By definition of ℓ_0 this ensures (6) with $\alpha \ge 1-\gamma(\gamma-1)/N$. In particular, for $N > \gamma(\gamma-1)$ we obtain $\alpha > 0$ and asymptotic stability can be concluded from Proposition 3.4.

Variant 1 essentially follows the arguments used in Grimm et al. [2005], where it should be noted that the setting in this reference is considerably more general than our setting here, cf. Section 3.3(e) and (f), below. In the present setting, we can considerably enlarge the lower bound for α — and thus decrease the upper bound for the minimal stabilizing horizon N — using the following alternative proof.

Variant 2: We use the same notation as in Variant 1. Since by the dynamic programming principle tails of optimal trajectories are again optimal trajectories, for each $p = 0,\dots,N-2$ we obtain

$$\sum_{k=p}^{N-1} \ell_k = V_{N-p}(x_{u^\star}(p,x)) \le \gamma\ell^\star(x_{u^\star}(p,x)) \le \gamma\ell_p$$

implying

$$\sum_{k=p+1}^{N-1} \ell_k \le (\gamma-1)\ell_p \quad \text{for all } p = 0,\dots,N-2 \tag{10}$$

which yields

$$\ell_p + \sum_{k=p+1}^{N-1} \ell_k \ge \frac{\sum_{k=p+1}^{N-1} \ell_k}{\gamma-1} + \sum_{k=p+1}^{N-1} \ell_k = \frac{\gamma}{\gamma-1} \sum_{k=p+1}^{N-1} \ell_k.$$

Using this inequality inductively for $p = 1,\dots,N-2$ yields

$$\sum_{k=1}^{N-1} \ell_k \ge \left(\frac{\gamma}{\gamma-1}\right)^{N-2} \ell_{N-1}.$$

Applying (10) for $p = 0$ we then obtain

$$(\gamma-1)\ell_0 \ge \sum_{k=1}^{N-1} \ell_j \ge \left(\frac{\gamma}{\gamma-1}\right)^{N-2} \ell_{N-1}$$

which finally leads to

$$\ell_{N-1} \le (\gamma-1)\left(\frac{\gamma-1}{\gamma}\right)^{N-2} \ell_0 = \gamma\left(\frac{\gamma-1}{\gamma}\right)^{N-1} \ell_0.$$

Replacing the second inequality in (8) by this inequality and continuing as in Variant 1 with $p = N-1$ we obtain

$$\begin{aligned} & V_N(f(x,\mu_N(x))) \\ \le\ & V_N(x) - \left(1 - \gamma(\gamma-1)\left(\frac{\gamma-1}{\gamma}\right)^{N-1}\right)\ell_0 \\ =\ & V_N(x) - \left(1 - \frac{(\gamma-1)^N}{\gamma^{N-2}}\right)\ell_0 \end{aligned}$$

which ensures (6) with $\alpha \ge 1 - (\gamma-1)^N/\gamma^{N-2}$. A little computation shows that the inequality $\alpha > 0$ needed to conclude stability from Proposition 3.4 is now ensured for

$$N > 2 + 2\frac{\ln\gamma}{\ln\gamma - \ln(\gamma-1)}.$$

For $\gamma \to \infty$, the expression on the right hand side grows like[2] $2\gamma\ln\gamma$ which shows that Variant 2 yields a much smaller bound on the optimization horizon N needed to guarantee stability than Variant 1. This bound and its derivation is similar to Tuna et al. [2006] and Grüne and Rantzer [2008].

Variant 3: The bound from Variant 2 can be further improved by using that (9) and Assumption 3.5 implies

$$\begin{aligned} V_N(x_{u^\star}(1,x)) &\le \sum_{k=1}^{p-1} \ell_k + V_{N-p+1}(x_{u^\star}(p,x)) \\ &\le \sum_{k=1}^{p-1} \ell_k + \gamma\ell_p \quad \text{for all } p = 1,\dots,N-1. \qquad (11) \end{aligned}$$

Although the derivation of a bound for α from these inequalities is not as simple as in Variants 1 and 2, an explicit expression can still be obtained: in Grüne [2009] it was observed that computing α in (6) under the inequalities (10) and (11) is a linear optimization problem and an explicit solution of this optimization problem was obtained in Grüne et al. [2010a]. This explicit solution reads

$$\alpha = 1 - \frac{(\gamma-1)^N}{\gamma^{N-1} - (\gamma-1)^{N-1}} \tag{12}$$

which is positive if the inequality

$$N > 2 + \frac{\ln(\gamma-1)}{\ln\gamma - \ln(\gamma-1)} \tag{13}$$

holds. For $\gamma \to \infty$ this bound behaves asymptotically like $\gamma\ln\gamma$ and is thus about half the size of the bound from

[2] By this we mean that

$$\left(2 + 2\frac{\ln\gamma}{\ln\gamma - \ln(\gamma-1)}\right) \Big/ (2\gamma\ln\gamma) \to 1 \quad \text{as } \gamma \to \infty.$$

Variant 2. We summarize the findings in the following theorem.

Theorem 3.6. Consider an NMPC problem satisfying Assumptions 3.2 and 3.5. Then the closed loop is asymptotically stable if the optimization horizon N satisfies (13). In this case, the inequality

$$J_\infty^{cl}(x, \mu_N) \le V_\infty(x)/\alpha$$

holds for all $x \in \mathbb{X}$ with α from (12).

As shown in Grüne [2009], in terms of stability this is the best possible bound that can be derived from Assumption 3.5: if we consider the class of all NMPC problems satisfying Assumptions 3.2 and 3.5 for a given $\gamma > 1$ and choose N smaller than the bound in (13) (resulting in $\alpha < 0$), then there is at least one system in this class for which the NMPC closed loop solutions do not converge to 0, i.e., for which asymptotic stability fails to hold.

We remark that computing γ in Assumption 3.5 is in general a difficult task, hence, rigorously ensuring stability of NMPC via this inequality may not always be feasible. However, there exist NMPC problems — including infinite dimensional ones — for which a derivation was carried out using the controllability condition (7), see, e.g., Altmüller et al. [2010a,b]. Moreover, even if γ in Assumption 3.5 is not explicitly computable, the estimates provided in this section can provide valuable guidelines for designing stage costs ℓ under which stabilization is possible with small N, see, e.g., [Grüne and Pannek, 2011, Section 6.6]. Note, however, that a stage cost particulary designed for stabilization purposes may not necessarily reflect desired performance criteria like, e.g., low energy consumption, hence there may be a tradeoff between good closed loop performance and stability. In this sense, the situation is similar as in terminal constrained MPC where the terminal constraint may also enforce stability at the price of lower performance. Finally, α in (6) can alternatively be computed numerically, see, e.g., Grüne and Pannek [2009] or [Grüne and Pannek, 2011, Section 7.7].

