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Some variational problems arising in the theory of nonlinear waves

Quelques problèmes variationnels issus de la théorie des ondes non-linéaires

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For my dear family!

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Summary

This thesis focuses on the study of special solutions (traveling wave and standing wave type) for nonlinear dispersive partial differential equations in $\mathbf{R}^N$. The considered problems have a variational structure, the solutions are critical points of some functionals. We demonstrate the existence of critical points using minimization methods. One of the main difficulties comes from the lack of compactness. To overcome this, we use some recent improvements of P.-L. Lions concentration-compactness principle.

In the first part of the dissertation, we show the existence of the least energy solutions to quasi-linear elliptic equations in $\mathbf{R}^N$. We generalize the results of Brézis and Lieb in the case of the Laplacian, and the results of Jeanjean and Squassina in the case of the p-Laplacian.

In the second part, we show the existence of subsonic travelling waves of finite energy for a Gross-Pitaevskii-Schrödinger system which models the motion of a non charged impurity in a Bose-Einstein condensate. The obtained results are valid in three and four dimensional space.

Keywords. Nonlinear elliptic equations · nonlinear Schrödinger equation · Gross-Pitaevskii-Schrödinger system · standing wave· travelling wave · minimization · constrained minimization · concentration-compactness principle.

Résumé

Cette thèse porte sur l'étude des solutions spéciales (de type onde progressive et onde stationnaire) pour des équations aux dérivées partielles dispersives non-linéaires dans $\mathbf{R}^N$. Les problèmes considérés ont une structure variationnelle, les solutions sont des points critiques de certaines fonctionnelles. Nous démontrons l'existence des points critiques en utilisant des méthodes de minimisation. Une des principales difficultés vient du manque de compacité. Pour y remédier, on utilise quelques raffinements récents du principe de concentration-compacité de P.-L. Lions.

Dans la première partie du mémoire on montre l'existence des solutions d'énergie minimale pour des équations elliptiques quasi-linéaires dans $\mathbf{R}^N$. Nous généralisons les résultats de Brézis et Lieb dans le cas du Laplacien, ainsi que les résultats de Jeanjean et Squassina dans le cas du p-Laplacien.

Dans la seconde partie on montre l'existence des ondes progressives subsoniques d'énergie finie pour un système de Gross-Pitaevskii-Schrödinger qui modélise le mouvement d'une impureté non chargée dans un condensat de Bose-Einstein. Les résultats obtenus sont valables en dimension trois et quatre d'espace.

Mots-clés. Équations elliptiques non-linéaires · équation de Schrödinger nonlinéaire · système de Gross-Pitaevskii-Schrödinger · onde stationnaire · onde progressive · minimisation · minimisation sous contrainte · principe de concentration-compacité.

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Contents

Summary v

Résumé vii

Remerciements ix

Introduction 1

1. Physical motivation and overview 2
2. Main results 7
3. Outline of the manuscript 9
4. Other results and perspectives 9

References 11

I Least energy solutions for general quasilinear elliptic systems 15

1. Introduction 15
2. The case $p < N$ 23
3. The case $p = N$ 30

References 43

II Traveling waves for a Gross-Pitaevskii-Schrödinger system 45

1. Introduction 45
2. The variational framework 48
3. The case $N = 4$ 53
4. The case $N = 3$ 61

References 65

Introduction

In the class of nonlinear dispersive partial differential equations, the nonlinear Schrödinger (NLS) equation is one of the most widely applicable equations in Physics. It describes a large class of phenomena e.g., modulational instability of water waves, propagation of heat pulses in anharmonic crystals, helical motion of a very thin vortex filament, nonlinear modulation of collision-less plasma waves, and self trapping of a light beam in a color dispersive system [65]. Additionally, the equation appears in the studies of the Bose-Einstein condensate confined to highly anisotropic cigar-shaped traps, in the mean-field regime, both with the attractive and repulsive nonlinearities, where the nonlinear term models the interaction between the particles and represents the Planck constant. It also arises in the description of nonlinear waves, for example light waves propagating in a medium whose index of refractionis sensitive to the wave amplitude, water waves at the free surface of an ideal fluid, and plasma waves [61]. More generally, the NLS equation appears as one of universal equations that describe the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear media that have dispersion.

In this thesis we are interested in the existence of the special solutions (standing waves and traveling waves) for NLS equations. The considered problems have a variational structure and their solutions are critical points of some functionals. The main difficulty in the problems is the lack of compactness which originates from the invariance of $\mathbf{R}^N$ under the action of the noncompact group of translations and dilations, and manifests itself in the noncompactness of the Sobolev imbedding $H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N)$. Thus, except for the special case of convex functionals, the standard convexity-compactness methods used in problems set in bounded domains fail to treat problems in unbounded domains. To overcome this, we use some recent improvements of P.-L. Lions concentration-compactness principle.

We present, in more detail, physical motivations of the considered problem and known results on the quasilinear elliptic equations, the defocusing nonlinear Schrödinger flow past an obstacle and the Gross-Pitaevskii-Schrödinger system. Then we introduce our results.

1. Physical motivation and overview

1.1. General quasilinear elliptic systems

Standing waves for the nonlinear Schrödinger equations.

It is well known that the NLS equation has not only been extensively used in the scalar case, but has also been largely studied in the vector case. For instance, several years after the discovery of Zakharov and Shabat [67] about the integrability of the scalar NLS equation, Manakov [45] showed that a special form of a two-component vector NLS equation with isotropic nonlinearity has the same property. More precisely, he considered the system

$$\begin{aligned} iu_t + \frac{1}{2}u_{xx} + (|u|^2 + |v|^2)u = 0 \\ iv_t + \frac{1}{2}v_{xx} + (|u|^2 + |v|^2)v = 0 \end{aligned}$$

as a model governing the propagation of the electric field in a waveguide, where each equation governs the evolution of one of the components of the field transverse to the direction of propagation. Generally, it can be used as a model for wave propagation under conditions similar to those where the NLS equation applies and there are two wave trains moving with nearly the same group velocity [58, 66].

Moreover, the NLS equation has been recently applied to the study of the propagation of pulses in a single-mode optical fiber. Nevertheless, according to Kaminow [40], a single-mode optical fiber is not really “single-mode” in the usuals, it is actually bimodal due to the presence of birefringence. While the linear birefringence makes a pulse split in the two polarization directions, nonlinear birefringence can trap them together against splitting. Menyuk [52] showed that the evolution of two polarization components in birefringent optical fiber satisfy the Coupled Nonlinear Schrödinger system, which is a generalization of Manakov's system,

$$\begin{aligned} iu_t + \frac{1}{2}u_{xx} + (|u|^2 + \beta|v|^2)u = 0 \\ iv_t + \frac{1}{2}v_{xx} + (\beta|u|^2 + |v|^2)v = 0. \end{aligned}$$

where β is a real positive constant which depends on the anisotropy of the fiber. This system also appears in the Hartree-Fock theory for a binary mixture of Bose-Einstein condensates in two different hyperfine states [26].

We refer to the m component vector generalization of the two-component system,

$$i\psi_t + \Delta\psi + f(\psi) = 0, \tag{1.1}$$

where $\psi : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^m$ and $f : \mathbf{R}^m \to \mathbf{R}^m$. Like the scalar case, the vector NLS equation plays an important role in many branches of physics. It has been used as models for the interaction of m waves, propagating in a regime that for one wave leads to the scalar nonlinear Schrödinger equation. These can be for example water waves in a deep fluid, interacting on the surface or propagating at different levels. Especially, in recent years, it has been widely applied for light-wave propagation in optical fibers ([27, 46, 52, 63]) where the components of ψ in eq. (1.1) correspond to components of the electric field transverse to the direction of wave propagation.

These components of the transverse field compose a basis of the polarization states. Besides that, the nonlinear Schrödinger equation and its vector generalizations have been shown to be gauge equivalent to the Heisenberg ferromagnet equations and their generalizations [2, 3].

The name "NLS equation" comes from a formal analogy with the Schrödinger equation of quantum mechanics where a nonlinear potential appears to describe interacting particles. In the wave context, the second-order linear operator describes the dispersion and diffraction of the wave-packet, and the nonlinearity presentes the sensitivity of the refractive index to the medium on the wave amplitude.

Heuristically, the nonlinear term in the NLS equation has a focusing effect which tends to concentrate the solution and compensates the dispersive effect of the linear terms, which tends to flatten the solution as time goes on. Therefore, it can be expected that the NLS equation has solitary waves as solutions whose energy travels as localized packets and which preserve their shape under perturbations. This kind of phenomena was first observed in 1834 by John Scott Russel on the Edinburgh to Glasgow canal, while he was working on the design of the keels of canal boats. Most of the scientists of that time did not believe the existence of such a wave, which does not disperse. In 1895, Korteweg and de Vries derived an equation for the motion of water admitting solitary wave solutions. Nonetheless, it was not until the 1960's and the advent of modern computers that the study of solitary wave evolution in a uniform medium began to be extended, not only because of their observations in experiments but also because of their diverse potential applications to ultrafast signal processing such as optical switching, computing, filtering, and beam splitting ([18, 61, 25, 24]). It is natural to ask whether similar solutions exist for the vector NLS equation. Their properties have been investigated experimentally [6, 9, 41].

The focusing vector NLS equation, where the sign of the nonlinear term is positive, has solitary wave solutions of "standing wave" type, which are solutions of the form

$$\psi(t,x) = e^{-iEt}u(x),$$

where E is the energy of the wave and $u : \mathbf{R}^N \to \mathbf{R}^m$ satisfies

$$-\Delta u = \nabla G(u) \text{ in } \mathcal{D}'(\mathbf{R}^N), \tag{1.2}$$

where $G : \mathbf{R}^m \longrightarrow \mathbf{R}$ is a C^1 function. The system (1.2) also appears in the study of standing waves for systems of coupled nonlinear Klein-Gordon equations. Moreover, it appears in various other contexts of physics, for example, the classical approximation in statistical mechanics, constructive field theory, false vacuum in cosmology, etc ([7, 8, 22, 29, 4]).

Variation methods and least energy solutions for some quasilinear elliptic equations. A natural method to solve (1.2) would be to look for critical points of the functional

$$E(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 dx - \int_{\mathbf{R}^N} G(u) dx.$$

This method was used by Strauss [59] in the case $m = 1$ (the scalar case) for $N \geqslant 3$, and was extended to the vector case and to the "zero mass" by Strauss and Vázquez [60]. But their work required severe restrictions on the function G and they did not explicitly consider the question whether or not their solution to (1.2) minimized the action E.

A first difficulty in this approach lies in the fact that for the focusing nonlinearity, E is not bounded from below. To see this it suffices to take u such that $\int_{\mathbf{R}^N} G(u)dx > 0$, to put $u_\sigma = u(\frac{\cdot}{\sigma})$ and to take the limit as $\sigma \to \infty$ of $E(u_\sigma)$. Another difficulty comes from the fact that E does not satisfy conditions of the type (PS^+) or (PS^-) in an obvious way.

On the other hand, physically meaningful solutions to (1.2) have finite energy and, if they exist, one is interested in the least energy solution u_0 which has the property $0 < E(u_0) \leq E(u)$ for any solution u to (1.2). Such a solution u_0 is also called a "ground state".

In 1978, Coleman et al. [23] not only made an important contribution to the problem by their "constrained minimization method" which yields a least energy solution for $N \geqslant 3$ but also discovered almost optimal assumptions on G so that the scalar problem (1.2) has a solution. However, their method was restricted in an essential way to $N \geqslant 3$ and $m = 1$. Berestycki and Lions presented some improvements of the Coleman, Glaser, Martin method, other theorems and related problems in [12, 13]. They also proved in [13] the existence of infinitely many finite action solutions to (1.2). However, all of these results were proven for $N \geqslant 3$ in the scalar case. In 1982, Berestycki, Gallouet and Kavian [10] solved the $N = 2, m = 1$ case.

One had to wait until the 1984 for the proof of the existence of least energy solutions to (1.2) in the vector case ($m > 1$) by Brezis and Lieb [16]. They assumed that G is a C^1 function on $\mathbf{R}^m \backslash \{0\}$, locally Lipschitz around the origin and having suitable subcritical bounds at the origin and at infinity. To our knowledge, the assumptions in [16] are still the most general in the literature. Unlike in the previous works, the problem of minimizing $\int_{\mathbf{R}^N} |\nabla u|^2 dx$ at fixed $\int_{\mathbf{R}^N} G(u)dx$ is solved in [16] without using symmetrization and some compactness properties for arbitrary minimizing sequences are given. Other interesting properties of solutions (general Pohozaev identities, behavior at infinity or compact support in some cases) are also proven in [16].

At the same time, P.-L. Lions introduced his celebrated concentration-compactness method [42, 43, 44], which allowed him (among many other important applications) to deal with the cases $N \geqslant 2$, $m \geqslant 1$. The author constructed a concentration-compactness lemma which was derived, at least heuristically, from the fact that, essentially, the loss of compactness may occur only if either the minimizing sequence slips to infinity, or the minimizing sequence breaks into at least two disjoint parts which are going infinitely far away from each other. Then he formulated a concentration-compactness principle which states that all minimizing sequences are relatively compact if and only if a subadditivity inequality is strict. In this way he could overcome the difficulties of loss of compactness coming from unbounded domains to solve different minimization problems. In many applications this method implies the precompactness of any minimizing sequence, which leads to the orbital stability of the minimizers, in the sense that a time-depending solution which starts near the set of global minimizers will remain close to it for all time.

Recently, in the vector case with the specific functions G, existence results have been established in many papers (e.g. [5, 56, 55]).

In the 2000's, many researchers have extended the existence results for the least energy solution to (1.2) to the p-Laplacian case, which has the following form:

$$-\Delta_p u = \nabla G(u) \text{ in } \mathcal{D}'(\mathbf{R}^N). \tag{1.3}$$

In [28, 31], the authors proved the existence of non-negative, nontrivial radial solutions vanishing at infinity of (1.3) but they require some regularity of the function G. The issue of least energy solutions is not considered in these papers. In [54], under assumptions on G allowing to work with regular functionals in $W^{1,p}(\mathbf{R}^N)$, the existence of a least energy solution is derived. For more general scalar equations the existence of solutions was shown by Jeanjean and Squassina in [38]. In this work we find least energy solutions to the general vector equations under nearly optimal assumptions.

1.2. The Gross-Pitaevskii-Schrödinger system

Nucleation by impurities. We are interested in an application of the NLS equation which concerns the modelling of vortex nucleation by an impurity, e.g. an electron [32]. In the Hartree approximation, Gross [35] and Clark [21] initially established that the one-particle wave function of the condensate ψ, and the wave function of the impurity ϕ, satisfy the coupled equations

$$\begin{aligned} ih\partial_t\Psi &= -\frac{h^2}{2M}\Delta\Psi + \left(U_0|\phi|^2 + V_0|\Psi|^2 - E\right)\Psi \\ ih\partial_t\Phi &= -\frac{h^2}{2\mu}\Delta\Phi + \left(U_0|\Psi|^2 - E_e\right)\Phi, \end{aligned} \tag{1.4}$$

where M and E are the mass and single-particle energy for the bosons, and μ and E_e are the mass and energy of the impurity. The normalization conditions on the wave functions are

$$N = \int |\psi|^2 dV, \qquad 1 = \int |\phi|^2 dV, \tag{1.5}$$

where N is the total number of bosons in the system. The interaction potentials between boson and electron and between bosons are here assumed to be of δ-function form $U_0\delta(x-x')$ and $V_0\delta(x-x')$ where x and x' are their positions. To lowest order, perturbation theory predicts such pseudopotentials, with

$$U_0 = \frac{2\pi l h^2}{\mu} \text{ and } V_0 = \frac{4\pi d h^2}{M},$$

where l is the boson-impurity scattering length, and d is the boson diameter. The healing length is defined by

$$a = h(2\rho_\infty V_0)^{\frac{-1}{2}} = (8\pi d\psi_\infty^2)^{\frac{-1}{2}}$$

where $\rho_\infty = M\psi_\infty^2 = \frac{EM}{V_0}$ is the mean condensate mass density. Using the system (1.4), Grant and Roberts studied the motion of a negative ion moving with speed v. They used an asymptotic expansion in $\frac{v}{v_s}$, where v_s is the speed of sound. Treating

$\varepsilon = \left(\frac{4\pi a^3 \psi_\infty^2 V_0}{U_0}\right)^{\frac{1}{5}} = \left(\frac{a\mu}{lM}\right)^{\frac{1}{5}}$ as a small parameter they calculated the effective (hydrodynamic) radius and effective mass of the electron bubble (in applications ε is about 0.2).

We introduce the transformations

$$r \to \frac{ar}{\epsilon}, \qquad t \to \frac{a^2 M}{h\varepsilon^2} t, \qquad \psi \to \psi_\infty \psi, \qquad \phi \to \frac{\epsilon^3}{4\pi a^3}\phi.$$

Then the system (1.4) becomes

$$\begin{aligned} 2i\partial_t \Psi &= -\Delta\Psi + \frac{1}{\epsilon^2}\left(\frac{1}{\epsilon^2}|\Phi|^2 + |\Psi|^2 - 1\right)\Psi \\ 2i\delta\partial_t \Phi &= -\Delta\Phi + \frac{1}{\epsilon^2}\left(q^2|\Psi|^2 - \epsilon^2 k^2\right)\Phi, \end{aligned} \tag{1.6}$$

where $\delta = \frac{\mu}{M}$ is the ratio of the mass of the impurity over the boson mass (δ), $q^2 = \delta\frac{U_0}{V_0}$, and $k^{10} = \frac{\mu^5 E_e^5 U_0^2}{2\pi^2 M^2 E^4 h^6}$ is a dimensionless measure for the single-particle impurity energy. Assuming that the condensate is at rest at infinity, the solutions Ψ and Φ must satisfy the "boundary conditions"

$$|\Psi(x)| \to 1, \qquad \Phi(x) \to 0 \text{ as } |x| \to \infty.$$

Based on formal asymptotic expansions and numerical experiments, Grant and Roberts [32] computed the effective radius and the induced mass of the uncharged impurity.

Traveling waves for a Gross-Pitaevskii-Schrödinger system. Traveling wave solutions for the system (1.6) are solutions of the form

$$\Psi(x,t) = \psi(x - cty), \qquad \Phi(x,t) = \phi(x - cty)$$

where y is the direction of propagation and c is the speed of the traveling wave. Without loss of generality, we may assume that $y = (1, 0, \ldots, 0)$, and then such solutions must satify the equations

$$\begin{aligned} 2ic\frac{\partial\psi}{\partial x_1} &= -\Delta\psi + \frac{1}{\varepsilon^2}\left(\frac{1}{\varepsilon^2}|\phi|^2\psi + |\psi|^2 - 1\right)\psi \\ 2ic\delta\frac{\partial\phi}{\partial x_1} &= -\Delta\phi + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\phi = 0. \end{aligned} \tag{1.7}$$

By extending the analysis of [33, 34] for the GP equation, Bouchel [15] showed decay estimates for finite energy traveling waves of the GP system and the nonexistence of supersonic traveling waves in dimension three. In space dimension one, Mariş [48] proved the existence of a global subcontinua of finite energy subsonic traveling waves. In [49] he proved the nonexistence of supersonic solutions in any space dimension.

To our knowledge, there have been no existence results in the literature for the system (1.7) in dimension $N \geqslant 2$.

2. Main results

2.1. Existence of least energy solutions for general quasilinear elliptic systems

We study the existence of solutions for nonlinear elliptic systems of the form

$$\partial_{u_i} a(u, \nabla u) - \operatorname{div}\left(\nabla_{\xi_i} a(u, \nabla u)\right) = g_i(u) \qquad \text{in } \mathcal{D}'(\mathbf{R}^N), \tag{2.1}$$

where $u = (u_1, \dots u_m) : \mathbf{R}^N \to \mathbf{R}^m$, $\nabla u = \left(\frac{\partial u_i}{\partial x_k}\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant k \leqslant N}}$, a is a real-valued function defined on $\mathbf{R}^m \times \mathbf{R}^{m \times N}$, where $\{\xi \longmapsto a(s, \xi)\}$ is p-homogeneous and $g_i(u) = \frac{\partial G}{\partial u_i}$, where $G : \mathbf{R}^m \longrightarrow \mathbf{R}$ is a C^1 function.

The system (2.1) admits a variational formulation. As in [13] or [16], we will find solutions to (2.1) by solving the minimization problem

$$(\mathcal{P}_\lambda) \qquad \text{minimize } \mathcal{A}(u) \text{ for } u \in \mathcal{X} \text{ under the constraint } \mathcal{V}(u) = \lambda,$$

where $\lambda \in \mathbf{R}$, the function space $\mathcal{X}$ will be introduced in Chapter I and

$$\mathcal{A}(u) = \int_{\mathbf{R}^N} a(u(x)), \nabla u(x))\, dx, \quad \mathcal{V}(u) = \int_{\mathbf{R}^N} G(u(x))\, dx, \quad E(u) = \mathcal{A}(u) - \mathcal{V}(u).$$

Under suitable assumptions, it can be shown that the solutions of (2.1) (or equivalently, the critical points of E) satisfy the Pohozaev identity $(N-p)\mathcal{A}(u) = N\mathcal{V}(u)$. Therefore, we should consider $(\mathcal{P}_\lambda)$ for $\lambda > 0$ in the case $p < N$ and for $\lambda = 0$ if $p = N$. In the case $p > N$, it has been shown in Remark 5 p. 486 in [11] that $(\mathcal{P}_\lambda)$ cannot have solutions (although (2.1) may sometimes have solutions and even least energy solutions, see e.g. [28]). Here, we focus on the minimization problem $(\mathcal{P}_\lambda)$ and we do not consider the supercritical case $p > N$.

Moreover, on the second term in (2.1), in the previous works it was assumed that $g(0) = 0$ (which means that 0 is an equilibrium point for (2.1)) and the authors were looking for solutions converging to 0 at infinity. However, many physical systems have more equilibria (often a manifold) and there is no *a priori* reason that the system would choose one equilibrium rather than another. This is the case, for instance, for models in condensed matter theory, superfluids, nonlinear optics or micromagnetics. Here, we assume that the set of *interesting* equilibria is a compact set $\mathcal{S} \subset \mathbf{R}^m$; we require no manifold structure on $\mathcal{S}$.

In the case $p < N$, under general assumptions on a and G that we will present in detail later, by using P.-L. Lions' concentration-compactness principle we show the existence of minimizers of $(\mathcal{P}_\lambda)$ and then we obtain the existence of least energy solutions for (2.1). In the particular case when $a(u, \nabla u) = |\nabla u|^2$ and $2 < N$, this existence result has been proven in [16]. However, even in that case we improve the results in [16] in the sense that we provide a more precise description of the behavior of minimizing sequences and we show that the technical assumption (2.5) p. 99 in [16] is unnecessary. That assumption has been used in [16] to perform an appropriate cut-off in the nonlinear term. In our proof we do not use a cut-off, but instead we use "localized scaling," that is, we "zoom in" or "zoom out" some regions in the Euclidean space. This is possible in view of the results in [51] which provide large regions in the space with small energy. In the case $p = N$ any (sufficiently smooth) solution $u \in \mathcal{X}$ of (2.1) must satisfy the Pohozaev identity $\int_{\mathbf{R}^N} G(u)\,dx = 0$. At first glance, one might think of finding solutions to (2.1) by solving the minimization problem $(\mathcal{P}_0)$. However, it is easily seen that the only solutions of $(\mathcal{P}_0)$ are the constant ones. In order to find nontrivial solutions to (2.1) it is natural to minimize $\mathcal{A}$ in the set $\mathcal{X}_0 = \{u \in \mathcal{X} \mid \mathcal{V}(u) = 0 \text{ and } u \text{ is not constant}\}$. This problem is significantly more difficult than the problem $(\mathcal{P}_\lambda)$ considered above for at least two reasons: firstly, one needs to prevent minimizing sequences to converge to constant functions (and this is not so obvious since constant functions are global minimizers in $\mathcal{X}$); and secondly, the problem is invariant by scaling.

It is well known that when using P.-L. Lions' concentration-compactness principle, the difficult part is to understand the dichotomy case. In [51], Mariş has improved the principle in the sense that whenever dichotomy occurs, we can choose a "dichotomizing subsequence" $\mu_{n_k} = \mu_{k,1} + \mu_{k,2} + o(1)$ such that $\mu_{k,1}$ and $\mu_{k,2}$ have supports far away from each other and, in adition, the sequence $\mu_{k,1}$ "concentrates." Iterating this argument he is able to prove a profile decomposition result for arbitrary sequences of bounded Borel measures. By using this method, we obtain the existence of least energy solutions to (2.1) in the case $p = N$.

We consider a more general problem: we denote $G_1 = G_-$, $G_2 = G_+$ and we minimize $\mathcal{A}$ in the set of functions $u \in \mathcal{X}$ satisfying $\int_{\mathbf{R}^N} G_2(u)\,dx = \lambda \int_{\mathbf{R}^N} G_1(u)\,dx > 0$, where $\lambda > 0$ is arbitrary. Then, of course, we will take $\lambda = 1$. However, it is important to consider the minimization problem for any $\lambda > 0$ in order to get an accurate information for $\lambda = 1$. We also allow a nontrivial set of equilibria $\mathcal{S}$ and we show the existence of minimizers as well as the precompactness of minimizing sequences under mild assumptions. For instance, we need only the continuity of G, while the corresponding result in [16] (see Theorem 3.1 p 106 there) requires either the differentiability of G on $\mathbf{R}^2 \setminus \{0\}$, or the assumption that $|G(tv)| \leqslant C|G(v)|$ for all $t \in [0,1]$ and $|v| \leqslant \varepsilon$, where C, ε are positive constants.

2.2. Existence of nontrivial finite energy traveling waves for a Gross-Pitaevskii-Schrödinger system

This work focuses on the study of the existence of traveling wave solutions to the system

$$
\begin{aligned}
2i\frac{\partial \Psi}{\partial t} &= -\Delta\Psi + \frac{1}{\varepsilon^4}|\Phi|^2\Psi - F(|\Psi|^2)\Psi,\\
2i\delta\frac{\partial \Phi}{\partial t} &= -\Delta\Phi + \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi
\end{aligned}
\tag{2.2}
$$

with "boundary conditions"

$$|\Psi(x)| \to 1, \qquad \Phi(x) \to 0 \text{ as } |x| \to \infty,$$

and $F(1) = 0, F'(1) < 0$. Denoting $V(s) = \int_s^1 F(\tau)d\tau$, at least formally, traveling waves are critical points of the functional

$$
\begin{aligned}
E_c(\psi,\phi) = \int_{\mathbf{R}^N} \left(|\nabla\psi|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\phi|^2 + V(|\psi|^2) + \frac{1}{\varepsilon^4}|\psi|^2|\Phi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\phi|^2 \right) dx \\
+ 2cQ(\psi) + 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi),
\end{aligned}
$$

where Q is the momentum with respect to the x_1 direction. Under suitable assumptions on F, it can be shown that any traveling wave $(\psi,\phi) \in \mathcal{E} \times H^1(\mathbf{R}^N)$ of (2.2) must satisfy the Pohozaev-type identity $P_c(\psi,\phi) = 0$, where

$$P_c(\psi,\phi) = \int_{\mathbf{R}^N} \left(\left| \frac{\partial \psi}{\partial x_1} \right|^2 + \frac{1}{q^2\varepsilon^2} \left| \frac{\partial \phi}{\partial x_1} \right|^2 \right) dx + \frac{N-3}{N-1} \sum_{k=2}^{N} \left(\left| \frac{\partial \psi}{\partial x_k} \right|^2 + \frac{1}{q^2\varepsilon^2} \left| \frac{\partial \phi}{\partial x_k} \right|^2 \right) dx$$
$$+ \int_{\mathbf{R}^N} V(|\psi|^2)dx + \frac{1}{\varepsilon^4} \int_{\mathbf{R}^N} |\psi|^2|\phi|^2 dx - \frac{k^2}{\varepsilon^2 q^2} \int_{\mathbf{R}^N} |\phi|^2 dx + 2cQ(\psi) + 2\frac{c\delta}{q^2\varepsilon^2} Q(\phi).$$

Using the concentration-compactness principle, we prove the existence of traveling waves in space dimension $N = 3$ and $N = 4$ by minimizing the action E_c under the Pohozaev constraint $P_c = 0$ under general conditions on the nonlinearity F and for any speed $c \in (0, v_s)$ satisfying $\varepsilon^2(c^2\delta^2 + k^2) < q^2$.

3. Outline of the manuscript

This thesis contains two chapters that correspond to the following contributions.

I. Mihai MARIŞ and Lien Thuy NGUYEN, *Least energy solutions for general quasilinear elliptic systems.*
We prove the existence of least energy solutions for a large class of quasilinear systems with variational structure. Our method consists in solving appropriate constrained minimization problems. We show a stability property of solutions in the sense that any minimizing sequence has convergent subsequences. We are able to deal with very general assumptions thanks to the concentration-compactness principle and the new results in [51].

II. Lien Thuy NGUYEN, *Traveling waves for a Gross-Pitaevskii-Schrödinger system.*
We show the existence of subsonic traveling waves of finite energy for a Gross-Pitaevskii-Schrödinger system which models the motion of a non charged impurity in a Bose-Einstein condensate. The obtained results are valid in three and four dimensional space.

4. Other results and perspectives

As mentioned above, NLS equations have been used as models for superconductivity, superfluid Helium II, Bose-Einstein condensation and nonlinear optics. When impurities or obstacles are present in the superfluid, the new elements may be modeled by adding an external potential U to the equation. Following [36] in the one-dimensional case, the flow of a superfluid past an obstacle moving at velocity $c > 0$ in the x_1 direction may be described by the NLS equation with an external repulsive potential U, namely

$$(4.1)\quad i\frac{\partial \Phi}{\partial t} = \Delta\Phi + F(|\Phi|^2)\Phi - U(x_1 - ct, x')\Phi = 0, \quad x \in \mathbf{R}^N, t \in \mathbf{R}, x' = (x_2, ..., x_N),$$

where Φ is a complex-valued function on $\mathbf{R}^N$ satisfying the "boundary condition" $|\Phi(x)| \to r_0$ as $|x| \to \infty$. The nonlinear term has a repulsive sign ($F'(r_0) < 0$), so that a constant density solution is stable away from the impurity.

In (4.1) the superfluid is supposed to be at rest at infinity. Equation (4.1) may be recast in the frame of the moving obstacle as

$$i\frac{\partial \Psi}{\partial t} - ic\frac{\partial \Psi}{\partial x_1} + \Delta\Psi + F(|\Psi|^2)\Psi - U(x)\Psi = 0, \tag{4.2}$$

where $\Psi(x,t) = \Phi(x_1 + ct, x', t)$. This equivalent formulation describes the flow of an NLS fluid injected with constant speed c at infinity past a fixed obstacle. A stationary solution ψ satisfies the elliptic equation

$$-ic\frac{\partial \psi}{\partial x_1} + \Delta\psi + F(|\psi|^2)\psi - U(x)\psi = 0. \tag{4.3}$$

Equation (4.2) remains formally Hamiltonian. Denoting $V(s) = \int_s^{r_0^2} F(\tau)d\tau$, the Hamiltonian can be written as

$$F_c^U(\Psi) = \int_{\mathbf{R}^N} |\nabla\Psi|^2 dx + \int_{\mathbf{R}^N} V(|\Psi|^2)dx + \int_{\mathbf{R}^N} U(x)(|\Psi|^2 - r_0^2)dx - c\int_{\mathbf{R}^N} \langle i\frac{\partial \Psi}{\partial x_1}, \Psi - r_0\rangle dx.$$

The study of superfluid flows past an obstacle and the nucleation of vortices (especially for the Gross-Pitaevskii (GP) equation, which is a particular case of the NLS equation with nonlinearity $F(s) = 1 - s$) have been considered in a series of papers (see e.g. [30, 37, 39, 53, 57]). For instance, Raman et al. have studied dissipation in a Bose-Einstein condensed gas by moving a blue detuned laser beam through the condensate at different velocities [57]. In the homogeneous two dimensional case, Frisch et al. [30] performed direct numerical simulations of the NLS equation to study the stability of the superflows around a disk. They observed a transition to a dissipative regime characterized by vortex nucleation that they interpreted in terms of a saddle-node bifurcation of the stationary solutions to the NLS equation. A saddle-node bifurcation was explicitly found by Hakim [36] when studying the stability of one-dimensional NLS flows across obstacles described by a potential.

In [36], below a critical velocity (which depends on the obstacle and is always less than the sound velocity), Hakim [36] performed a formal and numerical analysis of stationary one-dimensional solutions to the GP equation in three cases: for weak potentials, for potentials of short range, and for slowly varying potentials. He showed the existence of two stationary solutions: one stable, the other unstable. The latter may be interpreted as the transition state towards the creation of gray solitons corresponding to vortices in the one-dimentional case. Moreover, in all three cases considered, at the critical velocity the two solutions become identical and no stationary solution exists above the critical velocity.

If the potential U is a bounded measure with small total variation, Mariş [47] proved the existence of solutions to the GP equation by minimizing the Hamiltonian F_c^U. When, in addition, U has compact support, he also established that (4.1) has exactly two solutions.

In dimension two, the existence of finite energy solutions was obtained in [14] by locally minimizing the Hamiltonian F_c^U near the constant solutions to the GP equation when $\|U\|_{L^2(\mathbf{R}^2)}$ is small. These solutions have small energy and momentum; in fact, they are perturbations of the constant solutions of (4.3) in the case when the potential is zero. To avoid the lack of compactness, the minimization problem is solved first on a torus. It is shown that there exists a minimizer on each torus and then one may pass to the limit when the size of the torus tends to infinity. Unfortunately this method cannot be used in higher dimensions.

In a work in progress (that we choose not to incorporate in this PhD thesis) we prove the existence of traveling wave solutions of (4.1) in dimension $N \geqslant 2$.

If the potential U (modeling the obstacle) is sufficiently localized and is not too large, we show that E_c attains a local minimum in a neighbourhood of the set of constant functions $\{\psi \in \mathcal{E} \,|\, E_{GL}(\psi) \leqslant k_1\}$. Any local minimizer ψ_0 satisfies (4.3) and we have $E_c(\psi_0) < 0$. The solutions obtained in this way have small energy and small momentum. We also show that there is a second family of solutions, with high energy and momentum, obtained by minimizing the energy at constant momentum.

