This is a postprint version of the following published document:

CYCLICITY IN DIRICHLET-TYPE SPACES AND EXTREMAL POLYNOMIALS

CATHERINE BÉNÉTEAU, ALBERTO A. CONDORI, CONSTANZE LIAW, DANIEL SECO, AND ALAN A. SOLA

ABSTRACT. For functions f in Dirichlet-type spaces D_α, we study how to determine constructively optimal polynomials p_n that minimize $\|pf - 1\|_\alpha$ among all polynomials p of degree at most n. Then we give upper and lower bounds for the rate of decay of $\|p_n f - 1\|_\alpha$ as n approaches ∞. Further, we study a generalization of a weak version of the Brown-Shields conjecture and some computational phenomena about the zeros of optimal polynomials.

1. INTRODUCTION

1.1. **Cyclicity in spaces of analytic functions.** In this paper, we study certain Hilbert spaces of analytic functions in the open unit disk $\mathbb{D}$, denoted D_α and referred to as *Dirichlet-type spaces of order α*. For $-\infty < \alpha < \infty$, the space D_α consists of all analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ whose Taylor coefficients in the expansion

$$f(z) = \sum_{k=0}^\infty a_k z^k, \quad z \in \mathbb{D},$$

satisfy

$$\|f\|_\alpha^2 = \sum_{k=0}^\infty (k+1)^\alpha |a_k|^2 < \infty.$$

It is easy to see that $D_\alpha \subseteq D_\beta$ when $\alpha \geq \beta$, and $f \in D_\alpha$ if and only if the derivative $f' \in D_{\alpha-2}$.

Three values of α correspond to spaces that have been studied extensively and are often defined in terms of integrability:

- $\alpha = -1$ corresponds to *the Bergman space B*, consisting of functions with

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty, \quad dA(z) = \frac{dxdy}{\pi},$$

- $\alpha = 0$ corresponds to *the Hardy space H^2*, consisting of functions with

$$\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$

- and $\alpha = 1$ corresponds to the usual *Dirichlet space D* of functions f with

$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.$$

A description similar to that of the Dirichlet space, in terms of an integral, is possible for the D_α spaces for $\alpha < 2$. Indeed, $f \in D_\alpha$ if and only if

$$D_\alpha(f) = \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} dA(z) < \infty. \tag{1.1}$$

This expression can be used to define an equivalent norm for $f \in D_\alpha$, which we use in Section 2. We refer the reader to the books [5], [6] and [9] for in-depth treatments of Hardy and Bergman spaces; recent surveys concerning the Dirichlet space D include [1] and [12].

A function $f \in D_\alpha$ is said to be *cyclic* in D_α if the closed subspace generated by polynomial multiples of f,

$$[f] = \overline{\text{span}\{z^k f \colon k = 0, 1, 2, \ldots\}},$$

coincides with D_α. Note that cyclicity in D_α implies cyclicity in D_β for all $\beta < \alpha$. The *multiplier space* $M(D_\alpha)$ consists of analytic functions ψ such that the induced operator $M_\psi \colon f \mapsto \psi f$ maps D_α into itself; such a function ψ is called a *multiplier*. Thus cyclic functions are precisely those that are cyclic with respect to the operator M_z. Since polynomials are dense in D_α, we have $[1] = D_\alpha$. It is well known (see [4]) that an equivalent (and more useful) condition for the cyclicity of f is that there exist a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ such that

$$\|p_n f - 1\|_\alpha \to 0, \quad \text{as } n \to \infty.$$

We note that for certain values of α, the multiplier spaces of D_α are relatively easy to determine. For $\alpha \leq 0$, we have $M(D_\alpha) = H^\infty$, and when $\alpha > 1$ the multiplier space coincides with D_α itself (see [4, p. 273]).