3.3 Extensions

The methods for obtaining stability and performance estimates described in the last section can be extended to more general settings in various ways. Here we describe some of these extensions.

(a) The values γ in Assumption 3.5 can be chosen to depend on N. Since the V_N are typically strictly increasing, this allows to use smaller γ_N for smaller N and thus better estimates for α and the stabilizing optimization horizons. The generalization of (12) to this setting is

$$\alpha = 1 - \frac{(\gamma_N - 1)\prod_{k=2}^N(\gamma_k - 1)}{\prod_{k=2}^N \gamma_k - \prod_{k=2}^N(\gamma_k - 1)}, \qquad (14)$$

for details see Grüne et al. [2010a] or [Grüne and Pannek, 2011, Section 6.4]. Particularly, if (7) holds, then we obtain the inequality

$$V_N(x) \le \gamma_N \ell^\star(x) \quad \text{with} \quad \gamma_N = C\frac{1-\sigma^N}{1-\sigma}.$$

An analysis of (14) for this case shows that, no matter what $\sigma \in (0,1)$ is, for C sufficiently close to 1 we can always obtain stability with $N = 2$, i.e., with the shortest possible prediction horizon, see, e.g., Grüne et al. [2010a] or [Grüne and Pannek, 2011, Section 6.6].

(b) If the system (1) is a sampled data system originating from a Lipschitz ODE, then the difference between $x_u(k,x)$ and $x_u(k+p,x)$ can be bounded by a Lipschitz estimate. This bound can be used as an additional growth condition for deriving the value for α in (12) or (14) and considerably improves the value of α for small sampling periods. For details see Grüne et al. [2010b], [Grüne and Pannek, 2011, Section 7.5] or [Worthmann, 2011, Section 5.3].

(c) Instead of using only the first element $u^\star(0)$ of the finite horizon optimal control sequence, one may also implement the first m elements. Formulas (12) and (14) can be adapted to this case and particularly show that when choosing $m = \lceil N/2 \rceil$ (i.e., when using the first half of elements of the finite horizon optimal control sequence) then the minimal N needed for ensuring stability grows only linearly in γ as $\gamma \to \infty$, as opposed to the asymptotic growth $\gamma \ln \gamma$ for the "classical" NMPC setting with $m = 1$, cf. Variant 3. For details we refer to Grüne et al. [2010a] and [Worthmann, 2011, Chapter 4]. In this setting, the advantage of Variant 3 over Variant 2 for computing α becomes even more significant, for a detailed comparison see [Worthmann, 2011, Section 5.4].

(d) The optimal control problem (3) can be modified by including terminal weights $\omega > 1$, leading to [3]

$$\underset{u\in\mathbb{U}^N(x)}{\text{minimize}} \;\; J_N(x,u) := \sum_{k=0}^{N-2} \ell(x_u(k,x), u(k)) + \omega\ell(x_u(N-1,x), u(N-1)). \quad (15)$$

The terminal weight ω can be included into the stability analysis. For instance, in the case $\gamma > \omega > 1$ Formula (13) becomes

$$N > 2 + \frac{\ln(\gamma - \omega)}{\ln\gamma - \ln(\gamma - 1)}.$$

Note however, that γ in Assumption 3.5 typically grows with ω when passing from (3) to (15). Moreover, the estimate for J_∞^{cl} in Proposition 3.4 in general no longer holds when introducing terminal weights. Nevertheless, the stability result in Proposition 3.4 remains valid and a judicious choice of ω can significantly reduce the optimization horizon N needed for stabilization, for details see Grüne et al. [2010a] or [Grüne and Pannek, 2011, Section 7.2]. This has also been exploited in the example in Section 3.1, where the terminal weight $\omega = 20$ was used in the simulations.

(e) Instead of the linear bound $\gamma\ell^\star(x)$ in Assumption 3.5 one could use a general nonlinear bound

$$V_N(x) \le \delta(\ell^\star(x))$$

for some $\delta \in \mathcal{K}_\infty$. It can then be shown that we still obtain semiglobal asymptotic stability (in the optimization horizon N) if $\delta(\cdot)$ has a linear upper bound on each interval

[3] Note that this approach is different from adding a Lyapunov function terminal cost as, e.g., in Mayne et al. [2000] (see also the references therein and the detailed analysis for linear MPC in Löfberg [2003]), since the terminal cost is a multiple of the stage cost here and does not need to be a Lyapunov function.

of the form $[0, R]$ and semiglobal practical asymptotic stability if $\delta(\cdot)$ has a linear upper bound on each interval of the form $[r, R]$, see [Grüne and Pannek, 2011, Section 6.7]. Since the latter holds for any $\mathcal{K}_\infty$-function, this proves semiglobal practical asymptotic stability for general nonlinear bounds, a fact already known from Grimm et al. [2005].

(f) The lower bound on $\ell^\star$ in Assumption 3.2(i) can be replaced by a detectability condition, thus allowing to use positive semidefinite stage costs. In this case, V_N will in general not be a Lyapunov function anymore. Rather, a Lyapunov function can be constructed from V_N and an auxiliary function from the detectability condition. For details see Grimm et al. [2005] or [Grüne and Pannek, 2011, Section 7.3].

(g) We finally remark that a continuous time version of the estimates from Variant 3 was derived in Reble and Allgöwer [2011]. An in depth comparison between the discrete time and the continuous time estimates has been carried out in Worthmann et al. [2012].

4. ECONOMIC NMPC

In the previous section we have considered NMPC problems in which the stage cost penalizes the distance to some desired equilibrium. There are, however, many optimal control problems with other optimization objectives.

Example 4.1. (i) For instance, one may look at the problem of keeping the state $x(k)$ of (1) inside the admissible set $\mathbb{X}$ with minimal control effort, which can, e.g., be modelled by the stage cost $\ell(x, u) = \|u\|^2$. In this case, ℓ does not satisfy the lower bound in Assumption 3.2(i). As a simple prototype system from this problem class, we may look at the (open loop unstable) 1d dynamics

$$x(k + 1) = 2x(k) + u(k)$$

with with $X = U = \mathbb{R}$, $\mathbb{X} = [-0.5, 0.5]$ and $\mathbb{U} = [-2, 2]$, cf. Grüne [2011] or Grüne [2012].

