Selection Strategies of Set-Valued Runge-Kutta Methods*

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Contents

1. Introduction

1.1. Differential Inclusions and Set-Valued Integral

Problem 1.1 Consider the nonlinear differential inclusion (DI)

$$x'(t) \in F(t, x(t)) \quad \text{(f. a. e. } t \in I := [t_0, T]) \ , \tag{1}$$
$$x(t_0) \in X_0 \tag{2}$$

with the nonempty set $X_0 \in \mathcal{C}(\mathbb{R}^n)$ and the set-valued mapping $F : I \times \mathbb{R}^n \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$.
Hereby, $\mathcal{C}(\mathbb{R}^n)$ denotes the set of nonempty, convex, compact subsets of $\mathbb{R}^n$ and $x : I \to \mathbb{R}^n$ fulfills $x(\cdot) \in \mathrm{AC}(I)$, i.e. $x(\cdot)$ is absolutely continuous.

Definition 1.2 The attainable set $\mathcal{R}(t, t_0, X_0)$ at a given time $t \in I$ for Problem 1.1 is defined as

$$\mathcal{R}(t, t_0, X_0) = \{x(t) \mid x(\cdot) \in \mathrm{AC}(I) \text{ is solution of (1)–(2)}\} \ .$$

Aim of the methods presented here:
approximation of the attainable set at time T by other sets

simplification for main parts of the talk:

Problem 1.3 The linear differential inclusion (LDI) is stated as follows:

$$x'(t) \in A(t)x(t) + B(t)U \quad \text{(f. a. e. } t \in I = [t_0, T]\text{)} \ , \tag{3}$$
$$x(t_0) \in X_0 \tag{4}$$

with matrix functions $A : I \to \mathbb{R}^{n \times n}$, $B : I \to \mathbb{R}^{n \times m}$ and sets $X_0 \in \mathcal{C}(\mathbb{R}^n)$, $U \in \mathcal{C}(\mathbb{R}^m)$.

Definition 1.4 The fundamental solution of the corresponding matrix differential equation

$$X'(t) = A(t)X(t) \quad \text{(f. a. e. } t \in I\text{)} \ ,$$
$$X(\tau) = I \ .$$

to Problem 1.3 is denoted by $\Phi(\cdot, \tau)$ for $\tau \in I$, where $I \in \mathbb{R}^{n \times n}$ is the unit matrix.

Definition 1.5 ([Aumann, 1965])
Consider a set-valued function $F : I \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$ which is measurable and integrably bounded, i.e. there exists $k(\cdot) \in L_1(I)$ with $F(t) \subset k(t)B_1(0)$ f. a. e. $t \in I$. Then, Aumann's integral is defined as

$$\int_{t_0}^{T} F(t)dt := \Big\{ \int_{t_0}^{T} f(t)dt \ \Big| \ f(\cdot) \text{ is an integrable selection of } F(\cdot) \Big\} \ .$$

1.2. Arithmetic Operations on Sets

Definition 1.6 Let $C, D \in \mathcal{C}(\mathbb{R}^n)$. The Hausdorff distance between C and D is defined as

$$\mathrm{d_H}(C, D) = \max\{\mathrm{d}(C, D), \mathrm{d}(D, C)\} \ ,$$

where

$$\mathrm{d}(C, D) = \sup_{c \in C} \mathrm{dist}(c, D) \ ,$$
$$\mathrm{dist}(c, D) = \inf_{d \in D} \|c - d\|_2 \quad (c \in C) \ .$$

Notation 1.7 The arithmetic operations of sets

$$\begin{aligned} \lambda \cdot C &:= \{ \lambda \cdot c \mid c \in C \} && \text{(scalar multiple)} \ , \\ C + D &:= \{ c + d \mid c \in C, \ d \in D \} && \text{(Minkowski sum)} \ , \\ A \cdot C &:= \{ A \cdot c \mid c \in C \} && \text{(image under a linear mapping)} \end{aligned}$$

are defined as usual for $C, D \in \mathcal{C}(\mathbb{R}^n)$, $A \in \mathbb{R}^{k \times n}$ and $\lambda \in \mathbb{R}$.

Definition 1.8 Let $C \in \mathcal{C}(\mathbb{R}^n)$, $l \in \mathbb{R}^n$. The support function resp. the supporting face for C in direction l is defined as

$$\delta^*(l, C) := \max_{c \in C} \langle l, c \rangle \quad \text{resp.} \quad Y(l, C) := \{c \in C \mid \langle l, c \rangle = \delta^*(l, C)\} \ .$$

Remark that

$$C = \bigcap_{\|l\|_2 = 1} \{x \in \mathbb{R}^n \ : \ \langle l, x \rangle \le \delta^*(l, x)\} \ .$$

Lemma 1.9 Let $C, D \in \mathcal{C}(\mathbb{R}^n)$,$A \in \mathbb{R}^{k \times n}$ and $\lambda \ge 0$. Then,

$$C \subset D \iff \delta^*(l, C) \le \delta^*(l, D) \quad \text{for all } l \in S_{n-1} \subset \mathbb{R}^n, \text{ i.e. } \|l\|_2 = 1$$

and the following calculus rules are valid for $l \in S_{n-1}$:

$$\begin{aligned} \delta^*(l, C+D) &= \delta^*(l, C) + \delta^*(l, D) \ , & Y(l, C+D) &= Y(l, C) + Y(l, D) \ , \\ \delta^*(l, \lambda C) &= \lambda \delta^*(l, C) \ , & Y(l, \lambda C) &= \lambda Y(l, C) \ , \\ \delta^*(l, AC) &= \delta^*(A^t l, C) \ , & Y(l, AC) &= A Y(A^t l, C) \ , \\ \mathrm{d_H}(C, D) &= \sup_{\|l\|_2 = 1} |\delta^*(l, C) - \delta^*(l, D)| \end{aligned} \tag{5}$$

and

$$\mathrm{d_H}(AU, BU) \le \|A - B\| \cdot \|U\| \qquad \text{with } \|U\| := \sup_{u \in U} \|u\|_2 \ , \tag{6}$$

$$\mathrm{d_H}((A+B)U, AU + BU) \le \|A - B\| \cdot \|U\| \ . \tag{7}$$

Theorem 1.10 ([Aumann, 1965])
Let $F : I \Rightarrow \mathbb{R}^n$ with nonempty, closed images be measurable and integrably bounded. Then, the Aumann integral of $F(\cdot)$ is compact, convex and nonempty with

$$\delta^*(l, \int_I F(t)dt) = \int_I \delta^*(l, F(t))dt \ .$$

Lemma 1.11 (e.g. [Sonneborn and van Vleck, 1965])
Given Problem 1.3, the attainable set at time $t \in I$ can be rewritten as

$$\mathcal{R}(T, t_0, X_0) = \Phi(T, t_0)X_0 + \int_{t_0}^{T} \Phi(T, t)B(t)U\,dt \ .$$

Scalarization by support functions resp. supporting faces yields for $l \in S_{n-1}$:

$$\delta^*(l, \mathcal{R}(T, t_0, X_0)) = \delta^*(\Phi(T, t_0)^t l, X_0) + \int_I \delta^*(B(t)^t \Phi(T, t)^t l, U)dt \ ,$$

$$Y(l, \mathcal{R}(T, t_0, X_0)) = \Phi(T, t_0) Y(\Phi(T, t_0)^t l, X_0) + \int_I \Phi(T, t)B(t) Y(B(t)^t \Phi(T, t)^t l, U)dt$$

1.3. Modulus of Smoothness

Definition 1.12 Let $f : I \longrightarrow \mathbb{R}^n$ be bounded. The averaged modulus of smoothness of order $k \in \mathbb{N}$ is defined as

$$\tau_k(f;h) := \|\omega_k(f;\cdot;h)\|_{L_1} \ ,$$
$$\omega_k(f;x;h) := \sup\{|\Delta_\delta^k f(t)| \ : \ t, t+k\delta \in [x-\frac{kh}{2}, x+\frac{kh}{2}] \cap I\} \quad \text{for } x \in I \ ,$$

where $\Delta_\delta^k f(t)$ is the k-th forward difference of $f(\cdot)$ in t with step-size δ.