In general, it is not an easy problem to characterize cyclic functions in a space of analytic functions. However, a complete answer to the cyclicity problem for H^2 (the case $\alpha = 0$) is given by a theorem of Beurling (see [5, Chapter 7]): f is cyclic if and only if f is an outer function. In particular, a cyclic function $f \in H^2$ cannot vanish in $\mathbb{D}$. In the Bergman space, the situation is considerably more complicated (see [9, Chapter 7]). A common

feature of all D_α is that cyclic functions have to be non-vanishing in $\mathbb{D}$. If $\alpha > 1$, to be non-vanishing in the *closed* unit disk, or equivalently,

$$|f(z)| > c > 0, \quad z \in \mathbb{D},$$

is a necessary and sufficient condition (see [4]) for cyclicity. However, when $\alpha \leq 1$, functions may still be cyclic if their zero set on the boundary, that is, the unit circle $\mathbb{T}$, is not too big. Here, we define the zero set in an appropriate sense via, for instance, non-tangential limits.

In [4], L. Brown and A.L. Shields studied the phenomenon of cyclicity in the Dirichlet space. In particular, they established the following equivalent condition for cyclicity: f is cyclic in D_α if and only if there exists a sequence of polynomials $\{p_n\}$ such that

$$\sup_n \|p_n f - 1\|_\alpha < \infty \tag{1.2}$$

and, pointwise as $n \to \infty$,

$$p_n(z)f(z) \to 1, \quad z \in \mathbb{D}. \tag{1.3}$$

Brown and Shields also obtained a number of partial results towards a characterization of cyclic vectors in the Dirichlet space D. Their starting point was a result of Beurling, stating that, for any $f \in D$, the non-tangential limit $f^*(\zeta) = \lim_{z \to \zeta} f(z)$ exists *quasi-everywhere*, that is, outside a set of logarithmic capacity zero. Brown and Shields proved that if the zeros of f^*,

$$\mathcal{Z}(f^*) = \{\zeta \in \mathbb{T} \colon f^*(\zeta) = 0\},$$

form a set of positive logarithmic capacity, then f cannot be cyclic. On the other hand, they also proved that $(1-z)^\beta$ is cyclic for any $\beta > 0$ and showed that any polynomial without zeros in $\mathbb{D}$ is cyclic. Hence, they asked if being outer and having $\mathrm{cap}(\mathcal{Z}(f^*)) = 0$ is sufficient for f to be cyclic. This problem remains open and is commonly referred to as the *Brown-Shields conjecture*; see however [8] for recent progress by El-Fallah, Kellay, and Ransford, and for background material. Subsequent to the Brown and Shields paper, Brown and Cohn showed (see [3]) that sets of logarithmic capacity zero do support zeros of cyclic functions, and later Brown (see [2]) proved that if $f \in D$ is invertible, that is $1/f \in D$, then f is cyclic. However, there are cyclic functions f for which $1/f \notin D$, e.g. $f(z) = 1 - z$.

The problem of cyclicity in D has been addressed in many papers. An incomplete list includes [10], where sufficient conditions for cyclicity are given in terms of Bergman-Smirnov exceptional sets; the paper [7], where these ideas are developed further, and examples of uncountable Bergman-Smirnov exceptional sets are found; and [11] where multipliers and invariant subspaces are discussed, leading, for instance, to a proof that non-vanishing univalent functions in the Dirichlet space are cyclic.

1.2. **Plan of the paper.** In this paper, we set out to improve understanding of cyclicity by studying certain classes of cyclic functions in detail. Many of the results in this paper are variations of the following questions: Given a cyclic function $f \in D_\alpha$, can we obtain an explicit sequence of polynomials $\{p_n\}$ such that

$$\|p_n f - 1\|_\alpha \to 0 \ \text{ as } n \to \infty?$$

Can we give an estimate on the rate of decay of these norms as $n \to \infty$? What can we say about the approximating polynomials?