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I

Least energy solutions for general quasilinear elliptic systems*

Mihai MARIŞ and Lien Thuy NGUYEN

Abstract. We prove the existence of least energy solutions for a large class of quasilinear systems with variational structure. Our method consists in solving appropriate constrained minimization problems. We show a stability property of solutions in the sense that any minimizing sequence has convergent subsequences. We are able to deal with very general assumptions thanks to the concentration-compactness principle and the new results in [22].

1. Introduction

In this paper we study the existence of solutions for nonlinear elliptic systems of the form

$$\partial_{u_i} a(u, \nabla u) - \operatorname{div}\left(\nabla_{\xi_i} a(u, \nabla u)\right) = g_i(u) \qquad \text{in } \mathcal{D}'(\mathbf{R}^N), \tag{1.1}$$

where $u = (u_1, \dots u_m) : \mathbf{R}^N \longrightarrow \mathbf{R}^m$, $\nabla u = \left(\frac{\partial u_i}{\partial x_k}\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant k \leqslant N}}$, and a is a real-valued function defined on $\mathbf{R}^m \times \mathbf{R}^{m \times N}$, where $\{\xi \longmapsto a(s, \xi)\}$ is p-homogeneous. Given a matrix $\xi = \left(\xi_i^k\right)_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant k \leqslant N}}$, we will write $\xi_i = (\xi_i^1, \dots, \xi_i^N)$, $1 \leqslant i \leqslant m$ for the rows of ξ and $\zeta^k = (\xi_1^k, \dots \xi_m^k)$ for its columns. If $s \in \mathbf{R}^m$ and $\xi \in \mathbf{R}^{m \times N}$ we write either $a(s, \xi)$ or $a(s, \xi_1, \dots, \xi_m)$ or $a(s, \zeta^1, \dots, \zeta^N)$. Given a function $u = (u_1, \dots, u_m) : \mathbf{R}^N \longrightarrow \mathbf{R}^m$, we write either $a(u, \nabla u)$ or $a(u_1, \dots, u_m, \nabla u_1, \dots, \nabla u_m)$ or $a(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N})$, all of this having the same (obvious) meaning. By $\nabla_{\xi_i} a$ we denote $\left(\frac{\partial a}{\partial \xi_1^i}, \dots, \frac{\partial a}{\partial \xi_N^i}\right)$. Finally, we assume that $g_i(u) = \frac{\partial G}{\partial u_i}$, where $G : \mathbf{R}^m \longrightarrow \mathbf{R}$ is a C^1 function.

Systems of the form (1.1) arise in a large variety of situations in physics and life sciences (see, for instance, the introduction of [2] for some standard applications of the simplest model case). In the model case $a(s, \xi) = |\xi|^p = \sum_{i=1}^m |\xi_i|^p =$

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$\sum_{i=1}^{m} \left((\xi_i^1)^2 + \cdots + (\xi_i^N)^2\right)^{\frac{p}{2}}$, the system (1.1) becomes

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \frac{1}{p} g_i(u) \qquad \text{in } \mathcal{D}'(\mathbf{R}^N) \tag{1.2}$$

where the differential operator $\Delta_p(v) = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)$ is the usual p-Laplacian. For $p = 2$ (1.2) reduces to the classical nonlinear vector field equations ($m \geqslant 2$) or scalar field equation ($m = 1$), which have been extensively studied in the literature.

The system (1.1) admits a variational formulation. Indeed, let us introduce the functionals

$$\mathcal{A}(u) = \int_{\mathbf{R}^N} a(u(x)), \nabla u(x))\, dx, \qquad \mathcal{V}(u) = \int_{\mathbf{R}^N} G(u(x))\, dx, \qquad E(u) = \mathcal{A}(u) - \mathcal{V}(u). \tag{1.3}$$

At least formally, solutions of (1.1) are critical points of the "energy functional" E. It is easy to see that the energy E cannot have global minimizers or maximizers. It is a natural idea to look for solutions of (1.1) as minimizers of $\mathcal{A}(u)$ under the constraint $\mathcal{V}(u) = \lambda$, where $\lambda \in \mathbf{R}$ is a constant. It has been shown in [5] that in the case $a(s, \xi) = |\xi|^p$ with $1 < p \leqslant N$ the solutions of (1.1) obtained in this way are precisely the *least energy solutions*, which means that they minimize the energy E among all solutions of (1.1).

The problem of minimizing $\mathcal{A}$ under the constraint that $\mathcal{V}$ is fixed was considered in a series of papers, starting with the pioneer work of Strauss [24], who used this approach to show the existence of radial solutions for the nonlinear scalar field equation. Almost at the same time, Coleman, Glaser and Martin [6] discovered almost optimal assumptions on the nonlinearity that guarantee the existence of least energy solutions for the nonlinear scalar field equation, but their method was limited in an essential way to $N \geqslant 3$ and $m = 1$. In the celebrated papers [2], the authors presented in detail the method of Coleman, Glaser and Martin with some improvements, gave further properties of the solutions and proved the existence of infinitely many solutions with energies going to infinity. The case $N = 2$, $m = 1$ has been solved in [1]. In their seminal work [4], Brezis and Lieb extended the previous results in several directions. They were able to find minimal energy solutions to the vector field equations ($N \geqslant 2$, $m \geqslant 1$) under nearly optimal assumptions (to our knowledge, the assumptions in [4] are still the most general in the literature). Unlike in the previous works, the problem of minimizing $\mathcal{A}$ at fixed $\mathcal{V}$ is solved in [4] without using symmetrization and some compactness properties for arbitrary minimizing sequences are given. Other interesting properties of solutions (general Pohozaev identities, behavior at infinity or compact support in some cases) are also proven in [4]. At the same time, P.-L. Lions introduced his celebrated concentration-compactness method [19], which allowed him (among many other important applications) to deal with the cases $N \geqslant 2$, $m \geqslant 1$. More recently, the problem (1.2) has been considered in [9], where the authors followed the approach in [2] and found sufficient conditions for the existence of radial solutions in the scalar case ($m = 1$). We also refer to [9] for an interesting discussion on the existence of radial ground states for (1.3) in the case of supercritical nonlinearities by using ODE methods. For more general scalar equations the existence of solutions was shown by Jeanjean and Squassina in [16]. It was proved in [5] that the minimum action solutions of (1.2) are radial; moreover, in the scalar case they are monotonic with respect to the radial variable. The proof easily extends to more general equations provided that the solutions of these equations are smooth (at least C^1).

As in [2] or [4], we will find solutions to (1.1) by solving the minimization problem

$$(\mathcal{P}_\lambda) \qquad \text{minimize } \mathcal{A}(u) \text{ for } u \in \mathcal{X} \text{ under the constraint } \mathcal{V}(u) = \lambda,$$

where $\lambda \in \mathbf{R}$ and the function space $\mathcal{X}$ will be introduced later (see (1.5) and (1.13) below). The situation is very different if $p < N$ (subcritical case), $p = N$ (critical case) or $p > N$ (supercritical case). Indeed, under suitable assumptions it can be shown that the solutions of (1.1) (or equivalently, the critical points of E) satisfy the Pohozaev identity $(N-p)\mathcal{A}(u) = N\mathcal{V}(u)$. (This is a consequence of the behavior of E with respect to dilations: if u is a critical point of E and $u_\sigma(x) = u(\frac{x}{\sigma})$, formally we should have $\frac{d}{d\sigma}_{|\sigma=1} E(u_\sigma) = 0$. Since $\mathcal{V}(u_\sigma) = \sigma^N \mathcal{V}(u)$ and assumption (a2) below implies $\mathcal{A}(u_\sigma) = \sigma^{N-p}\mathcal{A}(u)$, the identity follows. Of course, this is only a formal argument because we do not know that the curve $\sigma \longmapsto u_\sigma$ is differentiable, but it can be made rigorous by using a cut-off argument provided that u is a little bit more regular than arbitrary functions in $\mathcal{X}$; see e.g. Lemma 2.4 p. 104 in [4] in the case of the Laplacian, or [23] for more general results). Therefore we should consider $(\mathcal{P}_\lambda)$ for $\lambda > 0$ in the case $p < N$ and for $\lambda = 0$ if $p = N$. In the case $p > N$ it has been shown in Remark 5 p. 486 in [5] that $(\mathcal{P}_\lambda)$ cannot have solutions (although (1.1) may sometimes have solutions and even least energy solutions, see e.g. [9]). In this paper we focus on the minimization problem $(\mathcal{P}_\lambda)$ and we do not consider the supercritical case $p > N$.

We will consider (1.1) and $(\mathcal{P}_\lambda)$ under general conditions. We will work only with the following set of assumptions on the function a:

(a1) The function $a : \mathbf{R}^m \times \mathbf{R}^{m\times N} \longrightarrow \mathbf{R}_+$ is Borel measurable, lower semicontinuous in (s, ξ) and convex in ξ, where $s \in \mathbf{R}^m$, $\xi \in \mathbf{R}^{m\times N}$.

(a2) Homogeneity: a is p-homogeneous in ξ, that is, $a(s, \lambda\xi) = \lambda^p a(s, \xi)$ for all $\lambda > 0$, $s \in \mathbf{R}^m$ and $\xi \in \mathbf{R}^{m\times N}$.

(a3) There are positive constants C_1, C_2 such that

$$C_1|\xi|^p \leqslant a(s,\xi) \leqslant C_2|\xi|^p \qquad \text{for all } (s,\xi) \in \mathbf{R}^m \times \mathbf{R}^{m\times N}.$$

(a4) Symmetry: a has a one-dimensional symmetry, for instance

$$a(s, -\zeta^1, \zeta^2, \ldots, \zeta^N) = a(s, \zeta^1, \zeta^2, \ldots, \zeta^N) \qquad \text{for all } s, \zeta^1, \ldots, \zeta^N \in \mathbf{R}^m.$$

(a5) Regularity: $a \in C^1\left(\mathbf{R}^m \times \mathbf{R}^{m\times N}\right)$ and there is $C > 0$ such that

$$\left|\frac{\partial a}{\partial s_i}(s,\xi)\right| \leqslant C(1 + |s|^q + |\xi|^p), \qquad \left|\frac{\partial a}{\partial \xi_k^i}(s,\xi)\right| \leqslant C(1 + |s|^q + |\xi|^p)$$

for all $s \in \mathbf{R}^m$ and $\xi \in \mathbf{R}^{m\times N}$, where $q = p^* = \frac{Np}{N-p}$ if $p < N$ and $q \in [1, \infty)$ if $p = N$.

Now let us discuss our assumptions on the second term in (1.1). In the previous works it was assumed that $g(0) = 0$ (which means that 0 is an equilibrium point for

(1.1)) and the authors were looking for solutions converging to 0 at infinity. However, many physical systems have more equilibria (often a manifold) and there is no *a priori* reason that the system would choose one equilibrium rather than another. This is the case, for instance, for models in condensed matter theory, superfluids, nonlinear optics or micromagnetics. In the present paper we assume that the set of *interesting* equilibria is a compact set $\mathcal{S} \subset \mathbf{R}^m$; we require no manifold structure on $\mathcal{S}$. Our assumptions on the nonlinear potential G as well as our results are slightly different if $p < N$ or $p = N$. We consider separately the two cases.

Notation. If $1 \leqslant p < N$ we denote by $p^* = \frac{Np}{N-p}$ the Sobolev exponent corresponding to p. By μ we denote the Lebesgue measure in $\mathbf{R}^N$ and $\operatorname{dist}(s, A) = \inf\{|t-s| \mid t \in A\}$ is the distance from $s \in \mathbf{R}^m$ to the set $A \subset \mathbf{R}^m$. Given a function u defined on $\mathbf{R}^N$ and $\sigma > 0$ we denote

$$u_\sigma(x) = u\left(\frac{x}{\sigma}\right). \tag{1.4}$$

The case $1 < p < N$**.** We assume that $G : \mathbf{R}^m \longrightarrow \mathbf{R}$ is continuous and there exists a compact, nonempty set $\mathcal{S} \subset \mathbf{R}^m$ such that $G = 0$ on $\mathcal{S}$. We consider the following set of assumptions:

(G1) $\displaystyle\limsup_{\operatorname{dist}(s,\mathcal{S})\to 0} \frac{G(s)}{\operatorname{dist}(s,\mathcal{S})^{p^*}} \leqslant 0.$

(G2) $\displaystyle\limsup_{|s|\to\infty} \frac{G(s)}{|s|^{p^*}} \leqslant 0.$

(G3) There exists $s_0 \in \mathbf{R}^m$ such that $G(s_0) > 0$.

(G4) The function G is C^1 on $\mathbf{R}^m$, $\frac{\partial G}{\partial s_i} = g_i$ and there exists $C > 0$ such that $|g_i(s)| \leqslant C(1 + |s|^{p^*})$ for all $s \in \mathbf{R}^m$, $i = 1, \dots, m$.

Denote $G_+ = \max(G, 0)$, $G_- = -\min(G, 0)$, so that $G = G_+ - G_-$. Notice that assumptions (G1) and (G2) are only on the behavior of G_+ in a neighborhood of $\mathcal{S}$ and of infinity, respectively; nothing is assumed about G_-, except that it is continuous.

In view of the above assumptions, the natural function space to study $(\mathcal{P}_\lambda)$ is

$$\begin{aligned} \mathcal{X} \;=\; &\big\{u \in L^1_{loc}(\mathbf{R}^N, \mathbf{R}^m) \mid \nabla u \in L^p(\mathbf{R}^N),\ G(u) \in L^1(\mathbf{R}^N), \\ &\text{and } \mu\left(\{x \in \mathbf{R}^N \mid \operatorname{dist}(u(x), \mathcal{S}) > \alpha\}\right) < \infty \text{ for all } \alpha > 0\big\}. \end{aligned} \tag{1.5}$$

Using the notation (1.4) it is easy to see that for any $u \in \mathcal{X}$ there holds

$$\mathcal{A}(u_\sigma) = \sigma^{N-p}\mathcal{A}(u) \qquad \text{and} \qquad \mathcal{V}(u_\sigma) = \sigma^N \mathcal{V}(u). \tag{1.6}$$

If (G3) is satisfied then for any $\lambda > 0$ there exists $u \in \mathcal{X}$ such that $\mathcal{V}(u) = \lambda$. To see this consider a function $I \in C^\infty(\mathbf{R})$ such that $I = 1$ on $(-\infty, 0]$, $I = 0$ on $[1, \infty)$ and $-2 \leqslant I' \leqslant 0$. Fix $s_1 \in \mathcal{S}$ and $s_0 \in \mathbf{R}^m$ such that $G(s_0) > 0$. For $R > 0$ let $w_R(x) = s_1 + I(|x| - R)(s_0 - s_1)$. It is obvious that $w_R \in \mathcal{X}$ and $\mathcal{V}(w_R) \longrightarrow \infty$

as $R \longrightarrow \infty$. Hence there is $R_0 > 0$ such that $\mathcal{V}(w_{R_0}) > 0$ and then $\mathcal{V}((w_{R_0})_\sigma) = \lambda$ for a some $\sigma > 0$.

For $\lambda > 0$, denote

$$\mathcal{A}_{\min}(\lambda) = \inf\{\mathcal{A}(u) \mid u \in \mathcal{X}, \mathcal{V}(u) = \lambda\}. \tag{1.7}$$

It is obvious that $0 \leqslant \mathcal{A}_{\min}(\lambda) < \infty$ and (1.6) implies that

$$\mathcal{A}_{\min}(\lambda) = \lambda^{\frac{N-p}{N}} \mathcal{A}_{\min}(1) \qquad \text{for all } \lambda > 0. \tag{1.8}$$

We have the following:

Theorem 1.1. *Assume that $1 < p < N$ and the assumptions (a1), (a2), (a3), (G1), (G2), (G3) are satisfied. Then $\mathcal{A}_{\min}(1) > 0$.*

Moreover, for any $\lambda > 0$ and any sequence $(u_n)_{n \geqslant 1} \subset \mathcal{X}$ satisfying $\mathcal{V}(u_n) \longrightarrow \lambda$ and $\mathcal{A}(u_n) \longrightarrow \mathcal{A}_{\min}(\lambda)$ there exist a subsequence $(u_{n_k})_{k \geqslant 1}$, a sequence of points $(x_k)_{k \geqslant 1} \subset \mathbf{R}^N$ and a function $u \in \mathcal{X}$ satisfying $\mathcal{V}(u) = \lambda$, $\mathcal{A}(u) = \mathcal{A}_{\min}(\lambda)$ and

$$\nabla u_{n_k}(\cdot + x_k) \rightharpoonup \nabla u \qquad \textit{weakly in } L^p(\mathbf{R}^N), \tag{1.9}$$

$$u_{n_k}(\cdot + x_k) \longrightarrow u \qquad \textit{a.e. and in } L^q_{loc}(\mathbf{R}^N) \textit{ for } 1 \leqslant q < p^*, \tag{1.10}$$

$$G((u_{n_k}(\cdot + x_k)) \longrightarrow G(u) \qquad \textit{in } L^1(\mathbf{R}^N). \tag{1.11}$$

Furthermore, for each $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that $\int_{\mathbf{R}^N \setminus B(x_k, R_\varepsilon)} |\nabla u_{n_k}|^p + |u_{n_k}|^{p^*} \, dx < \varepsilon$ for all k.

In particular, the problem $(\mathcal{P}_\lambda)$ admits solutions.

In addition, if assumptions (a5) and (G4) hold, then any minimizer of $(\mathcal{P}_\lambda)$ satisfies

$$\partial_{u_i} a(u, \nabla u) - div(\nabla_{\xi_i} a(u, \nabla u)) = \alpha g_i(u) \qquad \textit{in } \mathcal{D}'(\mathbf{R}^N), \tag{1.12}$$

where $\alpha = \frac{N-p}{N} \lambda^{-\frac{p}{N}} \mathcal{A}_{\min}(1)$, and there is a unique $\lambda > 0$ such that minimizers of $(\mathcal{P}_\lambda)$ satisfy (1.1).

The case $p = N$. In the case $p = N$ any (sufficiently smooth) solution $u \in \mathcal{X}$ of (1.1) must satisfy the Pohozaev identity $\int_{\mathbf{R}^N} G(u) \, dx = 0$. At first glance, one might think of finding solutions to (1.1) by solving the minimization problem $(\mathcal{P}_0)$. However, it is easily seen that the only solutions of $(\mathcal{P}_0)$ are the constant ones, $u(x) = s_*$ with $s_* \in \mathcal{S}$. In order to find nontrivial solutions to (1.1) it is natural to minimize $\mathcal{A}$ in the set $\mathcal{X}_0 = \{u \in \mathcal{X} \mid \mathcal{V}(u) = 0 \text{ and } u \text{ is not constant}\}$. This problem is significantly more difficult than the problem $(\mathcal{P}_\lambda)$ considered above for at least two reasons: firstly, one needs to prevent minimizing sequences to converge to constant functions (and this is not so obvious since constant functions are global minimizers in $\mathcal{X}$); and secondly, the problem is invariant by scaling (see (1.6)). To overcome these difficulties, we notice that under suitable assumptions on G we may write $\mathcal{X}_0 = \{u \in \mathcal{X} \mid \int_{\mathbf{R}^N} G_+(u) \, dx = \int_{\mathbf{R}^N} G_-(u) \, dx > 0\}$. In fact, we will consider a more general problem: we denote $G_1 = G_-$, $G_2 = G_+$ and we minimize $\mathcal{A}$ in the set of functions $u \in \mathcal{X}$ satisfying $\int_{\mathbf{R}^N} G_2(u) \, dx = \lambda \int_{\mathbf{R}^N} G_1(u) \, dx > 0$, where $\lambda > 0$ is arbitrary. Then, of course, we will take $\lambda = 1$. However, it is important to consider the minimization problem for any $\lambda > 0$ in order to get an accurate information for $\lambda = 1$.

As in the case $1 < p < N$, let $\mathcal{S} \subset \mathbf{R}^m$ be a compact, nonempty set. Let $G_1, G_2 : \mathbf{R}^m \longrightarrow [0, \infty)$ be two continuous functions (one might think of as $G_1 = G_-$, $G_2 = G_+$, but we do not need to restrict ourselves to this case; in particular we do not need that $G_1 G_2 = 0$). We assume that G_1 and G_2 satisfy the following properties:

(g1) $G_1 = 0$ on $\mathcal{S}$ and there is an open set $U \supset \mathcal{S}$ such that $G_1 > 0$ on $U \setminus \mathcal{S}$.

(g2) There is $s \in \mathbf{R}^m$ with $G_2(s) > 0$, there is an open set $V \supset \mathcal{S}$ such that $G_2 = 0$ on V and there are $C, q > 0$ verifying $0 \leqslant G_2(s) \leqslant C(1 + |s|^q)$ for all $s \in \mathbf{R}^m$.

We consider the function space

$$\begin{aligned} \mathscr{X} = \big\{ u \in L^1_{loc}(\mathbf{R}^N, \mathbf{R}^m) \mid \nabla u \in L^N(\mathbf{R}^N), G_1(u), G_2(u) \in L^1(\mathbf{R}^N) \\ \text{and } \mu(\{x \in \mathbf{R}^N \mid \operatorname{dist}(u(x), \mathcal{S}) \geqslant \alpha\}) < \infty \text{ for all } \alpha > 0 \big\}. \end{aligned} \tag{1.13}$$

For all $u \in \mathscr{X}$ such that $\displaystyle\int_{\mathbf{R}^N} G_1(u(x))\, dx > 0$ we define $\mathcal{K}(u) = \dfrac{\int_{\mathbf{R}^N} G_2(u(x))\, dx}{\int_{\mathbf{R}^N} G_1(u(x))\, dx}$.
For any $\lambda > 0$, let

$$\mathscr{A}_{\min}(\lambda) = \inf \left\{ \mathcal{A}(u) \;\middle|\; u \in \mathscr{X}, \int_{\mathbf{R}^N} G_1(u(x))\, dx > 0, \ \mathcal{K}(u) = \lambda \right\}.$$

Let $\Lambda = \sup_{s \in \mathbf{R}^N} \frac{G_2(s)}{G_1(s)}$. It is easy to see that the set

$$\mathscr{X}_\lambda = \left\{ u \in \mathscr{X} \;\middle|\; \int_{\mathbf{R}^N} G_1(u(x))\, dx > 0, \ \mathcal{K}(u) = \lambda \right\} \tag{1.14}$$

is not empty and $\mathscr{A}_{\min}(\lambda) < \infty$ if and only if $0 \leqslant \lambda < \Lambda$. Indeed, fix $s_1 \in \mathcal{S}$ and $s_2 \in \mathbf{R}^m$ such that $G_2(s_2) > 0$. For $R > 0$ define $w_R(x) = s_1 + I(|x| - R)(s_2 - s_1)$, where $I \in C^\infty(\mathbf{R})$ satisfies $I = 1$ on $(-\infty, 0]$, $I = 0$ on $[1, \infty)$ and $-2 \leqslant I' \leqslant 0$. The mappings $R \longmapsto \int_{\mathbf{R}^N} G_i(w_R(x))\, dx$ are continuous for $i = 1, 2$, $\mathcal{K}(w_R)$ is well-defined for R sufficiently large and $\mathcal{K}(w_R) \longrightarrow \frac{G_2(s_2)}{G_1(s_2)}$ as $R \longrightarrow \infty$. Then we infer easily that $\mathscr{X}_\lambda \neq \emptyset$ if $\lambda < \Lambda$. It is clear that $\mathscr{X}_\lambda = \emptyset$ if $\lambda > \Lambda$. We have always $\mathscr{X}_\Lambda = \emptyset$. Indeed, if $u \in \mathscr{X}_\Lambda$ we should have $G_2(u(x)) = \Lambda G_1(u(x))$ for a.e. $x \in \mathbf{R}^N$; using (g1), (g2) and Lemma 3.2 below it is not hard to see that this is impossible.

Our main results in the case $p = N$ are as follows.

Theorem 1.2. *Assume that $p = N$ and the conditions (a1), (a2) (a3), (a4), (g1), (g2) hold. Let $(u_n)_{n \geqslant 1} \subset \mathscr{X}$ be a sequence of functions satisfying $\displaystyle\int_{\mathbf{R}^N} G_1(u_n)\, dx > 0$ for all n and*

$$\mathcal{K}(u_n) \longrightarrow \lambda > 0 \qquad \text{and} \qquad \mathcal{A}(u_n) \longrightarrow \mathscr{A}_{\min}(\lambda) \qquad \text{as } n \longrightarrow \infty. \tag{1.15}$$

There exist a subsequence $(u_{n_k})_{k \geqslant 1}$, a sequence $(\sigma_k)_{k \geqslant 1} \subset (0, \infty)$, a sequence of points $(x_k)_{k \geqslant 1} \subset \mathbf{R}^N$ and $u \in \mathscr{X}$ such that $\mathcal{K}(u) = \lambda$, $\mathcal{A}(u) = \mathscr{A}_{\min}(\lambda)$ and

$$\nabla (u_{n_k})_{\sigma_k}(\cdot + x_k) \rightharpoonup \nabla u \qquad \text{weakly in } L^N(\mathbf{R}^N), \tag{1.16}$$

$$(u_{n_k})_{\sigma_k}(\cdot + x_k) \longrightarrow u \qquad \text{a.e. and in } L^r(\mathbf{R}^N),\ 1 \leqslant r < \infty, \tag{1.17}$$

$$(1.18) \qquad G_i((u_{n_k})_{\sigma_k}(\cdot + x_k)) \longrightarrow G_i(u) \qquad \text{in } L^1(\mathbf{R}^N) \text{ for } i = 1, 2,$$

and for any $\varepsilon > 0$ *there is* $R_\varepsilon > 0$ *such that* $\int_{\mathbf{R}^N \setminus B(x_k, R_\varepsilon)} |\nabla (u_n)_{\sigma_k}|^N \, dx < \varepsilon$ *for all* k.

Corollary 1.3. *Assume that* $p = N$, *(a1)-(a5) hold, the function* $G \in C^1(\mathbf{R}^N)$ *satisfies (G4) (with some* $q < \infty$ *instead of* p^**), and* $G_1 = G_-$, $G_2 = G_+$ *satisfy (g1) and (g2).*

Let $(u_n)_{n \geqslant 1} \subset \mathcal{X}$ *be a sequence such that* $\mathcal{K}(u_n) \longrightarrow 1$ *and* $\mathcal{A}(u_n) \longrightarrow \mathscr{A}_{\min}(1)$. *Then there exists* $u \in \mathscr{X}$ *with* $\mathcal{K}(u) = 1$, $\mathcal{A}(u) = \mathscr{A}_{\min}(1)$ *and there are a subsequence* $(u_{n_k})_{k \geqslant 1}$, *a sequence* $(\sigma_k)_{k \geqslant 1} \subset (0, \infty)$ *and a sequence of points* $(x_k)_{k \geqslant 1} \subset \mathbf{R}^N$ *such that (1.16)-(1.18) hold.*

Any u *as above minimizes* $\mathcal{A}$ *in the set* $\{w \in \mathcal{X} \mid \mathcal{V}(w) \geqslant 0$ *and* $\int_{\mathbf{R}^N} |G(w)| \, dx > 0\}$, *and conversely. Moreover, there exists* $\alpha \geqslant 0$ *such that* u *solves (1.12). If* $\alpha > 0$, *then* u_σ *solves (1.1) for* $\sigma = \alpha^{-\frac{1}{N}}$.

To prove Theorem 1.2 we need the following proposition, which is of independent interest.

Proposition 1.4. *Let* $p = N$ *and suppose that the assumptions (a1), (a2) (a3), (g1), (g2) are satisfied. The function* $\mathscr{A}_{\min}$ *has the following properties:*

i) There exists $C > 0$, *independent of* λ, *such that for all* $\lambda > 0$ *there holds*

$$\mathscr{A}_{\min}(\lambda) \geqslant C \lambda^{\frac{N(N-1)}{N+(N-1)q}},$$

where q *is as in (g2). In particular, we have* $\mathscr{A}_{\min}(\lambda) > 0$ *for all* $\lambda > 0$.

ii) $\mathscr{A}_{\min}(\lambda) \longrightarrow 0$ *as* $\lambda \longrightarrow 0$.

iii) Assume, moreover, that a *has a one-dimensional symmetry, that is, (a4) holds. Consider* $u \in \mathscr{X}$ *such that* $\int_{\mathbf{R}^N} G_1(u) \, dx > 0$ *and* $\mathcal{K}(u) = \lambda > 0$. *Then for all* $\tilde{\lambda} \in (0, \lambda)$ *we have*

$$\mathscr{A}_{\min}(\tilde{\lambda}) < \mathcal{A}(u).$$

In particular, $\mathscr{A}_{\min}$ *is nondecreasing.*

Theorem 1.1 will be proven in the next section. We prove Proposition 1.4, Theorem 1.2 and Corollary 1.3 in Section 3. We end this section by discussing our assumptions and the relationship between our results and the existing literature.

Remark 1.5. In most applications one may get a much stronger convergence of minimizing (sub)sequences than provided by Theorems 1.1 and 1.2. For instance, if $\mathcal{A}(u) = \int_{\mathbf{R}^N} |\nabla u|^p \, dx$ the weak convergence (1.9) together with the convergence of norms $\|\nabla u_{n_k}(\cdot + x_k)\|^p_{L^p(\mathbf{R}^N)} = \mathcal{A}(u_{n_k}) \longrightarrow \mathcal{A}(u) = \|\nabla u\|^p_{L^p(\mathbf{R}^N)}$ implies the strong convergence $\nabla u_{n_k}(\cdot + x_k) \longrightarrow \nabla u$ in $L^p(\mathbf{R}^N)$. We get the same strong convergence whenever $\mathcal{A}^{\frac{1}{p}}$ is a uniformly convex norm on $\dot{W}^{1,p}(\mathbf{R}^N)$. It is beyond the scope of the present article to investigate optimal "abstract" conditions on the integrand a that guarantee the strong convergence of minimizing subsequences.

Remark 1.6. In the case $1 < p < N$ it follows from Lemma 7 p. 774 and Remark 4.2 p. 775 in [11] that for any $u \in \mathcal{X}$ there exists $c(u) \in \mathbf{R}^m$ such that $u - c(u) \in L^{p^*}(\mathbf{R}^N)$, that is, $u - c(u) \in \dot{W}^{1,p}(\mathbf{R}^N)$ (see also Theorem 4.5.9 in [14] for a different proof). Since G is continuous and $G(u) \in L^1(\mathbf{R}^N)$, it is easy to see that necessarily $c(u) \in \mathcal{S}$. Denoting $\mathcal{X}_s = \{u \in \mathcal{X} \mid u - s \in L^{p^*}(\mathbf{R}^N)\}$, we have thus $\mathcal{X} = \cup_{s \in \mathcal{S}} \mathcal{X}_s$. For each $s \in \mathcal{S}$ we may consider the problem $(\mathcal{P}_\lambda)$ in the space $\mathcal{X}_s$ instead of $\mathcal{X}$. Theorem 1.1 applies and we get minimizers of $\mathcal{A}$ in the set $\{u \in \mathcal{X}_s \mid \mathcal{V}(u) = \lambda\}$ for any $s \in \mathcal{S}$, as well as the precompactness of minimizing sequences in $\mathcal{X}_s$. In particular, for any $s \in \mathcal{S}$ there exist solutions u_s of (1.1) which minimize $\mathcal{A}$ at fixed $\mathcal{V}$ in the set $\mathcal{X}_s$. However, given $s_1, s_2 \in \mathcal{S}$, $s_1 \neq s_2$, and two such solutions u_{s_1} and u_{s_2} it is not clear whether there is a relationship between $\mathcal{A}(u_{s_1})$, $\mathcal{V}(u_{s_1})$, $E(u_{s_1})$ and $\mathcal{A}(u_{s_2})$, $\mathcal{V}(u_{s_2})$, $E(u_{s_2})$, respectively. The advantage of Theorem 1.1 is that it provides minimizers for $(\mathcal{P}_\lambda)$ as well as the precompactness of minimizing sequences in the whole space $\mathcal{X}$.

If $1 < p < N$ and we replace (G1) by the weaker assumption $\limsup\limits_{t \to s} \dfrac{G(t)}{|t - s|^{p^*}} \leqslant 0$ for any $s \in \mathcal{S}$ we still get minimizers for $(\mathcal{P}_\lambda)$ and solutions for (1.1) in each $\mathcal{X}_s$. However, this weaker assumption is not enough to guarantee that $\mathcal{A}_{\min}(1) > 0$. For instance, let $m = 2$, $1 < p < N$ and $\mathcal{A}(u) = \int_{\mathbf{R}^N} |\nabla u|^p \, dx$. Taking $G(s_1, s_2) = \min\left(1, \max\left((|s_1| - 1)_+, \min\left(|s_1|^{p^*+1}, \left|\frac{s_2}{s_1}\right|^{p^*+1}\right)\right)\right)$ and $\mathcal{S} = [-1, 1] \times \{0\}$, we can prove that $\mathcal{A}_{\min}(\lambda) = 0$ for any $\lambda > 0$. We can "mollify" G to get a C^1 function $\widetilde{G}$ with the same property.

The situation is more complicated if $p = N$: functions in $\mathcal{X}$ may oscillate at infinity and it is no longer true that $\mathcal{X} = \cup_{s \in \mathcal{S}} \mathcal{X}_s$. See Remark 4.2 p. 775 in [11] for a simple example. Our results concerning the existence of minimizers and the precompactness of minimizing sequences are thus far more interesting if $p = N$ and it seems that they cannot be deduced in a simple way from the corresponding results in the case when $\mathcal{S}$ is reduced to a single point.

Remark 1.7. Under various additional assumptions one can prove the $C^{0,\alpha}$ or even the $C^{1,\alpha}$ regularity of solutions to (1.1) provided by Theorem 1.1 or Corollary 1.3 above. We refer to [7], [12], [13], [20], [26] and references therein for regularity results. However, the counterexamples in the literature (see e.g. [10], [25]) indicate that one should not expect a good regularity theory for solutions of (1.1) with only the general assumptions considered in this paper.