(ii) Another situation which does not fit the assumptions of the previous section occurs if ℓ satisfies the lower bound in Assumption 3.2(i) but 0 is not an equlibrium of the dynamics for any $u \in \mathbb{U}$. An example from this class, taken from Diehl et al. [2011], is a linearized model of a continuously stirred tank reactor with two dimensional affine linear dynamics

$$x(k + 1) = \begin{pmatrix} 0.8353 & 0 \\ 0.1065 & 0.9418 \end{pmatrix} x(k) + \begin{pmatrix} 0.00457 \\ -0.00457 \end{pmatrix} u(k) + \begin{pmatrix} 0.5559 \\ 0.5033 \end{pmatrix}$$

and stage cost $\ell(x, u) = \|x\|^2 + 0.05u^2$. We use the state and control constraints $\mathbb{X} = [-100, 100]^2$ and $\mathbb{U} = [-10, 10]$. Here the stage cost ℓ tries to force the system to the origin $(0, 0)^T$ with control 0. However, since $(x, u) = ((0, 0)^T, 0)$ is not an equilibrium for the dynamics, stabilization at this point will not be possible.

Instead, one may aim at the "best compromise", i.e., at the equilibrium (x^e, u^e) which yields the smallest value of ℓ among all equilibria of the dynamics. This equilibrium can be computed as $x^e \approx (3.546, 14.653)^T$ with $u^e \approx 6.163$ and cost $\ell(x^e, u^e) \approx 229.1876$.

In fact, Example 3.3 is similar to Example 4.1(ii) in the sense that the desired states x_1^e and x_2^e are not equilibria of the system under the imposed state constraints. However, in this example the set of minimizers $\{y \in \mathbb{X} \,|\, \ell(y) = \min_{x\in\mathbb{X}} \ell(x)\}$ w.r.t. the state constraint set $\mathbb{X}$ still consists of equilibria and the stage cost is positive definite with respect to this set, which is why this example still fits the framework of Section 3 (in the generalized setting discussed after Definition 3.1). In contrast to this, for Example 4.1(ii) the optimal equilibrium stage cost $\ell(x^e, u^e) \approx 229.1876$ does not coincide with the minimal value of ℓ over all admissible $x \in \mathbb{X}$ and $u \in \mathbb{U}$, which is $\ell(0, 0) = 0$.

4.1 Terminal constrained economic NMPC

Problems like those in Example 4.1 are referred to as *economic (N)MPC* in the literature since the stage cost reflects an economic criterion rather than a mere distance to some desired equilibrium. In a series of papers (Angeli et al. [2009], Angeli and Rawlings [2010], Diehl et al. [2011], Amrit et al. [2011]), a theory of economic NMPC with terminal constraints has been developed. We briefly sketch some of the main results of these papers for the special case where $x^e \in \mathbb{X}$ is an equilibrium, i.e., $f(x^e, u^e) = x^e$ holds for some $u^e \in \mathbb{U}$ (note that some of these references also discuss the case of periodic solutions which we will not treat here). For any equilibrium x^e it is shown that if we use the NMPC approach from Section 2 but impose the terminal constraint $x_u(N, x) = x^e$ in (3) (assuming that this constraint is feasible for the given initial value $x \in \mathbb{X}$), then the inequality $\overline{J}^{cl}_\infty(x, \mu_N) \le \ell(x^e, u^e)$ holds, where

$$\overline{J}^{cl}_\infty(x, \mu) := \limsup_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \ell(x_\mu(k, x), \mu(x_\mu(k, x)))$$

denotes the infinite horizon averaged functional. Particularly, if $\ell(x^e, u^e)$ is an optimal equilibrium (in the sense that the equilibrium cost $\ell(x^e, u^e)$ is less or equal than the infinite horizon averaged functional along any other trajectory) then optimal performance of the closed loop follows. In the Examples 4.1(i) and (ii) it can be shown that optimal equilibria in this sense exist. In Example 4.1(i) this is given by $(x^e, u^e) = (0, 0)$ and in Example 4.1(ii) the "best compromise" equilibrium $x^e \approx (3.546, 14.653)^T$, $u^e \approx 6.163$ meets this condition.

Note that even in the case of an optimal equilibrium convergence of the closed loop trajctories to x^e does not necessarily follow. In order to ensure convergence, the following condition (cf. Angeli and Rawlings [2010]) can be employed. We define a modified cost

$$\tilde{\ell}(x, u) := \ell(x, u) + \lambda(x) - \lambda(f(x, u)) \qquad (16)$$

for a given function $\lambda : \mathbb{X} \to \mathbb{R}$. Then the inequality $\min_{x\in\mathbb{X},u\in\mathbb{U}} \tilde{\ell}(x, u) \le \tilde{\ell}(x^e, u^e) = \ell(x^e, u^e)$ holds. Additionally, we make the following assumption.

Assumption 4.2. The function λ in (16) is bounded on $\mathbb{X}$ and there exists an equilibrium $(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ and $\alpha_\ell \in \mathcal{K}_\infty$ such that

$$\min_{u\in\mathbb{U}} \tilde{\ell}(x, u) \ge \ell(x^e, u^e) + \alpha_\ell(\|x - x^e\|)$$

holds for all $x \in \mathbb{X}$ with $\tilde{\ell}$ from (16).

One checks that Assumption 4.2, which can be interpreted as a dissipativity property, is satisfied for Examples 4.1(i) and (ii) for $\lambda(x) = -x^2/2$ and $\lambda(x) = c^T x$ with $c^T \approx (-368.6684, -503.5415)^T$, respectively. Assumption 4.2 is sufficient for the equilibrium (x^e, u^e) to be optimal and also ensures that NMPC closed loop solutions for the terminal constrained NMPC scheme converge to x^e.

4.2 Economic NMPC without terminal constraints

For exactly the same reasons as outlined at the beginning of Section 3 it is now interesting to investigate whether these properties remain true if we do not impose the terminal constraint $x_u(N, x) = x^e$ in (3). In order to show what kind of performance is reasonable to expect for such schemes, let us first show simulations for the Examples 4.1(i) and (ii).