A natural first guess is to take $\{p_n\}$ as the Taylor polynomials of the function $1/f$. Since $1/f$ is analytic in $\mathbb{D}$ by the cyclicity assumption, we have $p_n \to 1/f$ pointwise, and hence (1.3) is satisfied. However, it may be the case that norm boundedness in (1.2) fails. This is certainly true for the Taylor polynomials $T_n(1/f)$ in the case $f(z) = 1 - z$; indeed, $1/f \notin B \supset H^2 \supset D$ and a computation shows that

$$\|T_n(1/f)f - 1\|_D^2 = \|z^{n+1}\|_D^2 = n + 2.$$

Much of the development that follows is motivated by our goal of finding concrete substitutes for the Taylor polynomials of $1/f$.

Definition 1.1. Let $f \in D_\alpha$. We say that a polynomial p_n of degree at most n is an *optimal approximant* of order n to $1/f$ if p_n minimizes $\|pf - 1\|_\alpha$ among all polynomials p of degree at most n. We call $\|p_n f - 1\|_\alpha$ the *optimal norm* of degree n associated with f.

In other words, p_n is an optimal polynomial of order n to $1/f$ if

$$\|p_n f - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n),$$

where $\mathcal{P}_n$ denotes the space of polynomials of degree at most n and

$$\text{dist}_X(x, A) = \inf\{\|x - a\|_X : a \in A\}$$

for any normed space X, $A \subseteq X$ and $x \in X$.

Notice that, given $f \in D_\alpha \setminus \{0\}$, the existence and uniqueness of an optimal approximant of order n to $1/f$ follows immediately from the fact that $f \cdot \mathcal{P}_n$ is a finite dimensional subspace of the Hilbert space D_α. Thus, f is cyclic if and only if the optimal approximants p_n of order n to $1/f$ satisfy $\|p_n f - 1\|_\alpha \to 0$ as $n \to \infty$. Furthermore, since $\|p_n f - 1\|_\alpha \leq \|f - 1\|_\alpha$, it follows from (1.2) and (1.3) that f is cyclic if and only if the sequence of optimal approximants $\{p_n\}_{n=1}^\infty$ converges pointwise to $1/f$.

In Section 2, we describe a constructive approach for computing the coefficients of the optimal approximant of order n to $1/f$ for a general function f. In particular, Theorem 2.1 below states that the coefficients of the optimal approximants can be computed as ratios of determinants of matrices whose entries can be explicitly computed via the moments of the derivative of f. When f itself is a polynomial, these matrices are banded (see Proposition 2.2). As a simple but fundamental example, we compute optimal approximants to the function $1/f$ when $f(z) = 1 - z$.

We are also interested in the rate of convergence of optimal norms. Since optimal norms decay exponentially for any function f such that $1/f$ is analytic in the closed unit disk, functions that have zeros on the unit circle are of particular interest. In Section 3, we examine the question of whether all functions with no zeros in the open unit disk but with zeros on the boundary, admitting an analytic continuation to the closed disk, have optimal norm achieving a similar rate of decay. In Theorem 3.6, we prove that this is indeed the case.

In Section 4, we deal with a generalization to all D_α of a subproblem of the Brown-Shields conjecture. We ask the question whether a function f satisfying $f \in D_\alpha$ and $\log f \in D_\alpha$, must be cyclic in D_α. We note that this is true in the simple cases of $\alpha = 0$ or $\alpha > 1$. In Theorem 4.4 we are able to answer affirmatively in the case $\alpha = 1$. Then, Theorem 4.5 shows that for the case $\alpha < 1, \alpha \neq 0$, the same holds with an additional technical condition. We do not know if this condition is necessary; however, it is satisfied by a large class of examples, namely, all of the functions constructed in Brown-Cohn ([3]).

We conclude, in Section 5, by presenting some open questions and basic computations connected to the zero sets $\mathcal{Z}(p_n)$ of the optimal approximants p_n of $1/f$ for cyclic functions f.

2. Construction of optimal approximants

The optimal approximants p_n of order n to $1/f$ are determined by the fact that $p_n f$ is the orthogonal projection of 1 onto the space $f \cdot \mathcal{P}_n$, and hence, in principle, if $f \in D_\alpha \setminus \{0\}$, they can be computed using the Gram-Schmidt process. More precisely, once a basis for $f \cdot \mathcal{P}_n$ is chosen, one can construct an orthonormal basis for $f \cdot \mathcal{P}_n$ and then compute the coefficients of p_n with respect to this orthonormal basis.