Lemma 1.13 (cf. [Sendov and Popov, 1988])
Let $f : I \longrightarrow \mathbb{R}^n$ be bounded and $p \in \mathbb{N}$. Then,

$$\tau_p(f;h) = \begin{cases} o(1), & \text{if } f(\cdot) \text{ is Riemann integrable} \\ \mathcal{O}(h), & \text{if } f(\cdot) \text{ has bounded variation} \\ o(h^{p-1}), & \text{if } p \geq 2 \text{ and } f^{p-2}(\cdot) \in \mathrm{AC}(I) \\ \mathcal{O}(h^p), & \text{if } p \geq 2,\ f^{p-2}(\cdot) \in \mathrm{AC}(I) \\ & \text{and } f^{p-1}(\cdot) \text{ has bounded variation} \end{cases}$$

2. Quadrature and Combination Methods

2.1. Quadrature Methods

Notation 2.1 Let $I := [t_0, T]$ and $f : I \to \mathbb{R}^n$ be given. We denote the point-wise quadrature formula by

$$Q(f; [t_0, T]) := \sum_{\mu=1}^{s} b_\mu f(t_0 + c_\mu (T - t_0)) \ ,$$

where $b_\mu \in \mathbb{R}$ are the weights and $c_\mu \in [0, 1]$ determine the nodes ($\mu = 1, \ldots, s$). Set $h = \frac{T - t_0}{N}$ as step-size for $N \in \mathbb{N}$ and define the iterated quadrature formula as

$$Q_N(f; [t_0, T]) := h \sum_{j=0}^{N-1} Q(f; [t_j, t_{j+1}]) = h \sum_{j=0}^{N-1} \sum_{\mu=1}^{s} b_\mu f(t_j + c_\mu h) \ .$$

$Q(f; I)$ has precision $p \in \mathbb{N}_0$, if all polynomials up to degree p are integrated exactly and there exists a polynomial f with degree $p + 1$ and $Q(f; I) \neq \int_I f(t) dt$.

Definition 2.2 Consider a point-wise quadrature formula of Notation 2.1 and $F : I \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$. The iterated set-valued quadrature method is defined with the usual arithmetic operations

$$Q_N(F; [t_0, T]) := h \sum_{j=0}^{N-1} \sum_{\mu=1}^{s} b_\mu F(t_j + c_\mu h) \ .$$

Proposition 2.3
(cf. [Polovinkin, 1975], [Balaban, 1982], [Donchev and Farkhi, 1990], [Veliov, 1989a]), [Krastanov and Kirov, 1994], [B. and Lempio, 1994b], [B., 1995])
Consider $N \in \mathbb{N}$ and a point-wise iterated quadrature formula of Notation 2.1 with non-negative weights $b_\mu \geq 0$ ($\mu = 1, \ldots, s$) and the remainder term

$$R_N(f; I) := \int\limits_I f(t)dt - Q_N(f; I) \ .$$

Then, the corresponding set-valued quadrature method fulfills for $F : I \Rightarrow \mathbb{R}^n$ with images in $\mathcal{C}(\mathbb{R}^n)$:

$$\mathrm{d_H}(\int\limits_I F(t)dt, Q_N(F; I)) = \sup_{\|l\|_2 = 1} |R_N(\delta^*(l, F(\cdot)); I)| \ .$$

2.2. Quadrature Method for the Approximation of Attainable Sets

Proposition 2.4
(cf. [Donchev and Farkhi, 1990], [B. and Lempio, 1994b], [B., 1995])
Consider $N \in \mathbb{N}$ and a point-wise iterated quadrature formula of Notation 2.1 with non-negative weights $b_\mu \geq 0$ ($\mu = 1, \ldots, s$). Assume that

- the values $\Phi(T, t_j + c_\mu h)$ are known for $j = 0, \ldots, N-1$ and $\mu = 1, \ldots, s$
- the quadrature method has precision $p-1$, $p \in \mathbb{N}$
- $\tau_p(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \leq Ch^p$ uniformly in $l \in S_{n-1}$

Then,

$$\mathrm{d_H}(\mathcal{R}(T, t_0, X_0), Q_N(\Phi(T, \cdot)B(\cdot)U); [t_0, T]) = \mathcal{O}(h^p).$$

Proof: In [Sendov and Popov, 1988, Theorem 3.4]:

$$|R_N(f)| = |\int\limits_I f(t)dt - Q_N(f; [t_0, T])| \leq \Big(1 + \sum_{\mu=1}^{s} \frac{b_\mu}{T - t_0}\Big) \cdot W_p \cdot \sup_{\|l\|_2 = 1} \tau_p\big(f, \frac{2}{p}h\big)$$

Since Lemma 1.11 and

$$\delta^*(l, Q_N(F; I)) = Q_N(\delta^*(l, F(\cdot)); I) \quad ,$$

one can apply the error estimation above to $f(\cdot) = \delta^*(l, F(\cdot))$ with $F(\cdot) = \Phi(T, \cdot)B(\cdot)U$.

■

Example 2.5 set-valued rectangular rule (special Riemannian sum) for $I = [t_0, T]$:

$$Q(F;I) = (T - t_0)F(t_0), \quad Q_N(F;I) = h \sum_{j=0}^{N-1} F(t_j),$$

$$Q_N(\Phi(T,\cdot)B(\cdot)U;I) = h \sum_{j=0}^{N-1} \Phi(T,t_j)B(t_j)U$$

in iterative form:

$$Q_{j+1}^N = \Phi(t_{j+1},t_j)Q_j^N + h\Phi(t_{j+1},t_j)B(t_j)U, \quad Q_0^N = X_0$$

Example 2.6 set-valued trapezoidal rule for $I = [t_0, T]$:

$$Q(F; I) = \frac{T - t_0}{2}\big(F(t_0) + F(T)\big), \quad Q_N(F; I) = \frac{h}{2}\sum_{j=0}^{N-1}\big(F(t_j) + F(t_{j+1})\big),$$

$$Q_N(\Phi(T, \cdot)B(\cdot)U; I) = \frac{h}{2}\sum_{j=0}^{N-1}\big(\Phi(T, t_j)B(t_j)U + \Phi(T, t_{j+1})B(t_{j+1})U\big)$$

in iterative form:

$$Q_{j+1}^N = \Phi(t_{j+1}, t_j)Q_j^N + \frac{h}{2}\big(\Phi(t_{j+1}, t_j)B(t_j)U + \Phi(t_{j+1}, t_{j+1})B(t_{j+1})U\big), \quad Q_0^N = X_0$$

Remark 2.7
problems with quadrature methods:

- no generalization for nonlinear differential inclusions possible
- values of fundamental solutions $\Phi(t_{j+1}, t_j)$ resp. $\Phi(T, t_j)$ must be known in advance

2.3. Combination Methods

Proposition 2.8 (cf. [B. and Lempio, 1994b], [B., 1995])
Consider $N \in \mathbb{N}$ and a point-wise iterated quadrature formula of Notation 2.1 with non-negative weights $b_\mu \geq 0$ ($\mu = 1, \dots, s$). Assume that

(i) the quadrature method has precision $p - 1$, $p \in \mathbb{N}$

(ii) $\tau_p(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \leq Ch^p$ uniformly in $l \in S_{n-1}$

(iii) $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h^p)$

and uniformly in $j = 0, \dots, N-1$ and $\mu = 1, \dots, s$

(iv) $\widetilde{\Phi}(t_{j+1}, t_j) = \Phi(t_{j+1}, t_j) + \mathcal{O}(h^{p+1})$

(v) $\mathrm{d_H}(\widetilde{U}_\mu(t_j + c_\mu h), \Phi(t_{j+1}, t_j + c_\mu h)B(t_j + c_\mu h)U) = \mathcal{O}(h^p)$

Then, the combination method defined as

$$X_{j+1}^N = \widetilde{\Phi}(t_{j+1}, t_j)X_j^N + h \sum_{\mu=1}^{s} b_\mu \widetilde{U}_\mu(t_j + c_\mu h) \quad (j = 0, \dots, N-1)$$

satisfies the global estimate

$$\mathrm{d_H}(\mathcal{R}(T, t_0, X_0), X_N^N) = \mathcal{O}(h^p) \ .$$

Especially, (iv) is satisfied for

$$\widetilde{U}_\mu(t_j + c_\mu h) := \widetilde{\Phi}_\mu(t_{j+1}, t_j + c_\mu h)B(t_j + c_\mu h)U \ ,$$

$$\text{if} \quad \widetilde{\Phi}_\mu(t_{j+1}, t_j + c_\mu h) = \Phi(t_{j+1}, t_j + c_\mu h) + \mathcal{O}(h^p) \ .$$

Proof: Define for $j = 0, \ldots, N-1$ the iterations

$$R^N_{j+1} = \Phi(t_{j+1}, t_j) R^N_j + \int\limits_{t_j}^{t_{j+1}} \Phi(t_{j+1}, \tau) B(\tau) U d\tau \ ,$$

$$Q^N_{j+1} = \Phi(t_{j+1}, t_j) Q^N_j + h \sum_{\mu=1}^{s} b_\mu \Phi(t_{j+1}, t_j + c_\mu h) B(t_j + \beta_\mu h) U \ ,$$

$$R^N_0 = Q^N_0 = X_0 \ .$$

Then,

$$R^N_N = \mathcal{R}(T, t_0, X_0) \ , \quad Q^N_N = Q_N(\Phi(T, \cdot) B(\cdot) U; [t_0, T]) \ .$$