Example 4.3. (i) We reconsider Example 4.1(i). For this problem, it is easily seen that an optimal way of keeping the system within the admissible set $\mathbb{X}$ in an infinite horizon averaged sense is to steer the system to the equilibrium $x^e = 0$ in a finite number of steps k' and set $u(k) = u^e = 0$ for $k \geq k'$. Moreover, $\ell(x, u) \geq 0$ implies $\overline{J}^{cl}_\infty(x, \mu) \geq 0$ for each feedback law μ.

Figure 2 shows the NMPC closed loop trajectory $x(k) = x_{\mu_N}(k, x)$ for $x = 0.5$ (solid) and the open loop optimal trajectories $x_{u^\star}(\cdot, x(k))$ for each k (dashed) for $\mathbb{X} = [-0.5, 0.5]$. One sees that while the open loop trajectories eventually move to the upper boundary of the admissible set, the closed loop trajectory tends towards a neighborhood of $x^e = 0$.

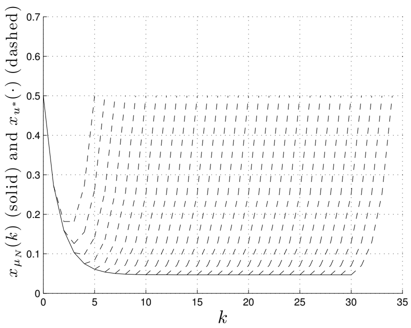

Fig. 2. Closed loop trajectory $x(k) = x_{\mu_N}(k, x_0)$ (solid) and optimal predictions $x_{u^\star}(\cdot, x(k))$ (dashed) along $x(k)$ for Example 4.1(i) with $N = 5$, $x = 0.5$ and $\mathbb{X} = [-0.5, 0.5]$

When increasing N, the closed loop solution ends up in smaller neighborhoods of x^e whose diameters shrink exponentially. This exponential decay is also reflected in the infinite horizon averaged value $\overline{J}^{cl}_\infty(x, \mu_N)$, which converges to the optimal equilibrium value $\ell(0, 0) = 0$ exponentially fast, i.e., the difference to 0 decays like $C\theta^N$ for constants $C > 0$ and $\theta \in (0, 1)$, as shown in Figure 3 (note that the scale on the $\overline{J}^{cl}_\infty(x, \mu_N)$-axis is logarithmic). This figure also shows that for the admissible set $\mathbb{X} = [-0.5, 0.5]$ the values $\overline{J}^{cl}_\infty(x, \mu_N)$ are smaller — and thus better — than for the larger set $\mathbb{X} = [-1, 1]$. This may be surprising at the first sight since the infinite horizon optimal equilibrium does not depend on the size of the state constraints. An explanation for this phenomenon is provided at the end of this section.

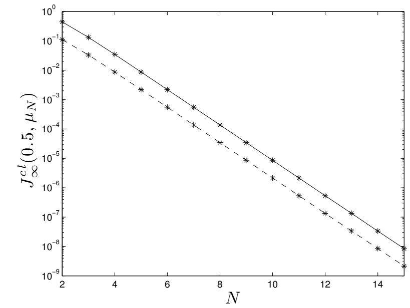

Fig. 3. $\overline{J}^{cl}_\infty(x, \mu_N)$ for Example 4.1(i) with $N = 2, \ldots, 15$, $x = 0.5$, $\mathbb{X} = [1, 1]$ (solid) and $\mathbb{X} = [-0.5, 0.5]$ (dashed)

(ii) We now perform similar simulations for Example 4.1(ii). As mentioned in Section 4.1, in this example the infinite horizon averaged performance is bounded from below by the optimal equilibrium value $\ell(x^e, u^e) \approx 229.1876$. The solutions exhibit a similar behavior as for Example 4.1(i): the open loop optimal trajectories first move towards x^e and then move away while the closed loop trajectories converge to an equilibrium close to x^e (Figure 4) and the closed loop performance $\overline{J}^{cl}_\infty(x, \mu_N)$ converges exponentially towards ℓ^e for $N \to \infty$ (Figure 5).

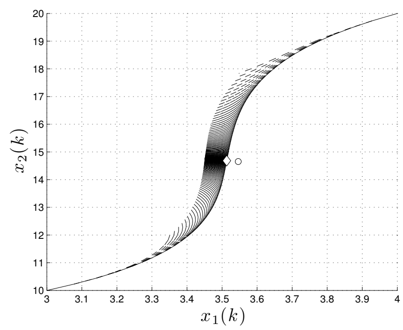

Fig. 4. Phase space plot of two closed loop trajectories $x(k) = x_{\mu_N}(k, x_0)$ (solid) and optimal predictions $x_{u^\star}(\cdot, x(k))$ (dashed) along $x(k)$ for Example 4.1(ii) with $N = 10$ and $x_0 = (4, 20)^T$ and $x_0 = (3, 10)^T$. The diamond indicates the equilibrium of the closed loop dynamics and the circle indicates the optimal equilibrium x^e.

In order to prove the behavior observed in these simulations, it turns out that one can identify the following counterpart to Proposition 3.4.

Proposition 4.4. Let $N \geq 2$, abbreviate $\ell^e = \ell(x^e, u^e)$ and assume that the optimal value function V_N and the NMPC feedback law μ_N satisfy the inequality

$$V_N(f(x, \mu_N(x))) - V_N(x) \leq \ell(x, \mu_N(x)) + \ell^e + \varepsilon(N) \quad (17)$$

for all $x \in \mathbb{X}$ and a function $\varepsilon : \mathbb{N} \to \mathbb{R}^+_0$ Then the inequality

$$\overline{J}^{cl}_\infty(x, \mu_N) \leq \ell^e + \varepsilon(N)$$

holds for all $x \in \mathbb{X}$.

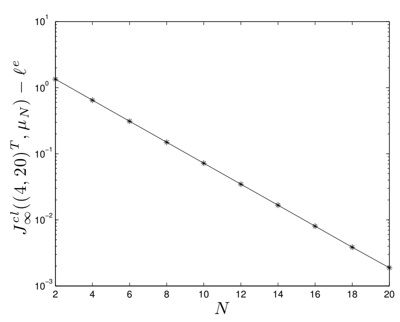

Fig. 5. $\overline{J}^{cl}_\infty(x, \mu_N) - \ell(x^e, u^e)$ for Example 4.1(ii) with $N = 2, \ldots, 20$ and $x = (4, 20)^T$

The proof of this proposition follows by observing that (17) is equivalent to

$$V_N(x) - V_{N-1}(x) \le \ell^e + \varepsilon(N).$$

Now the assertion follows as in [Grüne, 2012, Proof of Proposition 4.1].