Not too much regularity is needed to show that any solution u of (1.1) satisfies the Pohozaev inequality $(N - p)\mathcal{A}(u) = N\mathcal{V}(u)$. Whenever all solutions of (1.1) satisfy this identity, it follows from Lemma 1 p. 484 in [5] that the minimizers of $(\mathcal{P}_\lambda)$ are (after scaling) precisely the least energy solutions of (1.1), i.e. they minimize E among all solutions of (1.1). The same is true about the solutions provided by Corollary 1.3, and this justifies the title of the paper.

In all cases when the solutions of $(\mathcal{P}_\lambda)$ are at least C^1 and the integrand a has some symmetry in the variable ξ, Theorem 2 p. 314 in [21] implies that all minimizers of $(\mathcal{P}_\lambda)$ inherit the symmetry properties of the functional $\mathcal{A}$. For instance, if a depends only on s and $|\xi|$, all minimizers of $(\mathcal{P}_\lambda)$ are radially symmetric. The same is true for the solutions given by Corollary 1.3 in the case $p = N$. Moreover, if we are in the scalar case and we assume that a depends only on $|\xi|$, the proof of Theorem 7 in [5] implies that the radial profile of any least energy solution is monotonic.

Remark 1.8. In the particular case when $\mathcal{A}(u) = \int_{\mathbf{R}^N} |\nabla u|^2 \, dx$, Theorem 1.1 has been proven in [4] (see the proof of Theorem 2.1 p. 100 there). However, even in that case we improve the results in [4] in the sense that we provide a more precise description of the behavior of minimizing sequences and we show that the technical assumption (2.5) p. 99 in [4] is unnecessary. That assumption has been used in [4] to perform an appropriate cut-off in the nonlinear term. In the present paper we do not use a cut-off, but instead we use "localized scaling," that is, we "zoom in" or "zoom out" some regions in the Euclidean space. This is possible in view of the results in [22] which provide large regions in the space with small energy.

The improvement is still greater in the case $p = N$. We solve a more general minimization problem (see Theorem 1.2) and we allow a nontrivial set of equilibria $\mathcal{S}$. We show the existence of minimizers as well as the precompactness of minimizing sequences under mild assumptions. For instance, we need only the continuity of G, while the corresponding result in [4] (see Theorem 3.1 p 106 there) requires either the differentiability of G on $\mathbf{R}^2 \setminus \{0\}$, or the assumption that $|G(tv)| \leqslant C|G(v)|$ for all $t \in [0, 1]$ and $|v| \leqslant \varepsilon$, where C, ε are positive constants.

If $\mathcal{S}$ is finite and a regularity theory is available for the solutions of (1.1) (with an additional term $h(x) \in L^\infty(\mathbf{R}^N)$ in the right side), we may relax the differentiability assumptions on G in Theorem 1.1 and Corollary 1.3 and we can show that the minimizers satisfy (1.1) if we require only $G \in C^1(\mathbf{R}^N) \setminus \mathcal{S}$. The proofs are the same as in [4].

2. The case $p < N$

Proof of Theorem 1.1. It follows from (G1) and (G2) that there is $C > 0$ such that

$$G_+(s) \leqslant C \operatorname{dist}(s, \mathcal{S})^{p^*} \qquad \text{for any } s \in \mathbf{R}^m. \tag{2.1}$$

Consider $u \in \mathcal{X}$ such that $\mathcal{V}(u) = 1$. By Remark 1.6 there is $c(u) \in \mathcal{S}$ such that $u - c(u) \in L^{p^*}(\mathbf{R}^N)$. Using (2.1), the Sobolev inequality and (a3) we get

$$1 = \int_{\mathbf{R}^N} G(u) \, dx \leqslant \int_{\mathbf{R}^N} G_+(u) \, dx \leqslant C \int_{\mathbf{R}^N} \operatorname{dist}(u, \mathcal{S})^{p^*} \, dx \leqslant C \int_{\mathbf{R}^N} |u - c(u)|^{p^*} \, dx$$

$$\leqslant C \left(\int_{\mathbf{R}^N} |\nabla u|^p dx \right)^{\frac{p^*}{p}} \leqslant C \left(\int_{\mathbf{R}^N} a(u, \nabla u) dx \right)^{\frac{p^*}{p}} = C \left(\mathcal{A}(u) \right)^{\frac{p^*}{p}}.$$

Passing to the infimum we get $\mathcal{A}_{\min}(1) > 0$.

Assume that $\lambda > 0$ and $(u_n)_{n \geqslant 1} \subset \mathcal{X}$ satisfies $\mathcal{V}(u_n) \longrightarrow \lambda$ and $\mathcal{A}(u_n) \longrightarrow \mathcal{A}_{\min}(\lambda)$. Let $c(u_n)$ be as in Remark 1.6. Since $\mathcal{A}(u_n)$ is bounded, using (a3) we infer that $\|\nabla u_n\|_{L^p(\mathbf{R}^N)}$ is bounded and then the Sobolev embedding implies that $\|u_n - c(u_n)\|_{L^{p^*}(\mathbf{R}^N)}$ is bounded. Using (2.1) we infer that $(G_+(u_n))_{n \geqslant 1}$ is bounded in $L^1(\mathbf{R}^N)$. Since $\mathcal{V}(u_n) = \int_{\mathbf{R}^N} G_+(u_n) \, dx - \int_{\mathbf{R}^N} G_-(u_n) \, dx \longrightarrow \lambda$, it follows that

$(G_-(u_n))_{n\geqslant 1}$ is bounded in $L^1(\mathbf{R}^N)$, and consequently $(G(u_n))_{n\geqslant 1}$ is bounded in $L^1(\mathbf{R}^N)$.

We will use the concentration-compactness principle for the sequence of functions

$$\rho_n = |\nabla u_n|^p + |u_n - c(u_n)|^{p^*} + |G(u_n)|. \tag{2.2}$$

Clearly, $(\rho_n)_{n\geqslant 1}$ is bounded in $L^1(\mathbf{R}^N)$ and $\liminf\limits_{n\to\infty} \int_{\mathbf{R}^N} \rho_n(x)\,dx \geqslant \lim\limits_{n\to\infty} \mathcal{V}(u_n) = \lambda$. Passing to a subsequence (still denoted $(u_n)_{n\geqslant 1}$) we may assume that

$$\int_{\mathbf{R}^N} \rho_n(x)\,dx \longrightarrow \alpha > 0 \qquad \text{as } n \longrightarrow \infty. \tag{2.3}$$

We denote by q_n the concentration function of ρ_n, that is,

$$q_n(t) = \sup_{y\in\mathbf{R}^N} \int_{B(y,t)} \rho_n(z)\,dz.$$

By Helly's theorem there are a subsequence of $(u_n, q_n)_{n\geqslant 1}$, still denoted $(u_n, q_n)_{n\geqslant 1}$, and a nondecreasing function $q : [0,\infty) \longrightarrow [0,\infty)$ such that $q_n(t) \longrightarrow q(t)$ as $n \longrightarrow \infty$ for all $t \geqslant 0$. Let $\beta = \lim\limits_{t\to\infty} q(t)$. It is clear that $\beta \in [0,\alpha]$. We will show that $\beta = \alpha$ (that is, the sequence $(\rho_n)_{n\geqslant 1}$ "concentrates"). In order to do that we rule out the cases $\beta = 0$ ("vanishing") and $\beta \in (0,\alpha)$ ("dichotomy").

Let us show that $\beta > 0$. It follows from (G1) and (G2) that for any $\varepsilon > 0$ there exist $d(\varepsilon) > 0$, $D(\varepsilon) > 0$ such that $\mathcal{S} \subset B\left(0, \frac{1}{2}D(\varepsilon)\right)$ and

$$G_+(s) \leqslant \varepsilon\cdot\mathrm{dist}(s,\mathcal{S})^{p^*} \qquad \text{for any } s\in\mathbf{R}^m \text{ satisfying } \mathrm{dist}(s,\mathcal{S}) \leqslant d(\varepsilon) \text{ or } |s| \geqslant D(\varepsilon). \tag{2.4}$$

Let $M_\varepsilon = \max\{G_+(s) \mid s\in\mathbf{R}^m,\ |s| \leqslant 2D(\varepsilon)\}$. If $s\in\mathbf{R}^m$ satisfies $\mathrm{dist}(s,\mathcal{S}) \leqslant \frac{3}{2}D(\varepsilon)$, then $G_+(s) \leqslant M_\varepsilon$, and if there is $s_1 \in \mathcal{S}$ such that $|s - s_1| \geqslant \frac{3}{2}D(\varepsilon)$ then s satisfies (2.4). Using (2.4), the Sobolev inequality and (a3) we infer that for any $u \in \mathcal{X}$ there holds

$$\begin{aligned}
\mathcal{V}(u) &= \int_{\mathbf{R}^N} G(u)\,dx \leqslant \int_{\mathbf{R}^N} G_+(u)\,dx \\
&\leqslant \int_{\{|u-c(u)|\leqslant d(\varepsilon)\}\cup\{|u-c(u)|\geqslant \frac{3}{2}D(\varepsilon)\}} \varepsilon\,|u-c(u)|^{p^*}\,dx + \int_{\{d(\varepsilon)<|u-c(u)|<\frac{3}{2}D(\varepsilon)\}} G_+(u)\,dx \\
&\leqslant \varepsilon\big\|u-c(u)\big\|^{p^*}_{L^{p^*}(\mathbf{R}^N)} + M_\varepsilon\mu\left(\{x\in\mathbf{R}^N \mid d(\varepsilon) < |u(x)-c(u)| < \tfrac{3}{2}D(\varepsilon)\}\right) \\
&\leqslant C\varepsilon\mathcal{A}(u)^{\frac{p^*}{p}} + M_\varepsilon\mu\left(\{x\in\mathbf{R}^N \mid d(\varepsilon) < |u(x)-c(u)| < \tfrac{3}{2}D(\varepsilon)\}\right).
\end{aligned} \tag{2.5}$$

We write (2.5) for each u_n. Since $\mathcal{A}(u_n)$ is bounded, we may fix $\varepsilon_0 > 0$ such that $C\varepsilon_0\mathcal{A}(u_n)^{\frac{p^*}{p}} \leqslant \frac{\lambda}{4}$ for all n, where C is as in (2.5). Using the fact that $\mathcal{V}(u_n) \longrightarrow \lambda > 0$, there exist $m_0 > 0$ and $n_0 \in \mathbf{N}$ such that

$$\mu\left(\{x\in\mathbf{R}^N \mid |u_n(x) - c(u_n)| > d(\varepsilon_0)\}\right) \geqslant m_0 \qquad \text{for all } n \geqslant n_0. \tag{2.6}$$

We will use the following lemma, due to E. H. Lieb ([18]):

Lemma 2.1 ([18]). *Let $p \in [1,\infty)$. Consider $w \in L^1_{loc}(\mathbf{R}^N,\mathbf{R})$ such that $\nabla w \in L^p(\mathbf{R}^N)$. Assume that w satisfies $\|\nabla w\|_{L^p(\mathbf{R}^N)} \leqslant M$ and $\mu\left(\{x\in\mathbf{R}^N \mid w(x)\geqslant\varepsilon\}\right) \geqslant \alpha$, where $M,\alpha,\varepsilon>0$. Let $\delta\in(0,\varepsilon)$. There exist a positive constant β depending only on $N,p,M,\alpha,\varepsilon,\delta$, but not on w, and $y\in\mathbf{R}^N$ such that*

$$\mu\left(\{x\in B(y,1) \mid w(x)\geqslant\delta\}\right)\geqslant\beta.$$

The proof is similar to Brezis' proof of Lemma 6 in [18] (see [18] p. 447-448), so we skip it.

Using Lemma 2.1 for $w_n = |u_n - c(u_n)|$, it follows that there exists $m_1>0$ and for any $n\geqslant n_0$ there is $y_n\in\mathbf{R}^N$ such that

$$\mu\left(\{x\in B(y_n,1) \mid |u_n(x)-c(u_n)|\geqslant \frac{1}{2}d(\varepsilon_0)\}\right)\geqslant m_1 \qquad \text{for all } n\geqslant n_0.$$

Hence for $n\geqslant n_0$ we have

$$q_n(1)\geqslant\int_{B(y_n,1)}\rho_n(x)\,dx\geqslant\int_{B(y_n,1)}|u_n(x)-c(u_n)|^{p^*}\,dx\geqslant 2^{-p^*}d(\varepsilon_0)^{p^*}m_1.$$

Passing to the limit as $n\longrightarrow\infty$ we get $q_n(1)\geqslant 2^{-p^*}d(\varepsilon_0)^{p^*}m_1$, thus necessarily $\beta>0$.

Next we show that we cannot have $\beta\in(0,\alpha)$. We argue by contradiction and we assume that $\beta\in(0,\alpha)$. Using Lemma 3 in [22] there is a sequence $R_n>1$, $R_n\longrightarrow\infty$ such that $q_n(R_n)\longrightarrow\beta$ and $q_n(R_n^3)\longrightarrow\beta$ as $n\longrightarrow\infty$. For each $n\geqslant 1$ we fix $x_n\in\mathbf{R}^N$ such that $\int_{B(x_n,R_n)}\rho_n(x)\,dx > q_n(R_n)-\frac{1}{n}$. Then

$$\int_{\mathbf{R}^N\setminus B(x_n,R_n^3)}\rho_n(x)\,dx\geqslant\int_{\mathbf{R}^N}\rho_n(x)\,dx-q_n(R_n^3)\longrightarrow\alpha-\beta \quad \text{as } n\longrightarrow\infty, \qquad \text{and}$$

$$\int_{\mathbf{R}^N\setminus B(x_n,R_n^3)}\rho_n(x)\,dx\leqslant\int_{\mathbf{R}^N}\rho_n(x)\,dx-\int_{B(x_n,R_n)}\rho_n(x)\,dx\leqslant\int_{\mathbf{R}^N}\rho_n(x)\,dx-q_n(R_n)+\frac{1}{n}.$$

Therefore

$$\int_{B(x_n,R_n)}\rho_n(x)\,dx\longrightarrow\beta, \qquad \int_{\mathbf{R}^N\setminus B(x_n,R_n^3)}\rho_n(x)\,dx\longrightarrow\alpha-\beta \quad \text{and} \tag{2.7}$$

$$\int_{B(x_n,R_n^3)\setminus B(x_n,R_n)}\rho_n(x)\,dx\longrightarrow 0 \quad \text{as } n\longrightarrow\infty. \tag{2.8}$$

After translation we may assume that $x_n=0$, and we will do so to simplify notation. Let

(2.9)

$$u_{n,1}(x)=\begin{cases} u_n(x) & \text{if } |x|\leqslant R_n,\\ u_n\left(\frac{|x|}{R_n}x\right) & \text{if } R_n<|x|<R_n^2,\\ u_n(R_nx) & \text{if } |x|\geqslant R_n^2,\end{cases} \qquad u_{n,2}(x)=\begin{cases} u_n(R_nx) & \text{if } |x|\leqslant 1,\\ u_n\left(\frac{R_nx}{|x|^{\frac{1}{2}}}\right) & \text{if } 1<|x|<R_n^2,\\ u_n(x) & \text{if } |x|\geqslant R_n^2.\end{cases}$$

For $a, b \in [0, \infty]$ denote $\Omega_{a,b} = \{x \in \mathbf{R}^N \mid a < |x| < b\}$. It is easy to see that $u_{n,1}, u_{n,2} \in \mathcal{X}$ and a straightforward computation gives

(2.10)
$$\int_{\Omega_{R_n,R_n^2}} |G(u_{n,1}(x))|\, dx = \int_{\Omega_{R_n,R_n^2}} \left| G\left(u_n\left(\frac{|x|}{R_n}x\right)\right)\right| dx$$

$$= \int_{\Omega_{R_n,R_n^3}} |G\left(u_n\left(y\right)\right)| \cdot \frac{1}{2}\left(\frac{R_n}{|y|}\right)^{\frac{N}{2}} dy \leqslant \frac{1}{2}\int_{\Omega_{R_n,R_n^3}} |G\left(u_n\left(y\right)\right)|\; dy \leqslant \frac{1}{2}\int_{\Omega_{R_n,R_n^3}} \rho_n(y)\, dy,$$

$$\int_{\mathbf{R}^N\setminus B(0,R_n^2)} |G(u_{n,1}(x))|\, dx = \frac{1}{R_n^N}\int_{\mathbf{R}^N\setminus B(0,R_n^3)} |G(u_n(y))|\, dy, \tag{2.11}$$

(2.12)
$$\int_{B(0,1)} |G(u_{n,2}(x))|\, dx = \int_{B(0,1)} |G(u_n(R_n x))|\, dx = \frac{1}{R_n^N}\int_{B(0,R_n)} |G(u_n(y))|\, dy,$$

(2.13)
$$\int_{\Omega_{1,R_n^2}} |G(u_{n,2}(x))|\, dx = \int_{\Omega_{1,R_n^2}} \left| G\left(u_n\left(\frac{R_n}{|x|^{1/2}}x\right)\right)\right| dx$$

$$= \int_{\Omega_{R_n,R_n^2}} |G\left(u_n\left(y\right)\right)| \cdot 2\left(\frac{|y|}{R_n^2}\right)^N dy \leqslant 2\int_{\Omega_{R_n,R_n^2}} \rho_n(y)\, dy.$$

Using (a2) we get

(2.14)
$$\int_{\mathbf{R}^N\setminus B(0,R_n^2)} a(u_{n,1}(x), \nabla u_{n,1}(x))\, dx = \int_{\mathbf{R}^N\setminus B(0,R_n^2)} a\big(u_n(R_n x), R_n \nabla u_n(R_n x)\big)\, dx$$

$$= R_n^{p-N}\int_{\mathbf{R}^N\setminus B(0,R_n^3)} a(u_n(y), \nabla u_n(y))\, dy,$$

and similarly

$$\int_{B(0,1)} a(u_{n,2}(x), \nabla u_{n,2}(x))\, dx = R_n^{p-N}\int_{B(0,R_n)} a(u_n(y), \nabla u_n(y))\, dy. \tag{2.15}$$

It is easy to see that there is a positive constant C such that

$$|\nabla u_{n,1}(x)| \leqslant C\frac{|x|}{R_n}\left|\nabla u_n\left(\frac{|x|}{R_n}x\right)\right| \qquad \text{for a.e. } x \in \Omega_{R_n,R_n^2}.$$

Using (a3) we get

(2.16)
$$\int_{\Omega_{R_n,R_n^2}} a(u_{n,1}, \nabla u_{n,1})\, dx \leqslant C\int_{\Omega_{R_n,R_n^2}} |\nabla u_{n,1}|^p\, dx \leqslant C\int_{\Omega_{R_n,R_n^2}} \tfrac{|x|^p}{R_n^p}\left|\nabla u_n\left(\tfrac{|x|}{R_n}x\right)\right|^p dx$$

$$\leqslant C\int_{\Omega_{R_n,R_n^3}} |\nabla u_n(y)|^p \cdot \tfrac{1}{2}\left(\tfrac{R_n}{|y|}\right)^{\frac{N-p}{2}} dy \leqslant C\int_{\Omega_{R_n,R_n^3}} \rho_n(y)\, dy.$$

In the same way, there is $C > 0$ such that

$$|\nabla u_{n,2}(x)| \leqslant C\frac{R_n}{|x|^{1/2}}\left|\nabla u_n\left(\frac{R_n}{|x|^{1/2}}x\right)\right| \qquad \text{a.e. in } \Omega_{1,R_n^2}.$$

Consequently,
(2.17)
$$\int_{\Omega_{1,R_n^2}} a(u_{n,2},\nabla u_{n,2})\,dx \leqslant C\int_{\Omega_{1,R_n^2}} |\nabla u_{n,2}|^p\,dx \leqslant C\int_{\Omega_{1,R_n^2}} \left|\nabla u_n\left(\frac{R_n}{|x|^{1/2}}x\right)\right|^p \left(\frac{R_n}{|x|^{1/2}}\right)^p dx$$
$$\leqslant C\int_{\Omega_{R_n,R_n^2}} |\nabla u_n(y)|^p\cdot 2\left(\frac{|y|}{R_n^2}\right)^{N-p} dy \leqslant C\int_{\Omega_{R_n,R_n^2}} \rho_n(y)\,dy.$$

From (2.7)-(2.17) it follows that as $n \longrightarrow \infty$ we have
(2.18)
$$\mathcal{V}(u_{n,1}) = \int_{B(0,R_n)} G(u_n)\,dx + o(1), \qquad \mathcal{V}(u_{n,2}) = \int_{\mathbf{R}^N\setminus B(0,R_n^2)} G(u_n)\,dx + o(1),$$
(2.19)
$$\mathcal{A}(u_{n,1}) = \int_{B(0,R_n)} a(u_n,\nabla u_n)\,dx + o(1), \quad \mathcal{A}(u_{n,2}) = \int_{\mathbf{R}^N\setminus B(0,R_n^2)} a(u_n,\nabla u_n)\,dx + o(1).$$
Passing again to a subsequence we may assume that
$$\mathcal{V}(u_{n,1}) \longrightarrow v_1, \quad \mathcal{V}(u_{n,2}) \longrightarrow v_2, \quad \mathcal{A}(u_{n,1}) \longrightarrow a_1, \quad \mathcal{A}(u_{n,2}) \longrightarrow a_2 \quad \text{as } n \longrightarrow \infty.$$
From (2.7), (2.8), (2.18), (2.19) and the fact that $\mathcal{V}(u_n) \longrightarrow \lambda$, $\mathcal{A}(u_n) \longrightarrow \mathcal{A}_{\min}(\lambda)$ we get
$$v_1 + v_2 = \lambda \qquad \text{and} \qquad a_1 + a_2 = \mathcal{A}_{\min}(\lambda) = \mathcal{A}_{\min}(v_1 + v_2). \tag{2.20}$$

It is clear that $a_1, a_2 \geqslant 0$. We claim that $a_1 > 0$ and $a_2 > 0$. To see this we argue by contradiction and we assume, for instance, that $a_1 = 0$. Using (a3) this implies $\|\nabla u_{n,1}\|_{L^p(\mathbf{R}^N)} \longrightarrow 0$ and the Sobolev embedding gives $\|u_{n,1} - c(u_{n,1})\|_{L^{p^*}(\mathbf{R}^N)} \longrightarrow 0$. By (2.1) we get $\int_{\mathbf{R}^N} G_+(u_{n,1})\,dx \longrightarrow 0$ as $n \longrightarrow \infty$. It is obvious that $c(u_n) = c(u_{n,1})$ and estimates similar to (2.10)-(2.11) imply that $\int_{B(0,R_n)} |u_n - c(u_n)|^{p^*}\,dx = \int_{\mathbf{R}^N} |u_{n,1} - c(u_{n,1})|^{p^*}\,dx + o(1) \longrightarrow 0$ as $n \longrightarrow \infty$. From (2.2) and (2.7) we infer that $\int_{B(0,R_n)} G_-(u_n)\,dx \longrightarrow \beta$ as $n \longrightarrow \infty$. Therefore
$$v_1 = \lim_{n\to\infty} \mathcal{V}(u_{n,1}) = \lim_{n\to\infty} \int_{B(0,R_n)} G_+(u_n) - G_-(u_n)\,dx = -\beta,$$
and consequently $v_2 = \lambda + \beta > \lambda$. It is clear that $\mathcal{A}(u_{n,2}) \geqslant \mathcal{A}_{\min}(\mathcal{V}(u_{n,2}))$. Passing to the limit and using the continuity of $\mathcal{A}_{\min}$ we get
(2.21)
$$\mathcal{A}_{\min}(\lambda) \geqslant a_2 = \lim_{n\to\infty} \mathcal{A}(u_{n,2}) \geqslant \limsup_{n\to\infty} \mathcal{A}_{\min}(\mathcal{V}(u_{n,2})) = \mathcal{A}_{\min}(v_2) = \mathcal{A}_{\min}(\lambda + \beta),$$
and (2.21) contradicts the fact that $\mathcal{A}_{\min}$ is increasing. We conclude that necessarily $a_1 > 0$. A similar argument shows that $a_2 > 0$.

We have $v_1, v_2 \in (0, \lambda)$. Indeed, if $v_1 \leqslant 0$ then (2.20) implies $v_2 \geqslant \lambda$ and then $a_2 \geqslant \mathcal{A}_{\min}(v_2) \geqslant \mathcal{A}_{\min}(\lambda)$, which contradicts $\mathcal{A}_{\min}(\lambda) > a_2$ since $a_1 > 0$. Thus $v_1 > 0$ and a similar argument shows that $v_2 > 0$.

Since $\mathcal{A}(u_{n,i}) \geqslant \mathcal{A}_{\min}(\mathcal{V}(u_{n,i}))$, $i = 1, 2$, passing to the limit as $n \longrightarrow \infty$ we get $a_1 \geqslant \mathcal{A}_{\min}(v_1)$ and $a_2 \geqslant \mathcal{A}_{\min}(v_2)$, hence
$$\mathcal{A}_{\min}(v_1 + v_2) = \mathcal{A}_{\min}(\lambda) = a_1 + a_2 \geqslant \mathcal{A}_{\min}(v_1) + \mathcal{A}_{\min}(v_2).$$

On the other hand, using (1.8) we obtain $\mathcal{A}_{\min}(v_1+v_2) < \mathcal{A}_{\min}(v_1)+\mathcal{A}_{\min}(v_2)$, a contradiction. We conclude that we cannot have $\beta\in(0,\alpha)$, thus necessarily $\beta=\alpha$.

Arguing as in [19] we see that there exists a sequence of points $(z_n)_{n\geqslant 1}\subset\mathbf{R}^N$ and for any $\varepsilon>0$ there are $R_\varepsilon>0$ and $n_\varepsilon\in\mathbf{N}$ such that

$$\int_{B(z_n,R_\varepsilon)}\rho_n\,dx\geqslant\alpha-\varepsilon,\qquad\int_{\mathbf{R}^N\setminus B(z_n,R_\varepsilon)}\rho_n\,dx\leqslant 2\varepsilon,\qquad\text{for all } n\geqslant n_\varepsilon. \tag{2.22}$$

Let $\tilde u_n=u_n(\cdot+z_n)$. Since ∇u_n is bounded in $L^p(\mathbf{R}^N)$ and u_n is bounded in $W^{1,p}_{loc}(\mathbf{R}^N)$, there exist a subsequence $(u_{n_k})_{k\geqslant 1}$ and $u\in\dot W^{1,p}(\mathbf{R}^N)$ that satisfy (1.9), (1.10), and $c(\tilde u_{n_k})\longrightarrow c\in\mathcal{S}$. Using (1.10), the Fatou lemma and the Sobolev embedding we get

$$\|u-c\|_{L^{p^*}(\mathbf{R}^N)}\leqslant\liminf_{k\to\infty}\|\tilde u_{n_k}-c(\tilde u_{n_k})\|_{L^{p^*}(\mathbf{R}^N)}<\infty.$$

Hence $c(u)=c\in\mathcal{S}$. Using again the Fatou lemma we obtain

$$\int_{\mathbf{R}^N}G_\pm(u)\,dx\leqslant\liminf_{k\to\infty}\int_{\mathbf{R}^N}G_\pm(\tilde u_{n_k})\,dx\leqslant\alpha. \tag{2.23}$$

In particular, we have $u\in\mathcal{X}$. We will show that

$$G_+(\tilde u_{n_k})\longrightarrow G_+(u)\qquad\text{in } L^1(\mathbf{R}^N). \tag{2.24}$$

To see this fix $\varepsilon>0$. Then take $R_\varepsilon>0$ such that (2.22) holds. Since $\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)}G_+(\tilde u_{n_k})\,dx\leqslant 2\varepsilon$ if $n_k\geqslant n_\varepsilon$, the Fatou lemma implies $\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)}G_+(u)\,dx\leqslant 2\varepsilon$. Take $D(\varepsilon)>0$ as in (2.4) and let $M_\varepsilon=\max\{G_+(s)\mid s\in\mathbf{R}^m,\ |s|\leqslant 2D(\varepsilon)\}$. Let $H_1(s)=\min(G_+(s),M_\varepsilon)$ and $H_2(s)=G_+(s)-H_1(s)$, so that H_1, H_2 are continuous, H_1 is bounded and $0\leqslant H_2(s)\leqslant\varepsilon\cdot\operatorname{dist}(s,\mathcal{S})^{p^*}$ for all $s\in\mathbf{R}^m$ by (2.4).

Since $\tilde u_{n_k}\longrightarrow u$ a.e., the Lebesgue dominated convergence theorem gives

$$\int_{B(0,R_\varepsilon)}|H_1(\tilde u_{n_k})-H_1(u)|\,dx\longrightarrow 0\qquad\text{as } k\longrightarrow\infty.$$

Hence there exists $k_\varepsilon\geqslant n_\varepsilon$ such that $\int_{B(0,R_\varepsilon)}|H_1(\tilde u_{n_k})-H_1(u)|\,dx<\varepsilon$ for all $k\geqslant k_\varepsilon$. On the other hand,

$$0\leqslant\int_{B(0,R_\varepsilon)}H_2(\tilde u_{n_k})\,dx\leqslant\varepsilon\int_{\mathbf{R}^N}|\tilde u_{n_k}-c(\tilde u_{n_k})|^{p^*}\,dx\leqslant\varepsilon\int_{\mathbf{R}^N}\rho_{n_k}(x)\,dx\leqslant 2\alpha\varepsilon$$

for all $k\geqslant k'_\varepsilon$, where $k'_\varepsilon\geqslant k_\varepsilon$. It is obvious that a similar estimate holds with u instead of $\tilde u_{n_k}$. Thus we find

$$\begin{aligned}\int_{\mathbf{R}^N}|G_+(\tilde u_{n_k})-G_+(u)|\,dx&\leqslant\int_{\mathbf{R}^N\setminus B(0,R_\varepsilon)}|G_+(\tilde u_{n_k})|+|G_+(u)|\,dx\\&\quad+\int_{B(0,R_\varepsilon)}|H_1(\tilde u_{n_k})-H_1(u)|\,dx+\int_{B(0,R_\varepsilon)}H_2(\tilde u_{n_k})+H_2(u)\,dx\leqslant(5+4\alpha)\varepsilon\end{aligned}$$

for all $k\geqslant k'_\varepsilon$. Since ε is arbitrary, (2.24) follows.

From (2.23) and (2.24) we get

$$\mathcal{V}(u)=\int_{\mathbf{R}^N}G_+(u)dx-\int_{\mathbf{R}^N}G_-(u)dx\geqslant\liminf_{k\to\infty}\int_{\mathbf{R}^N}G(\tilde u_{n_k})\,dx=\lambda. \tag{2.25}$$

We will use the following result which is a consequence of Theorem 1 p. 522 and Theorem 3 p. 524 in [15]:

Lemma 2.2 ([15])**.** *Assume that the function $i : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \longrightarrow \mathbf{R}_+$, $(s, \xi) \longmapsto i(s, \xi)$ is nonnegative, Borel measurable on $\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$, lower semicontinuous in (s, ξ) and convex in $\xi \in \mathbf{R}^{m_2}$. Let $A \subset \mathbf{R}^N$ be a Borel measurable set with $\mu(A) < \infty$. For $f \in L^q(A, \mathbf{R}^{m_1})$ and $g \in L^q(A, \mathbf{R}^{m_1})$ (where $q, r \in [1, \infty]$) we define*

$$I(f, g) = \int_A i(f(x), g(x))\, dx.$$

Assume that I is finite at one point $(f, g) \in L^q(A, \mathbf{R}^{m_1}) \times L^r(A, \mathbf{R}^{m_2})$.

Then I is (sequentially) lower semicontinuous on $L^q(A, \mathbf{R}^{m_1}) \times L^r_w(A, \mathbf{R}^{m_2})$, where $L^r_w(A, \mathbf{R}^{m_2})$ denotes $L^r(A, \mathbf{R}^{m_2})$ endowed with the weak topology.

It follows from (1.9), (1.10) and Lemma 2.2 that for any Borel measurable set A of finite measure,

$$\int_A a(u, \nabla u)\, dx \leqslant \liminf_{k \to \infty} \int_A a(\tilde{u}_{n_k}, \nabla \tilde{u}_{n_k})\, dx \leqslant \liminf_{k \to \infty} \int_{\mathbf{R}^N} a(u_{n_k}, \nabla u_{n_k})\, dx.$$

Taking $A = B(0, n)$, then passing to the limit as $n \longrightarrow \infty$ we get

$$\mathcal{A}(u) \leqslant \liminf_{k \to \infty} \mathcal{A}(u_{n_k}) = \lim_{k \to \infty} \mathcal{A}(u_{n_k}) = \mathcal{A}_{\min}(\lambda). \tag{2.26}$$

From (2.25), (2.26) and the fact that $\mathcal{A}_{\min}$ is increasing we infer that necessarily $\mathcal{V}(u) = \lambda$, $\mathcal{A}(u) = \mathcal{A}_{\min}(\lambda)$, that is, u is a solution of $(\mathcal{P}_\lambda)$. Moreover, we have

$$\int_{\mathbf{R}^N} G_-(u)\, dx = \lim_{k \to \infty} \int_{\mathbf{R}^N} G_-(\tilde{u}_{n_k})\, dx.$$

Since $G_-(\tilde{u}_{n_k}) \geqslant 0$ and $G_-(\tilde{u}_{n_k}) \longrightarrow G_-(u)$ a.e. in $\mathbf{R}^N$ as $k \longrightarrow \infty$, we deduce that $G_-(\tilde{u}_{n_k}) \longrightarrow G_-(u)$ in $L^1(\mathbf{R}^N)$. Combined with (2.24), this gives $G(\tilde{u}_{n_k}) \longrightarrow G(u)$ in $L^1(\mathbf{R}^N)$.