Show that X^N_j is bounded uniformly in $j = 0, \ldots, N$ and that

$$\begin{aligned} \mathrm{d_H}(R^N_{j+1}, Q^N_{j+1}) &\leq \|\Phi(t_{j+1}, t_j)\| \cdot \mathrm{d_H}(R^N_j, Q^N_j) \\ &\quad + \mathrm{d_H}\Big(\int\limits_{t_j}^{t_{j+1}} \Phi(t_{j+1}, t) B(t) U dt, h \sum_{\mu=1}^{s} b_\mu \Phi(t_{j+1}, t_j + c_\mu h) B(t_j + c_\mu h) U\Big) \\ &\leq (1 + h\widetilde{C}) \, \mathrm{d_H}(R^N_j, Q^N_j) + \mathcal{O}(h^{p+1}) \end{aligned}$$

$$\Rightarrow \mathrm{d_H}(Q^N_j, X^N_j) \leq (1 + h\widetilde{C})^j \, \mathrm{d_H}(R^N_0, Q^N_0) + j\mathcal{O}(h^{p+1}) \leq N\mathcal{O}(h^{p+1}) = \mathcal{O}(h^p) \ .$$

Furthermore,

$$
\begin{aligned}
\mathrm{d_H}(Q_{j+1}^N, X_{j+1}^N) \le & \|\Phi(t_{j+1}, t_j)\| \cdot \mathrm{d_H}(Q_j^N, X_j^N) \\
& + \|\Phi(t_{j+1}, t_j) - \widetilde{\Phi}(t_{j+1}, t_j)\| \cdot \|X_j^N\| \\
& + h \sum_{\mu=1}^{s} b_\mu \, \mathrm{d_H}(\Phi(t_{j+1}, t_j + c_\mu h) B(t_j + c_\mu h) U, \widetilde{U}_\mu(t_j + c_\mu h)) \\
\le & (1 + h\widetilde{C}) \, \mathrm{d_H}(Q_j^N, X_j^N) + \mathcal{O}(h^{p+1}) \\
\Rightarrow \mathrm{d_H}(Q_j^N, X_j^N) \le & (1 + h\widetilde{C})^j \, \mathrm{d_H}(Q_0^N, X_0^N) + j\mathcal{O}(h^{p+1}) \\
\le & (1 + h\widetilde{C})^N \, \mathrm{d_H}(X_0, X_0^N) + N\mathcal{O}(h^{p+1}) \\
\le & e^{(T-t_0)\widetilde{C}} \mathcal{O}(h^p) + \mathcal{O}(h^p) = \mathcal{O}(h^p) \quad . \\
\Rightarrow \mathrm{d_H}(R_j^N, X_j^N) \le & \mathrm{d_H}(R_j^N, Q_j^N) + \mathrm{d_H}(Q_j^N, X_j^N) = \mathcal{O}(h^p)
\end{aligned}
$$

uniformly in $j = 0, \ldots, N$. ∎

Example 2.9 combination method: iter. Riemannian sum/Euler for matrix differ. equation

$$X'(t) = A(t)X(t) \quad (t \in [t_j, t_{j+1}]) \ ,$$
$$X(t_j) = I$$

$$X_{j+1}^N = \widetilde{\Phi}(t_{j+1}, t_j)X_j^N + h\widetilde{\Phi}_1(t_{j+1}, t_j)B(t_j)U \ , \quad (j = 0, \ldots, N-1)$$
$$\widetilde{\Phi}(t_{j+1}, t_j) = \widetilde{\Phi}(t_j, t_j) + hA(t_j)\widetilde{\Phi}(t_j, t_j) \ ,$$
$$\widetilde{\Phi}_1(t_{j+1}, t_j) = \widetilde{\Phi}(t_{j+1}, t_j) \ .$$

Hence,

$$X_{j+1}^N = (I + hA(t_j))X_j^N + h(I + hA(t_j))B(t_j)U \quad (j = 0, \ldots, N-1).$$

Other possibility for calculation: Euler for adjoint equation

$$Y'(t) = -Y(t)A(t) \quad (t \in [t_0, T]) \ ,$$
$$Y(T) = I$$

gives

$$X_N^N = \widetilde{\Phi}(T, t_j)X_0^N + h\sum_{j=0}^{N-1}\widetilde{\Phi}_1(T, t_j)B(t_j)U \ ,$$
$$\widetilde{\Phi}(T, t_j) = N - j \text{ (backward) steps of Euler for adjoint equation,}$$
$$\widetilde{\Phi}_1(T, t_j) = \widetilde{\Phi}(T, t_j) \ .$$

Example 2.10 usual combination of set-valued quadrature method and pointwise DE solver which provides approximations to the values of the fundamental solution at the quadrature nodes:

set-valued quadrature method	solver for differential equations	step-size of DE solver	overall order
iter. Riemannian sum	Euler	h	$\mathcal{O}(h)$
iter. trapezoidal rule	Euler-Cauchy/Heun	h	$\mathcal{O}(h^2)$
iter. midpoint rule	modified Euler	$\frac{h}{2}$	$\mathcal{O}(h^2)$
iter. Simpson's rule	classical RK(4)	$\frac{h}{2}$	$\mathcal{O}(h^4)$
Romberg's method	extrapolation of midpoint rule (with Euler as starting procedure)	$h_i = \frac{T-t_0}{2^i}$	$\mathcal{O}(\prod_{\nu=0}^{j} h_{i-\nu}^2)$

(under suitable smoothness assumptions)

Remark 2.11
problems with these combination methods:

- no generalization for nonlinear differential inclusions possible
- values of fundamental solutions $\Phi(t_{j+1}, t_j)$, $\Phi_\mu(t_j + c_\mu h, t_j)$ resp. $\Phi(T, t_j)$, $\Phi_\mu(T, t_j + c_\mu h)$ must be calculated additionally
- approximation for $\Phi_\mu(t_j + c_\mu h, t_j)$ resp. $\Phi_\mu(T, t_j + c_\mu h)$ is calculated too accurately ($\mathcal{O}(h^{p+1})$ instead of $\mathcal{O}(h^p)$)

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3. Set-Valued Runge-Kutta Methods

Runge-Kutta methods could be expressed by the Butcher array (cf. [Butcher, 1987]):

$$\begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & \ldots & a_{1,s-2} & a_{1,s-1} & a_{1,s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2,s-2} & a_{2,s-1} & a_{2,s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{s-1} & a_{s-1,1} & a_{s-1,2} & \ldots & a_{s-1,s-2} & a_{s-1,s-1} & a_{1,s} \\ c_s & a_{s,1} & a_{s,2} & \ldots & a_{s,s-2} & a_{s,s-1} & a_{s,s} \\ \hline & b_1 & b_2 & \ldots & b_{s-2} & b_{s-1} & b_s \end{array} \quad \text{with } c_1 := 0 \enspace .$$

Explicit Runge-Kutta methods satisfy $a_{\mu,\nu} = 0$, if $\mu \leq \nu$ and $c_1 = 0$.
The set-valued Runge-Kutta method for LDI is defined as follows:
Choose a starting set $X_0^N \in \mathcal{C}(\mathbb{R}^n)$ and define for $j = 0, \ldots, N-1$ and $\mu = 1, \ldots, s$:

$$\eta_{j+1}^N = \eta_j^N + h \sum_{\mu=1}^{s} b_\mu \xi_j^{(\mu)} \enspace , \tag{8}$$

$$\xi_j^{(\mu)} = A(t_j + c_\mu h)\big(\eta_j^N + h \sum_{\nu=1}^{\mu-1} a_{\mu,\nu} \xi_j^{(\nu)}\big) + B(t_j + c_\mu h) u_j^{(\mu)} \enspace , \tag{9}$$

$$u_j^{(\mu)} \in U \enspace , \tag{10}$$

$$\eta_0^N \in X_0^N \enspace , \tag{11}$$

$$X_{j+1}^N = \{\eta_{j+1}^N \mid \eta_{j+1}^N \text{ is defined by (8)–(11)}\} \enspace . \tag{12}$$

Remark 3.1 If nonlinear DIs are considered with $F(t,x) = \bigcup_{u \in U} \{f(t,x,u)\}$, equation (9) must be replaced by

$$\xi_j^{(\mu)} = f(t_j + c_\mu h, \eta_j^N + h \sum_{\nu=1}^{\mu-1} a_{\mu,\nu} \xi_j^{(\nu)}, u_j^{(\mu)}) \ .$$

For some selection strategies, some of the selections $u_j^{(\mu)}$ depend on others (e.g., they could be all equal).
If $f(t,x,u) = f(t,u)$, i.e. $F(t,x) = F(t)$, and $X_0^N = \{0_{\mathbb{R}^n}\}$, we arrive at the underlying quadrature method

$$\eta_{j+1}^N = \eta_j^N + h \sum_{\mu=1}^{s} b_\mu f(t_j + c_\mu h, u_j^{(\mu)}), \quad u_j^{(\mu)} \in U \ ,$$

$$X_{j+1}^N = X_j^N + h \sum_{\mu=1}^{s} b_\mu F(t_j + c_\mu h) \ ,$$

$$X_N^N = h \sum_{j=0}^{N-1} \sum_{\mu=1}^{s} b_\mu F(t_j + c_\mu h) = Q_N(F; [t_0, T])$$

of the Runge-Kutta method.
If $f(t,x,u) = f(t,x)$, i.e. $F(t,x) = \{f(t,x)\}$, then $X_j^N = \{\eta_j^N\}$ coincides with the pointwise Runge-Kutta method.