Proposition 4.4 means that we can prove *value convergence* for the closed loop. If, moreover, Assumption 4.2 holds and $N\varepsilon(N) \to 0$ as $N \to \infty$, then also convergence of the closed loop solution $x_\mu(k, x)$ to a neighborhood of x^e can be shown, where the size of this neighborhood shrinks to 0 as $N \to \infty$, cf. [Grüne, 2012, Theorem 7.6]. Hence, under the additional condition $N\varepsilon(N) \to 0$ we can also conclude *trajectory convergence*. Note that the numerical results from Example 4.1 indicate that this additional condition holds for these examples, since the observed exponential decay $\varepsilon(N) \le C\theta^N$, $\theta \in (0, 1)$, implies $N\varepsilon(N) \to 0$.

The central question is thus whether we can ensure the inequality (17), preferably with $N\varepsilon(N) \to 0$ as $N \to \infty$. Inequality (17) can be concluded by a modification of the construction in Variant 1 from Section 3.2. Similar to the construction before (9), the idea is to obtain an approximate control sequence $\tilde{u}$ for initial value $f(x, \mu_N(x))$ by suitably prolonging the tail $(u^\star(1), \ldots, u^\star(N-1))$ of the optimal control sequence $u^\star$ for initial value x. This is done by inserting an additional control value $\hat{u} \approx u^e$ at a time k at which the trajectory $x_{u^\star}$ satisfies $x_{u^\star}(k, x) \approx x^e$ and shifting the remainder of the sequence $u^\star$ to the right by one time index, for details we refer to [Grüne, 2012, Proof of Theorem 4.2]. Besides some continuity and boundedness conditions on f, ℓ and V_N, which ensure that $\ell(\hat{u}, x_{u^\star}(k, x)) \approx \ell^e$ and that the value along the tail of the optimal trajectory does not change too much after inserting $\hat{u}$, the main requirement for this proof to work is that the open loop optimal trajectory for horizon N satisfies $x_{u^\star}(k, x) \approx x^e$ for some $k \in \{0, \ldots, N\}$. In quantitative terms, this leads to the following assumption.

Assumption 4.5. There exists $\sigma : \mathbb{N} \to \mathbb{R}^+_0$ with $\sigma(N) \to 0$ as $N \to \infty$ and $N_1 \in \mathbb{N}$ such that for each $x \in \mathbb{X}$ and each $N \ge N_1$ there exists an optimal trajectory $x_{u^\star}(\cdot, x)$ satisfying

$$\|x_{u^\star}(k_x, x) - x^e\| \le \sigma(N)$$

for some $k_x \in \{0, \ldots, N\}$.

Assumption 4.5 is a particular form of a so called *turnpike property* which is a classical tool in optimal control, see [Carlson et al., 1991, Section 4.4], particularly for understanding the optimal dynamics of economic control problems, cf. McKenzie [1986] and the references therein.

An inspection of the proof of [Grüne, 2012, Theorem 4.2] shows that the value $\varepsilon(N)$ in (17) can be obtained from $\sigma(N)$. More precisely, under suitable continuity and boundedness assumptions on f, ℓ and V_N in a neighborhood of x^e, the estimate $\varepsilon(N) \le p(\sigma(N))$ can be obtained, where p is a polynomial with $p(0) = 0$. Particularly, this shows that $\sigma(N) \to 0$ implies $\varepsilon(N) \to 0$ and if $\sigma(N)$ converges to 0 exponentially fast, then $\varepsilon(N)$ will do so, too.

In what follows we will thus investigate conditions for Assumption 4.5 to hold. In order to simplify the computations, for the subsequent considerations we will assume $\ell(x^e, u^e) = 0$ and $\lambda(x^e) = 0$ which also implies $\tilde{\ell}(x^e, u^e) = 0$. These assumptions can be made without loss of generality by adding suitable constants to ℓ and λ. Note that adding such constants does neither change the optimal trajectories and control sequences nor does it affect the validity of Assumption 4.2 and the function α in this assumption. Moreover, we define the modified cost functional

$$\widetilde{J}_N(x, u) := \sum_{k=0}^{N-1} \tilde{\ell}(x_u(k, x), u(k)).$$

Observe that by definition of the modified cost $\tilde{\ell}$ the functionals J_N from (3) and $\widetilde{J}_N$ are related via

$$\widetilde{J}_N(x, u) = J_N(x, u) + \lambda(x) - \lambda(x_u(N, x)). \qquad (18)$$

With these assumptions and notations we now present two variants for proving Assumption 4.5.

Variant 1: Let Assumption 4.2 hold and let $C := 2\sup_{x\in\mathbb{X}} |\lambda(x)| < \infty$. Assume moreover that $V_N(x)$ is bounded from above on $\mathbb{X}$, i.e., $V_N(x) \le M$ holds for all $x \in \mathbb{X}$ and some $M \in \mathbb{R}$. Then from (18) we obtain

$$\widetilde{J}_N(x, u^\star) \le J_N(x, u^\star) + C = V_N(x) + C \le M + C.$$

for all $N \in \mathbb{N}$. Like in Variant 1 from Section 3.2 this implies $\tilde{\ell}(x_{u^\star}(k, x), u^\star(k)) \le (M + C)/N$ for some $k \in \{0, \ldots, N-1\}$. Assumption 4.2 then implies

$$\|x_{u^\star}(k, x) - x^e\| \le \alpha^{-1}((M + C)/N) =: \sigma(N)$$

with α from Assumption 4.2 which shows Assumption 4.5.

In our examples the function $\tilde{\ell}$ is quadratic around x^e which means that α is quadratic and hence α^{-1} behaves like a square root near 0. Hence, the proof shows that $\sigma(N)$ is of the order of $\sqrt{N}$ which converges to 0 much slower than the exponential convergence we observe in the examples. Generally, unless α^{-1} happens to be very "flat" near 0 (which appears to be an exceptional case), the proof just sketched will not yield exponential convergence of $\sigma(N)$ to 0. Consequently, this proof (which follows [Grüne, 2012, Theorem 5.3]) shows value convergence but in general we cannot conclude trajectory convergence. In order to improve the estimate, we present another variant to estimate $\sigma(N)$ which, however, needs stronger assumptions.

