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ARE WEIGHTED GAMES SUFFICIENTLY GOOD FOR BINARY VOTING?

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ABSTRACT: Binary "yes"-"no" decisions in a legislative committee or a shareholder meeting are commonly modeled as a weighted game. However, there are noteworthy exceptions. E.g., the voting rules of the European Council according to the Treaty of Lisbon use a more complicated construction. Here we want to study the question if we loose much from a practical point of view, if we restrict ourselves to weighted games. To this end, we invoke power indices that measure the influence of a member in binary decision committees. More precisely, we compare the achievable power distributions of weighted games with those from a reasonable superset of weighted games. It turns out that the deviation is relatively small.

JEL classification: C61, C71

Keywords: power measurement; weighted games

1. INTRODUCTION

Consider a family, consisting of mother Ann, father Bob, and the two kids Cathrin and Dave, deciding on their joint weekend activities by binary voting. In a *weighted game* each voter i has a non-negative weight w_i and a proposal is accepted if the sum of the weights of its supporters meets or exceeds a positive quota q. As an abbreviation we write $[q; w_1, \dots, w_n]$ for a weighted game with n voters. The example $[3; 3, 2, 1, 1]$ (where we number in alphabetical order) might model a slightly parents biased, especially mother biased, decision rule. Another voting rule might be that either both parents or both kids have to agree. It can be shown that no representation as a weighted game exists. Since all family members have equal opportunities to influence the final decision, all reasonable measures of voting power assign equal power to all members. This is also true for other weighted games such as $[2; 1, 1, 1, 1]$ or $[3; 1, 1, 1, 1]$ (but not for $[3; 3, 2, 1, 1]$). If we only care about the resulting power distribution we can also choose a weighted game in our situation. Even more practically, we may accept a weighted game as a plausible replacement of the original voting rule if the corresponding power distribution does not differ too much. Here we want to study the question how large this difference can be in the worst case.

A related problem is the so-called *inverse power index problem*, where one wants to determine the game whose power distribution is closest to a predefined target power distribution. For more details see e.g. [De et al.(2017)] and the reference cited therein. [Alon and Edelman(2010)] have shown that some target power distributions, where most players have negligible or even zero power, like e.g. $(0.75, 0.25, 0, \ldots, 0)$, cannot be approximated too closely by the power distribution of any game.[1] Our setting differs as follows. Instead of all non-negative vectors summing to one, we only consider the power distributions attained by a superset of weighted games as possible target power distributions and ask to what extend they can be approximated by the power distribution of a weighted game.

2. Preliminaries

By $N = \{1, \ldots, n\}$ we denote the set of voters. A *simple game* is a surjective and monotone mapping $v\colon 2^N \to \{0,1\}$ from the set of subsets of N into a binary output $\{0,1\}$. *Monotone* means $v(S) \le v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$. A simple game v is weighted if there exist weights $w_1, \ldots, w_n \in \mathbb{R}_{\ge 0}$ and a quota $q \in \mathbb{R}_{>0}$ such that $v(S) = 1$ iff $w(S) := \sum_{i \in S} w_i \ge q$. As stated in the introduction, we abbreviate a weighted game by $[q; w_1, \ldots, w_n]$, Two voters i and j are called *symmetric*, in a given simple game v, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $\emptyset \subseteq S \subseteq N \backslash \{i,j\}$. Voter $i \in N$ is a *null voter* if $v(S) = v(S \cup \{i\})$ for all $\emptyset \subseteq S \subseteq N \backslash \{i\}$.

Given two simple games v and v' we define their *intersection*[2] $v \wedge v'$ via $(v \wedge v')(S) = \min\{v(S), v'(S)\}$ for all $S \subseteq N$. Similarly, the *union*[3] is given by $(v \vee v')(S) = \max\{v(S), v'(S)\}$ for all $S \subseteq N$. The non-weighted decision rule from the introduction can be written as $[2; 2, 0, 1, 1] \wedge [2; 0, 2, 1, 1]$ or $[2; 1, 1, 0, 0] \vee [2; 0, 0, 1, 1]$. It is well known, see e.g. [Taylor and Zwicker(1999)], that every simple game can be written as the intersection (or union) of a finite list of weighted games. Also combinations of $\wedge$ and $\vee$ are used in practice.