Remark 3.2 Grouping in equation (8) by matrices multiplied by η_j^N and $u_j^{(\mu)}$, $\mu = 1, \ldots, s$ we arrive at the form

$$X_{j+1}^N = \widetilde{\Phi}(t_{j+1}, t_j) X_j^N + h \bigcup_{u_j^{(\mu)} \in U} \{ \sum_{\mu=1}^{s} b_\mu \widetilde{\Psi}_\mu(t_{j+1}, t_j + c_\mu h) u_j^{(\mu)} \}$$

with suitable matrices $\widetilde{\Phi}(t_{j+1}, t_j)$ (involving matrix values of $A(\cdot)$) and $\widetilde{\Psi}_\mu(t_{j+1}, t_j + c_\mu h)$ (involving matrix values of $A(\cdot)$ and $B(\cdot)$).
$\widetilde{\Phi}(t_{j+1}, t_j)$ is the same matrix as in the pointwise case for $f(t, x, u) = A(t)x$, hence it approximates $\Phi(t_{j+1}, t_j)$ from the same order as in the pointwise case.

Questions:

- What is the order of the set-valued Runge-Kutta method,
  i.e. $\mathrm{d_H}(\mathcal{R}(T, t_0, X_0), X_0^N) = \mathcal{O}(h^p)$?
  Does the order coincide with the single-valued case?
- What selection strategy is preferrable?
- Should the chosen selection strategy depend on the Runge-Kutta method?
- What smoothness assumptions do we need?

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Answers in the literature:

set-valued RK-method	iter. quadrature method	global order	disturbance term for . . .	local order of disturbance	overall global order
Euler	Riemannian sum	$\mathcal{O}(h)$	η_j^N	$\mathcal{O}(h^2)$	$\mathcal{O}(h)$
			$u_j^{(1)}$	$\mathcal{O}(h)$	
Euler/Cauchy (constant sel.)	midpoint rule	$\mathcal{O}(h^2)$	η_j^N	$\mathcal{O}(h^3)$	$\mathcal{O}(h^2)$
			$u_j^{(1)}$	$\mathcal{O}(h^2)$	
Euler/Cauchy (2 free sel.)	trapezoidal rule	$\mathcal{O}(h^2)$	η_j^N	$\mathcal{O}(h^3)$	$\mathcal{O}(h^2)$
			$u_j^{(1)}$	$\mathcal{O}(h^2)$	
			$u_j^{(2)}$	$\mathcal{O}(h^2)$	

Euler's method (see Subsection 3.1):
  cf. [Nikol'skiĭ, 1988], [Dontchev and Farkhi, 1989], [Wolenski, 1990] for nonlinear DIs,
  for extensions see [Artstein, 1994], [Grammel, 2003])
Euler-Cauchy method (see Subsection 3.2):
  cf. [Veliov, 1992] as well as [Veliov, 1989b]
  for strongly convex nonlinear DIs
modified Euler method (see Subsection 3.3)
Runge-Kutta(4) method (see Subsection 3.4)

3.1. Euler's Method

Remark 3.3 Consider Euler's method, i.e. the Butcher array

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \quad .$$

underlying quadrature method = special Riemannian sum:

$$Q_N(F;[t_0,T]) = h\sum_{j=0}^{N-1} F(t_j)$$

Grouping by η_j^N and the single selection $u_j^{(1)}$ yields

$$X_{j+1}^N = \big(I + hA(t_j)\big)X_j^N + hB(t_j)U \quad (j = 0,\dots,N-1) \ .$$

Proposition 3.4 Euler's method is a combination method with the following settings:

$$Q_N(F;[t_0,T]) = h\sum_{j=0}^{N-1} F(t_j) \ ,$$

$$\widetilde{\Phi}(t_{j+1},t_j) = I + hA(t_j) \ ,$$

$$\widetilde{\Phi}_1(t_{j+1},t_j) = I \ .$$

Proposition 3.5 (cf. [Nikol'skiĭ, 1988], [Dontchev and Farkhi, 1989], [Wolenski, 1990], see also [Artstein, 1994], [Grammel, 2003])
If

- $A(\cdot)$ is Lipschitz,
- $B(\cdot)$ is bounded,
- $\tau_1(\delta^*(l, \Phi(T,\cdot)B(\cdot)U), h) \leq Ch$ uniformly in $l \in S_{n-1}$, e.g., if $B(\cdot)$ is Lipschitz,
- $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h)$,

then Euler's method converges at least with order $\mathcal{O}(h)$.

Proof: The quadrature method has precision 0.
If $B(\cdot)$ is Lipschitz, then $\Phi(T,\cdot)B(\cdot)$ and hence also $\delta^*(l, \Phi(T,\cdot)B(\cdot)U)$ (uniformly in $l \in S_{n-1}$) are Lipschitz.
The following estimations are valid:

$$\|\widetilde{\Phi}(t_{j+1}, t_j) - \Phi(t_{j+1}, t_j)\| = \|\big(I + hA(t_j)\big) - \Phi(t_{j+1}, t_j)\| = \mathcal{O}(h^2) \;,$$
$$\|\widetilde{\Phi}_1(t_{j+1}, t_j) - \Phi(t_{j+1}, t_j)\| = \|I - \Phi(t_{j+1}, t_j)\| = \mathcal{O}(h) \;.$$

Hence, Proposition 2.8 can be applied yielding $\mathcal{O}(h)$. ■

For order of convergence 1, it is sufficient that $A(\cdot)$ and $B(\cdot)$ (resp. $\delta^*(l, \Phi(T,\cdot)B(\cdot)U)$, uniformly in $l \in S_{n-1}$) have bounded variation.

3.2. Euler-Cauchy Method (or Heun's Method)

Remark 3.6 Consider method of Euler-Cauchy (or Heun's method), i.e. the Butcher array

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \quad .$$

underlying quadrature method = iterated trapezoidal rule:

$$Q_N(F;[t_0,T]) = \frac{h}{2}\sum_{j=0}^{N-1}\big(F(t_j)+F(t_{j+1})\big)$$

Grouping by η_j^N and the two selections $u_j^{(1)}$ and $u_j^{(2)}$ yields

$$\begin{aligned} X_{j+1}^N = {} & \Big(I + \frac{h}{2}\big(A(t_j)+A(t_{j+1})\big) + \frac{h^2}{2}A(t_{j+1})A(t_j)\Big)X_j^N \\ & + \frac{h}{2}\bigcup_{u_j^{(1)},u_j^{(2)}\in U}\Big(\big(I+hA(t_{j+1})\big)B(t_j)u_j^{(1)} + B(t_{j+1})u_j^{(2)}\Big) \end{aligned}$$

for $j = 0, \ldots, N-1$.

Proposition 3.7 The method of Euler-Cauchy with two free selections "$u_j^{(1)}, u_j^{(2)} \in U$" is a combination method with the following settings:

$$
\begin{aligned}
Q_N(F;[t_0,T]) &= \frac{h}{2}\sum_{j=0}^{N-1}\big(F(t_j)+F(t_{j+1})\big) \ , \\
\widetilde{\Phi}(t_{j+1},t_j) &= I+\frac{h}{2}\big(A(t_j)+A(t_{j+1})\big)+\frac{h^2}{2}A(t_{j+1})A(t_j) \ , \\
\widetilde{\Phi}_1(t_{j+1},t_j) &:= I+hA(t_{j+1}) \ , \\
\widetilde{\Phi}_2(t_{j+1},t_{j+1}) &:= I \ .
\end{aligned}
$$

Proposition 3.8 The method of Euler-Cauchy with constant selection strategy "$u_j^{(1)} = u_j^{(2)}$" is a combination method with the following settings:

$$
\begin{aligned}
Q_N(F;[t_0,T]) &= h\sum_{j=0}^{N-1}F(t_j+\frac{h}{2}) \ , \\
\widetilde{\Phi}(t_{j+1},t_j) &= I+\frac{h}{2}\big(A(t_j)+A(t_{j+1})\big)+\frac{h^2}{2}A(t_{j+1})A(t_j) \ , \\
\widetilde{U}_1(t_j+\frac{h}{2}) &:= \frac{1}{2}\big(B(t_j)+B(t_{j+1})+hA(t_{j+1})B(t_j)\big)U \ .
\end{aligned}
$$

Proposition 3.9
(cf. [Veliov, 1992] as well as [Veliov, 1989b] for strongly convex nonlinear DIs)
If

- $A'(\cdot)$ and $B(\cdot)$ are Lipschitz,
- $\tau_2(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \leq Ch^2$ uniformly in $l \in S_{n-1}$, e.g., if $B'(\cdot)$ is Lipschitz,
- $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h^2)$,

then the method of Euler-Cauchy with constant or with two free selections converges at least with order $\mathcal{O}(h^2)$.