Assume that, in addition, assumptions (a5) and (G4) are satisfied. Let $u \in \mathcal{X}$ be a minimizer for the problem $(\mathcal{P}_\lambda)$. By (G4) there is $C > 0$ such that for any $w_1, w_2 \in \mathbf{R}^m$ and any $t \in [-1, 1]$ there holds

$$|G(w_1 + t w_2) - G(w_1)| \leqslant C|t| \left(1 + |w_1|^{p^*} + |w_2|^{p^*}\right) |w_2|.$$

Hence for any $\phi \in C^1_c(\mathbf{R}^N, \mathbf{R}^m)$ and $t \in [-1, 1]$, $t \neq 0$ we have $\frac{1}{|t|}|G(u + t\phi) - G(u)| \leqslant C \left(1 + |u|^{p^*} + |\phi|^{p^*}\right) |\phi|$ and the right hand side is in $L^1(\mathbf{R}^N)$. Using the dominated convergence theorem we get

$$\frac{1}{t}\left(\mathcal{V}(u + t\phi) - \mathcal{V}(u)\right) \longrightarrow \int_{\mathbf{R}^N} \sum_{i=1}^m \frac{\partial G}{\partial u_i}(u) \phi_i\, dx$$

as $t \longrightarrow 0$, or equivalently,

$$\mathcal{V}(u + t\phi) = \mathcal{V}(u) + t\Phi_u(\phi) + o(t), \qquad \text{where } \Phi_u(\phi) = \int_{\mathbf{R}^N} \sum_{i=1}^m g_i(u) \phi_i\, dx. \tag{2.27}$$

Similarly, assumption (a5) implies that for any $\phi \in C_c^1(\mathbf{R}^N, \mathbf{R}^m)$ and any $t \in [-1, 1]$, $t \neq 0$ there holds

$$\left| \frac{a(u + t\phi, \nabla u + t\nabla\phi) - a(u, \nabla u)}{t} \right| \leqslant C \left(1 + |u|^{p^*} + |\phi|^{p^*} + |\nabla u|^p + |\nabla\phi|^p\right)(|\phi| + |\nabla\phi|)$$

and the right hand side in the above inequality is in $L^1(\mathbf{R}^N)$. Using again the dominated convergence theorem we obtain $\mathcal{A}(u + t\phi) = \mathcal{A}(u) + t\Psi_u(\phi) + o(t)$, where

$$\Psi_u(\phi) = \int_{\mathbf{R}^N} \sum_{i=1}^m \frac{\partial a}{\partial s_i}(u, \nabla u)\phi_i + \sum_{i=1}^m \sum_{k=1}^N \frac{\partial a}{\partial \xi_i^k}(u, \nabla u)\frac{\partial \phi_i}{\partial x_k}\, dx. \tag{2.28}$$

In particular, we see that $u + t\phi \in \mathcal{X}$. By the definition of $\mathcal{A}_{\min}$ and (1.8) we have

$$\mathcal{A}(v) \geqslant \mathcal{A}_{\min}(1)\mathcal{V}(v)^{\frac{N-p}{N}} \qquad \text{for any } v \in \mathcal{X} \text{ with } \mathcal{V}(v) > 0.$$

Writing the above inequality for $u + t\phi$ instead of v, using (2.27) and (2.28) and recalling that $\mathcal{V}(u) = \lambda$, $\mathcal{A}(u) = \mathcal{A}_{\min}(\lambda) = \mathcal{A}_{\min}(1)\lambda^{\frac{N-p}{N}}$ we get

$$t\Psi_u(\phi) + o(t) \geqslant t\mathcal{A}_{\min}(1)\frac{N-p}{N}\lambda^{-\frac{p}{N}}\Phi_u(\phi) + o(t)$$

for small t. Taking $-\phi$ instead of ϕ we infer that necessarily

$$\Psi_u(\phi) = \mathcal{A}_{\min}(1)\frac{N-p}{N}\lambda^{-\frac{p}{N}}\Phi_u(\phi)$$

for any $\phi \in C_c^1(\mathbf{R}^N, \mathbf{R}^m)$ and (1.12) is proven. □

3. The case $p = N$

In the proof of Theorem 1.2 we will use the following lemmas.

Lemma 3.1 ([4]). *Let $q \in [1, \infty)$. There is $C_q > 0$ such that for any function $\phi \in L^1_{loc}(\mathbf{R}^N, \mathbf{R})$ satisfying $\nabla\phi \in L^N(\mathbf{R}^N)$ and $\mu\left(\{x \in \mathbf{R}^N \mid \phi(x) \neq 0\}\right) < \infty$ there holds*

$$\|\phi\|_{L^q(\mathbf{R}^N)} \leqslant C_q \|\nabla\phi\|_{L^N(\mathbf{R}^N)} \mu\left(\{x \in \mathbf{R}^N \mid \phi(x) \neq 0\}\right)^{\frac{1}{q}}.$$

The proof of Lemma 3.1 is very similar to the proof of inequality (3.10) p. 107 in [4] and we omit it.

Lemma 3.2. *Assume that $v \in L^1_{loc}(\mathbf{R}^N, \mathbf{R})$ is nonnegative, $\mu\left(\{x \in \mathbf{R}^N \mid v(x) \geqslant \varepsilon\}\right) < \infty$ for all $\varepsilon > 0$ and $\nabla v \in L^p(\mathbf{R}^N)$, where $p \in (1, \infty)$. Let $h : [0, \infty) \longrightarrow [0, \infty)$ be a continuous function and let $H(t) = \int_0^t h(\tau)^{1-\frac{1}{p}}\, d\tau$. Then we have for all $a > 0$,*

$$H(a)|S^{N-1}|^{\frac{1}{N}} \mu\left(\{x \in \mathbf{R}^N \mid v(x) \geqslant a\}\right)^{\frac{N-1}{N}} \leqslant \left(\int_{\{v<a\}} |\nabla v|^p\, dx\right)^{\frac{1}{p}} \left(\int_{\{v<a\}} h(v(x))\, dx\right)^{1-\frac{1}{p}}, \tag{3.1}$$

where $|S^{N-1}|$ is the surface measure of the unit sphere $S^{N-1} \subset \mathbf{R}^N$.

Proof. Fix $a > 0$. Let $v_a = \min(v, a)$. Then $\nabla v_a = \mathbf{1}_{\{v<a\}}\nabla v$ a.e. Denote by $w(x) = \varphi(|x|)$ the Schwarz rearrangement of v_a. Let $A = \{x \in \mathbf{R}^N \mid v(x) \geqslant a\}$ and let $A^* = B(0, R_a)$ be the Schwarz rearrangement of A. If $\mu(A) = 0$ then (3.1) is obvious, hence we may assume that $\mu(A) > 0$. We have $\varphi(R_a) = a$ and $\varphi(t) < a$ for $t > R_a$. The assumption $\mu\left(\{x \in \mathbf{R}^N \mid v(x) \geqslant \varepsilon\}\right) < \infty$ for all $\varepsilon > 0$ implies

$\varphi(r) \longrightarrow 0$ as $r \longrightarrow \infty$. By the Pólya-Szegö inequality we have $\|\nabla w\|_{L^p(\mathbf{R}^N)} \leqslant \|\nabla v_a\|_{L^p(\mathbf{R}^N)} = \left(\int_{\{v<a\}} |\nabla v|^p \, dx\right)^{\frac{1}{p}}$. Theorem 1 p. 163 in [8] implies $\varphi \in W^{1,p}_{loc}(\mathbf{R})$. Therefore

$$\left(\int_{\{v<a\}} |\nabla v|^p \, dx\right)^{\frac{1}{p}} \left(\int_{\{v<a\}} h(v(x)) \, dx\right)^{1-\frac{1}{p}} \geqslant \|\nabla w\|_{L^p(\mathbf{R}^N)} \left(\int_{\{w<a\}} h(w(x)) \, dx\right)^{1-\frac{1}{p}}$$

$$= |S^{N-1}| \left(\int_0^\infty |\varphi'(r)|^p \, r^{N-1} \, dr\right)^{\frac{1}{p}} \left(\int_{R_a}^\infty h(\varphi(r)) r^{N-1} \, dr\right)^{1-\frac{1}{p}}$$

$$\geqslant |S^{N-1}| \int_{R_a}^\infty |\varphi'(r)| \, (h(\varphi(r)))^{1-\frac{1}{p}} \, r^{N-1} \, dr \qquad \text{(by Hölder's inequality)}$$

$$\geqslant |S^{N-1}| R_a^{N-1} \int_{R_a}^\infty |\varphi'(r)| \, (h(\varphi(r)))^{1-\frac{1}{p}} \, dr$$

$$\geqslant |S^{N-1}| R_a^{N-1} \int_{R_a}^\infty -(H(\varphi(r)))' \, dr = |S^{N-1}| R_a^{N-1} H(a) = H(a) |S^{N-1}|^{\frac{1}{N}} \mu(A)^{\frac{N-1}{N}}.$$

□

Proof of Proposition 1.4. For $\alpha > 0$ we denote $\mathcal{S}_\alpha = \{x \in \mathbf{R}^m \mid \operatorname{dist}(x, \mathcal{S}) < \alpha\}$. Fix $\delta > 0$ such that $G_1 > 0$ on $\mathcal{S}_{32\delta} \backslash \mathcal{S}$ and $G_2 = 0$ on $\mathcal{S}_{32\delta}$ (this is possible in view of (g1) and (g2)). Choose $R > 100\delta$ such that $\mathcal{S}_{100\delta} \subset B(0, R)$ and a function $\rho \in C_c^\infty(\mathbf{R}^m)$ such that $\rho \geqslant 0$, $\int_{\mathbf{R}^m} \rho(x) \, dx = 1$ and $\operatorname{supp}(\rho) \subset B(0, \delta)$. Let $d_0 = \rho * \operatorname{dist}(\cdot, \overline{\mathcal{S}_{2\delta}})$. Fix a nondecreasing function $I \in C^\infty(\mathbf{R})$ such that $I = 0$ on $(-\infty, R]$ and $I = 1$ on $[2R, \infty)$. Define $d : \mathbf{R}^m \longrightarrow [0, \infty)$ by

$$d(s) = I(|s|)|s| + (1 - I(|s|)) d_0(s), \tag{3.2}$$

so that $d \in C^\infty(\mathbf{R}^m)$, $d \geqslant 0$, $d = 0$ on $\overline{\mathcal{S}_\delta}$, $d > 0$ on $\mathbf{R}^m \setminus \mathcal{S}_{3\delta}$ and $d(s) = |s|$ if $|s| \geqslant 2R$.

By (g2) there is $C > 0$ such that

$$0 \leqslant G_2(s) \leqslant C\big(d(s) - 16\delta\big)_+^q \leqslant C\big(d(s) - 9\delta\big)_+^q \qquad \text{for all } s \in \mathbf{R}^m. \tag{3.3}$$

Let $D = \inf\{G_1(x) \mid x \in \overline{\mathcal{S}_{8\delta}} \setminus \mathcal{S}_\delta\}$. By (g1) and the choice of δ we have $D > 0$. Choose $h \in C_c^\infty(\mathbf{R})$ such that $0 \leqslant h \leqslant D$, $h = 0$ on $\mathbf{R} \setminus [2\delta, 5\delta]$ and $h = D$ on $[3\delta, 4\delta]$. We claim that

$$G_1(s) \geqslant h(d(s)) \qquad \text{for all } s \in \mathbf{R}^m. \tag{3.4}$$

It is obvious if $d(s) \leqslant 2\delta$ or $d(s) \geqslant 5\delta$. If $2\delta < d(s) < 5\delta$ we have $\delta < \operatorname{dist}(s, \overline{\mathcal{S}_{2\delta}}) < 6\delta$. Indeed, if $\operatorname{dist}(s, \overline{\mathcal{S}_{2\delta}}) \leqslant \delta$ then $\operatorname{dist}(t, \overline{\mathcal{S}_{2\delta}}) \leqslant 2\delta$ for any $t \in B(s, \delta)$ and $d(s) = \int_{B(s,\delta)} \operatorname{dist}(t, \overline{\mathcal{S}_{2\delta}}) \rho(s - t) \, dt \leqslant 2\delta$, a contradiction; a similar argument shows that we cannot have $\operatorname{dist}(s, \overline{\mathcal{S}_{2\delta}}) \geqslant 6\delta$. The estimate $\delta < \operatorname{dist}(s, \overline{\mathcal{S}_{2\delta}}) < 6\delta$ implies $\delta < \operatorname{dist}(s, \mathcal{S}) < 8\delta$, hence $G_1(s) \geqslant D \geqslant h(d(s))$.

We have

$$\{s \in \mathbf{R}^m \mid d(s) \geqslant 13\delta\} \subset \{s \in \mathbf{R}^m \mid \operatorname{dist}(s, \overline{\mathcal{S}_{2\delta}}) \geqslant 12\delta\} \subset \{s \in \mathbf{R}^m \mid \operatorname{dist}(s, \mathcal{S}) \geqslant 12\delta\}$$

$$\subset \{s \in \mathbf{R}^m \mid \operatorname{dist}(s, \overline{\mathcal{S}_{2\delta}}) \geqslant 10\delta\} \subset \{s \in \mathbf{R}^m \mid d(s) \geqslant 9\delta\} \subset \{s \in \mathbf{R}^m \mid \operatorname{dist}(s, \mathcal{S}) \geqslant 8\delta\}.$$

Since $G_2(s) = 0$ if $\operatorname{dist}(s, \mathcal{S}) < 16\delta$, for any function $u \in \mathscr{X}$ with $\int_{\mathbf{R}^N} G_2(u(x))\, dx > 0$ there holds

$$0 < \mu\left(\{x \in \mathbf{R}^N \mid d(u(x)) \geqslant 9\delta\}\right) < \infty. \tag{3.5}$$

i) Denote $\mathscr{X}_\lambda = \{u \in \mathscr{X} \mid \int_{\mathbf{R}^N} G_1(u(x))\, dx > 0 \text{ and } \mathcal{K}(u) = \lambda\}$. If $\mathscr{X}_\lambda = \emptyset$ then $\mathscr{A}_{\min}(\lambda) = \infty$ and (i) is obvious. Otherwise consider an arbitrary $u \in \mathscr{X}_\lambda$ and denote $v = d \circ u$, where d is as in (3.2). Proceeding as in the proof of Proposition IX.5 p. 155 in [3] it is easy to see that $v \in W^{1,N}(\mathbf{R}^N)$ and $\dfrac{\partial v}{\partial x_j} = \displaystyle\sum_{k=1}^m \frac{\partial d}{\partial s_k}(u) \cdot \frac{\partial u_k}{\partial x_j}$ a.e. in $\mathbf{R}^N$. In particular, there is $C > 0$ (depending only on d) such that $|\nabla v| \leqslant C|\nabla u|$ a.e. in $\mathbf{R}^N$.

We have $\int_{\mathbf{R}^N} G_2(u(x))\, dx = \lambda \int_{\mathbf{R}^N} G_1(u(x))\, dx > 0$, hence u satisfies (3.5). Using Lemma 3.2 with $v = d \circ u$, $p = N$, $a = 9\delta$, the function h introduced above and $H(t) = \int_0^t h(\tau)^{\frac{N-1}{N}}\, d\tau$ we get $H(9\delta) > 0$ and

(3.6)

$$0 < \mu\left(\{x \in \mathbf{R}^N \mid d(u(x)) \geqslant 9\delta\}\right) \leqslant C \left(\int_{\{v<9\delta\}} |\nabla v|^N\, dx\right)^{\frac{1}{N-1}} \int_{\{v<9\delta\}} h(v(x))\, dx$$

$$\leqslant C \left(\int_{\{v<9\delta\}} |\nabla v|^N\, dx\right)^{\frac{1}{N-1}} \int_{\mathbf{R}^N} G_1(u(x))\, dx \qquad \text{(by (3.4))}$$

$$= \frac{C}{\lambda} \left(\int_{\{v<9\delta\}} |\nabla v|^N\, dx\right)^{\frac{1}{N-1}} \int_{\mathbf{R}^N} G_2(u(x))\, dx$$

$$\leqslant \frac{C}{\lambda} \left(\int_{\{v<9\delta\}} |\nabla v|^N\, dx\right)^{\frac{1}{N-1}} \int_{\mathbf{R}^N} (v - 9\delta)_+^q\, dx \qquad \text{(by (3.3))}$$

$$\leqslant \frac{C}{\lambda} \left(\int_{\{v<9\delta\}} |\nabla v|^N\, dx\right)^{\frac{1}{N-1}} \|\nabla\left((v-9\delta)_+\right)\|_{L^N(\mathbf{R}^N)}^q \mu\left(\{x \in \mathbf{R}^N \mid d(u(x)) \geqslant 9\delta\}\right).$$

Here the last inequality is obtained due to Lemma 3.1. From (3.6) it follows that

$$C \left(\int_{\{v<9\delta\}} |\nabla v|^N\, dx\right)^{\frac{1}{N-1}} \left(\int_{\{v>9\delta\}} |\nabla v|^N\, dx\right)^{\frac{q}{N}} \geqslant \lambda. \tag{3.7}$$

In particular, (3.7) implies $\|\nabla v\|_{L^N(\mathbf{R}^N)}^{\frac{N}{N-1}+q} \geqslant C\lambda$. Therefore

$$\mathcal{A}(u) = \int_{\mathbf{R}^N} a(u, \nabla u)\, dx \geqslant C \int_{\mathbf{R}^N} |\nabla u|^N\, dx \geqslant C \int_{\mathbf{R}^N} |\nabla v|^N\, dx \geqslant C\lambda^{\left(\frac{1}{N-1} + \frac{q}{N}\right)^{-1}},$$

where the constants do not depend on λ and u. Since the above inequality is true for any $u \in \mathscr{X}_\lambda$ we get (i).

ii) Choose $\alpha \in (0, 1 - \frac{1}{N})$. For $R > 0$ we define $w_R : \mathbf{R}^N \longrightarrow \mathbf{R}_+$ by

$$w_R(x) = \begin{cases} 1 & \text{if } |x| \leqslant e^{-(R+1)^{\frac{1}{\alpha}}}, \\ |\ln|x||^{\alpha} - R & \text{if } e^{-(R+1)^{\frac{1}{\alpha}}} < |x| < e^{-(R)^{\frac{1}{\alpha}}}, \\ 0 & \text{if } |x| \geqslant e^{-(R)^{\frac{1}{\alpha}}}. \end{cases}$$

It is easy to see that $w_R \in W^{1,N}(\mathbf{R}^N)$ and

$$|\nabla w_R(x)| = \begin{cases} 0 & \text{if } |x| \leqslant e^{-(R+1)^{\frac{1}{\alpha}}} \text{ or } |x| \geqslant e^{-(R)^{\frac{1}{\alpha}}}, \\ \frac{\alpha|\ln|x||^{\alpha-1}}{|x|} & \text{if } e^{-(R+1)^{\frac{1}{\alpha}}} < |x| < e^{-(R)^{\frac{1}{\alpha}}}, \end{cases}$$

hence

$$\|\nabla w_R\|^N_{L^N(\mathbf{R}^N)} = \alpha^N |S^{N-1}| \int_{e^{-(R+1)^{\frac{1}{\alpha}}}}^{e^{-(R)^{\frac{1}{\alpha}}}} \frac{|\ln r|^{N(\alpha-1)}}{r}\, dr \longrightarrow 0 \qquad \text{as } R \longrightarrow \infty.$$

Choose $m_0 \in \mathcal{S}$ and $s_0 \in R^m$ such that $G_2(s_0) > 0$ and define $u_R(x) = m_0 + w_R(x)(s_0 - m_0)$. It is easy to see that $u_R \in \mathscr{X}$, $\mathcal{A}(u_R) \leqslant C\|\nabla w_R\|^N_{L^N(\mathbf{R}^N)} \longrightarrow 0$ as $R \longrightarrow \infty$ and the mappings $R \longmapsto \int_{\mathbf{R}^N} G_i(u_R(x))\, dx$ are continuous, $i = 1, 2$. We have $\int_{\mathbf{R}^N} G_2(u_R(x))\, dx \geqslant |S^{N-1}| e^{-N(R+1)^{\frac{1}{\alpha}}} G_2(s_0) > 0$ for all $R > 0$. The following observation is very useful.

Remark 3.3. Whenever $u \in \mathscr{X}$ satisfies $\int_{\mathbf{R}^N} G_2(u(x))\, dx > 0$ it also satisfies (3.5) and the third inequality in (3.6) implies $\int_{\mathbf{R}^N} G_1(u(x))\, dx > 0$.

We infer that the mapping $\Lambda(R) = \mathcal{K}(u_R)$ is well-defined, positive and continuous on $(0, \infty)$. We claim that $\Lambda(R) \longrightarrow 0$ as $R \longrightarrow \infty$. For otherwise, there would be $\lambda_0 > 0$ and a sequence $R_n \longrightarrow \infty$ such that $\mathcal{K}(u_{R_n}) \geqslant \lambda_0$ for all n, and part (i) would imply that $\mathcal{A}(u_{R_n})$ is bounded from below by a positive constant, a contradiction. Let $(\varepsilon_n)_{n\geqslant 1} \subset (0, \Lambda(1)]$ be a sequence such that $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. There exists a sequence $R_n \longrightarrow \infty$ such that $\Lambda(R_n) = \varepsilon_n$ (take $R_n = \max \Lambda^{-1}(\{\varepsilon_n\})$). We have $\mathcal{K}(u_{R_n}) = \varepsilon_n$, hence $\mathscr{A}_{\min}(\varepsilon_n) \leqslant \mathcal{A}(u_{R_n}) \longrightarrow 0$ as $n \longrightarrow \infty$. We conclude that $\mathcal{A}_{\min}(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow 0$.

iii) Given a function f defined on $\mathbf{R}^N$ and $t \in \mathbf{R}$ we define

$$S_t^- f(x) = \begin{cases} f(x) & \text{if } x_1 \leqslant t, \\ f(2t - x_1, x_2, \ldots, x_N) & \text{if } x_1 > t, \end{cases}$$

$$S_t^+ f(x) = \begin{cases} f(x) & \text{if } x_1 \geqslant t, \\ f(2t - x_1, x_2, \ldots, x_N) & \text{if } x_1 < t. \end{cases}$$

If $u \in \mathscr{X}$ it is easy to see that $S_t^+ u, S_t^- u \in \mathscr{X}$ and

$$\int_{\mathbf{R}^N} G_i(S_t^- u)\, dx = 2\int_{\{x_1 \leqslant t\}} G_i(u)\, dx, \quad \int_{\mathbf{R}^N} G_i(S_t^+ u)\, dx = 2\int_{\{x_1 > t\}} G_i(u)\, dx, i = 1, 2,$$

$$\mathcal{A}(S_t^- u) = 2\int_{\{x_1 \leqslant t\}} a(u, \nabla u)\, dx, \quad \mathcal{A}(S_t^+ u) = 2\int_{\{x_1 > t\}} a(u, \nabla u)\, dx$$

(for the last equalities we use (a4)).

Let $u \in \mathscr{X}$ be as in Proposition 1.4 (iii). We claim that there is $\tilde{u} \in \mathscr{X}$ such that $\int_{\mathbf{R}^N} G_1(\tilde{u})\, dx > 0$, $\mathcal{K}(\tilde{u}) \geqslant \mathcal{K}(u)$, $\mathcal{A}(\tilde{u}) = \mathcal{A}(u)$ and $\tilde{u}$ is symmetric with respect to x_1. Indeed, after a translation we may assume that $\mathcal{A}(S_0^- u) = \mathcal{A}(S_0^+ u) = \mathcal{A}(u)$. We have $\int_{\mathbf{R}^N} G_i(S_0^-(u)\,dx + \int_{\mathbf{R}^N} G_i(S_0^+(u)\,dx = 2\int_{\mathbf{R}^N} G_i(u)\,dx$ for $i = 1, 2$. We infer that there is $\tilde{u} \in \{S_0^- u, S_0^+ u\}$ satisfying $\int_{\mathbf{R}^N} G_2(\tilde{u})\,dx \geqslant \mathcal{K}(u)\int_{\mathbf{R}^N} G_1(\tilde{u})\,dx > 0$ (here we use Remark 3.3). From now we replace u by $\tilde{u}$ (and drop the $\tilde{}$).

Since $G_i(u) \in L^1(\mathbf{R}^N)$, the mappings $t \longmapsto \int_{\{x_1 \leqslant t\}} G_i(u)\,dx = \frac{1}{2}\int_{\mathbf{R}^N} G_i(S_t^- u)\,dx$ are continuous, nonnegative and nondecreasing on $\mathbf{R}$. Let

$$T_1 = \inf\left\{ t \in \mathbf{R} \;\Big|\; \int_{\{x_1 \leqslant t\}} G_1(u)\,dx > 0 \right\}.$$

It is clear that $T_1 < 0$ and the mapping $t \longmapsto \mathcal{K}(S_t^- u)$ is well-defined and continuous on (T_1, ∞). We claim that

$$\lim_{t \downarrow T_1} \mathcal{K}(S_t^- u) = 0. \tag{3.8}$$

To see this we argue by contradiction and we assume that (3.8) is false. Then there exist a decreasing sequence $(t_n)_{n \geqslant 1} \subset (T_1, 0)$, $t_n \longrightarrow T_1$ and $\lambda_0 > 0$ such that $\mathcal{K}(S_{t_n}^- u) \geqslant \lambda_0$ for all n. Let $v = d \circ u$ and $v_n = d \circ (S_{t_n}^- u) = S_{t_n}^- v$. For all $n \geqslant 1$ we have $\int_{\mathbf{R}^N} |\nabla v_n|^N\,dx \leqslant \int_{\mathbf{R}^N} |\nabla v|^N\,dx$. Each v_n satisfies (3.7) with $\lambda_n = \mathcal{K}(S_{t_n}^- u) \geqslant \lambda_0$ instead of λ. Since $\|\nabla v_n\|_{L^N(\mathbf{R}^N)}$ is bounded, we infer that there is a constant $k_0 > 0$ such that

$$k_0 \leqslant \int_{\{v_n > 9\delta\}} |\nabla v_n|^N\,dx = 2\int_{\{v > 9\delta\} \cap \{x_1 \leqslant t_n\}} |\nabla v|^N\,dx.$$

We have $|\nabla v|^N \in L^1(\mathbf{R}^N)$ and the absolute continuity of the Lebesgue integral implies that there is $\mu_0 > 0$ such that

$$\mu\left(\{v > 9\delta\} \cap \{x_1 \leqslant t_n\}\right) \geqslant \mu_0 \qquad \text{for all } n.$$

We use the first three inequalities in (3.6) for $S_{t_n}^- u$ to get

$$\mu_0 \leqslant \mu\left(\{x \in \mathbf{R}^N \mid v_n(x) \geqslant 9\delta\}\right) \leqslant C\left(\int_{\{v < 9\delta\}} |\nabla v|^N\,dx\right)^{\frac{1}{N-1}} \int_{\mathbf{R}^N} G_1(S_{t_n}^- u(x))\,dx$$

for all n. In particular, this implies $T_1 > -\infty$; then passing to the limit as $n \longrightarrow \infty$ in the above inequality we find $\int_{\mathbf{R}^N} G_1(S_{T_1}^- u)\,dx = 2\int_{\{x_1 \leqslant T_1\}} G_1(u(x))\,dx > 0$. Then we infer that $\int_{\{x_1 \leqslant t\}} G_1(u(x))\,dx > 0$ for some $t < T_1$, t close to T_1, and this contradicts the definition of T_1. The claim (3.8) is thus proven.

Fix $\tilde{\lambda} \in (0, \mathcal{K}(u))$. The continuity of $t \longmapsto \mathcal{K}(S_t^- u)$ implies that there exists $t_* \in (T_1, 0)$ such that $\mathcal{K}(S_{t_*}^- u) = \tilde{\lambda}$. We have

$$\mathcal{A}(S_{t_*}^- u) = 2\int_{\{x_1 \leqslant t_*\}} a(u, \nabla u)\,dx \leqslant 2\int_{\{x_1 \leqslant 0\}} a(u, \nabla u)\,dx = \mathcal{A}(u). \tag{3.9}$$

If $\mathcal{A}(S_{t_*}^- u) < \mathcal{A}(u)$ we get $\mathscr{A}_{\min}(\tilde{\lambda}) \leqslant \mathcal{A}(S_{t_*}^- u) < \mathcal{A}(u)$, as desired. Otherwise we must have equality throughout in (3.9) and assumption (a3) implies $|\nabla u| = 0$ a.e. in the strip $\{(x_1, \dots, x_N) \in \mathbf{R}^N \mid t_* < x_1 < 0\}$, hence u must be constant on that strip, say $u(x) = s_1 \in \mathbf{R}^m$. Since $\mu\left(\{x \in \mathbf{R}^N \mid \operatorname{dist}(u(x), \mathcal{S}) > \alpha\}\right) < \infty$ for all $\alpha > 0$, we infer that necessarily $s_1 \in \mathcal{S}$. Then we have

$$\int_{\{x_1 \leqslant t_*\}} G_i(u)\,dx = \int_{\{x_1 \leqslant 0\}} G_i(u)\,dx$$

for $i = 1, 2$ and this implies $\mathcal{K}(S_{\bar{t}}^- u) = \mathcal{K}(u)$, a contradiction. Thus we have always $\mathcal{A}(S_*^- u) < \mathcal{A}(u)$ and the proof of Proposition 1.4 is complete. $\square$

Lemma 3.4. *Let $M, \lambda > 0$, let the function d be as in (3.3) and q as in (g2).*

i) There exist $R_0, \eta, D_1 > 0$ such that for any function $u \in \mathscr{X}$ satisfying $\|\nabla u\|_{L^N(\mathbf{R}^N)} \leqslant M$, $\mathcal{K}(u) \geqslant \lambda$ and $\mu\big(\{x \in \mathbf{R}^N \mid d(u(x)) \geqslant 9\delta\}\big) = 1$ there is $x_0 \in \mathbf{R}^N$ such that

$$\mu\big(\{x \in B(x_0, 1) \mid d(u(x)) > 12\delta\}\big) \geqslant \eta \qquad \text{and} \qquad \int_{B(x_0,R_0)} |\nabla u|^N \, dx \geqslant D_1.$$

ii) There exists $D_2 = D_2(M, \lambda) > 0$ such that for any function $u \in \mathscr{X}$ satisfying $\|\nabla u\|_{L^N(\mathbf{R}^N)} \leqslant M$, $\int_{\mathbf{R}^N} G_1(u(x)\, dx > 0$ and $\mathcal{K}(u) \geqslant \lambda$ there holds

$$\sup_{x \in \mathbf{R}^N} \int_{B(x,1)} |\nabla u|^N + \big(d(u) - 9\delta\big)_+^q dx \geqslant D.$$

Proof. i) Let u be as in (i) and let $v = d \circ u$. By the first inequalities in (3.6) we have $\int_{\mathbf{R}^N} G_1(u)\, dx \geqslant C$ and we infer that $\int_{\mathbf{R}^N} G_2(u)\, dx \geqslant \lambda C$. On the other hand, using (3.2) and Lemma 3.1 we get

$$\begin{aligned} \int_{\mathbf{R}^N} G_2(u)\, dx &\leqslant C \int_{\mathbf{R}^N} \big(v(x) - 13\delta\big)_+^q \, dx \\ &\leqslant C \|\nabla v\|_{L^N(R^N)}^q \mu\big(\{x \in \mathbf{R}^N \mid d(u(x)) > 13\delta\}\big) \\ &\leqslant CM^q \mu\big(\{x \in \mathbf{R}^N \mid d(u(x)) > 13\delta\}\big). \end{aligned} \tag{3.10}$$

and we infer that $\mu\big(\{x \in \mathbf{R}^N \mid d(u(x)) > 13\delta\}\big) \geqslant C\lambda$. It follows from Lemma 2.1 that there are $\eta > 0$ (with η independent of u) and $x_0 \in \mathbf{R}^N$ such that

$$\mu\big(\{x \in B(x_0, 1) \mid d(u(x)) > 12\delta\}\big) \geqslant \eta. \tag{3.11}$$

Using Lemma 3.1 we get

$$\int_{\mathbf{R}^N} \big(v(x) - 9\delta\big)_+ \, dx \leqslant C \|\nabla v\|_{L^N(\mathbf{R}^N)} \mu\big(\{x \in \mathbf{R}^N \mid d(u(x)) > 9\delta\}\big) \leqslant CM.$$

We denote by $m(f, A)$ the mean value of an integrable function f on a set A, that is, $m(f; A) = \frac{1}{\mu(A)} \int_A f(x)\, dx$. For any $x \in \mathbf{R}^N$ and $R > 0$ we have

$$m\left(\big(v(x) - 9\delta\big)_+; B(x, R)\right) \leqslant \frac{1}{\mu(B(x,R))} \int_{\mathbf{R}^N} \big(v(x) - 9\delta\big)_+ \, dx \leqslant \frac{C}{R^N}.$$

Hence there is $R_0 \geqslant 1$ (depending only on $N, G_1, G_2, \delta, M, \lambda$) such that for any u as in Lemma 3.4 (i) and for any $x \in \mathbf{R}^N$ there holds $m\left(\big(v(x) - 9\delta\big)_+; B(x, R_0)\right) \leqslant \delta$, where $v = d \circ u$. Then (3.11) implies

$$\int_{B(x_0,R_0)} \left| \big(v(x) - 9\delta\big)_+ - m\left(\big(v(x) - 9\delta\big)_+; B(x_0, R_0)\right) \right| dx \geqslant 2\eta\delta.$$

Using the above inequality and the Poincaré inequality we get

$$2\eta\delta \leqslant \int_{B(x_0,R_0)} \left| \left(v(x)-9\delta\right)_+ - m\left(\left(v(x)-9\delta\right)_+; B(x_0,R_0)\right)\right| dx$$

$$\leqslant C_P \left\| \nabla\left(\left(v(x)-9\delta\right)_+\right)\right\|_{L^N(B(x_0,R_0))} \leqslant C\|\nabla u\|_{L^N(B(x_0,R_0))}$$

and (i) follows.

ii) Let $u \in \mathscr{X}$ be as in (ii). Let $\mu_0 = \mu\big(\{x \in \mathbf{R}^N \mid d(u(x)) \geqslant 9\delta\}\big)$. By (3.5) we have $0 < \mu_0 < \infty$. Denote $u_\sigma(x) = u\left(\frac{x}{\sigma}\right)$. For $\sigma_0 = \mu_0^{-\frac{1}{N}}$, the function u_{σ_0} satisfies the assumptions of Lemma 3.4 (i). By (i) there exists $x_0 \in \mathbf{R}^N$ such that

$$D_1 \leqslant \int_{B(x_0,R_0)} |\nabla(u_{\sigma_0})|^N\,dx = \int_{B(\mu_0^{\frac{1}{N}}x_0,\ \mu_0^{\frac{1}{N}}R_0)} |\nabla u|^{\frac{1}{N}}\,dx. \tag{3.12}$$

If $\mu_0^{\frac{1}{N}}R_0 \leqslant 1$, (3.12) implies the desired conclusion. If $\mu_0^{\frac{1}{N}}R_0 > 1$ we have $\mu_0 > R_0^{-N}$, where $R_0 > 0$ is independent of u. Proceeding as in the proof of (3.10) and (3.11) above we see that there are $x_1 \in \mathbf{R}^N$ and $\tilde{\eta} > 0$, $\tilde{\eta}$ independent of u, such that $\mu\big(\{x \in B(x_1,1) \mid d(u(x)) > 12\delta\}\big) \geqslant \tilde{\eta}$, and this implies $\int_{B(x_1,1)} \left(d(u(x)) - 9\delta\right)_+^q dx \geqslant 3\delta\tilde{\eta}$. Hence Lemma 3.4 (ii) holds true with $D_2 = \min(D_1, 3\delta\tilde{\eta})$. □

Proof of Theorem 1.2. Eliminating a finite number of terms, we may assume that $\mathcal{K}(u_n) > \frac{\lambda}{2}$ for all n. Then for each n we choose $\sigma_n > 0$ such that

$$\int_{\mathbf{R}^N} G_1((u_n)_{\sigma_n})\,dx = 1 \qquad \text{and} \qquad \int_{\mathbf{R}^N} G_2((u_n)_{\sigma_n})\,dx = \mathcal{K}(u_n) = \lambda_n. \tag{3.13}$$

To simplify notation, from now on we replace $(u_n)_{\sigma_n}$ by u_n, where σ_n is as above. For convenience we split the proof into several steps.