An example is given by the voting system of the European Council according to the Treaty of Lisbon. For $n = 27$ (or $n = 28$) countries the voting system can be written as $v = ([0.55n; 1, \ldots, 1] \wedge [0.65; p_1, \ldots, p_n]) \vee [n-3; 1, \ldots, 1]$, where p_i denotes the relative population of country i. As remarked by [Kirsch and Langner(2011)], dropping the union with $[n-3; 1, \ldots, 1]$ has almost no impact on the characteristic function v or corresponding power distributions. Consisting of a Boolean combination, i.e., $\wedge$'s and $\vee$'s, of three weighted games the stated representation of the voting system of the European Council (according to the Treaty of Lisbon) is relatively compact. For a general simple game for n voters an exponential number of weighted games can be necessary in the worst case, see [Faliszewski et al.(2009)]. Writing down the characteristic function v explicitly also has exponential complexity, while a weighted game can be written by listing n integer weights and a quota. Framed differently, the number of simple games is many orders of magnitudes larger than the number of weighted games.

As a class of binary voting systems between simple games and weighted games we consider *complete simple games*, see [Carreras and Freixas(1996)]. They are based on *Isbell's desirability relation*, see [Isbell(1956)], where we write $i \succeq j$ if $v(S \cup \{i\}) \ge v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$ for two voters $i, j \in N$. A simple game v is called

[1] More precisely, [Alon and Edelman(2010)] show such a result for the Banzhaf index. Results for other power indices have been obtained by [Kurz(2016)].

[2] conjunction

[3] disjunction

complete if this relation is complete, i.e., if for all $i, j \in N$ we have $i \succeq j$ or $j \succeq i$. Two players $i, j \in N$ are symmetric iff $i \succeq j$ and $j \succeq i$. The relation $\succeq$ induces an ordering of the players, which is satisfied in many practical applications. E.g. the voting systems of the European Council (according to the Treaty of Lisbon and also those before) are complete simple games. Here we use the standard assumption $1 \succeq 2 \succeq \cdots \succeq n$ and note that Shapley-Shubik index $\operatorname{SSI}(v)$ and the Penrose-Banzhaf index $\operatorname{PBI}(v)$, see the definitions below, are non-increasing vectors for every complete simple game v. In order to uniquely characterize a complete simple game v we can list all subsets $S \subseteq N$ such that $v(S) = 1$ and for every $i \in S$, $j \notin S$ with $i < j$ (using the usual ordering in $\mathbb{N}$) we have $v(S \backslash \{i\} \cup \{j\}) = 0$. For our example $[3; 3, 2, 1, 1]$ those subsets are given by $\{1\}$ and $\{2, 4\}$. In our example $[2; 2, 0, 1, 1] \wedge [2; 0, 2, 1, 1]$ the voters 1 and 2 as well as voters 3 and 4 are symmetric. For all other pairs of different voters we neither have $i \succeq j$ nor $j \succeq i$, i.e., the game is not complete.

A power index p is a mapping from the set of simple (or weighted) games on n voters into $\mathbb{R}^n$. By $p_i(v)$ we denote the ith component of $p(v)$, i.e., the power of voter i. Here we consider two of the most commonly used power indices, i.e., the *Shapley-Shubik index*, see [Shapley and Shubik(1954)],

$$\operatorname{SSI}_i(v) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot (v(S \cup \{i\}) - v(S))$$

and the *Penrose-Banzhaf index*, see [Penrose(1946), Banzhaf(1964)],

$$\operatorname{PBI}_i(v) = \frac{\sum_{S \subseteq N \backslash \{i\}} (v(S \cup \{i\}) - v(S))}{\sum_{j \in N} \sum_{S \subseteq N \backslash \{j\}} (v(S \cup \{j\}) - v(S))}.$$

For our first example $v = [3; 3, 2, 1, 1]$ we have

$$\operatorname{SSI}(v) = \tfrac{1}{12} \cdot (7, 3, 1, 1) \approx (0.5833, 0.25, 0.0833, 0.0833)$$

and

$$\operatorname{PBI}(v) = \tfrac{1}{10} \cdot (5, 3, 1, 1) = (0.5, 0.3, 0.1, 0.1).$$

As a measure for the distance between two different power distributions $x, y \in \mathbb{R}^i$ we use the *Manhattan distance* $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ and the *Chebyshev distance* $d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|$. For the above two power distributions the Manhattan distance is $\frac{1}{6} \approx 0.1667$ and the Chebyshev distance is $\frac{1}{12} \approx 0.0833$.

3. RESULTS

In the introduction we have noticed that $[2; 1, 1, 1, 1]$ as well as $[3; 1, 1, 1, 1]$ yield the power distribution $(0.25, 0.25, 0.25, 0.25)$ both for the Shapley-Shubik and the Banzhaf indices. In Table 1 we state the number of different power distributions for the Shapley-Shubik and the Banzhaf indices that are attained by weighted games with $n \le 8$ voters. The corresponding numbers for complete simple games are listed in Table 2.