For order of convergence 2, it is sufficient that $A'(\cdot)$ and $B'(\cdot)$ (resp. $\frac{d}{dt}\delta^*(l, \Phi(T, \cdot)B(\cdot)U)$, uniformly in $l \in S_{n-1}$) have bounded variation.

3.3. Modified Euler Method

Remark 3.10 Consider modified Euler method, i.e. the Butcher array

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \quad .$$

underlying quadrature method = iterated midpoint rule:

$$Q_N(F;[t_0,T]) = h\sum_{j=0}^{N-1} F(t_j + \frac{h}{2})$$

Grouping by η_j^N and the two selections $u_j^{(1)}$ and $u_j^{(2)}$ yields

$$\begin{aligned} X_{j+1}^N = &\Big(I + hA(t_j+\frac{h}{2}) + \frac{h^2}{2}A(t_j+\frac{h}{2})A(t_j)\Big)X_j^N \\ &+ h \bigcup_{u_j^{(1)},u_j^{(2)}\in U} \Big(\frac{h}{2}A(t_j+\frac{h}{2})B(t_j)u_j^{(1)} + B(t_j+\frac{h}{2})u_j^{(2)}\Big) \end{aligned}$$

for $j = 0, \ldots, N-1$.

Proposition 3.11 Modified Euler method with constant selection strategy "$u_j^{(1)} = u_j^{(2)}$" is a combination method with the following settings:

$$Q_N(F; [t_0, T]) = h \sum_{j=0}^{N-1} F(t_j + \frac{h}{2}) \; ,$$

$$\widetilde{\Phi}(t_{j+1}, t_j) = I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2} A(t_j + \frac{h}{2}) A(t_j) \; ,$$

$$\widetilde{U}_1(t_j + \frac{h}{2}) := \Big(B(t_j + \frac{h}{2}) + \frac{h}{2} A(t_j + \frac{h}{2}) B(t_j) \Big) U \; .$$

constant approximation by the quadrature method (midpoint rule) on $[t_j, t_{j+1}]$
$\Rightarrow$ constant selection in modified Euler is appropriate

Proposition 3.12 If

- $A'(\cdot)$ and $B(\cdot)$ are Lipschitz,
- $\tau_2(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \leq Ch^2$ uniformly in $l \in S_{n-1}$, e.g., if $B'(\cdot)$ is Lipschitz,
- $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h^2)$,

then modified Euler method with constant selection strategy converges at least with order $\mathcal{O}(h^2)$.

For order of convergence 2, it is sufficient that $A'(\cdot)$ and $B'(\cdot)$ (resp. $\frac{d}{dt}\delta^*(l, \Phi(T, \cdot)B(\cdot)U)$, uniformly in $l \in S_{n-1}$) have bounded variation.

Proof: The quadrature method has precision 1.
Careful Taylor expansion shows (as in the pointwise case) that

$$\begin{aligned}&\|\widetilde{\Phi}(t_{j+1},t_j)-\Phi(t_{j+1},t_j)\|\\=&\|\big(I+hA(t_j+\frac{h}{2})+\frac{h^2}{2}A(t_j+\frac{h}{2})A(t_j)\big)-\Phi(t_{j+1},t_j)\|=\mathcal{O}(h^3)\ .\end{aligned}$$

The following estimations are valid:

$$\mathrm{d_H}(\widetilde{U}_1(t_j+\frac{h}{2}),\big(I+\frac{h}{2}A(t_j+\frac{h}{2})\big)B(t_j+\frac{h}{2})U)=\mathcal{O}(h^2)\ ,$$

$$\mathrm{d_H}(\big(I+\frac{h}{2}A(t_j+\frac{h}{2})\big)B(t_j+\frac{h}{2})U,\Phi(t_{j+1},t_j+\frac{h}{2})B(t_j+\frac{h}{2})U)=\mathcal{O}(h^2)$$

Hence,

$$\mathrm{d_H}(\widetilde{U}_1(t_j+\frac{h}{2}),\Phi(t_{j+1},t_j+\frac{h}{2})B(t_j+\frac{h}{2})U)=\mathcal{O}(h^2)$$

follows with (6).
Alltogether, Proposition 2.8 can be applied yielding $\mathcal{O}(h^2)$. ■

Proposition 3.13 Modified Euler method with two free choices $u_j^{(1)}, u_j^{(2)} \in U$ is a combination method with the following settings:

$$\text{(i)} \quad Q_N(F;[t_0,T]) = h \sum_{j=0}^{N-1} F(t_j + \frac{h}{2}) \ ,$$

$$\widetilde{\Phi}(t_{j+1},t_j) = I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2} A(t_j + \frac{h}{2}) A(t_j) \ ,$$

$$\widetilde{U}_1(t_j + \frac{h}{2}) = B(t_j + \frac{h}{2})U + \frac{h}{2} A(t_j + \frac{h}{2}) B(t_j) U$$

resp.

$$\text{(ii)} \quad Q_N(F;[t_0,T]) = \frac{h}{2} \sum_{j=0}^{N-1} \big(F(t_j) + F(t_{j+1})\big) \ ,$$

$$\widetilde{\Phi}(t_{j+1},t_j) = I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2} A(t_j + \frac{h}{2}) A(t_j) \ ,$$

$$\widetilde{U}_1(t_j) = B(t_j + \frac{h}{2})U + hA(t_j + \frac{h}{2}) B(t_j) U \ ,$$

$$\widetilde{U}_2(t_{j+1}) = B(t_j + \frac{h}{2})U \ .$$

problem in (i): Minkowski sum of 2 sets in $\widetilde{U}_1(t_j + \frac{h}{2})$, hence disturbance term $\mathcal{O}(h)$

problem in (ii): Minkowski sum of 2 sets and $B(t_j + \frac{h}{2})$ instead of $B(t_j)$ in $\widetilde{U}_1(t_j)$
resp. $B(t_j + \frac{h}{2})$ instead of $B(t_{j+1})$ in $\widetilde{U}_2(t_{j+1})$

The problem with two selections was also observed in the approximation of nonlinear optimal controls (cf. [Dontchev et al., 2000]).

Proposition 3.14 If

- $A(\cdot)$ is Lipschitz and $B(\cdot)$ is bounded,
- $\tau_1(\delta^*(l, \Phi(T,\cdot)B(\cdot)U), h) \le Ch$ uniformly in $l \in S_{n-1}$, e.g., if $B(\cdot)$ is Lipschitz,
- $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h)$,

then modified Euler method with two free selections converges at least with order $\mathcal{O}(h)$.

Proof: The quadrature method has precision 1, hence also 0.
Careful Taylor expansion shows (as in the pointwise case) that

$$\|\widetilde{\Phi}(t_{j+1},t_j) - \Phi(t_{j+1},t_j)\| = \|\big(I + hA(t_j+\frac{h}{2}) + \frac{h^2}{2}A(t_j+\frac{h}{2})A(t_j)\big) - \Phi(t_{j+1},t_j)\|$$
$$\le \|\big(I + hA(t_j+\frac{h}{2})\big) - \Phi(t_{j+1},t_j)\| + \frac{h^2}{2}\|A(t_j+\frac{h}{2})\| \cdot \|A(t_j)\| = \mathcal{O}(h^2) \ .$$

The following estimations for (i) in Proposition 3.13 are valid:

$$\begin{aligned}
&\mathrm{d_H}(\widetilde{U}_1(t_j+\frac{h}{2}), \Phi(t_{j+1}, t_j+\frac{h}{2})B(t_j+\frac{h}{2})U) \\
=&\,\mathrm{d_H}(B(t_j+\frac{h}{2})U + \frac{h}{2}A(t_j+\frac{h}{2})B(t_j)U, \Phi(t_{j+1}, t_j+\frac{h}{2})B(t_j+\frac{h}{2})U) \\
\le&\,\mathrm{d_H}(B(t_j+\frac{h}{2})U, \Phi(t_{j+1}, t_j+\frac{h}{2})B(t_j+\frac{h}{2})U) + \frac{h}{2}\|A(t_j+\frac{h}{2})B(t_j)U\| \\
\le&\,\|I - \Phi(t_{j+1}, t_j+\frac{h}{2})\| \cdot \|B(t_j+\frac{h}{2})\| \cdot \|U\| + \mathcal{O}(h) = \mathcal{O}(h) \ .
\end{aligned}$$