Step 1. Selection of an appropriate subsequence.

Since $\mathcal{A}(u_n)$ is bounded, (a3) implies that ∇u_n is bounded in $L^N(\mathbf{R}^N)$ Let $v_n = d \circ u_n$, where d is as in (3.2). The first inequalities in (3.6) (with u_n instead of u) imply that there is a positive constant K such that $\mu\big(\{x \in \mathbf{R}^N \mid v_n(x) \geqslant 9\delta\}\big) \leqslant K$ for all n. Then Lemma 3.1 implies that $\left(d(u_n) - 9\delta\right)_+$ is bounded in all spaces $L^r(\mathbf{R}^N)$, $1 \leqslant r < \infty$. Using (3.10) with u_n instead of u we see that there is $\kappa > 0$ such that

$$\kappa \leqslant \mu\big(\{x \in \mathbf{R}^N \mid v_n(x) \geqslant 13\delta\}\big) \leqslant \mu\big(\{x \in \mathbf{R}^N \mid v_n(x) \geqslant 9\delta\}\big) \leqslant K \quad \text{for all } n. \tag{3.14}$$

Let

$$\rho_n = |\nabla u_n|^N + G_1(u_n) + \left(d(u_n) - 9\delta\right)_+^q. \tag{3.15}$$

It is clear that $(\rho_n)_{n\geqslant 1}$ is bounded in $L^1(\mathbf{R}^N)$ and $\int_{\mathbf{R}^N} \rho_n\,dx \geqslant \int_{\mathbf{R}^N} G_1(u_n)\,dx = 1$. Passing to a subsequence if necessary we may assume that

$$\int_{\mathbf{R}^N} \rho_n\,dx \longrightarrow \alpha \qquad \text{as } n \longrightarrow \infty, \quad \text{where } \alpha \in [1,\infty). \tag{3.16}$$

We will use a refined version of the concentration-compactness principle (Theorem in [22]) for the sequence $(\rho_n)_{n\geqslant 1}$

Let $M = \sup_{n\geqslant 1} \|\nabla u_n\|_{L^N(\mathbf{R}^N)}$ and let $D = \min\left(\frac{1}{2}D_1\left(M+1, \frac{\lambda}{4}\right), \frac{1}{2}D_2(M, \frac{\lambda}{4})\right)$. By Theorem in [22] there exist an increasing mapping $j : \mathbf{N}^* \longrightarrow \mathbf{N}^*$, an integer $k \geqslant 0$, k sequences of points $(x_n^i)_{n\geqslant 1} \subset \mathbf{R}^N$ and k increasing sequences of positive numbers $(R_n^i)_{n\geqslant 1}$, $i = 1, \dots, k$, such that $R_n^i \longrightarrow \infty$ as $n \longrightarrow \infty$ and positive numbers $\alpha_1, \dots, \alpha_k$ with the following properties:

(P1) For each n the balls $B(x_n^i, (R_n^i)^3)$, $i \in \{1, \dots, k\}$ are disjoint.

(P2) For each $i \in \{1, \dots, k\}$ we have

$$\int_{B(x_n^i, R_n^i)} \rho_{j(n)}(x)\,dx \longrightarrow \alpha_i \quad \text{and} \quad \int_{B(x_n^i,(R_n^i)^3)\setminus B(x_n^i,R_n^i)} \rho_{j(n)}(x)\,dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

(P3) For each $i \in \{1, \dots, k\}$ the sequence $\rho_{j(n)}\mathbf{1}_{B(x_n^i,(R_n^i)^3)}$ "concentrates around $(x_n^i)_{n\geqslant 1}$," which means that for any $\varepsilon > 0$ there exist $R_\varepsilon^i \in (0,\infty)$ and $n(\varepsilon, i) \in \mathbf{N}$ such that

$$\int_{B(x_n^i,(R_n^i)^3)\setminus B(x_n^i,R_\varepsilon^i)} \rho_{j(n)}(x)\,dx < \varepsilon \qquad \text{for all } n \geqslant n(\varepsilon, i).$$

(P4) The sequence $\tilde{\rho}_n = \rho_{j(n)}\mathbf{1}_{\mathbf{R}^N\setminus\cup_{i=1}^k B(x_n^i,(R_n^i)^3)}$ satisfies

$$\lim_{R\to\infty}\left(\limsup_{n\to\infty}\left(\sup_{x\in\mathbf{R}^N}\int_{B(x,R)} \tilde{\rho}_n(y)\,dy\right)\right) < D. \tag{3.17}$$

If $k = 0$ we have $\tilde{\rho}_n = \rho_{j(n)}$ and from property (P4) above we get

$$\sup_{x\in\mathbf{R}^N}\int_{B(x,1)} \tilde{\rho}_n(y)\,dy < 2D \leqslant D_2(M, \frac{\lambda}{4}) \text{ for all sufficiently large } n.$$

Then Lemma 3.4 (ii) implies $\mathcal{K}(u_{j(n)}) \leqslant \frac{\lambda}{4}$ for n large, contradicting the fact that $\mathcal{K}(u_n) > \frac{\lambda}{2}$. Thus necessarily $k \geqslant 1$.

The sequences $\left(\left(d(u_n) - 9\delta\right)_+\right)_{n\geqslant 1}$ and $(\nabla u_n)_{n\geqslant 1}$ are bounded in $L^N(\mathbf{R}^N)$, hence

$$\sup_{n\in\mathbf{N}^*}\sup_{x\in\mathbf{R}^N} \|u_n\|_{W^{1,N}(B(x,1))} < \infty.$$

Using the diagonal extraction procedure we see that there is an increasing mapping $\ell : \mathbf{N}^* \longrightarrow \mathbf{N}^*$ and there are functions $u_1, \dots u_k \in W^{1,N}_{loc}(\mathbf{R}^N, \mathbf{R}^m)$ such that for any $i \in \{1, \dots, k\}$ we have

$$u_{j(\ell(n))}(\cdot + x_{\ell(n)}^i) \rightharpoonup u^i \qquad \text{weakly in } W^{1,N}(B(0,R)) \quad \text{for any } R > 0, \tag{3.18}$$

$$u_{j(\ell(n))}(\cdot + x_{\ell(n)}^i) \longrightarrow u^i \quad \text{a.e. and in } L^p(B(0,R)) \text{ for any } R > 0 \text{ and any } p \in [1,\infty). \tag{3.19}$$

Step 2. Properties of the limit functions $u^1, \dots, u^k$.

It follows from Lemma 2.2 that for any fixed $R > 0$,

$$\begin{aligned}\int_{B(0,R)} a(u^i, \nabla u^i)\, dx &\leqslant \liminf_{n\to\infty} \int_{B(0,R)} a(u_{j(\ell(n))}(x + x^i_{\ell(n)}), \nabla u_{j(\ell(n))}(x + x^i_{\ell(n)}))\, dx \\ &\leqslant \liminf_{n\to\infty} \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} a(u_{j(\ell(n))}, \nabla u_{j(\ell(n))}\, dx\end{aligned}$$

Letting $R \longrightarrow \infty$ and using the monotone convergence theorem we find

$$\int_{\mathbf{R}^N} a(u^i, \nabla u^i)\, dx \leqslant \liminf_{n\to\infty} \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} a(u_{j(\ell(n))}, \nabla u_{j(\ell(n))})\, dx. \tag{3.20}$$

Using the Fatou lemma and proceeding as above we discover

$$\int_{\mathbf{R}^N} G_1(u^i)\, dx \leqslant \liminf_{n\to\infty} \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} G_1(u_{j(\ell(n))})\, dx \tag{3.21}$$

and for all $R \geqslant 0$,

$$\int_{\mathbf{R}^N \setminus B(0,R)} G_2(u^i)\, dx \leqslant \liminf_{n\to\infty} \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)}) \setminus B(x^i_{\ell(n)}, R)} G_2(u_{j(\ell(n))})\, dx. \tag{3.22}$$

We claim that for any $i \in \{1, \ldots, k\}$,

$$\int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} G_2(u_{j(\ell(n))})\, dx \longrightarrow \int_{\mathbf{R}^N} G_2(u^i)\, dx \qquad \text{as } n \longrightarrow \infty. \tag{3.23}$$

To see this fix $\varepsilon > 0$. Using property (P3) in step 1 and the fact that $0 \leqslant G_2(u_{j(n)}(x)) \leqslant C\big(d(u_{j(n)}(x)) - 9\delta\big)_+^q \leqslant C\rho_{j(n)}(x)$ we may find $R^i_\varepsilon > 0$ and $n(\varepsilon, i) \in \mathbf{N}^*$ such that

$$\int_{B(x^i_{\ell(n)}, R^i_{\ell(n)}) \setminus B(x^i_{\ell(n)}, R^i_\varepsilon)} G_2(u_{j(\ell(n))})\, dx < \varepsilon \qquad \text{for all } n \geqslant n(\varepsilon, i). \tag{3.24}$$

Then (3.22) and (3.24) imply $\int_{\mathbf{R}^N \setminus B(0,R^i_\varepsilon)} G_2(u^i)\, dx \leqslant \varepsilon$.

Since G_2 is continuous, $G_2(s) \leqslant C(1 + |s|^q)$ and (3.19) holds, we have

$$\int_{B(x^i_{\ell(n)}, R^i_\varepsilon)} G_2(u_{j(\ell(n))})\, dx = \int_{B(0,R^i_\varepsilon)} G_2(u_{j(\ell(n))}(x + x^i_{\ell(n)}))\, dx \longrightarrow \int_{B(0,R^i_\varepsilon)} G_2(u^i)\, dx$$

as $n \longrightarrow \infty$ (see, for instance, Lemma 16.1 p. 60 in [17] or Theorem A2 p. 133 in [28]). Hence there is $n'(\varepsilon, i) \geqslant n(\varepsilon, i)$ such that for all $n \geqslant n'(\varepsilon, i)$,

$$\left| \int_{B(x^i_{\ell(n)}, R^i_\varepsilon)} G_2(u_{j(\ell(n))})\, dx - \int_{B(0,R^i_\varepsilon)} G_2(u^i)\, dx \right| < \varepsilon.$$

From the above estimate, (3.24) and the same for u^i we get

$$\left| \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} G_2(u_{j(\ell(n))})\, dx - \int_{\mathbf{R}^N} G_2(u^i)\, dx \right| \leqslant 3\varepsilon$$

for all $n \geqslant n'(\varepsilon, i)$ and (3.23) is proven.

Let us show that $u^i \in \mathscr{X}$. Denoting $A^i_\beta = \{x \in \mathbf{R}^N \mid \operatorname{dist}(u^i(x), \mathcal{S}) > \beta\}$, it only remains to prove that $\mu(A^i_\beta) < \infty$ for any $\beta > 0$. Using the Fatou lemma we get, as in (3.21),

$$\begin{aligned}\int_{\mathbf{R}^N} \big(d(u^i(x)) - 9\delta\big)^q_+ \, dx &\leqslant \liminf_{n\to\infty} \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} \big(d(u_{j(\ell(n))}(x)) - 9\delta\big)^q_+ \, dx \\ &\leqslant \lim_{n\to\infty} \int_{\mathbf{R}^N} \rho_n \, dx = \alpha.\end{aligned}$$

From the above we infer that $\mu\big(\{x \in \mathbf{R}^N \mid d(u^i(x)) \geqslant 10\delta\}\big) < \infty$. We have $A^i_\beta \subset \{x \in \mathbf{R}^N \mid d(u^i(x)) \geqslant 10\delta\} \cup \{x \in \mathbf{R}^N \mid \beta < \operatorname{dist}(u(x), \mathcal{S}) < 13\delta\} = A_1 \cup A_2$. We already know that $\mu(A_1) < \infty$. Let us prove that $\mu(A_2)$ is finite. Denoting $K_\beta = \inf\{G_1(s) \mid \beta \leqslant \operatorname{dist}(s, \mathcal{S}) \leqslant 13\delta\}$ we have

$$K_\beta \cdot \mu(A_2) \leqslant \int_{\mathbf{R}^N} G_1(u^i)\, dx \leqslant 1,$$

hence $\mu(A_2) < \infty$, as desired.

Step 3. Estimates on $u_{j(\ell(n))}$ outside the balls $B(x^i_{\ell(n)}, R^i_{\ell(n)})$.

We define

(3.25)
$$w_n(x) = \begin{cases} u_{j(\ell(n))}\Big(x^i_{\ell(n)} + R^i_{\ell(n)}(x - x^i_{\ell(n)})\Big) & \text{if } x \in B(x^i_{\ell(n)}, 1), \\[2ex] u_{j(\ell(n))}\Bigg(x^i_{\ell(n)} + \frac{R^i_{\ell(n)}}{|x - x^i_{\ell(n)}|^{\frac12}}(x - x^i_{\ell(n)})\Bigg) & \text{if } 1 \leqslant |x - x^i_{\ell(n)}| \leqslant (R^i_{\ell(n)})^2, \\[2ex] u_{j(\ell(n))}(x) & \text{if } x \notin \cup_{i=1}^k B(x^i_{\ell(n)}, (R^i_{\ell(n)})^2). \end{cases}$$

It is easy to see that $w_n \in \mathscr{X}$. Since $\int_{B(x^i_n, (R^i_n)^3)\setminus B(x^i_n, R^i_n)} \rho_{j(n)}(x)\, dx \longrightarrow 0$ as $n \longrightarrow \infty$ (see (P2) above), a straightforward computation gives

(3.26)
$$\int_{\mathbf{R}^N} G_\gamma(w_n(x))\, dx = \int_{\mathbf{R}^N \setminus \cup_{i=1}^k B(x^i_{\ell(n)}, R^i_{\ell(n)})} G_\gamma(u_{j(\ell(n))}(y))\, dy + o(1), \qquad \gamma = 1, 2,$$

(3.27) $\mu\big(\{x \in \mathbf{R}^N \mid d(w_n(x)) \geqslant 9\delta\}\big) \leqslant 2\mu\big(\{x \in \mathbf{R}^N \mid d(u_{j(\ell(n))}(x)) \geqslant 9\delta\}\big) \leqslant 2K,$

(3.28)
$$\mathcal{A}(w_n) = \mathcal{A}(u_{j(\ell(n))}) + o(1) \quad \text{and} \quad \int_{\mathbf{R}^N} |\nabla w_n|^N \, dx = \int_{\mathbf{R}^N} |\nabla u_{j(\ell(n))}|^N \, dx + o(1)$$

as $n \longrightarrow \infty$. In particular, for all sufficiently large n we have

(3.29)
$$\int_{\mathbf{R}^N} G_1(w_n(x))\, dx \leqslant 2, \quad \int_{\mathbf{R}^N} G_2(w_n(x))\, dx \leqslant \lambda + 1 \quad \text{and} \quad \int_{\mathbf{R}^N} |\nabla w_n|^N \, dx \leqslant M + 1.$$

We prove that

$$\int_{\mathbf{R}^N} G_2(w_n(x))\, dx \leqslant \frac{\lambda}{2} \int_{\mathbf{R}^N} G_1(w_n(x))\, dx + o(1). \tag{3.30}$$

To do this we argue by contradiction and we assume that (3.30) does not hold. Then there exists a positive constant K such that, passing to a subsequence (still denoted the same) of $(w_n)_{n\geqslant 1}$ we may assume that for all n, (3.30) is satisfied and we have

$$\int_{\mathbf{R}^N} G_2(w_n(x))\,dx \geqslant K \quad \text{and} \quad \int_{\mathbf{R}^N} G_2(w_n(x))\,dx \geqslant \frac{\lambda}{4}\int_{\mathbf{R}^N} G_1(w_n(x))\,dx \tag{3.31}$$

Using (3.10) for w_n we infer that there is $\kappa_1 > 0$ such that for any n,

$$\kappa_1 \leqslant \mu(\{x \in \mathbf{R}^N \mid d(w_n(x)) \geqslant 13\delta\}) \leqslant \mu(\{x \in \mathbf{R}^N \mid d(w_n(x)) \geqslant 9\delta\}) \leqslant 2K. \tag{3.32}$$

Hence for any n there is $\sigma_n \in [(2K)^{-\frac{1}{N}}, \kappa_1^{-\frac{1}{N}}]$ such that $\mu(\{x \in \mathbf{R}^N \mid d((w_n)_{\sigma_n}(x)) \geqslant 9\delta\}) = 1$. Moreover, by (3.29) and (3.31) we have $\int_{\mathbf{R}^N} |\nabla((w_n)_{\sigma_n})|^N\,dx \leqslant M+1$ and $\mathcal{K}((w_n)_{\sigma_n}) \geqslant \frac{\lambda}{4}$. We may thus use Lemma 3.4 (i) for $(w_n)_{\sigma_n}$ and we infer that there exists $x_0^n \in \mathbf{R}^N$ such that

$$\mu\left(\left\{x \in B\left(x_0^n, \frac{1}{\sigma_n}\right) \;\Big|\; d(w_n(x)) \geqslant 12\delta\right\}\right) \geqslant \sigma_n^{-N}\eta\left(M+1, \frac{\lambda}{4}\right) \geqslant \kappa_1\eta\left(M+1, \frac{\lambda}{4}\right) \tag{3.33}$$

and

$$\int_{B\left(x_0^n, \frac{R_0(M+1,\frac{\lambda}{4})}{\sigma_n}\right)} |\nabla w_n|^N\,dx \geqslant D_1\left(M+1, \frac{\lambda}{4}\right). \tag{3.34}$$

For each n there are only two possibilities:

Case A: $B\left(x_0^n, (2K)^{\frac{1}{N}}(1+R_0)\right) \cap B(x^i_{\ell(n)}, 1) \neq \emptyset$ for some $i \in \{1, \ldots, k\}$.

Case B: $B\left(x_0^n, (2K)^{\frac{1}{N}}(1+R_0)\right) \cap B(x^i_{\ell(n)}, 1) = \emptyset$ for all $i \in \{1, \ldots, k\}$.

Assume that we are in case A and n is sufficiently large, so that $1 + 2(2K)^{\frac{1}{N}}(1 + R_0) < (R^i_{\ell(n)})^2$. Then we have $B(x_0^n, \frac{1}{\sigma_n}) \subset B(x_0^n, (2K)^{\frac{1}{N}}) \subset B(x^i_{\ell(n}, (R^i_{\ell(n)})^2)$. By (3.33) we get

$$\int_{B(x^i_{\ell(n)}, (R^i_{\ell(n)})^2)} (d(w_n) - 9\delta)_+^q\,dx \geqslant \int_{B(x_0^n, \frac{1}{\sigma_n})} (d(w_n) - 9\delta)_+^q\,dx \geqslant 3^q\kappa_1\eta\delta^q.$$

On the other hand, using the definition of w_n in (3.25) we get

$$\int_{B(x^i_{\ell(n)}, (R^i_{\ell(n)})^2)} (d(w_n) - 9\delta)_+^q\,dx \leqslant \frac{1}{(R^i_{\ell(n)})^N}\int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} \left(d(u_{j(\ell(n))}) - 9\delta\right)_+^q\,dx$$

$$+2\int_{B(x^i_{\ell(n)}, (R^i_{\ell(n)})^2)\setminus B(x^i_{\ell(n)}, R^i_{\ell(n)})} \left(d(u_{j(\ell(n))}) - 9\delta\right)_+^q\,dx$$

and the right hand side in the above inequality tends to 0 as $n \longrightarrow \infty$ by (P2) and the fact that $R^i_n \longrightarrow \infty$. We conclude that here is $n_1 \in \mathbf{N}^*$ such that the property in Case A cannot be satisfied for $n \geqslant n_1$.

Assume that we are in case B. Then $B\left(x_0^n, \dfrac{R_0(M+1,\frac{\lambda}{4})}{\sigma_n}\right) \subset B(x_0^n, (2K)^{\frac{1}{N}}R_0) \subset \mathbf{R}^N \setminus \cup_{i=1}^k B(x^i_{\ell(n)}, 1)$. Denoting $\widetilde{R}_0 = (2K)^{\frac{1}{N}}R_0$, using (3.34) and the definition of

w_n we get

$$D_1\left(M+1,\frac{\lambda}{4}\right) \leqslant \int_{B(x_0,\widetilde{R}_0)} |\nabla w_n|^N\,dx$$

$$\leqslant \sum_{i=1}^{k} \int_{B(x^i_{\ell(n)},(R^i_{\ell(n)})^2)\setminus B(x^i_{\ell(n)},1)} |\nabla w_n|^N\,dx + \sup_{z\in\mathbf{R}^N} \int_{B(z,\widetilde{R}_0)\setminus\cup_{i=1}^k B(x^i_{\ell(n)},(R^i_{\ell(n)})^2)} |\nabla w_n|^N\,dx$$

$$\leqslant C\sum_{i=1}^{k} \int_{B(x^i_{\ell(n)},(R^i_{\ell(n)})^2)\setminus B(x^i_{\ell(n)},R^i_{\ell(n)})} |\nabla u_{j(\ell(n))}|^N\,dx + \sup_{z\in\mathbf{R}^N} \int_{B(z,\widetilde{R}_0)} \tilde{\rho}_{\ell(n)}\,dx,$$

where $\tilde{\rho}_{\ell(n)}$ is as in (P4). By (P2) and the choice of D in (P4) (recall that $D \leqslant \frac{1}{2}D_1\left(M+1,\frac{\lambda}{4}\right)$), the last quantity above is smaller than $\frac{3}{4}D_1\left(M+1,\frac{\lambda}{4}\right)$ for all n sufficiently large. We infer that there is $n_2 \geqslant n_1$ such that Case B cannot hold either for $n \geqslant n_2$. Hence we have got a contradiction and the proof of (3.30) is complete.

Step 4. There is $i \in \{1,\dots,k\}$ such that

$$\int_{\mathbf{R}^N} G_2(u^i)\,dx \geqslant \lambda \int_{\mathbf{R}^N} G_1(u^i)\,dx. \tag{3.35}$$

We argue by contradiction and we assume that (3.35) is false. Taking into account Remark 3.3, it follows that for each $i \in \{1,\dots,k\}$ there is $\lambda_i < \lambda$ such that $\int_{\mathbf{R}^N} G_2(u^i)\,dx \leqslant \lambda_i \int_{\mathbf{R}^N} G_1(u^i)\,dx$. Then using (3.21) and (3.23) we find

$$\int_{B(x^i_{\ell(n)},R^i_{\ell(n)})} G_2(u_{j(\ell(n))}(x))\,dx \leqslant \lambda_i \int_{B(x^i_{\ell(n)},R^i_{\ell(n)})} G_1(u_{j(\ell(n))}(x))\,dx + o(1). \tag{3.36}$$

From (3.30) and (3.26) we get

$$\begin{aligned} &\int_{\mathbf{R}^N\setminus\cup_{i=1}^k B(x^i_{\ell(n)},R^i_{\ell(n)})} G_2(u_{j(\ell(n))}(x))\,dx \\ &\leqslant \frac{\lambda}{2}\int_{\mathbf{R}^N\setminus\cup_{i=1}^k B(x^i_{\ell(n)},R^i_{\ell(n)})} G_1(u_{j(\ell(n))}(y))\,dy + o(1). \end{aligned} \tag{3.37}$$

Fix $\tilde{\lambda}$ such that $\max(\frac{\lambda}{2},\lambda_1,\dots,\lambda_k) < \tilde{\lambda} < \lambda$. Adding (3.36) (for $i = 1,\dots,k$) and (3.37) we find

$$\int_{\mathbf{R}^N} G_2(u_{j(\ell(n))}(x))\,dx \leqslant \tilde{\lambda}\int_{\mathbf{R}^N} G_1(u_{j(\ell(n))}(x))\,dx + o(1).$$

Since $\int_{\mathbf{R}^N} G_1(u_n(x))\,dx = 1$ for all n (see (3.13), the above inequality implies $\mathcal{K}(u_{j(\ell(n))}) \leqslant \tilde{\lambda} + o(1)$, contradicting the fact that $\mathcal{K}(u_{j(\ell(n))}) \longrightarrow \lambda > \tilde{\lambda}$ by (1.15). The proof of (3.35) is thus complete.

Step 5. Conclusion.

We may assume that (3.35) holds for $i = 1$. Then we have $u^1 \in \mathscr{X}$ and $\mathcal{K}(u^1) \geqslant \lambda$. From Proposition 1.4 (iii) it follows that $\mathcal{A}(u^1) \geqslant \mathscr{A}_{\min}(\lambda)$. On the other hand, by (3.20) and (1.15) we get

$$\mathcal{A}(u^1) \leqslant \liminf_{n\to\infty} \int_{B(x^i_{\ell(n)}, R^i_{\ell(n)})} a(u_{j(\ell(n))}, \nabla u_{j(\ell(n))})\, dx \leqslant \lim_{n\to\infty} \mathcal{A}(u_{j(\ell(n))}) = \mathscr{A}_{\min}(\lambda). \tag{3.38}$$

Therefore $\mathcal{A}(u^1) = \mathscr{A}_{\min}(\lambda)$. If $\mathcal{K}(u^1) > \lambda$ by Proposition 1.4 (iii) we would have $\mathcal{A}(u^1) > \mathscr{A}_{\min}(\lambda)$, a contradiction. Hence $\mathcal{K}(u^1) = \lambda$. From (3.38) and the fact that $\mathcal{A}(u_{j(\ell(n))}) \longrightarrow \mathcal{A}(u^1)$ as $n \longrightarrow \infty$ (see (1.15)) we infer that

$$\int_{\mathbf{R}^N \setminus B(x^1_{\ell(n)}, R^1_{\ell(n)})} a(u_{j(\ell(n))}, \nabla u_{j(\ell(n))})\, dx \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.39}$$

Next we claim that

$$\int_{\mathbf{R}^N \setminus B(x^1_{\ell(n)}, R^1_{\ell(n)})} G_2(u_{j(\ell(n))})\, dx \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{3.40}$$

To see this we argue again by contradiction. If (3.40) is not true, there is a subsequence of $(u_{j(\ell(n))})_{n\geqslant 1}$, denoted the same, such that along this subsequence the integral in (3.40) tends to some $\beta > 0$. Define $\tilde{w}_n$ as in (3.25), except that $k = 1$. Then $\tilde{w}_n \in \mathscr{X}$ and, as in (3.26), we have

$$\int_{\mathbf{R}^N} G_2(\tilde{w}_n(x))\, dx = \int_{\mathbf{R}^N \setminus B(x^1_{\ell(n)}, R^1_{\ell(n)})} G_2(u_{j(\ell(n))}(y))\, dy + o(1) = \beta + o(1)$$

and

$$\int_{\mathbf{R}^N} G_1(\tilde{w}_n(x))\, dx = \int_{\mathbf{R}^N \setminus B(x^1_{\ell(n)}, R^1_{\ell(n)})} G_1(u_{j(\ell(n))}(y))\, dy + o(1) \leqslant 1 + o(1).$$

Using Remark 3.3 we see that for n sufficiently large $\mathcal{K}(\tilde{w}_n)$ is well-defined and $\mathcal{K}(\tilde{w}_n) \geqslant \frac{\beta}{2}$. It is also easy to see that (3.27) - (3.29) hold with $\tilde{w}_n$ instead of w_n. Using (3.10) we see that there exists $\kappa_2 > 0$ such that (3.32) holds with κ_1 and w_n replaced by κ_2 and $\tilde{w}_n$, respectively. Then we proceed exactly as in the proof of (3.30) to get a contradiction (notice that we use (3.39), which is equivalent to $\int_{\mathbf{R}^N \setminus B(x^1_{\ell(n)}, R^1_{\ell(n)})} |\nabla u_{j(\ell(n))}|^N\, dx = o(1)$, to eliminate the case B).

Using (3.40) and (3.13) we get $\int_{B(x^1_{\ell(n)}, R^1_{\ell(n)})} G_2(u_{j(\ell(n))})\, dx \longrightarrow \lambda$, and then (3.23) implies $\int_{\mathbf{R}^N} G_2(u^1(x))\, dx = \lambda$. Then we infer that $\int_{\mathbf{R}^N} G_1(u^1(x))\, dx = 1$ because $\mathcal{K}(u^1) = \lambda$. For $\gamma = 1, 2$ we know that the functions $G_\gamma(u_{j(\ell(n))})$ are nonnegative, belong to $L^1(\mathbf{R}^N)$, converge to $G_\gamma(u^1)$ a.e. on $\mathbf{R}^N$ and $\int_{\mathbf{R}^N} G_\gamma(u_{j(\ell(n))}(x))\, dx \longrightarrow \int_{\mathbf{R}^N} G_\gamma(u^1(x))\, dx$ as $n \longrightarrow \infty$; this implies that $G_\gamma(u_{j(\ell(n))}) \longrightarrow G_\gamma(u^1)$ in $L^1(\mathbf{R}^N)$, as desired. $\square$

Proof of Corollary 1.3. The precompactness of minimizing sequences follows directly from Theorem 1.2.

Let u be a minimizer of $\mathscr{A}_{\min}(1)$. Then $\int_{\mathbf{R}^N} G_+(u)\, dx = \int_{\mathbf{R}^N} G_-(u)\, dx > 0$ and $\mathcal{V}(u) = 0$. For any $w \in \mathscr{X}$ with $\int_{\mathbf{R}^N} |G(w)|\, dx > 0$ we have $\int_{\mathbf{R}^N} G_-(w)\, dx > 0$ by Remark 3.3. If $\mathcal{V}(w) \geqslant 0$ we have $\int_{\mathbf{R}^N} G_+(w)\, dx \geqslant \int_{\mathbf{R}^N} G_-(w)\, dx > 0$, hence $\mathcal{K}(w) \geqslant 1$ and $\mathcal{A}(w) \geqslant \mathscr{A}_{\min}(\mathcal{K}(w)) \geqslant \mathscr{A}_{\min}(1) = \mathcal{A}(u)$, with equality if and only if $\mathcal{K}(w) = 1$ (that is, $\mathcal{V}(w) = 0$) and $\mathcal{A}(w) = \mathscr{A}_{\min}(1)$.

Let $\phi \in C_c^1(\mathbf{R}^N, \mathbf{R}^m)$. Exactly as in the proof of Theorem 1.1 we show that $u + t\phi \in \mathscr{X}$ and

$$\mathcal{V}(u+t\phi) = \mathcal{V}(u) + t\Phi_u(\phi) + o(t), \qquad \mathcal{A}(u+t\phi) = \mathcal{A}(u) + t\Psi_u(\phi) + o(t) \tag{3.41}$$

as $t \longrightarrow 0$, where Φ_u and Ψ_u are as in (2.27) and (2.28), respectively. By dominated convergence we have $\int_{\mathbf{R}^N} G_-(u+t\phi)\,dx \longrightarrow \int_{\mathbf{R}^N} G_-(u)\,dx > 0$ as $t \longrightarrow 0$, hence $\int_{\mathbf{R}^N} G_-(u+t\phi)\,dx > 0$ for t sufficiently small. If $\Phi_u(\phi) > 0$ we have $\mathcal{V}(u+t\phi) > 0$ and $\mathcal{K}(u+t\phi) > 1$ for t small, $t > 0$ and we infer that $\mathcal{A}(u+t\phi) > \mathcal{A}(u)$ for t close to 0, $t > 0$, which implies $\Psi_u(\phi) \geqslant 0$. Similarly if $\Phi_u(\phi) < 0$ we deduce that $\Psi_u(\phi) \leqslant 0$. It is not hard to see that $\Phi_u \not\equiv 0$ and $\mathrm{Ker}(\Psi_u) \subset \mathrm{Ker}(\Phi_u)$. Hence there exists $\alpha \in \mathbf{R}$ such that $\Psi_u = \alpha\Phi_u$. Since $\Phi_u(\phi) > 0$ implies $\Psi_u(\phi) \geqslant 0$ we have necessarily $\alpha \geqslant 0$. $\square$

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II

Traveling waves for a Gross-Pitaevskii-Schrödinger system*

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Abstract. We show the existence of subsonic traveling waves of finite energy for a Gross-Pitaevskii-Schrödinger system which models the motion of a non charged impurity in a Bose-Einstein condensate. The obtained results are valid in three- and four-dimensional space.
Keywords. traveling wave, constrained minimization, nonlinear Schrödinger equation, Gross-Pitaevskii equation, Gross-Pitaevskii-Schrödinger system, Ginzburg-Landau energy.

1. Introduction

This paper focuses on the study of the system

$$
\begin{aligned}
2i\frac{\partial \Psi}{\partial t} &= -\Delta\Psi + \frac{1}{\varepsilon^4}|\Phi|^2\Psi - F(|\Psi|^2)\Psi,\\
2i\delta\frac{\partial \Phi}{\partial t} &= -\Delta\Phi + \frac{1}{\varepsilon^2}(q^2|\Psi|^2 - \varepsilon^2 k^2)\Phi
\end{aligned}
\tag{1.1}
$$

which describes the motion of an uncharged impurity in a Bose condensate. Here, Ψ and Φ are the wavefunctions for bosons, respectively for the impurity, $\delta = \frac{\mu}{M}$ is the ratio of the mass of the impurity over the boson mass ($\delta \ll 1$), $q^2 = \frac{l}{2d}$, l is the boson-impurity scattering length and d is the boson diameter, k is a dimensionless measure for the single-particle impurity energy, and ε is a dimensionless constant ($\varepsilon = \left(\frac{a\mu}{lM}\right)^{\frac{1}{5}}$, where a is the "healing length"; in applications, $\varepsilon \approx 0.2$).