Variant 2: Let Assumption 4.2 hold, assume that $\tilde{\ell}$ is bounded on $\mathbb{X} \times \mathbb{U}$ and consider the following terminal constrained optimal value function

$$\widetilde{V}_N^t(x_0,x_T) := \inf_{\substack{u\in\mathbb{U}^N(x_0)\\ x_u(N,x_0)=x_T}} \widetilde{J}_N(x_0,u).$$

Assume that there exists $\gamma \geq 1$ such that for all $x_0, x_T \in \mathbb{X}$ for which a trajectory from x_0 to x_T exists the inequality

$$\widetilde{V}_N^t(x_0,x_T) \leq \gamma\ell^\star(x_0) + (\gamma-1)\ell^\star(x_T) \qquad (19)$$

holds. This inequality generalizes Assumption 3.5 to the terminal constrained problem and we can use it in a similar way as we have exploited Assumption 3.5 in Variant 2 from Section 3.2: We first observe that by virtue of (18) for each optimal trajectory $x_{u^\star}(\cdot,x)$ for $V_N(x)$ and all $p,q \in \{0,\dots,N\}$ with $p<q$ the trajectory piece from $k=p$ to $k=q$ is also an optimal trajectory for $\widetilde{V}_{q-p}^t(x_0,x_T)$ if we set $x_0 = x_{u^\star}(p,x)$ and $x_T = x_{u^\star}(q,x)$. Hence, defining $\tilde{\ell}_k := \ell(x_{u^\star}(k,x),u^\star(k))$, $k=0,\dots,N-1$ and $\tilde{\ell}_k := \ell^\star(x_{u^\star}(N,x))$, from (19) we obtain the inequalities

$$\sum_{k=p}^{q} \tilde{\ell}_k \leq \gamma\tilde{\ell}_p + \gamma\tilde{\ell}_q.$$

An inspection of this inequality yields that now we can proceed similarly to Variant 2 from Section 3.2, either in forward direction for $k=0,\dots,\lfloor N/2\rfloor$ or in backward direction for $k=N,\dots,\lceil N/2\rceil$. This yields either

$$\tilde{\ell}_{\lfloor N/2\rfloor} \leq \gamma\left(\frac{\gamma-1}{\gamma}\right)^{\lfloor N/2\rfloor-1}\tilde{\ell}_0$$

or

$$\tilde{\ell}_{\lceil N/2\rceil} \leq \gamma\left(\frac{\gamma-1}{\gamma}\right)^{\lfloor N/2\rfloor-1}\tilde{\ell}_N.$$

Since we assumed $\tilde{\ell}$ to be bounded on $\mathbb{X}$ (say, by a constant M), this implies Assumption 4.5 with

$$\sigma(N) = \alpha^{-1}\left(M\gamma\left(\frac{\gamma-1}{\gamma}\right)^{\lfloor N/2\rfloor-1}\right).$$

If α has at least polynomial growth near 0 (which is true in Examples 4.1(i) and (ii) since $\tilde{\ell}$ and thus α is quadratic), this σ indeed decays exponentially.

This analysis also yields an explanation for the observation from Figure 3, namely that the error $\varepsilon(N)$ increases when the constraint set $\mathbb{X}$ is enlarged. Indeed, as $\mathbb{X}$ is enlarged the bound M on $\tilde{\ell}$ increases and thus $\sigma(N)$ and consequently also the error bound $\varepsilon(N)$ becomes larger.

We summarize our findings in the following theorem.

Theorem 4.6. Consider an economic NMPC problem satisfying Assumption 4.2 where without loss of generality we assume $\ell(x^e,u^e)=0$.

(i) Assume that f, ℓ and V_N are locally Lipschitz with Lipschitz constant independent of N in case of V_N [4] and that λ from Assumption 4.2 and V_N are bounded on $\mathbb{X}$ with a bound independent of N in case of V_N. Then the value convergence

$$\overline{J}_\infty^{cl}(x,\mu_N) \to \ell(x^e,u^e)$$

holds for all $x\in\mathbb{X}$ as $N\to\infty$.

(ii) If, moreover, $\tilde{\ell}$ is bounded on $\mathbb{X}\times\mathbb{U}$, α from Assumption 4.2 has at least polynomial growth near 0 and (19) holds, then the value convergence is exponentially fast and for each $\delta>0$ there exists $N>0$ such that

$$\|x_{\mu_N}(k,x) - x^e\| \leq \delta$$

holds for all $x\in\mathbb{X}$ and all sufficiently large $n\in\mathbb{N}$.

Inequality (19) used in Part (ii) of this theorem is a natural extension of Assumption 3.5. However, unlike the by now well investigated Assumption 3.5, it is currently not completely clear how restrictive (19) is in terms of the dynamics of the system and the stage cost. Investigations on this are subject of ongoing research and will — together with a detailed version of the proof sketched in Variant 2 — appear in Damm et al. [2012].

5. FEASIBILITY

In order to ensure that an NMPC scheme is well defined, it is important that at each time k there exists an admissible control sequence for the closed loop state $x = x_{\mu_N}(k,x_0)$, i.e., that $\mathbb{U}^N(x)\neq\emptyset$. In the previous sections we have circumvented this problem by simply assuming $\mathbb{U}^N(x)\neq\emptyset$ for all $x\in\mathbb{X}$. However, in general it is a difficult task to construct a state constraint set $\mathbb{X}$ meeting this assumption. It is thus of interest to look at other ways to ensure $\mathbb{U}^N(x)\neq\emptyset$ for all points $x=x_{\mu_N}(k,x_0)$, $k\in\mathbb{N}$.

Defining the *feasible sets*

$$\mathbb{X}_N := \{x\in\mathbb{X} \mid \mathbb{U}^N(x)\neq\emptyset\}$$

and

$$\mathbb{X}_\infty := \{x\in\mathbb{X} \mid \mathbb{U}^\infty(x)\neq\emptyset\}$$

the question is thus whether we can ensure $x_{\mu_N}(k,x_0)\in\mathbb{X}_N$ for all $k\in\mathbb{N}$. To this end, we use the following definitions.

Definition 5.1. Consider an NMPC scheme with optimization horizon N and a set $A\subseteq\mathbb{X}_N$.