Hence, Proposition 2.8 can be applied yielding $\mathcal{O}(h)$. ■

Remark 3.15 If we assume that $A'(\cdot)$ is Lipschitz, it would be valid that

$$
\begin{aligned}
&\|\widetilde{\Phi}(t_{j+1}, t_j) - \Phi(t_{j+1}, t_j)\| \\
=&\|(I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2} A(t_j + \frac{h}{2}) A(t_j)) - \Phi(t_{j+1}, t_j)\| = \mathcal{O}(h^3) \ .
\end{aligned}
$$

But the disturbances in $\widetilde{U}_1(t_j + \frac{h}{2})$ are not of order $\mathcal{O}(h^2)$.
Please notice that in (i)

$$
\begin{aligned}
\widetilde{U}_1(t_j + \frac{h}{2}) &= B(t_j + \frac{h}{2}) U + \frac{h}{2} A(t_j + \frac{h}{2}) B(t_j) U \\
&\neq \big(B(t_j + \frac{h}{2}) + \frac{h}{2} A(t_j + \frac{h}{2}) B(t_j)\big) U \\
&= \Phi(t_{j+1}, t_j + \frac{h}{2}) B(t_j + \frac{h}{2}) U + \mathcal{O}(h^2)
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
\mathrm{d_H}(&B(t_j + \frac{h}{2}) U + \frac{h}{2} A(t_j + \frac{h}{2}) B(t_j) U, \\
&\big(B(t_j + \frac{h}{2}) + \frac{h}{2} A(t_j + \frac{h}{2}) B(t_j)\big) U) = \mathcal{O}(h) \ .
\end{aligned}
$$

constant approximation by the quadrature method (midpoint rule) on $[t_j, t_{j+1}]$
$\Rightarrow$ two free selections in modified Euler do not fit well, possible order breakdown

Proposition 3.16 Modified Euler method with linear interpolated selections $u_j^{(1)}, u_j^{(3)} \in U$ and $u_j^{(2)} = \frac{1}{2}(u_j^{(1)} + u_j^{(3)})$ is a combination method with the following settings:

$$Q_N(F;[t_0,T]) = \frac{h}{2}\sum_{j=0}^{N-1}\big(F(t_j) + F(t_{j+1})\big) \quad ,$$

$$\widetilde{\Phi}(t_{j+1},t_j) = I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2}A(t_j + \frac{h}{2})A(t_j) \quad ,$$

$$\widetilde{U}_1(t_j) = \big(B(t_j + \frac{h}{2}) + hA(t_j + \frac{h}{2})B(t_j)\big)U \quad ,$$

$$\widetilde{U}_2(t_{j+1}) = B(t_j + \frac{h}{2})U \quad .$$

problem: $B(t_j + \frac{h}{2})$ instead of $B(t_j)$ resp. $B(t_j + \frac{h}{2})$ instead of $B(t_{j+1})$
This strategy was used in the approximation of the value function of Hamilton-Jacobi-Bellman equations in [Ferretti, 1994] and caused two unexpected results in one test example.

Proposition 3.17 If

- $A(\cdot)$ is Lipschitz and $B(\cdot)$ is bounded,
- $\tau_1(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \leq Ch$ uniformly in $l \in S_{n-1}$, e.g., if $B(\cdot)$ is Lipschitz,
- $d_H(X_0, X_0^N) = \mathcal{O}(h)$,

then modified Euler method with linear interpolated selections converges at least with order $\mathcal{O}(h)$.

Proof: The quadrature method has precision 1, hence also 0.
Careful Taylor expansion shows as for two free selections that

$$\|\widetilde{\Phi}(t_{j+1}, t_j) - \Phi(t_{j+1}, t_j)\| = \mathcal{O}(h^2) \quad .$$

The following estimations in Proposition 3.16 are valid:

$$\begin{aligned}
& d_H(\widetilde{U}_1(t_j), \Phi(t_{j+1}, t_j)B(t_j)U) \\
= & d_H\Big(\big(B(t_j + \frac{h}{2}) + hA(t_j + \frac{h}{2})B(t_j)\big)U, \big(B(t_j) + hA(t_j)B(t_j)\big)U\Big) \\
& + d_H((I + hA(t_j))B(t_j)U, \Phi(t_{j+1}, t_j)B(t_j)U) \\
\leq & \Big(\|B(t_j + \frac{h}{2}) - B(t_j)\| + h\|A(t_j + \frac{h}{2}) - A(t_j)\| \cdot \|B(t_j)\|\Big) \cdot \|U\| \\
& + \|(I + hA(t_j)) - \Phi(t_{j+1}, t_j)\| \cdot \|B(t_j)\| \cdot \|U\| = \mathcal{O}(h) \quad , \\
& d_H(\widetilde{U}_2(t_{j+1}), \Phi(t_{j+1}, t_{j+1})B(t_{j+1})U) \leq \|B(t_j + \frac{h}{2}) - B(t_{j+1})\| \cdot \|U\| = \mathcal{O}(h)
\end{aligned}$$

Hence, Proposition 2.8 can be applied yielding $\mathcal{O}(h)$. ■

Remark 3.18 Assuming more smoothness, we could show that

$$\|\widetilde{\Phi}(t_{j+1}, t_j) - \Phi(t_{j+1}, t_j)\| = \mathcal{O}(h^3) \quad ,$$

for time-independent situations it is valid that

$$\begin{aligned} \mathrm{d_H}(\widetilde{U}_1(t_j), \Phi(t_{j+1}, t_j)B(t_j)U) &= \mathcal{O}(h^2) \quad , \\ \mathrm{d_H}(\widetilde{U}_2(t_{j+1}), \Phi(t_{j+1}, t_{j+1})B(t_{j+1})U) &= \mathcal{O}(h^2) \end{aligned}$$

and hence global order of convergence $\mathcal{O}(h^2)$.

Example 3.19 (cf. [Veliov, 1992]) Let $n = 2$, $m = 1$, $I = [0, 1]$ and set

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad U = [-1, 1].$$

Since (13) is fulfilled here, both selection strategies for modified Euler differ.

→ image: modif. Euler (constant selections) → image: modif. Euler (2 free selections)

data for the pictures:

- reference set (black) = combination method "iterated trapezoidal rule and Euler/Cauchy" with $N = 10000$ subintervals
- calculated supporting points in $M = 200$ directions
- different stepsizes: $h = 1$ (red), 0.5 (blue), 0.25 (green), 0.125 (magenta), 0.0625 (cyan)

computed estimations of the order of convergence:

N	Hausdorff distance to reference set	estimated order of convergence	Hausdorff distance to reference set	estimated order of convergence
1	0.21434524	———	0.75039466	———
2	0.05730861	1.90311	0.36454336	1.04156
4	0.01517382	1.91717	0.17953522	1.02182
8	0.00384698	1.97979	0.08841414	1.02192
16	0.00096510	1.99498	0.04419417	1.00042
	(constant selections)		(2 free selections)	

Possible order breakdown to $\mathcal{O}(h)$ in Proposition 3.14 for modified Euler with two free selections can occur!

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Example 3.20 data as in Example 3.21, only $A(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Both selection strategies for modified Euler coincide, since

$$(B + \frac{h}{2}AB)U = (\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{h}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix})\,\mathrm{co}\{-1, 1\} = \mathrm{co}\{\begin{pmatrix} 0 \\ -1-h \end{pmatrix}, \begin{pmatrix} 0 \\ 1+h \end{pmatrix}\}\ ,$$

$$BU + \frac{h}{2}ABU = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{co}\{-1, 1\} + \frac{h}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix}\mathrm{co}\{-1, 1\}$$

$$= \mathrm{co}\{\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} + \mathrm{co}\{\begin{pmatrix} 0 \\ -h \end{pmatrix}, \begin{pmatrix} 0 \\ h \end{pmatrix}\} = \mathrm{co}\{\begin{pmatrix} 0 \\ -1-h \end{pmatrix}, \begin{pmatrix} 0 \\ 1+h \end{pmatrix}\}\ .$$

→ image: modif. Euler (constant selections) → image: modif. Euler (2 free selections)

computed estimations of the order of convergence:

N	Hausdorff distance to reference set	estimated order of convergence	Hausdorff distance to reference set	estimated order of convergence
1	1.19452805	—	1.19452805	—
2	0.56952805	1.06860	0.56952805	1.06860
4	0.20807785	1.45264	0.20807785	1.45264
8	0.06340445	1.71447	0.06340445	1.71447
16	0.01748660	1.85833	0.01748660	1.85833
32	0.00458787	1.93035	0.00458787	1.93035
64	0.00117462	1.96562	0.00117462	1.96562
	(constant selections)		(2 free selections)	

Possible order breakdown to $\mathcal{O}(h)$ in Proposition 3.14 for modified Euler with two free selections does not occur always!