Assuming that the condensate is at rest at infinity, the solutions Ψ and Φ must satisfy the "boundary conditions"

$$|\Psi(x)| \to 1, \qquad \Phi(x) \to 0 \text{ as } |x| \to \infty.$$

The first equation in (1.1) can be recast into a hydrodynamical form by using the Madelung transformation $\Psi(x,t) = \sqrt{\rho(x,t)}e^{i\theta(x,t)}$, which is meaningful where ρ

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does not vanish. By a straightforward computation, we find that ρ and θ satisfy the equations

$$\rho_t + \operatorname{div}(\rho\nabla\theta) = 0, \tag{1.2}$$

$$2\theta_t + |\nabla\theta|^2 - \frac{\Delta\rho}{2\rho} + \frac{|\nabla\rho|^2}{4\rho^2} + \frac{1}{\varepsilon^4}|\Phi|^2 - F(\rho) = 0. \tag{1.3}$$

These equations are similar to a system of Euler equations for a compressible inviscid fluid of density ρ and velocity $\nabla\theta$. If F is C^1, taking the derivative with respect to t of (1.3) and substituting ρ_t from (1.2) we obtain

$$2\theta_{tt} + F'(\rho)(\rho\Delta\theta + \nabla\rho.\nabla\theta) + \frac{\partial}{\partial t}\left(|\nabla\theta|^2 - \frac{\Delta\rho}{2\rho} + \frac{|\nabla\rho|^2}{4\rho^2} + \frac{1}{\varepsilon^4}|\Phi|^2\right) = 0. \tag{1.4}$$

For a small oscillatory motion (i.e. a sound wave), all the nonlinear terms appearing in (1.4), except $\rho\Delta\theta$, may be neglected and in a neighborhood of infinity, the velocity potential θ satisfies the wave equation $2\theta_{tt} + F'(1)\Delta\theta + \frac{2}{\varepsilon^4}|\phi|\phi_t = 0$. We find that sound waves propagate with velocity $\frac{\sqrt{-2F'(1)}}{2}$ and therefore the sound velocity at infinity associated to the first equation of (1.1) is $v_s = \frac{\sqrt{-2F'(1)}}{2}$.

Let $V(s) = \int_s^1 F(\tau)\,d\tau$. As in the case of the Gross-Pitaevskii equation, the following Hamiltonian is formally conserved by (1.1):
(1.5)

$$E(\Psi,\Phi) = \int_{\mathbf{R}^N}\left(|\nabla\Psi|^2 + \frac{1}{\varepsilon^2 q^2}|\nabla\Phi|^2 + V(|\Psi|^2) + \frac{1}{\varepsilon^4}|\Psi|^2|\Phi|^2 - \frac{k^2}{\varepsilon^2 q^2}|\Phi|^2\right) dx.$$

We are interested in traveling wave solutions for the system (1.1), i.e., solutions of the form $\Psi(x,t) = \psi(x - cty), \Phi(x,t) = \phi(x - cty)$, where y is the direction of propagation and c is the speed of the traveling wave. Without loss of generality, we may assume that $y = (1, 0, \dots, 0)$. Such solutions must satisfy the equations

$$\begin{aligned} &-2ic\frac{\partial\psi}{\partial x_1} + \Delta\psi - \frac{1}{\varepsilon^4}|\phi|^2\psi + F(|\psi|^2)\psi = 0, \\ &-2ic\delta\frac{\partial\phi}{\partial x_1} + \Delta\phi - \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\phi = 0. \end{aligned} \tag{1.6}$$

It is clear that (ψ,ϕ) satisfies (1.6) for some velocity c if and only if $(\overline{\psi}(-x_1,x'), \overline{\phi}(-x_1,x'))$ satisfies (1.6) for the velocity $-c$. Thus it suffices to consider the case $c > 0$. We say that (ψ,ϕ) has finite energy if $\nabla\psi \in L^2(\mathbf{R}^N)$, $V(|\psi|^2) \in L^1(\mathbf{R}^N)$ and $\phi \in H^1(\mathbf{R}^N)$.

In the particular case where $F(s) = \frac{1}{\varepsilon^2}(1-s)$, the system (1.1) has been considered in several papers. Base on formal asymptotic expansions and numerical experiments, Grant and Roberts [2] computed the effective radius and the induced mass of the uncharged impurity. In space dimension one, Mariş [7] proved the existence of a global subcontinua of finite energy subsonic traveling waves. In higher dimensions he proved the nonexistence of supersonic solutions to the system (1.1) in [8].

Here, our goal is to prove the existence of nontrivial finite energy traveling waves of the system (1.1) in space dimension $N = 3$ and $N = 4$, under general conditions on the nonlinearity F and for any speed $c \in (0, v_s)$ and $\varepsilon^2(c^2\delta^2 + k^2) < q^2$.

Notation. Throughout the paper, $\mathcal{L}^N$ is the Lebesgue measure on $\mathbf{R}^N$. For $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$, we put $x' = (x_2, \ldots, x_N) \in \mathbf{R}^{N-1}$. Given a function f on $\mathbf{R}^N$ and $\lambda, \sigma > 0$, we define dilations of f as

$$f_{\lambda,\sigma}(x) = f\left(\frac{x_1}{\lambda}, \frac{x'}{\sigma}\right).$$

We consider the following set of assumptions:

(A1) The function F is continuous on $[0, \infty), C^1$ in a neighborhood of $1, F(1) = 0$ and $F'(1) < 0$.
(A2) There exist $C > 0$ and $p_0 < \frac{2}{N-2}$ such that $F(s) \leqslant C(1 + s^{p_0})$ for any $s \geqslant 0$.
(A3) There exist $C, \alpha > 0$ and $r_* > 1$ such that $F(s) \geqslant -Cs_0^\alpha$ for any $s \geqslant r_*$.

If (A1) is satisfied, denote $V(s) = \int_s^1 F(\tau)d\tau$ and $a = \sqrt{-\frac{1}{2}F'(1)}$, then the sound velocity at infinity associated to the first equation of (1.1) is $v_s = a$, and using Taylor's formula for s in a neighborhood of 1 we have

$$V(s) = \frac{1}{2}V''(1)(s-1)^2 + (s-1)^2\varepsilon(s-1) = a^2(s-1)^2 + (s-1)^2\varepsilon(s-1),$$

where $\varepsilon \to 0$ as $t \to 0$. Hence, for $|\psi|$ close to 1, $V(|\psi|^2)$ can be approximated by the Ginzburg-Landau potential $a^2(|\psi|^2 - 1)^2$. If (A1) and (A3) are satisfied, it follows from Proposition 2.2 p. 1078 in [8] that $|\psi|$ is bounded by a universal constant; hence we can modify F in a neighborhood of infinity such that the modified function $\tilde{F}$ satisfies (A1), (A2), (A3) and the first equation of (1.1) has the same traveling waves as the equation obtained from it by replacing F by $\tilde{F}$ (see the introduction of [9] for more details).

Fix an odd function φ of class C^∞ such that $\varphi(s) = s$ for $s \in [0, 2], 0 \leqslant \varphi' \leqslant 1$ on $\mathbf{R}$ and $\varphi(s) = 3$ for $s \geqslant 4$. Denote

$$\mathcal{E} = \{\psi \in L^1_{\text{loc}}(\mathbf{R}^N) \mid \nabla\psi \in L^2(\mathbf{R}^N),\ \varphi^2(|\psi|) - 1 \in L^2(\mathbf{R}^N)\},$$

then according to [1],

$$\mathcal{E} = \{\psi : \mathbf{R}^N \to \mathbf{C} \mid \nabla\psi \in L^2(\mathbf{R}^N),\ |\psi| - 1 \in L^2(\mathbf{R}^N)\}.$$

Moreover, if $N \leq 4$ it can be proved that

$$\mathcal{E} = \{\psi : \mathbf{R}^N \to \mathbf{C} \mid \nabla\psi \in L^2(\mathbf{R}^N),\ |\psi|^2 - 1 \in L^2(\mathbf{R}^N)\}. \tag{1.7}$$

We claim that for $\phi \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, there holds $\phi \in L^2(\mathbf{R}^N)$ if and only if $\varphi(\phi) \in L^2(\mathbf{R}^N)$. Indeed, if $|\phi| \leqslant 1$ then $\varphi(\phi) = \phi$. If $|\phi| > 1$ then

$$0 \leqslant |\phi| - |\varphi(\phi)| < |\phi| < |\phi|^{\frac{2^*}{2}}. \tag{1.8}$$

Since $\phi \in \mathcal{D}^{1,2}(\mathbf{R}^N)$, by the Sobolev embedding we have $|\phi|^{\frac{2^*}{2}} \in L^2(\mathbf{R}^N)$ and the claim follows.

Our main result is as follows.

Theorem 1.1. *Assume that $N \in \{3, 4\}$, $0 < c < v_s$ and $\varepsilon^2(c^2\delta^2 + k^2) < q^2$. Suppose that the function F satisfies the conditions ((A1) and (A2)) or ((A1) and (A3)). Then the system (1.6) admits a nontrivial finite energy solution $(\psi, \phi) \in \mathcal{E} \times H^1(\mathbf{R}^N)$.*

At least formally, traveling waves are critical points of the functional

$$E_c(\psi,\phi) = E(\psi,\phi) + 2cQ(\psi) + 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi),$$

where Q is the momentum with respect to the x_1 direction. A rigorous definition of the momentum for all functions in $\mathcal{E}$ has been given in [9], section 2 if $N \geq 3$. Notice that in the case when $\langle i\frac{\partial\psi}{\partial x_1}, \psi\rangle \in L^1(\mathbf{R}^N)$ we have $Q(\psi) = \int_{\mathbf{R}^N} \langle i\frac{\partial\psi}{\partial x_1}, \psi\rangle dx = \int_{\mathbf{R}^N} \mathrm{Re}\left(i\frac{\partial\psi}{\partial x_1}\bar{\psi}\right) dx$. If $\phi \in H^1(\mathbf{R}^N)$, the momentum of ϕ is simply $Q(\phi) = \int_{\mathbf{R}^N} \langle i\frac{\partial\phi}{\partial x_1}, \phi\rangle dx$. To simplify notation, we denote by Q the momentum on both $\mathcal{E}$ and $H^1(\mathbf{R}^N)$.

If assumptions (A1) and (A2) above are verified, arguing as in the proof of [8, Proposition 4.1] it can be shown that any traveling wave $(\psi,\phi) \in \mathcal{E} \times H^1(\mathbf{R}^N)$ of (1.1) must satisfy the Pohozaev-type identity $P_c(\psi,\phi) = 0$, where

$$P_c(\psi,\phi) = \int_{\mathbf{R}^N} \left(\left|\frac{\partial\psi}{\partial x_1}\right|^2 + \frac{1}{q^2\varepsilon^2}\left|\frac{\partial\phi}{\partial x_1}\right|^2 \right) dx + \frac{N-3}{N-1}\sum_{k=2}^{N} \left(\left|\frac{\partial\psi}{\partial x_k}\right|^2 + \frac{1}{q^2\varepsilon^2}\left|\frac{\partial\phi}{\partial x_k}\right|^2 \right) dx$$

$$+ \int_{\mathbf{R}^N} V(|\psi|^2)dx + \frac{1}{\varepsilon^4}\int_{\mathbf{R}^N} |\psi|^2|\phi|^2 dx - \frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} |\phi|^2 dx + 2cQ(\psi) + 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi).$$

Following the approach in [9] we will prove the existence of traveling waves by minimizing the action E_c under the Pohozaev constraint $P_c = 0$.

In the next section we present the main tools used in the proof of Theorem 1.1. In section 3 we prove the existence of traveling waves in space dimension $N = 4$, and the last section is devoted to the case $N = 3$ (which is more difficult because the minimization problem is invariant by dilations).

2. The variational framework

In this section we study the properties of the functional E_c.

Let $\psi \in \mathcal{E}$. The modified Ginzburg-Landau energy of ψ in $\mathbf{R}^N$ is defined as

$$E_{GL}(\psi) = \int_{\mathbf{R}^N} |\nabla\psi|^2 dx + a^2 \int_{\mathbf{R}^N} (\varphi^2(|\psi|) - 1)^2 dx.$$

If $\Omega \subset \mathbf{R}^N$ is a measurable set, we define

$$E_{GL}^{\Omega}(\psi) = \int_{\Omega} |\nabla\psi|^2 dx + a^2 \int_{\Omega} (\varphi^2(|\psi|) - 1)^2 dx.$$

The modified Ginzburg-Landau energy is finite for any $\psi \in \mathcal{E}$ in any space dimension N. If $N \leq 4$, it follows from (1.7) that the Ginzburg-Landau energy $\mathcal{E}_{GL}(\psi)$ is finite, where

$$\mathcal{E}_{GL}(\psi) = \int_{\mathbf{R}^N} |\nabla\psi|^2 dx + a^2 \int_{\mathbf{R}^N} (|\psi|^2 - 1)^2 dx.$$

Moreover, if $N \in \{3,4\}$ denoting $2^* = \frac{2N}{N-2}$ and using the Sobolev inequality we have

$$
\begin{aligned}
0 \leqslant \mathcal{E}_{GL}(\psi) - E_{GL}(\psi) &= a^2 \int_{\mathbf{R}^N} (|\psi|^2 - \varphi^2(|\psi|))(|\psi|^2 + \varphi^2(|\psi|) - 2)\,dx \\
&\leqslant C \int_{\mathbf{R}^N} |\psi|^4 \mathbf{1}_{\{|\psi|\geq 2\}}\,dx \leqslant C \int_{\mathbf{R}^N} |\psi|^{2^*} \mathbf{1}_{\{|\psi|\geq 2\}}\,dx \leqslant C\|\nabla\psi\|^{2^*}_{L^2(\mathbf{R}^N)}.
\end{aligned} \tag{2.1}
$$

By [9, Lemma 4.3] we have $V(|\psi|^2) \in L^1(\mathbf{R}^N)$ whenever $\psi \in \mathcal{E}$, hence we may define

$$
\tilde{E}(\psi) = \int_{\mathbf{R}^N} |\nabla\psi|^2\,dx + \int_{\mathbf{R}^N} V(|\psi|^2)\,dx.
$$

We will always assume that $q^2 - \varepsilon^2 k^2 > 0$. On the space $H^1(\mathbf{R}^N)$ we will consider the norm

$$
\|\phi\|^2_{H^1} = \int_{\mathbf{R}^N} |\nabla\phi|^2\,dx + \frac{q^2 - \varepsilon^2 k^2}{\varepsilon^2} \int_{\mathbf{R}^N} |\phi|^2\,dx. \tag{2.2}
$$

Then we have

$$
E(\psi,\phi) = \tilde{E}(\psi) + \frac{1}{\varepsilon^4} \int_{\mathbf{R}^N} (|\psi|^2 - 1)|\phi|^2\,dx + \frac{1}{\varepsilon^2 q^2} \|\phi\|^2_{H^1(\mathbf{R}^N)}. \tag{2.3}
$$

We list some useful results from [9].

Lemma 2.1 ([9, Lemma 4.3])**.** *Assume that $N \geqslant 3$, $0 \leqslant c < v_s$, and let $\varepsilon_1 \in (0, 1 - \frac{c}{v_s})$. There exists a constant $K_1 = K_1(F, N, c, \varepsilon_1) > 0$ such that for any $\psi \in \mathcal{E}$ satisfying $E_{GL}(\psi) < K_1$, we have*

$$
\tilde{E}(\psi) - 2c|Q(\psi)| \geqslant \varepsilon_1 E_{GL}(\psi).
$$

Lemma 2.2 ([9, Lemma 4.5])**.** *For any $k > 0$, the functional Q is bounded on the set $\{\psi \in \mathcal{E} \mid E_{GL}(\psi) \leqslant k\}$.*

Lemma 2.3 ([9, Lemma 3.3])**.** *Let $A > A_3 > A_2 > \frac{5}{4}$. Assume that $(R_n)_{n\geq 1} \subset [1,\infty)$, $(y_n)_{n\geq 1} \subset \mathbf{R}^N$ and $(\psi_n)_{n\geq 1} \subset \mathcal{E}$ are sequences verifying*

$$
E_{GL}^{B(y_n,AR_n)\setminus B(y_n,R_n)}(\psi_n) = \int_{B(y_n,AR_n)\setminus B(y_n,R_n)} \Big(|\nabla\psi_n|^2 + a^2\left(\varphi^2(|\psi_n|) - 1\right)^2\Big)\,dx \to 0
$$

as $n \to \infty$. Then for each n there exist two functions $\psi_{n,1}, \psi_{n,2} \in \mathcal{E}$ and a constant $\theta_{n,0} \in [0, 2\pi)$ satisfying the following properties:

(i) *$\psi_{n,1} = \psi_n$ on $B(y_n, \frac{5}{4}R_n)$ and $\psi_{n,1} = e^{i\theta_{n,0}}$ on $\mathbf{R}^N \setminus B(y_n, A_2R_n)$,*

(ii) *$\psi_{n,2} = \psi_n$ on $\mathbf{R}^N \setminus B(y_n, AR_n)$ and $\psi_{n,2} = e^{i\theta_{n,0}} =$ constant on $B(y_n, A_3R_n)$,*

(iii) $\displaystyle\int_{\mathbf{R}^N} \left| \left|\frac{\partial\psi_n}{\partial x_j}\right|^2 - \left|\frac{\partial\psi_{n,1}}{\partial x_j}\right|^2 - \left|\frac{\partial\psi_{n,2}}{\partial x_j}\right|^2 \right| dx \to 0$ *as $n \to \infty$ for $j = 2, \ldots, N$,*

(iv) $\displaystyle\int_{\mathbf{R}^N} \left|(\varphi^2(|\psi_n|) - 1)^2 - (\varphi^2(|\psi_{n,1}|) - 1)^2 - (\varphi^2(|\psi_{n,2}|) - 1)^2\right| dx \to 0$ *as $n \to \infty$,*

(v) *$|Q(\psi_n) - Q(\psi_{n,1}) - Q(\psi_{n,2})| \to 0$ as $n \to \infty$,*

(vi) *If assumptions (A1) and (A2) hold then*

$$
\int_{\mathbf{R}^N} \left|V(|\psi_n|^2) - V(|\psi_{n,1}|^2) - V(|\psi_{n,2}|^2)\right| dx \to 0 \text{ as } n \to 0.
$$

We have the following result for $E(\psi,\phi)$, quite similar to Lemma 2.1 above.

Lemma 2.4. *Assume that $N\in\{3,4\}$, $0\leqslant c<v_s$ and $\varepsilon^2(c^2\delta^2+k^2)<q^2$. Let $\bar\varepsilon>0$ be such that $\bar\varepsilon<\min\left\{1-\frac{c}{v_s},1-\frac{c\delta\varepsilon}{\sqrt{q^2-\varepsilon^2k^2}}\right\}$. Then, there exists a constant $\bar K=\bar K(N,c,\bar\varepsilon,q,k,\delta)$ such that for any $(\psi,\phi)\in\mathcal{E}\times H^1(\mathbf{R}^N)$ satisfying $E_{GL}(\psi)<\bar K$ and $\|\phi\|^2_{H^1(\mathbf{R}^N)}<\bar K$ we have*

$$E_c(\psi,\phi)\geqslant\bar\varepsilon\left(\mathcal{E}_{GL}(\psi)+\frac{1}{\varepsilon^2q^2}\|\phi\|^2_{H^1(\mathbf{R}^N)}\right).$$

Proof. Fix ε_1 such that $\bar\varepsilon<\varepsilon_1<\min\left\{1-\frac{c}{v_s},1-\frac{c\delta\varepsilon}{\sqrt{q^2-\varepsilon^2k^2}}\right\}$. Using Lemma 2.1, there exists a constant $K_1=K_1(N,c,\varepsilon_1)$ such that

$$\tilde E_c(\psi)=\int_{\mathbf{R}^N}|\nabla\psi|^2dx+\int_{\mathbf{R}^N}V(|\psi|^2)dx+2cQ(\psi)\geqslant\varepsilon_1E_{GL}(\psi),\tag{2.4}$$

for any $\psi\in\mathcal{E}$ satisfying $E_{GL}(\psi)<K_1$.

Using the Cauchy-Schwarz inequality and (2.2) we have for any $\phi\in H^1(\mathbf{R}^N)$

$$|Q(\phi)|\leqslant\|\phi\|_{L^2}\left\|\frac{\partial\phi}{\partial x_1}\right\|_{L^2(\mathbf{R}^N)}\leqslant\frac{\varepsilon}{2\sqrt{q^2-\varepsilon^2k^2}}\|\phi\|^2_{H^1(\mathbf{R}^N)}.\tag{2.5}$$

If $N\in\{3,4\}$ using the Cauchy-Schwarz inequality, then the Sobolev inequality we get

$$\begin{aligned}&\int_{\mathbf{R}^N}\left||\psi|^2-1\right||\phi|^2\,dx\leqslant\left\||\psi|^2-1\right\|_{L^2(\mathbf{R}^N)}\|\phi\|^2_{L^4(\mathbf{R}^N)}\\&\leqslant C\left(\mathcal{E}_{GL}(\psi)\right)^{\frac12}\|\phi\|^2_{H^1}\leqslant C\|\phi\|_{H^1(\mathbf{R}^N)}\left(\mathcal{E}_{GL}(\psi)+\|\phi\|^2_{H^1(\mathbf{R}^N)}\right).\end{aligned}\tag{2.6}$$

From (2.1)-(2.6) we obtain

$$\begin{aligned}&E_c(\psi,\phi)-\bar\varepsilon\left(\mathcal{E}_{GL}(\psi))+\tfrac{1}{\varepsilon^2q^2}\|\phi\|^2_{H^1(\mathbf{R}^N)}\right)\\&=\left(\tilde E(\psi)+2cQ(\psi)-\varepsilon_1E_{GL}(\psi)\right)+\left(\varepsilon_1E_{GL}(\psi)-\bar\varepsilon\mathcal{E}_{GL}(\psi)\right)\\&\qquad+\tfrac{1}{\varepsilon^2q^2}\left((1-\bar\varepsilon)\|\phi\|^2_{H^1(\mathbf{R}^N)}+2c\delta Q(\phi)\right)+\tfrac{1}{\varepsilon^4}\textstyle\int_{\mathbf{R}^N}\left(|\psi|^2-1\right)|\phi|^2\,dx\\&\geqslant\left((\varepsilon_1-\bar\varepsilon)E_{GL}(\psi)-\bar\varepsilon C\|\nabla\psi\|^{2^*}_{L^2}\right)+\tfrac{1}{\varepsilon^2q^2}\left(1-\bar\varepsilon-\tfrac{c\delta\varepsilon}{\sqrt{q^2-\varepsilon^2k^2}}\right)\|\phi\|^2_{H^1(\mathbf{R}^N)}\\&\qquad-\tfrac{C}{\varepsilon^4}\|\phi\|_{H^1(\mathbf{R}^N)}\left(\mathcal{E}_{GL}(\psi)+\|\phi\|^2_{H^1(\mathbf{R}^N)}\right).\end{aligned}$$

It is clear that the last quantity is nonnegative if $E_{GL}(\psi)$ and $\|\phi\|_{H^1(\mathbf{R}^N)}$ are sufficiently small and Lemma 2.4 is proven. □

By combining (2.3), (2.1), (2.5), (2.6), [9, Lemma 4.1] and Lemma 2.2, we infer that for any $k>0$, the function E_c is bounded from below on the set $\{(\psi,\phi)\in$

$\mathcal{E} \times H^1(\mathbf{R}^N) | E_{GL}(\psi) + \frac{1}{\varepsilon^2 q^2} \|\phi\|^2_{H^1(\mathbf{R}^N)} = k\}$. For $k > 0$, we define

$$E_{c,\min}(k) = \inf\{E_c(\psi,\phi) \mid (\psi,\phi) \in \mathcal{E} \times H^1(\mathbf{R}^N), E_{GL}(\psi) + \frac{1}{\varepsilon^2 q^2} \|\phi\|^2_{H^1(\mathbf{R}^N)} = k\}.$$

Lemma 2.5. *Assume that $N \in \{3,4\}$, $0 < c < v_s$ and $\varepsilon^2(c^2\delta^2 + k^2) < q^2$. The function $E_{c,\min}$ has the following properties:*

(i) *There is $k_0 > 0$ such that $E_{c,\min}(k) > 0$ for all $k \in (0,k_0)$.*
(ii) $\lim_{k\to\infty} E_{c,\min}(k) = -\infty$.
(iii) *$E_{c,\min}(k) < k$ for any $k > 0$.*

Proof. Assertion (i) follows from Lemma 2.4. We have

$$\begin{aligned} E_{c,\min}(k) &\leqslant \inf\{E_c(\psi,0) \mid \psi \in \mathcal{E}, E_{GL}(\psi) = k\} \\ &= \inf\{\tilde{E}(\psi) + 2cQ(\psi) \mid \psi \in \mathcal{E}, E_{GL}(\psi) = k\}. \end{aligned}$$

This together with [9, Lemma 4.6] gives (ii)–(iii). □

Let

$$S_c = \sup_{k>0} E_{c,\min}(k). \tag{2.7}$$

By Lemma 2.5 (i) we have $S_c > 0$.

Lemma 2.6. *Assume that $N = 3,4$, $0 < c < v_s$ and $\varepsilon^2(c^2\delta^2 + k^2) < q^2$. Then, the set $\mathscr{C} = \{(\psi,\phi) \in \mathcal{E} \times (\mathbf{R}^N) \mid (|\psi|,|\phi|) \neq (1,0), P_c(\psi,\phi) = 0\}$ is not empty and*

$$T_c := \inf\{E_c(\psi,\phi) \mid (\psi,\phi) \in \mathscr{C}\} > 0.$$

Proof. Let $(\psi,\phi) \in \mathcal{E} \times (\mathbf{R}^N)$ and denote

$$A(\psi,\phi) = \sum_{k=2}^{N} \int_{\mathbf{R}^N} \left(\left|\frac{\partial \psi}{\partial x_k}\right|^2 + \frac{1}{q^2\varepsilon^2}\left|\frac{\partial \phi}{\partial x_k}\right|^2 \right) dx, \tag{2.8}$$

(2.9)

$$\begin{aligned} B_c(\psi,\phi) \quad &= \quad \int_{\mathbf{R}^N} \left(\left|\frac{\partial \psi}{\partial x_1}\right|^2 + \frac{1}{q^2\varepsilon^2}\left|\frac{\partial \phi}{\partial x_1}\right|^2 \right) dx + \int_{\mathbf{R}^N} V(|\psi|^2) dx \\ &+ \frac{1}{\varepsilon^4}\int_{\mathbf{R}^N} |\psi|^2|\phi|^2 dx - \frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} |\phi|^2 dx + 2cQ(\psi) + 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi). \end{aligned}$$

Then we have $P_c(\psi,\phi) = \frac{N-3}{N-1}A(\psi,\phi) + B_c(\psi,\phi)$ and $E_c(\psi,\phi) = A(\psi,\phi) + B_c(\psi,\phi)$. We first show that $\mathscr{C} \neq \varnothing$. By Lemma 2.5, there exists $(u,v) \in \mathcal{E} \times (\mathbf{R}^N)$ such that $E_c(u,v) < 0$, and then $P_c(u,v) < 0$. Moreover,

$$P_c((u,v)_{\sigma,1}) = \frac{1}{\sigma}\int_{\mathbf{R}^N} \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \frac{1}{q^2\varepsilon^2}\left|\frac{\partial v}{\partial x_1}\right|^2 \right) dx + \sigma\frac{N-3}{N-1}A(u,v) \tag{2.10}$$

$$+ \sigma\int_{\mathbf{R}^N} V(|u|^2)dx + \frac{\sigma}{\varepsilon^4}\int_{\mathbf{R}^N} |u|^2|v|^2 dx - \sigma\frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} |v|^2 dx + 2cQ(u) + 2\frac{c\delta}{q^2\varepsilon^2}Q(v).$$

Since $\lim_{\sigma\to 0} P_c((u,v)_{\sigma,1})) = +\infty$ and $P_c((u,v)_{1,1}) < 0$, there is $\sigma_0 \in (0,1)$ satisfying $P_c(u_{\sigma_0,1}, v_{\sigma_0,1}) = 0$, and so $\mathscr{C} \neq \varnothing$.

Next we prove that $T_c > 0$ in the case $N = 4$. Let $(\psi, \phi) \in \mathscr{C}$. We have

$$E_c((\psi,\phi)_{1,\sigma}) = \sigma^{N-3}A(\psi,\phi) + \sigma^{N-1}B_c(\psi,\phi),$$

which implies that

$$\frac{d}{d\sigma}(E_c((\psi,\phi)_{1,\sigma})) = (N-3)\sigma^{N-4}A(\psi,\phi) + (N-1)\sigma^{N-2}B_c(\psi,\phi).$$

Since $A(\psi,\phi) > 0$ and $B_c(\psi,\phi) = -\frac{N-3}{N-1}A(\psi,\phi) < 0$, we see that $\frac{d}{d\sigma}(E_c((\psi,\phi)_{1,\sigma}))$ is positive for $\sigma \in (0,1)$ and negative for $\sigma \in (1,\infty)$. It follows that $E_c((\psi,\phi)_{1,\sigma}) \leqslant E_c((\psi,\phi)_{1,1}) = E_c(\psi,\phi)$ for any $\sigma > 0$. In addition,

$$E_{GL}(\psi_{1,\sigma}) + \frac{1}{\varepsilon^2 q^2}\|\phi_{1,\sigma}\|^2_{H^1} = \sigma^{N-3}A(\psi,\phi) + \sigma^{N-1}D(\psi,\phi),$$

where
(2.11)

$$D(\psi,\phi) = \int_{R^N}\left[\left|\frac{\partial\psi}{\partial x_1}\right|^2 + \frac{1}{q^2\varepsilon^2}\left|\frac{\partial\phi}{\partial x_1}\right|^2 + a^2\left(\varphi^2(|\psi|)-1\right)^2 + \frac{q^2-\varepsilon^2k^2}{q^2\varepsilon^4}|\phi|^2\right]dx.$$

We see that the mapping $\sigma \longmapsto E_{GL}(\psi_{1,\sigma}) + \frac{1}{\varepsilon^2q^2}\|\phi_{1,\sigma}\|^2_{H^1(\mathbf{R}^N)}$ is strictly increasing and one-to-one from $(0,\infty)$ to $(0,\infty)$. Thus, for any $k > 0$, there exists a unique $\sigma(k,(\psi,\phi)) > 0$ such that $E_{GL}(\psi_{1,\sigma(k,(\psi,\phi))}) + \frac{1}{\varepsilon^2q^2}\|\phi_{1,\sigma(k,(\psi,\phi))}\|^2_{H^1(\mathbf{R}^N)} = k$. Therefore,

$$E_{c,\min}(k) \leqslant E_c((\psi,\phi)_{1,\sigma(k,(\psi,\phi))}) \leqslant E_c(\psi,\phi).$$

Since the last inequality holds for any $(\psi,\phi) \in \mathscr{C}$ and any $k > 0$, using Lemma 2.4 we infer that $T_c \geqslant \sup_{k>0} E_{c,\min}(k) = S_c > 0$.

Next consider the case $N = 3$. Let $(\psi,\phi) \in \mathscr{C}$. Then $P_c(\psi,\phi) = B_c(\psi,\phi) = 0$ and $E_c(\psi,\phi) = A(\psi,\phi) > 0$. For any $\sigma > 0$ we have $(E_c(\psi_{1,\sigma},\phi_{1,\sigma})) = A(\psi,\phi)$ and

$$E_{GL}(\psi_{1,\sigma}) + \frac{1}{\varepsilon^2q^2}\|\phi_{1,\sigma}\|^2_{H^1(\mathbf{R}^N)} = A(\psi,\phi) + \sigma^2 D(\psi,\phi).$$

It is easy to see that the mapping $\sigma \longmapsto E_{GL}((\psi,\phi)_{1,\sigma})$ is increasing and one-to-one from $(0,+\infty)$ to $(A(\psi,\phi),+\infty)$. Fix $\tilde{\varepsilon} > 0$. By the definition of S_c (see (2.7)) there exists $k_{\tilde{\varepsilon}} > 0$ such that $E_{c,\min}(k_{\tilde{\varepsilon}}) > S_c - \tilde{\varepsilon}$. If $A(\psi,\phi) \geqslant k_{\tilde{\varepsilon}}$, using Lemma 2.5 (iii) we get

$$E_c(\psi,\phi) = A(\psi,\phi) \geqslant k_{\tilde{\varepsilon}} > E_{c,\min}(k_{\tilde{\varepsilon}}) > S_c - \tilde{\varepsilon}.$$

If $A(\psi,\phi) < k_{\tilde{\varepsilon}}$, there exists $\sigma(k_{\tilde{\varepsilon}},(\psi,\phi)) > 0$ such that $E_{GL}(\psi_{1,\sigma(k_{\tilde{\varepsilon}},(\psi,\phi))}) + \frac{1}{\varepsilon^2q^2}\|\phi_{1,\sigma(k_{\tilde{\varepsilon}},(\psi,\phi))}\|^2_{H^1(\mathbf{R}^N)} = k_{\tilde{\varepsilon}}$. Then

$$E_c(\psi,\phi) = A(\psi,\phi) = E_c((\psi,\phi)_{1,\sigma(k_{\tilde{\varepsilon}},(\psi,\phi))}) \geqslant E_{c,\min}(k_{\tilde{\varepsilon}}) > S_c - \tilde{\varepsilon}.$$

Taking limit as $\tilde{\varepsilon} \to 0$ we obtain $T_c \geqslant S_c > 0$.