(i) The scheme is called *strongly feasible* on A, if for each $x\in A$ and $u\in\mathbb{U}^N(x)$ the condition $x_u(1,x)\in A$ holds.

(ii) The scheme is called *recursively feasible* on A, if for each $x\in A$ and each optimal control sequence $u^\star\in\mathbb{U}^N(x)$ for (3) the condition $x_{u^\star}(1,x)\in A$ holds.

The notion of strong feasibility demands that for any admissible trajectory starting in A remains in A for at least one step. Note that strong feasibility (the name goes back to Kerrigan [2000]) is weaker than the well known property of strong forward invariance: in contrast to strong forward invariance we do not require the admissible trajectories $x_u(k,x)$ to remain inside A for all $k=1,\dots,N-1$ but only for $k=1$. The notion of recursive feasibility is weaker in the sense that it requires the same property but not for all admissible trajectories starting in A but only for the optimal ones. Many authors have observed the importance of invariance in control (see, e.g., Blanchini [1999], Blanchini and Miani [2008] and the references therein) and more specifically in MPC. Some variants of MPC (like, e.g., the so called tube based MPC method, see Langson et al. [2004]) even rely on the explicit construction of invariant sets. In contrast to this, here we will only use them as an analysis tool whose explicit knowledge is not needed in order to run the NMPC algorithm.

[4] The Lipschitz conditions can be replaced by weaker but more technical continuity conditions, see Conditions (a) and (b) of [Grüne, 2012, Theorem 4.2]. Moreover, the Lipschitz and the boundedness condition on V_N independent of N can be ensured by a controllability condition [Grüne, 2012, Theorem 6.4].

Using the identity $x_{\mu_N}(1,x) = x_{u^\star}(1,x)$, it is easily seen by induction that recursive feasibility ensures $x_{\mu_N}(k,x) \in A \subseteq \mathbb{X}_N$ for all $x \in A$ and all $k \in \mathbb{N}$ and is thus sufficient for the NMPC scheme being well defined. Since this means that $x_{\mu_N}(k,x)$ is an admissible trajectory of infinite length, we immediately obtain the inclusion $A \subseteq \mathbb{X}_\infty$. The stronger condition of strong feasibility is of interest if we run an NMPC scheme with non-optimal (but admissible) u instead of the optimal $u^\star$. Although not treated in detail in this paper, this is, for instance, important if we cannot obtain an optimal solution of (3), e.g., if because of time constraints the iterative optimization algorithm used for solving (3) cannot be iterated until convergence.

One of the main advantages of terminal constrained NMPC schemes is that including a terminal constraint set $\mathbb{X}_0$ with the properties described at the beginning of Section 3 "automatically" leads to strong feasibility, if we take into account that the terminal constraint condition $x_u(N,x) \in \mathbb{X}_0$ needs to be incorporated in the definition of $\mathbb{X}_N$. For a detailed analysis we refer to [Kerrigan, 2000, Chapter 5]. It is also easily seen that strong feasibility holds without terminal constraints if the set $\mathbb{X}$ is controlled forward invariant, as assumed in the previous sections.

In the following two sections we sketch two ways which can be used in order to conclude strong or recursive feasibility if $\mathbb{X}$ is not controlled forward invariant and no terminal constraints are imposed.

5.1 Strong feasibility via stationarity

Strong feasibility for NMPC without terminal constraints can be ensured if the question whether a point x lies in $\mathbb{X}_\infty$ can be determined in a finite number of steps. In order to make this statement precise, we look at the dependence of the sets $\mathbb{X}_N$ on N, following [Kerrigan, 2000, Theorem 5.3] and [Grüne and Pannek, 2011, Section 8.2] (the latter references essentially rephrases the results from the earlier one using a different notation, a fact which unfortunately escaped us when writing Grüne and Pannek [2011]). Here we give a self-contained and actually quite short proof of the main result in these references.

The definition of the feasible set $\mathbb{X}_N$ immediately implies the inclusion $\mathbb{X}_{N_2} \subseteq \mathbb{X}_{N_1}$ for all $N_2 > N_1 \geq 2$. We say that the sequence of sets $\mathbb{X}_N$ becomes *stationary*, if there exists $N_0 \geq 0$ such that $\mathbb{X}_{N_2} = \mathbb{X}_{N_1}$ holds for all $N_1, N_2 \geq N_0$.

Theorem 5.2. Assume that the feasible sets become stationary for some $N_0 \in \mathbb{N}$. Then the NMPC scheme is strongly feasible on $A = \mathbb{X}_N$ for all $N \geq N_0 + 1$.

Proof: From the definition of the $\mathbb{X}_N$ it follows that for each $x \in A = \mathbb{X}_N$ and $u \in \mathbb{U}^N(x)$ the relation $x_u(1,x) \in \mathbb{X}_{N-1}$ holds. Since the stationarity assumption implies $\mathbb{X}_{N-1} = \mathbb{X}_N = A$ for all $N \geq N_0 + 1$, we obtain $x_u(1,x) \in A$ which shows the claim. □

Example 5.3. Consider the zero order hold sampled data model of a double integrator with sampling time $T = 1$, i.e., $x(k+1) = f(x(k), u(k)$ with

$$f(x,u) = \begin{pmatrix} x_1 + x_2 + u/2 \\ x_2 + u \end{pmatrix}.$$

We use the state constraints $\mathbb{X} = [-1,1]^2$ and the control constraints as $\mathbb{U}(x) = \mathbb{U} = [-\bar{u}, \bar{u}]$ with $\bar{u} > 0$.

A straightforward but somewhat tedious computation shows that feasible sets $\mathbb{X}_N$ are given by

$$\mathbb{X}_N = \mathbb{X} \setminus \bigcup_{j=1}^{N-2} \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x_1 > -jx_2 + 1 + j^2\bar{u}/2 \text{ or} \\ x_1 < jx_2 - 1 - j^2\bar{u}/2 \end{array} \right\}.$$

Since the two inequalities in this set are never satisfied for $x \in [-1,1]^2$ if $j \geq 1/\bar{u}$ holds, the sets $\mathbb{X}_N$ become stationary for $N_0 = \lceil 1/\bar{u} + 1 \rceil$.

Even though this has not been checked rigorously, we conjecture that the same is true for the more complicated dynamics and state constraints in Example 3.3, which explains why we did not encounter feasibility problems in the simulations for this example.