Example 3.21 (cf. [B. and Lempio, 1994a]) Let $n = m = 2$, $I = [0, 1]$ and set

$$A(t) = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1-t & t \cdot e^t \\ 3-2t & (-1+2t) \cdot e^t \end{pmatrix} \quad \text{and} \quad U = [-1, 1]^2.$$

$\longrightarrow$ image: modif. Euler (linear interpolation) $\longrightarrow$ image: modif. Euler (constant selections)

data for the pictures:

- reference set (black) = combination method ”iterated Simpson’s rule and RK(4)” with $N = 100000$ subintervals
- calculated supporting points in $M = 200$ directions
- different stepsizes: $h = 1$ (red), 0.5 (blue), 0.25 (green), 0.125 (magenta)

computed estimations of the order of convergence:

N	Hausdorff distance to reference set	estimated order of convergence	Hausdorff distance to reference set	estimated order of convergence
1	2.47539809	———	0.67713923	———
2	0.42619535	2.53807	0.12998374	2.38112
4	0.12006081	1.82775	0.02271635	2.51653
8	0.05540102	1.11578	0.00498557	2.18790
16	0.02687764	1.04351	0.00119539	2.06027
32	0.01321630	1.02409	0.00029294	2.02881
64	0.00655070	1.01260	0.00007252	2.01407
	(selections by linear interpolation)		(constant selections)	

Possible order breakdown to $\mathcal{O}(h)$ in Proposition 3.17 for modified Euler with linear interpolated selections can occur!

3.4. Runge-Kutta (4)

Remark 3.22 Consider the classical Runge-Kutta (4) method, i.e. the Butcher array

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} .$$

underlying quadrature method = iterated Simpson's rule:

$$Q_N(F;[t_0,T]) = \frac{h}{6}\sum_{j=0}^{N-1}\Big(F(t_j) + 4F(t_j + \frac{h}{2}) + F(t_{j+1})\Big)$$

Grouping by η_j^N and the four selections $u_j^{(\mu)}$, $\mu = 1, \ldots, 4$, yields

$$
\begin{aligned}
X_{j+1}^N = \Big(& I + \frac{h}{6}\Big(A(t_j) + 4A(t_j + \frac{h}{2}) + A(t_{j+1})\Big) \\
& + \frac{h^2}{6}\Big(A(t_j + \frac{h}{2})A(t_j) + A(t_j + \frac{h}{2})^2 + A(t_{j+1})A(t_j + \frac{h}{2})\Big) \\
& + \frac{h^3}{12}\Big(A(t_j + \frac{h}{2})^2 A(t_j) + A(t_{j+1})A(t_j + \frac{h}{2})^2\Big) + \frac{h^4}{24}\Big(A(t_{j+1})A(t_j + \frac{h}{2})^2 A(t_j)\Big)\Big) X_j^N \\
& + \frac{h}{6} \bigcup_{\substack{u_j^{(\nu)} \in U \\ \nu = 1,\ldots,4}} \Big\{ \Big(I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2}A(t_j + \frac{h}{2})^2 + \frac{h^3}{4}A(t_{j+1})A(t_j + \frac{h}{2})^2\Big) \cdot B(t_j)u_j^{(1)} \\
& \qquad + 2\Big(I + \frac{h}{2}A(t_j + \frac{h}{2}) + \frac{h^2}{4}A(t_{j+1})A(t_j + \frac{h}{2})\Big)B(t_j + \frac{h}{2})u_j^{(2)} \\
& \qquad + 2\Big(I + \frac{h}{2}A(t_{j+1})\Big)B(t_j + \frac{h}{2})u_j^{(3)} + B(t_{j+1})u_j^{(4)} \Big\}
\end{aligned}
$$

for $j = 0, \ldots, N-1$.

Remark 3.23 4 different selection strategies:

- constant selections: $u_j^{(1)} \in U$ and $u_j^{(\mu)} = u_j^{(1)}$ for $\mu = 2, 3, 4$
- linear interpolated selections: $u_j^{(1)}, u_j^{(4)} \in U$ and $u_j^{(2)} = \frac{1}{2}(u_j^{(1)} + u_j^{(4)})$, $u_j^{(3)} = u_j^{(2)}$
- 3 free selections: $u_j^{(1)}, u_j^{(2)}, u_j^{(4)} \in U$ and $u_j^{(3)} = u_j^{(2)}$
- 4 free selections: $u_j^{(1)}, u_j^{(2)}, u_j^{(3)}, u_j^{(4)} \in U$

Proposition 3.24 Runge-Kutta(4) method with 3 free selections ”$u_j^{(1)}, u_j^{(2)}, u_j^{(4)} \in U$” and ”$u_j^{(2)} = u_j^{(3)}$” is a combination method with the following settings:

$$Q_N(F;[t_0,T]) = \frac{h}{6}\sum_{j=0}^{N-1}\left(F(t_j) + 4F(t_j+\frac{h}{2}) + F(t_{j+1})\right) \ ,$$

$$\begin{aligned}\widetilde{\Phi}(t_{j+1},t_j) = \Bigg(I &+ \frac{h}{6}\Big(A(t_j) + 4A(t_j+\frac{h}{2}) + A(t_{j+1})\Big)\\ &+ \frac{h^2}{6}\Big(A(t_j+\frac{h}{2})A(t_j) + A(t_j+\frac{h}{2})^2 + A(t_{j+1})A(t_j+\frac{h}{2})\Big)\\ &+ \frac{h^3}{12}\Big(A(t_j+\frac{h}{2})^2A(t_j) + A(t_{j+1})A(t_j+\frac{h}{2})^2\Big)\\ &+ \frac{h^4}{24}\Big(A(t_{j+1})A(t_j+\frac{h}{2})^2A(t_j)\Big)\Bigg) \ ,\end{aligned}$$

$$\widetilde{\Phi}_1(t_{j+1},t_j) := I + hA(t_j+\frac{h}{2}) + \frac{h^2}{2}A(t_j+\frac{h}{2})^2 + \frac{h^3}{4}A(t_{j+1})A(t_j+\frac{h}{2})^2 \ ,$$

$$\widetilde{\Phi}_2(t_{j+1},t_j+\frac{h}{2}) := I + \frac{h}{4}\Big(A(t_j+\frac{h}{2}) + A(t_{j+1})\Big) + \frac{h^2}{8}A(t_{j+1})A(t_j+\frac{h}{2}) \ ,$$

$$\widetilde{\Phi}_3(t_{j+1},t_{j+1}) := I \ .$$

three sets involved in the quadrature method (Simpson’s rule) on $[t_j, t_{j+1}]$

$\Rightarrow$ three free selections in Runge-Kutta(4) is appropriate

$\frac{4h}{6}$ as weight in Simpson’s rule, $\frac{2h}{6}$ as weights for $u_j^{(2)}$ and $u_j^{(3)}$

$\Rightarrow u_j^{(2)} = u_j^{(3)}$

Proposition 3.25 If

- $A''(\cdot)$ is Lipschitz,
- $\tau_3(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \leq Ch^3$ uniformly in $l \in S_{n-1}$,
- $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h^3)$,

then Runge-Kutta(4) method with the three selection strategy converges at least with order $\mathcal{O}(h^3)$.