□

Lemma 2.7. *Let $(\psi_n,\phi_n)_{n\geqslant1} \subset \mathcal{E} \times H^1(\mathbf{R}^N)(\mathbf{R}^N)$ be a sequence such that $(E_{GL}(\psi_n)_{n\geqslant1}$ and $(\|\phi_n\|_{H^1(\mathbf{R}^N)})_{n\geqslant1}$ are bounded and $\lim_{n\to\infty} P_c(\psi_n,\phi_n) = m < 0$. Then*

$$\liminf_{n\to\infty} A(\psi_n,\phi_n) > \frac{N-1}{2}T_c.$$

Proof. Since $\lim_{n\to\infty} P_c(\psi_n, \phi_n) = m < 0$, we have $(|\psi_n|, |\phi_n|) \neq (1, 0)$ and

$$\int_{\mathbf{R}^N} \left(\left| \frac{\partial \psi_n}{\partial x_1} \right|^2 + \frac{1}{q^2\varepsilon^2} \left| \frac{\partial \phi_n}{\partial x_1} \right|^2 \right) dx > 0 \text{ for } n \text{ sufficiently large.}$$

It follows from (2.10) that $\lim_{\sigma\to 0} P_c((\psi_n, \phi_n)_{\sigma,1}) = +\infty$ for each n. Since $P_c((\psi_n, \phi_n)_{1,1}) = P_c(\psi_n, \phi_n) < 0$ for n large enough, there exists $\sigma_n \in (0, 1)$ such that $P_c((\psi_n, \phi_n)_{\sigma_n,1}) = 0$. From the definition of T_c, $E_c((\psi_n, \phi_n)_{\sigma_n,1}) \geqslant T_c$, and so

$$A((\psi_n, \phi_n)_{\sigma_n,1}) = \frac{N-1}{2}(E_c((\psi_n, \phi_n)_{\sigma_n,1}) - P_c((\psi_n, \phi_n)_{\sigma_n,1})) \geqslant \frac{N-1}{2} T_c.$$

This gives

$$A(\psi_n, \phi_n) \geqslant \frac{N-1}{2} \frac{1}{\sigma_n} T_c. \tag{2.12}$$

We claim that $\limsup_{n\to\infty} \sigma_n < 1$. Indeed, assume on the contrary that there exists a subsequence $(\sigma_{n_k})_{k\geqslant 1} \to 1$ as $k \to \infty$. Using Lemma 2.2, (2.5) and the boundedness of $(E_{GL}(\psi_{n_k})_{k\geqslant 1}$ and $(\|\phi_n\|_{H^1(\mathbf{R}^N)})_{n\geqslant 1}$ we obtain that

$$(Q(\psi_{n_k}))_{k\geqslant 1}, (Q(\phi_{n_k}))_{k\geqslant 1}, (A(\psi_{n_k}, \phi_{n_k}))_{k\geqslant 1} \text{ are bounded.}$$

Moreover, from [9, Lemma 4.1] it follows that $\int_{\mathbf{R}^N} V(|\psi_{n_k}|^2) dx$ is bounded, too. Using (2.6) and (2.1), we obtain the boundedness of $\left(\int_{\mathbf{R}^N} |\psi_{n_k}|^2 |\phi_{n_k}|^2 dx\right)_{k\geqslant 1}$. Passing to a subsequence (still denoted $(\psi_{n_k}, \phi_{n_k})_{k\geqslant 1}$) and using (2.10), we get

$$\lim_{n\to\infty} P_c(\psi_{n_k}, \phi_{n_k}) = \lim_{n\to\infty} P_c((\psi_{n_k}, \phi_{n_k})_{\sigma_{n_k},1}) = 0,$$

contradicting the assumption $\lim_{n\to\infty} P_c(\psi_n, \phi_n) = m < 0$. Thus $\limsup_{n\to\infty} \sigma_n < 1$, which together with (2.12) implies that $\liminf_{n\to\infty} A(\psi_n, \phi_n) > \frac{N-1}{2} T_c$. □

3. The case $N = 4$

Lemma 3.1. *Assume that $N \geqslant 4$. Let $(\psi_n, \phi_n)_{n\geqslant 1} \subset \mathcal{E} \times H^1(\mathbf{R}^N)$ be a sequence satisfying:*

- (i) *There exist M_1, $M_2 > 0$ such that $M_1 \leqslant E_{GL}(\psi_n) + \frac{1}{\varepsilon^2 q^2} \|\phi_n\|^2_{H^1(\mathbf{R}^N)}$ and $A(\psi_n, \phi_n) \leqslant M_2$ for any $n \geqslant 1$.*
- (ii) *$P_c(\psi_n, \phi_n) \to 0$ as $n \to \infty$.*

Then $\liminf_{n\to\infty} E_c(\psi_n, \phi_n) \geqslant T_c$, where T_c is as in Lemma 2.6.

Proof. For any $\sigma > 0$,

$$\begin{aligned} P_c((\psi_n, \phi_n)_{1,\sigma}) &= \sigma^{N-3} \frac{N-3}{N-1} A(\psi_n, \phi_n) + \sigma^{N-1} B_c(\psi_n, \phi_n) \\ &= \sigma^{N-3} \left(\frac{N-3}{N-1} A(\psi_n, \phi_n) + \sigma^2 B_c(\psi_n, \phi_n) \right). \end{aligned} \tag{3.1}$$

We claim that

$$M := \liminf_{n\to\infty} A(\psi_n, \phi_n) > 0.$$

We argue by contradiction and assume that there is a subsequence, still denoted $(\psi_n, \phi_n)_{n\geqslant 1}$ such that $A(\psi_n, \phi_n) \to 0$ as $n \to \infty$. Fix $k_0 > 0$ such that $E_{c,\min}(k_0) > 0$.

As in the proof of Lemma 2.6, we see that for each n there exists a unique $\sigma_n > 0$ such that

$$E_{GL}((\psi_n)_{1,\sigma_n}) + \frac{1}{\varepsilon^2 q^2}\|(\phi_n)_{1,\sigma_n}\|^2_{H^1(\mathbf{R}^N)} = \sigma_n^{N-3}A(\psi_n,\phi_n) + \sigma_n^{N-1}D(\psi_n,\phi_n) = k_0,$$

where $D(\psi_n,\phi_n)$ is as in (2.11). Thus $E_c((\psi_n,\phi_n)_{1,\sigma_n}) \geqslant E_{c,\min}(k_0) > 0$. Moreover, $(\sigma_n)_{n\geq 1}$ is bounded because

$$\begin{aligned} k_0 &= E_{GL}((\psi_n)_{1,\sigma_n}) + \frac{1}{\varepsilon^2 q^2}\|(\phi_n)_{1,\sigma_n}\|^2_{H^1(\mathbf{R}^N)} \\ &\geqslant \min(\sigma_n^{N-3},\sigma_n^{N-1})\left(E_{GL}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(\mathbf{R}^N)}\right) \geqslant \min(\sigma_n^{N-3},\sigma_n^{N-1})M_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} E_c((\psi_n,\phi_n)_{1,\sigma_n}) &= \sigma_n^{N-3}A(\psi_n,\phi_n) + \sigma_n^{N-1}B_c(\psi_n,\phi_n) \\ &= \sigma_n^{N-3}A(\psi_n,\phi_n) + \sigma_n^{N-1}\left(P_c(\psi_n,\phi_n) - \frac{N-3}{N-1}A(\psi_n,\phi_n)\right). \end{aligned}$$

Passing to the limit as $n \to \infty$ we get $0 \geqslant E_{c,min}(k_0) > 0$, a contradiction. Hence $\liminf_{n\to\infty} A(\psi_n,\phi_n) > 0$. Using the assumption (ii) and the fact that $B_c(\psi_n,\phi_n) = P_c(\psi_n,\phi_n) - \frac{N-3}{N-1}A(\psi_n,\phi_n)$, we obtain

$$\limsup_{n\to\infty} B_c(\psi_n,\phi_n) = -\frac{N-3}{N-1}M.$$

Thus, for n large enough, we can choose $\bar\sigma_n = \left(-\frac{\frac{N-3}{N-1}A(\psi_n,\phi_n)}{B_c(\psi_n,\phi_n)}\right)^{\frac12}$. Then (3.1) implies that $P_c((\psi_n,\phi_n)_{1,\bar\sigma_n}) = 0$, and so

$$E_c((\psi_n,\phi_n)_{1,\bar\sigma_n}) \geqslant T_c,$$

that is

(3.2)
$$E_c(\psi_n,\phi_n)+(\bar\sigma_n^{N-3}-1)A(\psi_n,\phi_n)+(\bar\sigma_n^{N-1}-1)\left(P_c(\psi_n,\phi_n) - \frac{N-3}{N-1}A(\psi_n,\phi_n)\right) \geqslant T_c.$$

We observe that $\bar\sigma_n \to 1$ as $n \to \infty$ because $\bar\sigma_n = \left(-\frac{P_c(\psi_n,\phi_n)}{B_c(\psi_n,\phi_n)} + 1\right)^{\frac12}$ and $P_c(\psi_n,\phi_n) \to 0$ as $n \to \infty$. Letting $n \to \infty$ in (3.2) and using the fact that $(A(\psi_n,\phi_n))_{n\geqslant 1}$ and $(P_c(\psi_n,\phi_n))_{n\geqslant 1}$ are bounded, we deduce

$$\liminf_{n\to\infty} E_c(\psi_n,\phi_n) \geqslant T_c.$$

□

Theorem 3.2. *Assume that $N = 4$, the conditions (A1) and (A2) are satisfied, $0 < c < v_s$ and $\varepsilon^2(c^2\delta^2 + k^2) < q^2$. Let $(\psi_n,\phi_n)_{n\geqslant 1} \subset \mathcal{E}\times H^1(\mathbf{R}^4)$ be a sequence such that $(|\psi_n|,|\phi_n|) \neq (1,0)$ for all n and*

$$P_c(\psi_n,\phi_n) \to 0 \text{ and } E_c(\psi_n,\phi_n) \to T_c \text{ as } n \to \infty.$$

Then there exist a subsequence $(\psi_{n_k},\phi_{n_k})_{k\geqslant 1}$, a sequence $(x_k)_{k\geqslant 1} \subset \mathbf{R}^4$ and $(\psi,\phi) \in \mathscr{C}$ such that

$$\nabla\psi_{n_k}(.+x_k) \to \nabla\psi, \qquad |\psi_{n_k}(.+x_k)| - 1 \to |\psi| - 1 \qquad \text{in } L^2(\mathbf{R}^4) \text{ and}$$
$$\phi_{n_k}(.+x_k) \to \phi \qquad \text{in } H^1(\mathbf{R}^4).$$

Moreover, $E_c(\psi,\phi) = T_c$ *and* (ψ,ϕ) *is a minimizer of* E_c *in* $\mathscr{C}$.

Proof. First, we see that $(E_{GL}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(\mathbf{R}^N)})_{n\geqslant 1}$ is bounded. Indeed, from the assumptions and the fact that $\frac{2}{N-1}A(\psi_n,\phi_n) = E_c(\psi_n,\phi_n) - P_c(\psi_n,\phi_n)$, we infer that $A(\psi_n,\phi_n)$ is bounded.

Next we show that $D(\psi_n,\phi_n)$ is bounded, where $D(\psi_n,\phi_n)$ is as in (2.11). Assume that there exists a subsequence, still denoted (ψ_n,ϕ_n), such that $D(\psi_n,\phi_n) \to \infty$ as $n \to \infty$. Fix $l_0 > 0$ such that $E_{c,\min}(l_0) > 0$. By the same argument as in the proof of Lemma 2.6, there is a sequence $(\sigma_n)_{n\geqslant 1}$ such that

$$E_{GL}((\psi_n)_{1,\sigma_n}) + \frac{1}{\varepsilon^2 q^2}\|(\phi_n)_{1,\sigma_n}\|^2_{H^1(\mathbf{R}^N)} = \sigma_n^{N-3}A(\psi_n,\phi_n) + \sigma_n^{N-1}D(\psi_n,\phi_n) = l_0. \tag{3.3}$$

Therefore, $\sigma_n \to 0$ as $n \to \infty$. It is easy to see that $(B_c(\psi_n,\phi_n))_{n\geqslant 1}$ is bounded since $B_c(\psi_n,\phi_n) = -\frac{N-3}{N-1}A(\psi_n,\phi_n) + P_c(\psi_n,\phi_n)$. Hence

$$E_c(\psi_{1,\sigma_n},\phi_{1,\sigma_n}) = \sigma_n^{N-3}A(\psi_n,\phi_n) + \sigma_n^{N-1}B_c(\psi_n,\phi_n) \to 0 \text{ as } n \to \infty,$$

contradicting the fact that $E_{c,\min}(l_0) > 0$. Thus there exists M such that $D(\psi_n,\phi_n) \leqslant M$ for n large enough. We conclude that $E_{GL}(\psi_n)$ and $\|\phi_n\|_{H^1(\mathbf{R}^N)}$ are bounded.

On the other hand,

$$\frac{2}{N-1}A(\psi_n,\phi_n) = E_c(\psi_n,\phi_n) - P_c(\psi_n,\phi_n) \to T_c \text{ as } n \to \infty, \tag{3.4}$$

and so

$$\liminf_{n\to\infty}\left(E_{GL}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(\mathbf{R}^N)}\right) \geqslant \lim_{n\to\infty} A(\psi_n,\phi_n) = \frac{N-1}{2}T_c.$$

We will use Lions' concentration-compactness principle. Let

$$q_n(t) = \sup_{y\in\mathbf{R}^N} E_{GL}^{B(y,t)}(\psi_n) + \int_{B(y,t)} \frac{1}{\varepsilon^2 q^2}|\nabla\phi_n|^2 + \frac{q^2 - \varepsilon^2 k^2}{\varepsilon^4 q^2}|\phi_n|^2\,dx. \tag{3.5}$$

Proceeding as in [9, Theorem 5.3], we infer that there exist a subsequence of $(((\psi_n,\phi_n),q_n))_{n\geqslant 1}$, still denoted $(((\psi_n,\phi_n),q_n))_{n\geqslant 1}$, a nondecreasing function $q : [0,\infty) \to \mathbf{R}$ and $\alpha \in [0,\alpha_0]$ such that

$$q_n(t) \to q(t) \text{ a.e on } [0,\infty) \text{ as } n \to \infty \text{ and } q(t) \to \alpha \text{ as } t \to \infty.$$

We will prove first that $\alpha > 0$. We argue by contradiction. Assume that there exists a subsequence (still denoted $(\psi_n,\phi_n)_{n\geqslant 1}$) such that

$$\lim_{n\to\infty}\sup_{y\in\mathbf{R}^N} q_n(1) = 0. \tag{3.6}$$

Arguing as in the proof of [9, Lemma 5.4] we get

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}\left|V(|\psi_n|^2) - a^2\left(\varphi^2(|\psi_n|) - 1\right)^2\right|dx = 0. \tag{3.7}$$

Since $E_{GL}(\psi_n)$ is bounded and $\sup_{y\in\mathbf{R}^N} E_{GL}^{B(y,1)}(\psi_n) \to 0$ as $n \to \infty$, using [9, Lemma 3.1 and Lemma 3.2] we infer that there exists a sequence $(\zeta_n)_{n\geqslant 1} \subset \mathcal{E}$ such that

$$\| |\zeta_n| - 1 \|_{L^\infty(\mathbf{R}^N)} \longrightarrow 0, \tag{3.8}$$

$$\|\psi_n - \zeta_n\|_{L^2(\mathbf{R}^N)} \to 0, \tag{3.9}$$

$$|Q(\psi_n) - Q(\zeta_n)| \to 0 \qquad \text{as } n \to \infty, \tag{3.10}$$

$$E_{GL}(\zeta_n) \leqslant E_{GL}(\psi_n) \qquad \text{for all } n. \tag{3.11}$$

By (3.8) we have $|\zeta_n| \leqslant 2$ a.e. on $\mathbf{R}^N$ for all n sufficiently large. For any such n we have $\varphi(\zeta_n) = \zeta_n$ a.e. and $|\varphi^2(|\psi_n|) - \varphi^2(|\zeta_n|)| \leqslant 6|\varphi(|\psi_n|) - \varphi(|\zeta_n|)| \leqslant 6|\psi_n - \zeta_n|$. Using (3.9) we infer that

$$\|\varphi^2(|\psi_n|) - |\zeta_n|^2\|_{L^2(\mathbf{R}^N)} = \|\varphi^2(|\psi_n|) - \varphi^2(|\zeta_n|)\|_{L^2(\mathbf{R}^N)} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

We have

$$\begin{aligned} \int_{\mathbf{R}^N} (|\psi_n|^2 - 1)|\phi_n|^2 \, dx &\geqslant \int_{\mathbf{R}^N} (\varphi^2(|\psi_n|) - 1)|\phi_n|^2 \, dx \\ &= \int_{\mathbf{R}^N} (\varphi^2(|\psi_n|) - |\zeta_n|^2)|\phi_n|^2 \, dx + \int_{\mathbf{R}^N} (|\zeta_n|^2 - 1)|\phi_n|^2 \, dx \end{aligned} \tag{3.12}$$

We know that $\|\phi_n\|_{L^4(\mathbf{R}^N)}$ is bounded because $N \leqslant 4$ and $\|\phi_n\|_{H^1(\mathbf{R}^N)}$ is bounded. Using the Cauchy-Schwarz inequality we get

$$\int_{\mathbf{R}^N} \left|\varphi^2(|\psi_n|) - |\zeta_n|^2\right| |\phi_n|^2 \, dx \leqslant \|\varphi^2(\psi_n) - |\zeta_n|^2\|_{L^2(\mathbf{R}^N)} \|\phi_n\|^2_{L^4(\mathbf{R}^N)} \to 0 \qquad \text{as } n \to \infty.$$

On the other hand we have

$$\int_{\mathbf{R}^N} \left||\zeta_n|^2 - 1\right| |\phi_n|^2 \, dx \leqslant \| |\zeta_n|^2 - 1 \|_{L^\infty(\mathbf{R}^N)} \|\phi_n\|^2_{L^2(\mathbf{R}^N)} \to 0 \qquad \text{as } n \to \infty.$$

Coming back to (3.12) we find

$$\liminf_{n\to\infty} \int_{\mathbf{R}^N} (|\psi_n|^2 - 1)|\phi_n|^2 \, dx \geqslant 0. \tag{3.13}$$

Let $(\bar\psi_n, \bar\phi_n) = (\psi_n, \phi_n)_{1,\sigma_0}$ and $\bar\zeta_n = (\zeta_n)_{1,\sigma_0}$, where $\sigma_0 = \sqrt{\frac{2(N-1)}{N-3}}$. It is clear that $\sup_{y \in \mathbf{R}^N} E_{GL}^{B(y,1)}(\bar\psi_n) \to 0$ as $n \to \infty$, $\|\phi_n\|_{H^1(\mathbf{R}^N)}$ is bounded and (3.7)–(3.11) hold with $\bar\psi_n$, $\bar\phi_n$ and $\bar\zeta_n$ instead of ψ_n ϕ_n and ζ_n, respectively.

Using (3.8), [9, Lemma 4.2] and the assumption that $0 < c < v_s$ we infer that $E_{GL}(\bar\zeta_n) + 2cQ(\bar\zeta_n) \geqslant 0$ for all sufficiently large n. Then using (3.10) and (3.11) we obtain

$$\liminf_{n\to\infty} \left(E_{GL}(\bar\zeta_n) + 2cQ(\bar\zeta_n)\right) \geqslant 0. \tag{3.14}$$

From (2.5) and the fact that $\varepsilon^2(c^2\delta^2 + k^2) < q^2$ we deduce that

$$\frac{1}{q^2\varepsilon^2} \int_{\mathbf{R}^N} |\nabla \bar\phi_n|^2 dx + \frac{q^2 - \varepsilon^2 k^2}{q^2 \varepsilon^4} \int_{\mathbf{R}^N} |\bar\phi_n|^2 dx + 2\frac{c\delta}{q^2\varepsilon^2} Q(\bar\phi_n) \geqslant 0. \tag{3.15}$$

It is obvious that

$$E_c(\bar\psi_n, \bar\phi_n) \geqslant E_{GL}(\bar\psi_n) + 2cQ(\bar\psi_n)$$

$$+\frac{1}{q^2\varepsilon^2}\int_{\mathbf{R}^N} |\nabla\bar\phi_n|^2 dx + \frac{q^2-\varepsilon^2k^2}{q^2\varepsilon^4}\int_{\mathbf{R}^N} |\bar\phi_n|^2 dx + 2\frac{c\delta}{q^2\varepsilon^2}Q(\bar\phi_n)$$

$$+\frac{1}{\varepsilon^4}\int_{\mathbf{R}^N} \left(|\bar\psi_n|^2 - 1\right)|\bar\phi_n|^2\, dx - \int_{\mathbf{R}^N} \left| V(|\bar\psi_n|^2) - a^2\left(\varphi^2(|\bar\psi_n|) - 1\right)^2 \right| dx.$$

Using (3.14), (3.15), (3.13) and (3.7) we infer that

$$\liminf_{n\to\infty} E_c(\bar\psi_n, \bar\phi_n) \geqslant 0. \tag{3.16}$$

On the other hand, $P_c(\psi_n, \phi_n) = \frac{N-3}{N-1}\sigma_0^{3-N}A(\bar\psi_n, \bar\phi_n) + \sigma_0^{1-N}B_c(\bar\psi_n, \bar\phi_n) \to 0$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty}\left(\frac{N-3}{N-1}\sigma_0^2 A(\bar\psi_n, \bar\phi_n) + B_c(\bar\psi_n, \bar\phi_n)\right) = \lim_{n\to\infty}\left(A(\bar\psi_n, \bar\phi_n) + E_c(\bar\psi_n, \bar\phi_n)\right) = 0.$$

Hence

$$\begin{aligned}\limsup_{n\to\infty} E_c(\bar\psi_n, \bar\phi_n) \leqslant -\liminf_{n\to\infty} A(\bar\psi_n, \bar\phi_n) &= -\sigma_0^{N-3}\liminf_{n\to\infty} A(\psi_n, \phi_n)\\ &\leqslant -\frac{N-1}{2}\sigma_0^{N-3}T_c < 0,\end{aligned}$$

which contradicts (3.16). We conclude that $\alpha > 0$.

Next assume that $\alpha \in (0, \alpha_0)$. Arguing as in [9, p. 156] there exist a sequence of points $(y_n)_{n\geqslant 1} \subset \mathbf{R}^N$ and a sequence $R_n \to \infty$ such that $\varepsilon_n = E_{GL}^{B(y_n,2R_n)\setminus B(y_n,R_n)}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(B(y_n,2R_n)\setminus B(y_n,R_n))} \to 0$ as $n \to \infty$. After a translation, we may suppose that $y_n = 0$.

Let $\chi_1 \in C_c^\infty$ such that $0 \leqslant \chi_1 \leqslant 1$, $\chi_1 = 1$ on $B(0,1)$ and $\mathrm{supp}(\chi_1) \subset B(0, \frac{5}{4})$ and let $\chi_2 \in C^\infty(\mathbf{R}^N)$ such that $\chi_2 = 0$ on $B(0, \frac{7}{4})$ and $\chi_2 = 1$ on $\mathbf{R}^N \setminus B(0,2)$. Denote $\phi_{n,1} = \chi_1(\frac{x}{R_n})\phi_n$ and $\phi_{n,2} = \chi_2(\frac{x}{R_n})\phi_n$. It is easily seen that, as $n \to \infty$,

$$\int_{\mathbf{R}^N} \left||\phi_n|^2 - |\phi_{n,1}|^2 - |\phi_{n,2}|^2\right| dx \to 0,$$

$$\int_{\mathbf{R}^N} \left| \left|\frac{\partial\phi_n}{\partial x_j}\right|^2 - \left|\frac{\partial\phi_{n,1}}{\partial x_j}\right|^2 - \left|\frac{\partial\phi_{n,2}}{\partial x_j}\right|^2 \right| dx \to 0 \qquad \text{for } j = 1, \dots, N,$$

$$|Q(\phi_n) - Q(\phi_{n,1}) - Q(\phi_{n,2})| \to 0.$$

We apply Lemma 2.3 with $A = 2$, $A_2 = \frac{7}{5}$ and $A_3 = \frac{7}{4}$. We infer that there exist two functions $\psi_{n,1}, \psi_{n,2} \in \mathcal{E}$ satisfying properties (i)-(vi) in Lemma 2.3. In particular, we have

$$|E_{GL}(\psi_n) - E_{GL}(\psi_{n,1}) - E_{GL}(\psi_{n,2})| \to 0,$$

$$\left|\tilde E(\psi_n) - \tilde E(\psi_{n,1}) - \tilde E(\psi_{n,2})\right| \to 0,$$

$$|Q(\psi_n) - Q(\psi_{n,1}) - Q(\psi_{n,2})| \to 0 \qquad \text{as } n \to \infty.$$

Moreover, by the choice of A_2, A_3 and χ_1, χ_2 above we get

$$\begin{aligned}\int_{\mathbf{R}^N} |\psi_n|^2|\phi_n|^2\,dx &\geqslant \int_{\mathbf{R}^N} |\psi_n|^2|\phi_n|^2(|\chi_1(\frac{x}{R_n})|^2+|\chi_2(\frac{x}{R_n})|^2)\,dx\\ &\geqslant \int_{\mathbf{R}^N} |\psi_{n,1}|^2|\phi_{n,1}|^2+|\psi_{n,2}|^2|\phi_{n,2}|^2\,dx.\end{aligned}$$

From the above we conclude that

$$\liminf_{n\to\infty}\,(E_c(\psi_n,\phi_n)-E_c(\psi_{n,1},\phi_{n,1})-E_c(\psi_{n,2},\phi_{n,2}))\geqslant 0 \qquad \text{and}$$

$$\liminf_{n\to\infty}\,(P_c(\psi_n,\phi_n)-P_c(\psi_{n,1},\phi_{n,1})-P_c(\psi_{n,2},\phi_{n,2}))\geqslant 0.$$

In addition, it is clear that the sequence $(E_c(\psi_{n,i},\phi_{n,i}))_{n\geqslant 1}$ and $(P_c(\psi_{n,i},\phi_{n,i}))_{n\geqslant 1})$ are bounded for $i=1,2$. Passing to a subsequence (still denoted $(\psi_n,\phi_n)_{n\geqslant 1}$), we may assume that $\lim_{n\to\infty} P_c(\psi_{n,1},\phi_{n,1})=p_1$ and $\lim_{n\to\infty} P_c(\psi_{n,2},\phi_{n,2})=p_2$, where $p_1+p_2=0$. We distinguish two cases.

Case 1. If $p_1=p_2=0$, then using Lemma 3.1 and the fact that there exists K_i such that $E_{GL}(\psi_{n,i})+\frac{1}{\varepsilon^2q^2}\|\phi_{n,i}\|^2_{H^1(\mathbf{R}^N)}\geqslant K_i$ for n large enough, $i=1,2$, we infer that $\liminf_{n\to\infty} E_c(\psi_{n,i},\phi_{n,i})\geqslant T_c$ for $i=1,2$, which yields $\liminf_{n\to\infty} E_c(\psi_n,\phi_n)\geqslant 2T_c$. This contradicts the assumption $E_c(\psi_n,\phi_n)\to T_c$.

Case 2. We have $p_1<0$ or $p_2<0$. If $p_i<0$, it follows from Lemma 2.7 that

$$\liminf_{n\to\infty} A(\psi_{n,i},\phi_{n,i})>\frac{N-1}{2}T_c.$$

Then $\liminf_{n\to\infty} A(\psi_n,\phi_n)\geqslant \liminf_{n\to\infty} A(\psi_{n,i},\phi_{n,i})>\frac{N-1}{2}T_c$, which is in contradiction with (3.4).

So far we have proved that $\alpha=\alpha_0$. Then there exists a sequence $(x_n)_{n\geqslant 1}$ such that for any $\bar\varepsilon>0$, there is $R_{\bar\varepsilon}>0$ satisfying $E_{GL}^{B(x_n,R_{\bar\varepsilon})}(\psi_n)+\frac{1}{\varepsilon^2q^2}\|\phi_n\|^2_{H^1(B(x_n,R_{\bar\varepsilon}))}>\alpha_0-\bar\varepsilon$ for all sufficiently large n. Denoting $\tilde\psi_n=\psi_n(.+x_n)$, $\tilde\phi_n=\phi_n(.+x_n)$, we see that for any $\bar\varepsilon>0$, there are $R_{\bar\varepsilon}>0$ and $n_{\bar\varepsilon}\in\mathbf{N}$ such that

$$E_{GL}^{\mathbf{R}^N\setminus B(0,\mathbf{R}_{\bar\varepsilon})}(\tilde\psi_n)+\frac{1}{\varepsilon^2q^2}\|\tilde\phi_n\|^2_{H^1(\mathbf{R}^N\setminus B(0,\mathbf{R}_{\bar\varepsilon}))}<\bar\varepsilon \qquad \text{for all } n\geqslant n_{\bar\varepsilon}. \tag{3.17}$$

Furthermore, $(\nabla\tilde\psi_n)_n$ is bounded in $L^2(\mathbf{R}^N)$ and $(\tilde\phi_n)_{n\geqslant 1}$ is bounded in $H^1(\mathbf{R}^N)$. Thus there exist functions $\psi\in H^1_{loc}(\mathbf{R}^N)$ with $\nabla\psi\in L^2(\mathbf{R}^N)$, $\phi\in H^1(\mathbf{R}^N)$ and a subsequence $(\tilde\psi_{n_k},\tilde\phi_{n_k})_{k\geqslant 1}$ such that

$$\begin{aligned}&\nabla\tilde\psi_{n_k}\rightharpoonup\nabla\psi \text{ weakly in } L^2(\mathbf{R}^N), \qquad \tilde\phi_{n_k}\rightharpoonup\phi \text{ weakly in } H^1(\mathbf{R}^N),\\ &\tilde\psi_{n_k}\to\psi \text{ in } L^p_{\rm loc}(\mathbf{R}^N), \qquad \tilde\phi_{n_k}\to\phi \text{ in } L^p_{\rm loc}(\mathbf{R}^N), \text{ for all } p\in[1,2^*),\\ &\tilde\psi_{n_k}\to\psi \quad\text{and}\quad \tilde\phi_{n_k}\to\phi \quad \text{a.e on } \mathbf{R}^N.\end{aligned}$$

By the proof of [9, Theorem 5.3] or by [1, Lemma 4.11 and Lemma 4.12] we have

$$\lim_{k\to\infty}\int_{\mathbf{R}^N} V(|\tilde\psi_{n_k}|^2)dx=\int_{\mathbf{R}^N} V(|\psi|^2)dx, \tag{3.18}$$

$$\lim_{k\to\infty}\Big\||\tilde\psi_{n_k}|-|\psi|\Big\|_{L^2(\mathbf{R}^N)}=0,$$

$$\lim_{k\to\infty} Q(\tilde\psi_{n_k}) = Q(\psi).$$

Using the a.e convergence $\tilde\psi_{n_k} \to \psi$, $\tilde\phi_{n_k} \to \phi$ and Fatou lemma we get

$$\int_{\mathbf{R}^N} |\psi|^2|\phi|^2 dx \leqslant \liminf_{k\to\infty} \int_{\mathbf{R}^N} |\tilde\psi_{n_k}|^2|\tilde\phi_{n_k}|^2 dx.$$

The weak convergence $\nabla\tilde\psi_{n_k} \rightharpoonup \nabla\psi$ and $\nabla\tilde\phi_{n_k} \rightharpoonup \nabla\phi$ in $L^2(\mathbf{R}^N)$ implies that

$$A(\psi,\phi) \leqslant \liminf_{k\to\infty} A(\psi_{n_k},\phi_{n_k}) = \frac{N-1}{2}T_c \tag{3.19}$$

and then using Lemma 2.7 we get $P_c(\psi,\phi) \geqslant 0$.

Next we see that $\lim_{k\to\infty} Q(\tilde\phi_{n_k}) = Q(\phi)$. Indeed, the fact that $\frac{\partial\tilde\phi_{n_k}}{\partial x_1} \rightharpoonup \frac{\partial\phi}{\partial x_1}$ weakly in $L^2(B(0,R_{\bar\varepsilon}))$ and $\tilde\phi_{n_k} \to \phi$ in $L^2(B(0,R_{\bar\varepsilon}))$ imply

$$\int_{B(0,R_{\bar\varepsilon})} \left\langle i\frac{\partial\tilde\phi_{n_k}}{\partial x_1}, \tilde\phi_{n_k}\right\rangle dx \to \int_{B(0,R_{\bar\varepsilon})} \left\langle i\frac{\partial\phi}{\partial x_1}, \phi\right\rangle dx.$$

Moreover,

$$\int_{\mathbf{R}^N\setminus B(0,R_{\bar\varepsilon})} \left|\left\langle i\frac{\partial\tilde\phi_{n_k}}{\partial x_1}, \tilde\phi_{n_k}\right\rangle\right| dx \leqslant \left\|\tilde\phi_{n_k}\right\|_{L^2(\mathbf{R}^N)} \left\|\frac{\partial\tilde\phi_{n_k}}{\partial x_1}\right\|_{L^2(\mathbf{R}^N\setminus B(0,R_{\bar\varepsilon}))} \leqslant M\sqrt{\bar\varepsilon}.$$

Thus $Q(\tilde\phi_{n_k}) \to Q(\phi)$ as $k \to \infty$. Hence $P_c(\psi,\phi) \leqslant \liminf_{k\to\infty} P_c(\tilde\psi_{n_k},\tilde\phi_{n_k}) = 0$. Combining with $P_c(\psi,\phi) \geqslant 0$, we obtain $P_c(\psi,\phi) = 0$.

We show that $(|\psi|,|\phi|) \neq (1,0)$. To see this, we observe that

$$\begin{aligned} &-2cQ(\tilde\psi_{n_k}) - 2\frac{c\delta}{q^2\varepsilon^2}Q(\tilde\phi_{n_k}) - \int_{\mathbf{R}^N} V(|\tilde\psi_{n_k}|)dx + \frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} |\phi_{n_k}|^2 dx \\ &\qquad\qquad \geqslant \frac{N-3}{N-1}A(\tilde\psi_{n_k},\tilde\phi_{n_k}) - P_c(\tilde\psi_{n_k},\tilde\phi_{n_k}). \end{aligned}$$

Passing to the limit as $k \to \infty$ we find

$$-2cQ(\psi) - 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi) - \int_{\mathbf{R}^N} V(|\psi|)dx + \frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N} |\phi|^2 dx \geqslant \frac{N-3}{2}T_c > 0,$$

which implies $(|\psi|,|\phi|) \neq (1,0)$. It follows that $(\psi,\phi) \in \mathscr{C}$, and so $A(\psi,\phi) \geqslant \frac{N-1}{2}T_c$. This together with (3.19) gives $A(\psi,\phi) = \frac{N-1}{2}T_c$. Therefore, $E_c(\psi,\phi) = \frac{2}{N-1}A(\psi,\phi) = T_c$ and (ψ,ϕ) is a minimizer of E_c in $\mathscr{C}$.