We observe that the counts coincide for $n \le 6$, which is no surprise for $n \le 5$, since every complete simple game consisting of at most 5 voters is weighted. However, for $n = 6$

n	3	4	5	6	7	8
# SSI	4	11	53	536	14188	1364907
# PBI	4	12	57	555	14720	1366032

TABLE 1. Number of different vectors $\mathrm{SSI}(v)$ and $\mathrm{PBI}(v)$ for weighted games v with n voters.

n	3	4	5	6	7	8
# SSI	4	11	53	536	17973	6314952
# PBI	4	12	57	555	18600	4616157

TABLE 2. Number of different vectors $\mathrm{SSI}(v)$ and $\mathrm{PBI}(v)$ for complete simple games v with n voters.

voters there exist $1171 - 1111 = 60$ complete simple games that are not weighted. Nevertheless, the power distributions according to the Shapley-Shubik index or the Banzhaf index of these 60 non-weighted complete simple games are also exactly attained by weighted games, respectively. So, if we are only interested in the resulting power distribution, then including complete non-weighted games comes with no benefit for $n = 6$ voters. For $n \in \{7, 8\}$ we do not have such a strong result. Here the number of attained power distributions for complete simple games is significantly larger. This goes in line with the fact that there are $44\,313 - 29\,373 = 14\,940$ and $16\,175\,188 - 2\,730\,164 = 13\,445\,024$ non-weighted complete simple games for $n = 7$ and $n = 8$ voters, respectively. There we can only give a worst-case bound for the minimum distance between the power distribution of a complete simple game and a weighted game. To this end, we denote the set of weighted games with n voters by $\mathcal{WG}(n)$ and the set of complete simple games with n voters by $\mathcal{CG}(n)$. Moreover, let

$$\omega_a^p(n) := \max\left\{\min\left\{d_a(p(c), p(v)) \,:\, v \in \mathcal{WG}(n)\right\} \,:\, c \in \mathcal{CG}(n)\right\},$$

where $a \in \{1, \infty\}$ and $p \in \{\mathrm{SSI}, \mathrm{PBI}\}$, be the worst-case distance between the power distribution $p(c)$ of a complete simple game c and the power distribution $p(v)$ of its best approximation by a weighted game v.

Proposition 1.

$$\begin{aligned}
\omega_1^{\mathrm{SSI}}(7) &= 0.0666667 & \omega_1^{\mathrm{SSI}}(8) &= 0.0666667\\
\omega_\infty^{\mathrm{SSI}}(7) &= 0.0166667 & \omega_\infty^{\mathrm{SSI}}(8) &= 0.0154762\\
\omega_1^{\mathrm{PBI}}(7) &= 0.0599700 & \omega_1^{\mathrm{PBI}}(8) &= 0.0567084\\
\omega_\infty^{\mathrm{PBI}}(7) &= 0.0173913 & \omega_\infty^{\mathrm{PBI}}(8) &= 0.0139124
\end{aligned}$$

Proof. The proof is obtained by a computer enumeration. First, we loop over all elements v in $\mathcal{WG}(n)$ and store the corresponding power distributions $p(v)$ in a k-d-tree (a data structure for storing multi-dimensional geometrical data). Afterwords, we loop over all elements c in $\mathcal{CG}(n)$, compute $p(c)$, and perform a nearest neighbor search within the previously computed k-d-tree. Let v denote the nearest neighbor that minimizes $d_a^p(p(v), p(c))$. Eventually update the worst-case distance with $d_a^p(p(v), p(c))$. □

As an example we state that the complete simple game attaining $\omega_\infty^{\mathrm{PBI}}(7) = 0.0173913$ is uniquely characterized by the subsets $\{3,4,5,6,7\}$, $\{2,3,5,6\}$, and $\{1,3,7\}$. For $n = 8$ the extremal complete simple games all contain a unique null voter. We remark that the same enumeration is computationally infeasible for $n = 9$ voters since the numbers $\#\mathcal{WG}(9) = 993\,061\,482$ and $\#\mathcal{CG}(9) = 284\,432\,730\,174$ are quite large. (See e.g. [Kartak et al.(2015)] and [Freixas and Molinero(2010)] for the details.) So, for $n \geq 9$ we can only state lower bounds for $\omega_a^p(n)$:

Proposition 2.