Proof: The quadrature method has precision 3, hence also 2.
Careful Taylor expansion shows (as in the pointwise case) that

$$\|\widetilde{\Phi}(t_{j+1}, t_j) - \Phi(t_{j+1}, t_j)\|$$
$$= \|\Big(I + hA(t_j + \frac{h}{2}) + \frac{h^2}{2}A(t_j + \frac{h}{2})A(t_j)\Big) - \Phi(t_{j+1}, t_j)\| = \mathcal{O}(h^4) \ .$$

The following estimations are valid:

$$\widetilde{\Phi}_3(t_{j+1}, t_{j+1}) = I = \Phi(t_{j+1}, t_{j+1}) \ ,$$

$$
\begin{aligned}
&\|\widetilde{\Phi}_1(t_{j+1},t_j)-\Phi(t_{j+1},t_j)\| \\
&=\Big\|\Big(I+hA(t_j+\frac{h}{2})+\frac{h^2}{2}A(t_j+\frac{h}{2})^2+\frac{h^3}{4}A(t_{j+1})A(t_j+\frac{h}{2})^2\Big) \\
&\qquad-\Big(I+hA(t_j)+\frac{h^2}{2}\big(A'(t_j)+A(t_j)^2\big)+\int_{t_j}^{t_{j+1}}(t_{j+1}-t)\big(\Phi''(t,t_j)-\Phi''(t_j,t_j)\big)dt\Big)\Big\| \\
&\le\Big\|h\big(A(t_j+\frac{h}{2})-A(t_j)\big)+\frac{h^2}{2}\big(A(t_j+\frac{h}{2})^2-A'(t_j)-A(t_j)^2\big)\Big\| \\
&\qquad+\frac{h^3}{4}\|A(t_{j+1})\|\cdot\|A(t_j+\frac{h}{2})\|^2+\int_{t_j}^{t_{j+1}}(t_{j+1}-t_j)\|\Phi''(t,t_j)-\Phi''(t_j,t_j)\|dt \\
&=\mathcal{O}(h^3)\ ,
\end{aligned}
$$

$$
\begin{aligned}
&\|\widetilde{\Phi}_2(t_{j+1}, t_j + \frac{h}{2}) - \Phi(t_{j+1}, t_j + \frac{h}{2})\| \\
&\quad = \|\Big(I + \frac{h}{4}\big(A(t_j + \frac{h}{2}) + A(t_j)\big) + \frac{h^2}{8} A(t_{j+1}) A(t_j + \frac{h}{2})\Big) \\
&\qquad - \Big(I + \frac{h}{2} A(t_j + \frac{h}{2}) + \frac{h^2}{8}\big(A'(t_j + \frac{h}{2}) + A(t_j + \frac{h}{2})^2\big) \\
&\qquad\qquad + \frac{h^2}{8} \int\limits_{t_j + \frac{h}{2}}^{t_{j+1}} (t_{j+1} - t)\big(\Phi''(t, t_j) - \Phi''(t_j, t_j)\big)\, dt\Big)\| \\
&\quad \le \|\frac{h}{4}\big(A(t_j + \frac{h}{2}) + A(t_{j+1}) - 2A(t_j + \frac{h}{2})\big) \\
&\qquad + \frac{h^2}{8}\big(A(t_{j+1}) A(t_j + \frac{h}{2}) - A'(t_j + \frac{h}{2}) - A(t_j + \frac{h}{2})^2\big)\| \\
&\qquad + \int\limits_{t_j + \frac{h}{2}}^{t_{j+1}} (t_{j+1} - t)\|\Phi''(t, t_j + \frac{h}{2}) - \Phi''(t_j + \frac{h}{2}, t_j + \frac{h}{2})\|\, dt = \mathcal{O}(h^3) \ .
\end{aligned}
$$

Hence, Proposition 2.8 can be applied yielding $\mathcal{O}(h^3)$. ■

Remark 3.26 For order of convergence 3, it is sufficient that $A''(\cdot)$ has bounded variation and $\frac{d}{dt}\delta^*(l, \Phi(T, \cdot)B(\cdot)U) \in \mathrm{AC}(I)$ and its derivative has bounded variation uniformly in $l \in S_{n-1}$).

Remark 3.27 The worse approximation of $\widetilde{\Phi}_2(t_{j+1}, t_j + \frac{h}{2})$ prevents the method of achieving globally $\mathcal{O}(h^4)$ as order of convergence.

Proposition 3.28 If

- $A'(\cdot)$ is Lipschitz,
- $\tau_2(\delta^*(l, \Phi(T, \cdot)B(\cdot)U), h) \le Ch^2$ uniformly in $l \in S_{n-1}$, e.g. if $B'(\cdot)$ is Lipschitz
- $\mathrm{d_H}(X_0, X_0^N) = \mathcal{O}(h^2)$,

then Runge-Kutta(4) method with the constant, linear interpolated or four selection strategy converges at least with order $\mathcal{O}(h^2)$.

Sketch of proof:
underlying quadrature method for the constant selections:
iterated midpoint rule
underlying quadrature method for the linear interpolated selections:
iterated trapezoidal rule
4 free selections:
consider this method as disturbed method with 3 free selections
of local order $\mathcal{O}(h^3)$ (use (7)) ∎

Example 3.29 Let $n = 2$, $m = 1$, $I = [0, 1]$ and set

$$A(t) = \frac{1}{t^4+1} \begin{pmatrix} t^4 + 2t^3 + 1 & 2t \\ -2t & t^4 + 2t^3 + 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} t^2 \\ 1 \end{pmatrix} \quad \text{and} \quad U = [-1, 1].$$

→ image: RK(4), 3 free selections → image: RK(4), 4 free selections

data for the pictures:

- reference set (black) = combination method "iterated Simpson's rule and RK(4)" with $N = 10000$ subintervals, $M = 200$ calculated supporting points
- different stepsizes: $h = 1$ (red), 0.5 (green), 0.25 (blue)

computed estimations of the order of convergence:

N	Hausdorff distance to reference set	estimated order of convergence	Hausdorff distance to reference set	estimated order of convergence
1	0.32495716	———	0.35441994	———
2	0.04104212	2.98507	0.07694989	2.20347
4	0.00535449	2.93828	0.02264766	1.76456
8	0.00065949	3.02132	0.00590203	1.94008
16	0.00008127	3.02061	0.00148039	1.99523
32	0.00001007	3.01255	0.00037051	1.99838
64	0.00000125	3.00679	0.00009275	1.99811
128	1.5623e-07	3.00352	0.00002320	1.99932
	(3 free selections)		(4 free selections)	

Hence, in general $\mathcal{O}(h^4)$ could not be expected in Proposition 3.25 for Runge-Kutta(4) with three free selections! With four free selections only $\mathcal{O}(h^2)$ is observed!

Example 3.30 Let $n = m = 2$, $I = [0, 2]$ and set

$$A(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = B_1(0).$$

$\longrightarrow$ image: RK(4), 3 free selections

$\longrightarrow$ image: RK(4), 4 free selections

computed estimations of the order of convergence:

N	Hausdorff distance to reference set	estimated order of convergence	Hausdorff distance to reference set	estimated order of convergence
1	12.15909236	———	16.31389286	———
2	0.57388484	4.40513	1.13925599	3.83994
4	0.01593964	5.17007	0.03880440	4.87573
8	0.00048901	5.02660	0.00134452	4.85106
16	0.00002391	4.35405	0.00006260	4.42478
32	0.00000136	4.14023	0.00000341	4.19891
64	8.1132e-08	4.06303	1.9924e-07	4.09663
128	4.9646e-09	4.03051	1.2045e-08	4.04797
	(3 free selections)		(4 free selections)	

Both selection strategies (3 resp. 4 free selections) lead to $\mathcal{O}(h^4)$.

Remark 3.31

Example	selection strategy	order of convergence
3.29	constant selections	4
	linear interpolated selections	3
	3 free selections	3
	4 free selections	2
3.30	3 free selections	4
	4 free selections	4
	constant selections	2
	linear interpolated selections	2

selection strategy	Example	order of convergence	minimal order
constant selections	3.29	4	2
	3.30	2	
linear interpolated selections	3.29	3	2
	3.30	2	
3 free selections	3.29	3	3
	3.30	4	
4 free selections	3.29	2	2
	3.30	4	

Hence, the use of 3 free selections is the best strategy for Runge-Kutta(4) in general.

4. Conclusions

- framework for convergence proof only suitable for linear differential inclusions
- selection strategy for linear differential inclusions could be transferred to nonlinear ones
- smoothness of $A(\cdot)$ and $B(\cdot)$ is not sufficient,
  smoothness of $\delta^*(l, \Phi(T, \cdot)B(\cdot)U)$ uniformly in $l \in S_{n-1}$ is additionally needed
- interpretation of set-valued Runge-Kutta method as quadrature method with disturbed matrices for fundamental solution is possible
- interpretation is not unique, but there exists a ”natural” choice for a Runge-Kutta method
- necessary for overall order $\mathcal{O}(h^p)$:
  global order $\mathcal{O}(h^p)$ for quadrature method,
  local order $\mathcal{O}(h^{p+1})$ for disturbance of matrix multiplied with the state η_j^N,
  local order $\mathcal{O}(h^p)$ for disturbance of matrices multiplied with the selections $u_j^{(\mu)}$
- convergence result gives minimal order of convergence,
  additional counter examples (numerically/theoretically) are necessary
- convergence proof does not depend on smoothness of optimal control function or corresponding solution
- selection strategies should fit to underlying quadrature method
- other selection strategies with restricted subsets of $U \times \ldots \times U$ are available, see e.g. [Ferretti, 1997], [Lempio and Veliov, 1998], [Grüne and Kloeden, 2001] and Krastanov (2004)
- few numerical implementations for nonlinear differential inclusions, see e.g. [Häckl, 1993], [Chahma, 2003]

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