Since $A(\psi,\phi) = \frac{N-1}{2}T_c = \lim_{k\to\infty} A(\tilde\psi_{n_k},\bar\phi_{n_k})$ and

$$P_c(\psi,\phi) = 0 = \lim_{k\to\infty} P_c(\tilde\psi_{n_k},\bar\phi_{n_k}),$$

we get $\int_{\mathbf{R}^N} |\frac{\partial\psi}{\partial x_i}|^2 dx = \lim_{k\to\infty}\int_{\mathbf{R}^N} |\frac{\partial\tilde\psi_{n_k}}{\partial x_i}|^2 dx$ and $\int_{\mathbf{R}^N} |\frac{\partial\phi}{\partial x_i}|^2 dx = \lim_{k\to\infty}\int_{\mathbf{R}^N} |\frac{\partial\tilde\phi_{n_k}}{\partial x_i}|^2 dx$ for $i = 1,\dots,N$. Thus

$$\lim_{k\to\infty}\left\|\nabla\tilde\psi_{n_k}\right\|^2_{L^2(\mathbf{R}^N)} = \|\nabla\psi\|^2_{L^2(\mathbf{R}^N)} \text{ and } \lim_{k\to\infty}\left\|\nabla\tilde\phi_{n_k}\right\|^2_{L^2(\mathbf{R}^N)} = \|\nabla\phi\|^2_{L^2(\mathbf{R}^N)}.$$

Together with the weak convergence this implies $\nabla\tilde\psi_{n_k} \to \nabla\psi$ and $\nabla\tilde\phi_{n_k} \to \nabla\phi$ strongly in $L^2(\mathbf{R}^N)$. This completes the proof of the theorem.

□

Next we show that the minimizers provided by Theorem 3.2 are solutions to (1.6). We start with a regularity result.

Lemma 3.3. *Assume that $N = 3$ and conditions (A1) and (A2) hold, or that N is arbitrary and the assumptions (A1) and (A3) hold. Let (ψ, ϕ) be a finite energy solution of (1.6). Then $\psi, \phi \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N)$ for any $p \in [1, \infty)$, $\nabla\psi, \nabla\phi \in W^{1,p}(\mathbf{R}^N)$ for any $p \in [2, \infty)$, ψ, ϕ and $\nabla\psi, \nabla\phi$ are bounded and $\psi, \phi \in C^{1,\alpha}(\mathbf{R}^N)$ for $\alpha \in [0, 1)$. Moreover, $|\psi(x)| \to 1$ as $|x| \to \infty$ and $\phi(x) \to 0$ as $|x| \to \infty$.*

Proof. If (A1) and (A3) are satisfied, this follows from [8, Proposition 2.2]. If (A1) and (A2) hold and $N \leqslant 3$ (which means that the system is subcritical), the proof is essentially the same as in [9, Lemma 5.5 and Proposition 5.6], so we omit it. □

Proposition 3.4. *Assume that $N = 4$ and the conditions (A1) and (A2) in the introduction are satisfied. Let $(\psi, \phi) \in \mathcal{E} \times H^1(\mathbf{R}^N)$ be a minimizer of E_c in $\mathscr{C}$. Then (ψ, ϕ) is a solution of the system (1.6).*

Proof. The proof is similar to the proof of [9, Proposition 5.6]. Let $\tilde{P}_c(v_1, v_2) = P_c(\psi + v_1, \phi + v_2)$. We proceed in four steps.

Step 1. There exists a function $\omega = (\omega_1, \omega_2) \in C^1(\mathbf{R}^N, \mathbf{C}^2)$ such that $\tilde{P}_c'(0, 0).\omega \neq 0$.

Step 2. Existence of a Lagrange multiplier α. We have

(3.20)
$$A'(\psi, \phi).v = \alpha P_c'(\psi, \phi).v \text{ for any } v = (v_1, v_2) \in H^1(\mathbf{R}^N, \mathbf{R}^2) \text{ with compact support.}$$

Step 3. We have $\alpha < 0$. Indeed, suppose that $\alpha > 0$. We may assume that $\nabla\tilde{P}_c(0, 0).\omega > 0$. Then for $t < 0$, t sufficiently close to 0, we have $P_c(\psi + t\omega_1, \phi + t\omega_2) < P_c(\psi, \phi) = 0$ and $A(\psi + t\omega_1, \phi + t\omega_2) < A(\psi, \phi) = \frac{N-1}{2}T_c$, contradicts Lemma 2.7. Thus $\alpha \leqslant 0$. If $\alpha = 0$, it follows that $A'(\psi, \phi).v = 0$ for any $v = (v_1, v_2) \in H^1(\mathbf{R}^N, \mathbf{R}^2)$. Let $\chi \in C_c^\infty(\mathbf{R}^N)$ be such that $\chi = 1$ on $B(0; 1)$ and $\mathrm{supp}(\chi) \subset B(0; 2)$. Put $v_n(x) = (\chi(\frac{x}{n})\psi(x), \chi(\frac{x}{n})\phi(x))$. Replacing v by v_n and passing to the limit as $n \to \infty$, we get $A(\psi, \phi) = 0$, which contradicts the fact that $A(\psi, \phi) = \frac{N-1}{2}T_c$ and so $\alpha < 0$.

Step 4. Conclusion. From (3.20), it follows that ψ, ϕ satisfy

(3.21)
$$-\frac{\partial^2\psi}{\partial x_1^2} - \left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right)\sum_{k=2}^{N}\frac{\partial^2\psi}{\partial x_k^2} + 2ic\psi_{x_1} - F(|\psi|^2)\psi + \frac{1}{\varepsilon^4}|\phi|^2\psi = 0 \text{ in } \mathcal{D}'(\mathbf{R}^N),$$

(3.22)
$$-\frac{\partial^2\phi}{\partial x_1^2} - \left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right)\sum_{k=2}^{N}\frac{\partial^2\phi}{\partial x_k^2} + 2ic\delta\phi_{x_1} + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2 k^2)\phi = 0 \text{ in } \mathcal{D}'(\mathbf{R}^N).$$

Then $(\psi_{1,\sigma_0}, \phi_{1,\sigma_0})$ satisfy (1.6), where $\sigma_0 = \left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right)^{-\frac{1}{2}}$. Therefore the conclusion of Lemma 3.3 holds for $(\psi_{1,\sigma_0}, \phi_{1,\sigma_0})$, and so for (ψ, ϕ). By the same argument as in the proof of [8, Proposition 4.1], we have $(\psi_{1,\sigma_0}, \phi_{1,\sigma_0})$ satisfies the Pohozaev identity

$$\frac{N-3}{N-1} A(\psi_{1,\sigma_0}, \phi_{1,\sigma_0}) + B_c(\psi_{1,\sigma_0}, \phi_{1,\sigma_0}) = 0,$$

which gives that $\frac{N-3}{N-1}\sigma_0^{N-3} A(\psi, \phi) + \sigma_0^{N-1} B_c(\psi, \phi) = 0$ and thus,

$$\frac{N-3}{N-1}\left(\frac{N-3}{N-1} - \frac{1}{\alpha}\right) A(\psi, \phi) + B_c(\psi, \phi) = 0.$$

Combining with the fact that $P_c(\psi, \phi) = \frac{N-3}{N-1} A(\psi, \phi) + B_c(\psi, \phi) = 0$ and $A(\psi, \phi) > 0$, we get $\frac{N-3}{N-1} - \frac{1}{\alpha} = 1$. Then (ψ, ϕ) satisfies (1.6).

□

4. The case $N = 3$

If $N = 3$ the minimization problem considered in the previous section is more difficult because it is invariant by scaling with respect to the (x_2, x_3) variables. Indeed, for $(\psi, \phi) \in \mathcal{E} \times H^1(\mathbf{R}^N)$ let $A(\psi, \phi)$, $B_c(\psi, \phi)$ and $D(\psi, \phi)$ be as in (2.8), (2.9) and (2.11), respectively. Note that for $N = 3$ we have $P_c = B_c$ and $E_c = A + B_c$. For any $\sigma > 0$ we have

$$A(\psi_{1,\sigma}, \phi_{1,\sigma}) = A(\psi, \phi), \quad B_c(\psi_{1,\sigma}, \phi_{1,\sigma}) = \sigma^2 B_c(\psi, \phi) \text{ and } D(\psi_{1,\sigma}, \phi_{1,\sigma}) = \sigma^2 D(\psi, \phi).$$

Since $E_c(\psi_{1,\sigma}, \phi_{1,\sigma}) = E_c(\psi, \phi)$ for all $\sigma > 0$ and all $(\psi, \phi) \in \mathscr{C}$, there exists a sequence $(\psi_n, \phi_n)_{n \geqslant 1}$ such that

$$D(\psi_n, \phi_n) = 1 \text{ and } E_c(\psi_n, \phi_n) = A(\psi_n, \phi_n) \to T_c \text{ as } n \to \infty. \tag{4.1}$$

We denote

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{there exists a sequence } (\psi_n, \phi_n)_{n \geqslant 1} \subset \mathcal{E} \times H^1(\mathbf{R}^N) \text{ such that } D(\psi_n, \phi_n) \geqslant 1, B_c(\psi_n, \phi_n) \to 0 \text{ and } A(\psi_n, \phi_n) \to \lambda \text{ as } n \to \infty\}.$$

It is easy to see that

$$\Lambda_c = \{\lambda \in \mathbf{R} \mid \text{there exists a sequence } (\psi_n, \phi_n)_{n \geqslant 1} \subset \mathcal{E} \times H^1(\mathbf{R}^N) \text{ and } C > 0 \text{ such that } D(\psi_n, \phi_n) \geqslant C, B_c(\psi_n, \phi_n) \to 0 \text{ and } A(\psi_n, \phi_n) \to \lambda \text{ as } n \to \infty\}.$$

Denote $\lambda_c = \inf \Lambda_c \geqslant 0$. Arguing as in [9, Lemma 6.1] we infer that $\lambda_c \geqslant S_c$, where S_c is given in (2.7). From (4.1) we have $T_c \in \Lambda_c$. Moreover, we see that Λ_c is closed in $\mathbf{R}$, and so $\lambda_c \in \Lambda_c$. Therefore

$$0 < S_c \leqslant \lambda_c \leqslant T_c.$$

Theorem 4.1. *Assume that $N = 3$, (A1) and (A2) are satisfied, $0 < c < v_s$ and $\varepsilon^2(c^2\delta^2 + k^2) < q^2$. Let $(\psi_n, \phi_n)_{n \geqslant 1} \subset \mathcal{E} \times H^1(\mathbf{R}^3)$ be a sequence such that*

$$D(\psi_n, \phi_n) \to 1, \qquad B_c(\psi_n, \phi_n) \to 0 \qquad \text{and} \qquad A(\psi_n, \phi_n) \to \lambda_c \qquad \text{as } n \to \infty.$$

Then there exist a subsequence $(\psi_{n_k}, \phi_{n_k})_{k \geqslant 1}$, a sequence $(x_k)_{k \geqslant 1} \subset \mathbf{R}^3$ and $(\psi, \phi) \in \mathscr{C}$ such that

$$\nabla \psi_{n_k}(. + x_k) \to \nabla \psi, \qquad |\psi_{n_k}(. + x_k)| - 1 \to |\psi| - 1 \qquad \text{in } L^2(\mathbf{R}^3) \text{ and}$$
$$\phi_{n_k}(. + x_k) \to \phi \qquad \text{in } H^1(\mathbf{R}^3).$$

Moreover, $E_c(\psi, \phi) = T_c$ and (ψ, ϕ) is a minimizer of E_c in $\mathcal{C}$.

Proof. We have $E_{GL}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(\mathbf{R}^3)} = A(\psi_n,\phi_n) + D(\psi_n,\phi_n) \to \lambda_c + 1$ as $n \to \infty$. Let $q_n(t) = \sup_{y\in\mathbf{R}^3}\left(E_{GL}^{B(y,t)}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(B(y,t))}\right)$. Proceeding as in [9, Theorem 5.3], there exist a subsequence of $((\psi_n,\phi_n),q_n)_{n\geqslant 1}$, still denoted $((\psi_n,\phi_n),q_n)_{n\geqslant 1}$, a nondecreasing function $q:[0,\infty)\to\mathbf{R}$ and $\alpha\in[0,\alpha_0]$ such that

$$q_n(t)\to q(t) \text{ a.e on } [0,\infty) \text{ as } n\to\infty \text{ and } q(t)\to\alpha \text{ as } t\to\infty.$$

First we show that $\alpha > 0$. Arguing by contradiction, we assume that there exists a subsequence (still denoted $(\psi_n,\phi_n)_{n\geqslant 1}$) such that

$$\lim_{n\to\infty}\sup_{y\in\mathbf{R}^3}\left(E_{GL}^{B(y,1)}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(B(y,1))}\right) = 0. \tag{4.2}$$

If the sequence $(\phi_n)_{n\geqslant 1}\subset H^1(\mathbf{R}^N)$ is bounded in $H^1(\mathbf{R}^N)$ and satisfies

$$\lim_{n\to\infty}\sup_{y\in\mathbf{R}^N}\|\phi_n\|^2_{H^1(B(y,1))} = 0,$$

it is standard to prove that $\phi_n \longrightarrow 0$ in $L^p(\mathbf{R}^N)$ for any $p\in(2,2^*)$. For $N=3$ we find $\phi_n \longrightarrow 0$ in $L^p(\mathbf{R}^3)$ for any $p\in(2,6)$. In particular, since $|\psi_n|^2-1$ is bounded in $L^2(\mathbf{R}^N)$ we get

$$\int_{\mathbf{R}^3}\big||\psi_n|^2-1\big|\,|\phi_n|^2\,dx \leqslant \|\,|\psi_n|^2-1\|_{L^2(\mathbf{R}^3)}\|\phi_n\|^2_{L^4(\mathbf{R}^3)} \to 0 \qquad \text{as } n\longrightarrow\infty. \tag{4.3}$$

As in the case $N=4$, we have

$$\lim_{n\to\infty}\int_{\mathbf{R}^3}\left|V(|\psi_n|^2) - a^2(\varphi^2(|\psi_n|)-1)^2\right|dx = 0. \tag{4.4}$$

From (4.3), (4.4) and the assumption that $B_c(\psi_n,\phi_n)\to 0$, $D(\psi_n,\phi_n)\to 1$ as $n\to\infty$ we get

$$\lim_{n\to\infty}\left[2cQ(\psi_n) + 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi_n)\right] = -1. \tag{4.5}$$

Fix $c_1\in(c,v_s)$ such that $\varepsilon^2(c_1^2\delta^2+k^2) < q^2$. Then fix $\sigma>0$ such that $\sigma^2 > \frac{(\lambda_c+2)c}{c_1-c}$. Let $(\bar\psi_n,\bar\phi_n) = (\psi_n,\phi_n)_{1,\sigma}$. Then $(\bar\psi_n,\bar\phi_n)_{n\geqslant 1}$ also satisfies (4.4). Moreover, $E_{GL}(\bar\psi_n)$ is bounded and $\lim_{n\to\infty}\sup_{y\in\mathbf{R}^3}E_{GL}^{B(y,1)}(\bar\psi_n) = 0$. By [9, Lemma 3.2], there is a sequence $(\zeta_n)_{n\geqslant 1}\subset\mathcal{E}$ such that (3.8)–(3.11) hold for $\bar\psi_n$ and ζ_n. Proceeding as in the proof of Theorem 3.2 we infer that

$$\liminf_{n\to\infty}E_{c_1}(\bar\psi_n,\bar\phi_n)\geqslant 0,$$

that is

$$\liminf_{n\to\infty}\left[A(\psi_n,\phi_n) + \sigma^2 B_c(\psi_n,\phi_n) + \sigma^2(c_1-c)\left(2Q(\psi_n) + 2\frac{\delta}{q^2\varepsilon^2}Q(\phi_n)\right)\right]\geqslant 0.$$

Taking into account (4.5), this implies $\lambda_c - \sigma^2\frac{c_1-c}{c}\geqslant 0$, contradicting the choice of σ. Thus we cannot have $\alpha = 0$.

Next suppose that $\alpha\in(0,\lambda_c+1)$. Then there exist a sequence of points $(y_n)_{n\geqslant 1}\subset\mathbf{R}^3$ and a sequence $R_n\to\infty$ such that $\varepsilon_n = E_{GL}^{B(y_n,2R_n)\setminus B(y_n,R_n)}(\psi_n) + \frac{1}{\varepsilon^2 q^2}\|\phi_n\|^2_{H^1(B(y_n,2R_n)\setminus B(y_n,R_n))}\to 0$ as $n\to\infty$. After a translation, we may assume that $y_n = 0$. Proceeding as in the proof of Theorem 3.2 in the case $N > 3$ and using the same cut-off functions χ_1 and χ_2, we infer that there exist functions $\psi_{n,1}, \psi_{n,2}$ and $\phi_{n,1}, \phi_{n,2}$ satisfying

(4.6)
$$E_{GL}(\psi_{n,1}) + \frac{1}{\varepsilon^2 q^2}\|\phi_{n,1}\|^2_{H^1(\mathbf{R}^3)} \to \alpha, \qquad E_{GL}(\psi_{n_2}) + \frac{1}{\varepsilon^2 q^2}\|\phi_{n,2}\|^2_{H^1(\mathbf{R}^3)} \to \lambda_c + 1 - \alpha,$$

(4.7)
$$|A(\psi_n, \phi_n) - A(\psi_{n,1}, \phi_{n,1}) - A(\psi_{n,2}, \phi_{n,2})| \to 0,$$
$$|D(\psi_n, \phi_n) - D(\psi_{n,1}, \phi_{n,1}) - D(\psi_{n,2}, \phi_{n,2})| \to 0.$$

Exactly as in the proof of Theorem 3.2 we find
$$\liminf_{n\to\infty} (B_c(\psi_n, \phi_n) - B_c(\psi_{n,1}, \phi_{n,1}) - B_c(\psi_{n,2}, \phi_{n,2})) \geqslant 0.$$

From the boundedness of $(E_{GL}(\psi_{n,i})_{n\geqslant 1}$ and of $(\|\phi_{n,i}\|_{H^1(\mathbf{R}^3)})_{n\geqslant 1}$ we obtain that $(B_c(\psi_{n,i}))_{n\geqslant 1}$ are bounded. Passing to a subsequence (still denoted $(\psi_n, \phi_n)_{n\geqslant 1}$), it may assume that
$$\lim_{n\to\infty} B_c(\psi_{n,1}, \psi_{n,1}) = b_1, \lim_{n\to\infty} B_c(\psi_{n,2}, \psi_{n,2}) = b_2, \text{ where } b_i \in \mathbf{R}, b_1 + b_2 \leqslant 0,$$
$$\lim_{n\to\infty} D(\psi_{n,1}, \psi_{n,1}) = d_1, \lim_{n\to\infty} D(\psi_{n,2}, \psi_{n,2}) = d_2, \text{ where } d_i \in \mathbf{R}, d_1 + d_2 = 1.$$

If one of the b_i's is negative, say $b_1 < 0$, proceeding as in the proof of [9, Lemma 4.8] it follows that
$$\liminf_{n\to\infty} A(\psi_{n,1}, \phi_{n,1}) > T_c \geqslant \lambda_c,$$
contradicting the assumption $\lim_{n\to\infty} A(\psi_n, \phi_n) = \lambda_c$. Thus necessarily $b_1 = b_2 = 0$. We distinguish two cases.

Case 1. If $d_i > 0$ for $i = 1, 2$, the definition of λ_c implies that $\liminf_{n\to\infty} A(\psi_{n,i}, \phi_{n_i}) \geqslant \lambda_c$ for $i = 1, 2$. By combining with (4.7),
$$\liminf_{n\to\infty} A(\psi_n, \phi_n) \geqslant \liminf_{n\to\infty} A(\psi_{n,1}, \phi_{n,1}) + \liminf_{n\to\infty} A(\psi_{n,1}, \phi_{n,1}) \geqslant 2\lambda_c,$$
contradicting the assumption $\lim_{n\to\infty} A(\psi_n, \phi_n) = \lambda_c$.

Case 2. If one of the d_i is zero, assume that $d_1 = 0$, then $d_2 = 1$. From (4.6) and the fact that $E_{GL}(\psi_{n,2}) + \frac{1}{\varepsilon^2 q^2}\|\phi_{n,2}\|^2_{H^1(\mathbf{R}^3)} = A(\psi_{n,2}, \phi_{n,2}) + D(\psi_{n,2}, \phi_{n,2})$, we obtain $A(\psi_{n,2}, \phi_{n,2}) \to \lambda_c - \alpha$. This together with $d_2 = 1, b_2 = 0$ implies that $\lambda_c - \alpha \in \Lambda_c$. This is a contradiction with the definition of λ_c.

Therefore we cannot have $\alpha \in (0, \lambda_c + 1)$ and so necessarily $\alpha = \lambda_c + 1$. As in the case $N \geqslant 4$, there exist a subsequence $(\psi_{n_k}, \phi_{n_k})_{k\geqslant 1}$, a sequence of points $(x_k)_{k\geqslant 1} \subset \mathbf{R}^3$ and $(\psi, \phi) \in \mathcal{E} \times H^1(\mathbf{R}^3)$ such that, denoting $(\tilde{\psi}_{n_k}(x), \tilde{\phi}_{n_k}(x)) = (\psi_{n_k}(x + x_k), \phi_{n_k}(x + x_k))$, (3.17) holds and we have
$$\nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi \text{ weakly in } L^2(\mathbf{R}^3), \qquad \tilde{\phi}_{n_k} \rightharpoonup \phi \text{ weakly in } H^1(\mathbf{R}^3),$$
$$\tilde{\psi}_{n_k} \to \psi \text{ in } L^p_{\text{loc}}, \qquad \tilde{\phi}_{n_k} \to \phi \text{ in } L^p_{\text{loc}}, \text{ where } p \in [1, 6),$$
$$\tilde{\psi}_{n_k} \to \psi \text{ a.e }, \qquad \tilde{\phi}_{n_k} \to \phi \text{ a.e on } \mathbf{R}^N.$$

Moreover,

(4.8)
$$\int_{\mathbf{R}^3} |\psi|^2|\phi|^2 dx \leqslant \liminf_{k\to\infty} \int_{\mathbf{R}^3} |\tilde{\psi}_{n_k}|^2|\tilde{\phi}_{n_k}|^2 dx \qquad \text{by Fatou Lemma,}$$

$$\lim_{k\to\infty}\int_{\mathbf{R}^3} V(|\tilde{\psi}_{n_k}|^2)dx = \int_{\mathbf{R}^3} V(|\psi|^2)dx \qquad \text{and} \tag{4.9}$$

$$\lim_{k\to\infty}\left\| |\tilde{\psi}_{n_k}| - |\psi| \right\|_{L^2(\mathbf{R}^3)} = 0 \qquad \text{by [1, Lemma 4.11]} \tag{4.10}$$

$$\lim_{k\to\infty} Q(\tilde{\psi}_{n_k}) = Q(\psi) \text{ and } \lim_{k\to\infty} Q(\tilde{\phi}_{n_k}) = Q(\phi) \qquad \text{by [1, Lemma 4.12].} \tag{4.11}$$

The weak convergences $\nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi$ and $\tilde{\phi}_{n_k} \rightharpoonup \phi$ in $L^2(\mathbf{R}^3)$ imply that

$$\int_{\mathbf{R}^3}\left|\frac{\partial\psi}{\partial x_j}\right|^2 dx \leqslant \liminf_{k\to\infty}\int_{\mathbf{R}^3}\left|\frac{\partial\tilde{\psi}_{n_k}}{\partial x_j}\right|^2 dx \text{ for } j = 1,2,3, \tag{4.12}$$

and

$$\int_{\mathbf{R}^3}\left|\frac{\partial\phi}{\partial x_j}\right|^2 dx \leqslant \liminf_{k\to\infty}\int_{\mathbf{R}^3}\left|\frac{\partial\tilde{\phi}_{n_k}}{\partial x_j}\right|^2 dx \text{ for } j = 1,2,3. \tag{4.13}$$

Thus

$$A(\psi,\phi) \leqslant \liminf_{k\to\infty} A(\tilde{\psi}_{n_k},\tilde{\phi}_{n_k}) = \lambda_c \leqslant T_c. \tag{4.14}$$

This together with Lemma 2.7 gives $B_c(\psi,\phi) \geqslant 0$. From (4.8), (4.9), (4.11), (4.12) and (4.13), we obtain $B_c(\psi,\phi) \leqslant \liminf_{k\to\infty} B_c(\tilde{\psi}_{n_k},\tilde{\phi}_{n_k}) = 0$ and so $B_c(\psi,\phi) = 0$. We claim that $(|\psi|,|\phi|) \neq (1,0)$. Indeed,

$$-2cQ(\tilde{\psi}_{n_k}) - 2\frac{c\delta}{q^2\varepsilon^2}Q(\tilde{\phi}_{n_k}) - \int_{\mathbf{R}^3} V(|\tilde{\psi}_{n_k}|^2)dx + \frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N}|\phi_{n_k}|^2 dx + a^2\int_{\mathbf{R}^3}\left(\varphi^2(|\tilde{\psi}_{n_k}|) - 1\right)^2 dx$$
$$\geqslant D(\tilde{\psi}_{n_k},\tilde{\phi}_{n_k}) - B_c(\tilde{\psi}_{n_k},\tilde{\phi}_{n_k}).$$

Passing to the limit as $k \to \infty$,

$$-2cQ(\psi) - 2\frac{c\delta}{q^2\varepsilon^2}Q(\phi) - \int_{\mathbf{R}^N} V(|\psi|^2)dx + \frac{k^2}{\varepsilon^2 q^2}\int_{\mathbf{R}^N}|\phi|^2 dx + a^2\int_{\mathbf{R}^3}\left(\varphi^2(|\psi|) - 1\right)^2 dx \geqslant 1,$$

which yields that $(|\psi|,|\phi|) \neq (1,0)$. Hence $A(\psi,\phi) \geqslant T_c$. Using (4.14), we obtain $A(\psi,\phi) = T_c = \lambda_c = \lim_{k\to\infty} A(\tilde{\psi}_{n_k},\tilde{\phi}_{n_k})$. Moreover, since $B_c(\psi,\phi) = 0 = \lim_{k\to\infty} B_c(\tilde{\psi}_{n_k},\tilde{\phi}_{n_k})$ it follows that

$$\int_{\mathbf{R}^3}\left|\frac{\partial\psi}{\partial x_1}\right|^2 dx = \lim_{k\to\infty}\int_{\mathbf{R}^3}\left|\frac{\partial\tilde{\psi}_{n_k}}{\partial x_1}\right|^2 dx,$$

and

$$\int_{\mathbf{R}^3}\left|\frac{\partial\phi}{\partial x_1}\right|^2 dx = \lim_{k\to\infty}\int_{\mathbf{R}^3}\left|\frac{\partial\tilde{\phi}_{n_k}}{\partial x_1}\right|^2 dx.$$

Thus $\lim_{k\to\infty}\left\|\nabla\tilde{\psi}_{n_k}\right\|^2_{L^2(\mathbf{R}^3)} = \|\nabla\psi\|^2_{L^2(\mathbf{R}^3)}$ and $\lim_{k\to\infty}\left\|\nabla\tilde{\phi}_{n_k}\right\|^2_{L^2(\mathbf{R}^3)} = \|\nabla\phi\|^2_{L^2(\mathbf{R}^3)}$. Together with the weak convergences $\nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi$ and $\nabla\tilde{\psi}_{n_k} \rightharpoonup \nabla\psi$ in $L^2(\mathbf{R}^3)$, this implies that $\nabla\tilde{\psi}_{n_k} \to \nabla\psi$ and $\nabla\tilde{\phi}_{n_k} \to \nabla\phi$ strongly in $L^2(\mathbf{R}^3)$. ☐

Proposition 4.2. *Assume that $N = 3$, the conditions (A1) and (A2) in the introduction are satisfied. Let $(\psi, \phi) \in \mathcal{E} \times H^1(\mathbf{R}^3)$ be a minimizer of E_c in $\mathscr{C}$. Then $\psi, \phi \in W^{2,p}_{\text{loc}}(\mathbf{R}^3)$ for any $p \in [1, \infty)$, $\nabla\psi, \nabla\phi \in W^{1,p}(\mathbf{R}^3)$ for any $p \in [2, \infty)$ and there exists $\sigma > 0$ such that $(\psi_{1,\sigma}, \phi_{1,\sigma})$ is a solution of system (1.6).*

Proof. We have $A(\psi, \phi) = E_c(\psi, \phi)$ and (ψ, ϕ) is a minimizer of A in $\mathscr{C}$. It is clear that for any $(\psi, \phi) \in \mathcal{E} \times H^1(\mathbf{R}^N)$ and for any $R > 0$, the functionals $\tilde{B}_c^{(\psi,\phi)}(\omega_1, \omega_2) := B_c(\psi + \omega_1, \phi + \omega_2)$ and $\tilde{A}(\omega_1, \omega_2) := A(\psi + \omega_1, \phi + \omega_2)$ are C^1 on $H^1_0(B(0, R), \mathbf{C}^2)$. We proceed in four steps.

Step 1. There exists a function $\omega = (\omega_1, \omega_2) \in C^1(\mathbf{R}^3, \mathbf{C}^2)$ such that $d\tilde{B}_c^{(\psi,\phi)}(0, 0).\omega \neq 0$, where d denotes the Gâteaux differential. To see this we argue as in the proof of [9, Lemma 6.4].

Step 2. Existence of a Lagrange multiplier α. We have

(4.15)

$$\tilde{A}'(0, 0).v = \alpha d\tilde{B_c^{(\psi,\phi)}}(0, 0).v \text{ for any } v = (v_1, v_2) \in H^1(\mathbf{R}^3, \mathbf{C}^2) \text{ with compact support.}$$

Step 3. We have $\alpha < 0$.

Note that the proof of Step 2 and 3 is the same as the proof of Step 2 and 3 in Proposition 3.4.

Step 4. Conclusion. From (4.15), it follows that ψ, ϕ satisfy

$$-\frac{\partial^2\psi}{\partial x_1^2} + \frac{1}{\alpha}\sum_{k=2}^{N}\frac{\partial^2\psi}{\partial x_k^2} + 2ic\psi_{x_1} - F(|\psi|^2)\psi + \frac{1}{\varepsilon^4}|\phi|^2\psi = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3), \tag{4.16}$$

$$-\frac{\partial^2\phi}{\partial x_1^2} + \frac{1}{\alpha}\sum_{k=2}^{N}\frac{\partial^2\phi}{\partial x_k^2} + 2ic\delta\phi_{x_1} + \frac{1}{\varepsilon^2}(q^2|\psi|^2 - \varepsilon^2k^2)\phi = 0 \text{ in } \mathcal{D}'(\mathbf{R}^3). \tag{4.17}$$

Then $(\psi_{1,\sigma_0}, \phi_{1,\sigma_0})$ satisfy (1.6), where $\sigma_0 = \left(-\frac{1}{\alpha}\right)^{-\frac{1}{2}}$. It is clear that $(\psi_{1,\sigma}, \phi_{1,\sigma}) \in \mathscr{C}$ and minimizes A (respectively E_c) in $\mathscr{C}$. From Lemma 3.3 we obtain the regularity of $(\psi_{1,\sigma}, \phi_{1,\sigma})$ and of (ψ, ϕ).

□

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Résumé

Cette thèse porte sur l'étude des solutions spéciales (de type onde progressive et onde stationnaire) pour des équations aux dérivées partielles dispersives non-linéaires dans $\mathbf{R}^N$. Les problèmes considérés ont une structure variationnelle, les solutions sont des points critiques de certaines fonctionnelles. Nous démontrons l'existence des points critiques en utilisant des méthodes de minimisation. Une des principales difficultés vient du manque de compacité. Pour y remédier, on utilise quelques raffinements récents du principe de concentration-compacité de P.-L. Lions.

Dans la première partie du mémoire on montre l'existence des solutions d'énergie minimale pour des équations elliptiques quasi-linéaires dans $\mathbf{R}^N$. Nous généralisons les résultats de Brézis et Lieb dans le cas du Laplacien, ainsi que les résultats de Jeanjean et Squassina dans le cas du p-Laplacien.

Dans la seconde partie on montre l'existence des ondes progressives subsoniques d'énergie finie pour un système de Gross-Pitaevskii-Schrödinger qui modélise le mouvement d'une impureté non chargée dans un condensat de Bose-Einstein. Les résultats obtenus sont valables en dimension trois et quatre d'espace.

Mots-clés. Équations elliptiques non-linéaires · équation de Schrödinger nonlinéaire · système de Gross-Pitaevskii-Schrödinger · onde stationnaire · onde progressive · minimisation · minimisation sous contrainte · principe de concentration-compacité.

Summary

This thesis focuses on the study of special solutions (traveling wave and standing wave type) for nonlinear dispersive partial differential equations in $\mathbf{R}^N$. The considered problems have a variational structure, the solutions are critical points of some functionals. We demonstrate the existence of critical points using minimization methods. One of the main difficulties comes from the lack of compactness. To overcome this, we use some recent improvements of P.-L. Lions concentration-compactness principle.

In the first part of the dissertation, we show the existence of the least energy solutions to quasi-linear elliptic equations in $\mathbf{R}^N$. We generalize the results of Brézis and Lieb in the case of the Laplacian, and the results of Jeanjean and Squassina in the case of the p-Laplacian.

In the second part, we show the existence of subsonic travelling waves of finite energy for a Gross-Pitaevskii-Schrödinger system which models the motion of a non charged impurity in a Bose-Einstein condensate. The obtained results are valid in three and four dimensional space.

Keywords. Nonlinear elliptic equations · nonlinear Schrödinger equation · Gross-Pitaevskii-Schrödinger system · standing wave· travelling wave · minimization · constrained minimization · concentration-compactness principle.