$$\begin{array}{ccc} \omega_1^{\mathrm{SSI}}(9) \geq 0.0634922 & \omega_1^{\mathrm{SSI}}(10) \geq 0.0634922 & \omega_1^{\mathrm{SSI}}(11) \geq 0.0591627 \\ \omega_\infty^{\mathrm{SSI}}(9) \geq 0.0130953 & \omega_\infty^{\mathrm{SSI}}(10) \geq 0.0123016 & \omega_\infty^{\mathrm{SSI}}(11) \geq 0.0109308 \\ \omega_1^{\mathrm{PBI}}(9) \geq 0.0562 & \omega_1^{\mathrm{PBI}}(10) \geq 0.0552 & \omega_1^{\mathrm{PBI}}(11) \geq 0.0552 \\ \omega_\infty^{\mathrm{PBI}}(9) \geq 0.0110 & \omega_\infty^{\mathrm{PBI}}(10) \geq 0.0106 & \omega_\infty^{\mathrm{PBI}}(11) \geq 0.0100 \end{array}$$

Proof. Let $a \in \{1, \infty\}$ and $p \in \{\mathrm{SSI}, \mathrm{PBI}\}$. In [Kurz(2012)] the inverse power index problem for the Shapley-Shubik index with respect to the Manhattan distance $d_1(\cdot,\cdot)$ and the Chebyshev distance $d_\infty(\cdot,\cdot)$ within the class of weighted, complete simple, or simple games was formulated as an integer linear programming (ILP) problem, which can be solved exactly even for $n > 9$, where the number of weighted games is unknown. For the Banzhaf index the problem whether a solution of the inverse power index problem with distance at most δ exists can be formulated as an ILP. Using the bisection method for δ the problem can be solved exactly by a sequence of ILPs, see [Kurz and Napel(2014), Appendix A] for the details. Thus, given a complete simple game c with n voters we can compute the corresponding power distribution $p(c)$ and exactly solve the inverse power index problem within $\mathcal{WG}(n)$. If v is a weighted game that minimizes $d_a(p(c), p(v))$, then $d_a(p(c), p(v))$ is a lower bound for $\omega_a^p(n)$. As heuristic candidates for the complete simple game c we have used the extremal ones of Proposition 1 and added a suitable number of null voters. □

We remark that we have also tried to use some randomly chosen complete simple games for c in Proposition 2. However, the resulting lower bounds for $\omega_a^p(n)$ are rather small. As an example, the value $\omega_1^{\mathrm{SSI}}(7) = 0.0666667$ is attained by the complete simple game c characterized by the subsets $\{4,5,6,7\}$, $\{2,4\}$, and $\{1\}$. If we add a null voter, the Shapley-Shubik index is given by

$$(0.5024, 0.1857, 0.1024, 0.1024, 0.03571, 0.03571, 0.03571, 0)$$

with best possible approximation $[84; 38, 27, 19, 16, 9, 9, 3, 0]$, which also shows $\omega_1^{\mathrm{SSI}}(8) \geq 0.0666667$.

For the voting system c of the European Council according to the Lisbon Treaty we cannot solve the inverse power index problem exactly. However, for all $a \in \{1, \infty\}$ and all $p \in \{\text{SSI}, \text{PBI}\}$ we can find a weighted game v with $d_a(p(c), p(v)) < 10^{-5}$, which goes in line with the computational experiments in [Kurz and Napel(2014)].

4. Conclusion

Does it pay off to use complete simple games instead of weighted games as binary voting systems? If only the resulting power distributions for the Shapley-Shubik or the Banzhaf index are relevant, then the answer is probably no. Whether the worst-case deviations stated in Proposition 1 can be regarded as negligible might depend on the application. For $n > 8$ voters our computational experiments suggest that the worst-case deviations might even go down with an increasing number of voters. Proving this claim rigorously might be a hard technical challenge.

We have chosen complete simple games as a reasonable superset of weighted games since the underlying ordering of the players can be assumed in many applications. Another reason is that the class of simple games is really large[4] and realizes a lot of power distributions. E.g., the parameterized target power distribution $\beta(n) = \frac{1}{2n-1} \cdot (2, \ldots, 2, 1) \in \mathbb{R}^n$ has been studied by [Kurz and Napel(2014)]. For $6 \leq n \leq 18$ there exists a simple game v_n such that $\text{SSI}(v_n) = \beta(n)$, while the best approximation within $\mathcal{WG}(n)$ seems to have a deviation of order $\Theta(\frac{1}{n})$. At the very least our values for $\omega_a^p(n)$ give a lower bound for the corresponding situation where we enlarge the possible target power distributions to those of simple games.

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