5.2 Recursive feasibility via stability

In case the condition of Theorem 5.2 is not satisfied, there is little hope to obtain strong feasibility without imposing terminal constraints. However, recursive feasibility can still be obtained if the assumptions needed for stability can be met. In this section we present a method to ensure recursive feasibility which goes back to Primbs and Nevistić [2000] for finite dimensional linear systems and was extended to nonlinear systems on general metric spaces in [Grüne and Pannek, 2011, Section 8.3].

Since we want to conclude feasibility from stability, we suppose that Assumptions 3.2 and 3.5 are satisfied, where we restrict ourselves to points x from the sets $\mathbb{X}_\infty$ and $\mathbb{X}_N$ on which the optimal value functions V_∞ and V_N are well defined. Moreover, we assume that there exists a neighborhood $\mathcal{N}$ of the origin for which $\mathcal{N} \cap \mathbb{X}$ is controlled forward invariant. Recall that controlled forward invariance means that for each $x \in \mathcal{N} \cap \mathbb{X}$ there exists $u \in \mathbb{U}$ with $f(x,u) \in \mathcal{N} \cap \mathbb{X}$. We emphasize that the set $\mathcal{N}$ is not needed in the implementation but only for the analysis of the NMPC scheme. Hence, there is no need to actually compute this set.

Defining $M := \inf_{x \notin \mathcal{N} \cap \mathbb{X}} \ell^\star(x)$, for each $x \in \mathbb{X}$ the inequality $\ell^*(x) < M$ implies $x \in \mathcal{N} \cap \mathbb{X}$. Moreover, if $\mathcal{N}$ contains a ball of radius ε around the origin then we can estimate $M \geq \alpha_1(\varepsilon)$ for α_1 from Assumption 3.2(i).

Following the computations from Variant 2 in Section 3.2, for each $x \in \mathbb{X}_N$ our assumptions ensure the inequality

$$\ell(x_{u^\star}(N-1,x), u^\star(N-1)) \leq \gamma \left(\frac{\gamma-1}{\gamma}\right)^{N-1} \ell(x, u^\star(0)).$$

Since $V_N(x) \geq \ell(x, u^\star(0))$ this implies

$$\ell(x_{u^\star}(N-1,x), u^\star(N-1)) \leq \gamma \left(\frac{\gamma-1}{\gamma}\right)^{N-1} V_N(x).$$

Hence, if for $c > 0$ we define the sublevel sets

$$A_{N,c} := \{x \in \mathbb{X}_N \,|\, V_N(x) \leq c\},$$

then for each $x \in A_{N,c}$ we obtain

$$\ell(x_{u^\star}(N-1,x), u^\star(N-1)) \leq \gamma \left(\frac{\gamma-1}{\gamma}\right)^{N-1} c. \quad (20)$$

Since the right hand side of this inequality decreases to 0, we can conclude that there exists $N_c > 0$ such that $\ell(x_{u^\star}(N-1,x), u^\star(N-1)) < M$ and thus $x_{u^\star}(N-1,x) \in \mathcal{N} \cap \mathbb{X}$ holds for all $x \in A_{N,c}$ and all $N \geq N_c$. Moreover, since the right hand side of (20) shrinks exponentially fast, N_c grows logarithmically (i.e., at very slow rate) with increasing c.

Now, for any $N \geq N_c$, since $x_{u^\star}(N-1,x) \in \mathcal{N} \cap \mathbb{X}$ and $\mathcal{N} \cap \mathbb{X}$ is controlled forward invariant, the trajectory can be extended to stay in $\mathbb{X}$ forever, and as a consequence all points on the trajectory $x_{u^\star}(k,x)$ lie in $\mathbb{X}_\infty$. Hence, we can now apply the estimates from Variant 3 in Section 3.2 which imply that V_N is a Lyapunov function for the NMPC closed loop if N satisfies (13). Hence, for all $N \geq N_c$ satisfying (13) we obtain

$$V_N(x_{u^\star}(1,x)) = V_N(x_{\mu_N}(1,x)) \leq V_N(x)$$

for all $x \in A_{N,c}$, implying $x_{u^\star}(1,x) \in A_{N,c}$. This proves that the NMPC scheme is recursively feasible on $A = A_{N,c}$.

Note that the upper bound $V_N(x) \leq \gamma\alpha_2(\|x\|)$ implies that $A_{N,c}$ covers each bounded subset of $\mathbb{X}_\infty$ for sufficiently large c. Particularly, if $\mathbb{X}$ itself is bounded then $\mathbb{X}_\infty$ will be recursively feasible for all sufficiently large N.

Again, we summarize the derivations from this section in a theorem.

Theorem 5.4. Consider an NMPC problem with a not necessarily controlled forward invariant state constrains set $\mathbb{X}$. Assume that Assumptions 3.2 and 3.5 hold on $\mathbb{X}_\infty$ and $\mathbb{X}_N$, respectively. Then for each $c > 0$ there exists $N_c > 0$ with $N_c \sim \ln c$ such that the NMPC algorithm is recursively feasible on the set $\{x \in \mathbb{X}_{N_c} \,|\, V_{N_c}(x) \leq c\}$. Particularly, for each bounded set $K \subseteq \mathbb{X}_\infty$ there exists $N_K > 0$ such that the NMPC algorithm is recursively feasible on a set $A \supseteq K$.

While the arguments outlined in this section essentially follow [Grüne and Pannek, 2011, Section 8.3] (to which we also refer for more details), the proof has been improved by using Variant 2 from Section 3.2 instead of Variant 1 as in [Grüne and Pannek, 2011, Section 8.3]. The benefits of using Variant 2 are the logarithmic growth of N_c with c (as opposed to a linear growth $N_c \sim c$) and a considerably simplified construction of the sublevel set $A_{N,c}$.

6. CONCLUSION

In this paper we have surveyed recent results on stability, performance and feasibility of NMPC without terminal constraints. We have shown that many properties of terminal constrained NMPC schemes can also be rigorously derived without terminal constraints. By means of Example 3.3 we have demonstrated that by avoiding terminal constraints NMPC can yield controllers with large operating regions even for very short optimization horizons. While a rigorous check of stability and feasibility conditions may be more involved than for terminal constrained schemes, the design is typically considerably more simple since no Lyapunov function terminal costs need to be computed. Hence, NMPC without terminal constraints can provide an attractive alternative to terminal constrained schemes.

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