In this section, we present a simple method which yields the optimal approximants p_n without the use of the Gram-Schmidt process, for $\alpha < 2$. To that end, we make use of the integral norm of D_α, namely,

$$\|f\|_\alpha^2 = |f(0)|^2 + D_\alpha(f),$$

where $D_\alpha(f)$ was defined in (1.1).

Recall that we seek an explicit solution to

Problem 1. Let $n \in \mathbb{N}$. Given $f \in D_\alpha \setminus \{0\}$ such that $1 \notin f \cdot \mathcal{P}_n$,

$$\text{minimize} \quad \|pf - 1\|_\alpha \quad \text{over } p \in \mathcal{P}_n.$$

2.1. **Construction of optimal approximants via determinants.** As mentioned in Section 1.2, there is a unique optimal approximant $p_n \in \mathcal{P}_n$ of order n to $1/f$ that solves Problem 1, that is,

$$\|p_n f - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n).$$

Observe that for any polynomial $p(z) = \sum_{k=0}^{n} c_k z^k \in \mathcal{P}_n$,

$$\|pf - 1\|_\alpha^2 = |p(0)f(0) - 1|^2 + \int_{\mathbb{D}} |(pf)'|^2 d\mu_\alpha$$

$$= |p(0)f(0) - 1|^2 + \int_{\mathbb{D}} \left| \sum_{k=0}^{n} c_k (z^k f)' \right|^2 d\mu_\alpha,$$

where $d\mu_\alpha(z) = (1 - |z|^2)^{1-\alpha} dA(z)$. It follows that if the optimal approximant of order n to $1/f$ vanishes at the origin, then $\|pf - 1\|_\alpha^2$ is minimal if and only if $c_0 = c_1 = \ldots = c_n = 0$. Consequently, we may assume without loss of generality that the optimal approximant p_n of order n to $1/f$ does not vanish at the origin. By replacing f with $p_n(0)f$, we may also assume that $p_n(0) = 1$ because the optimal approximant of order n to $1/(p_n(0)f)$ is $[p_n(0)]^{-1}p_n$. Hence, under this latter assumption, $p_n(z) = 1 + \sum_{k=1}^{n} c_n^* z^k$ is the optimal approximant of order n to $1/f$ if and only if $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to

Problem 2. Let $n \in \mathbb{N}$. Given $f \in D_\alpha \setminus \{0\}$ such that $1 \notin f \cdot \mathcal{P}_n$,

$$\text{minimize} \quad \int_{\mathbb{D}} \left| f' + \sum_{k=1}^{n} c_k (z^k f)' \right|^2 d\mu_\alpha \text{ over } (c_1, \ldots, c_n) \in \mathbb{C}^n.$$

It is evident that $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to Problem 2 if and only if

$$g = \sum_{k=1}^{n} c_k^* (z^k f)' \text{ satisfies } \|f' + g\|_{L^2(\mu_\alpha)} = \text{dist}_{L^2(\mu_\alpha)}(f', Y),$$

where $Y = \text{span}\{(z^k f)' : 1 \leq k \leq n\}$. Equivalently, $f' + g$ is orthogonal to Y with respect to the $L^2(\mu_\alpha)$ inner product; that is, for each j, $1 \leq j \leq n$,

$$\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \langle g, (z^j f)' \rangle_{L^2(\mu_\alpha)}.$$

Hence, $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to Problem 2 if and only if it is the solution to the non-homogeneous system of linear equations

$$\sum_{k=1}^{n} c_k \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}, \quad 1 \leq k \leq n, \qquad (2.1)$$

with $(c_1, \ldots, c_n) \in \mathbb{C}^n$.

Theorem 2.1. *Let $n \in \mathbb{N}$ and $f \in D_\alpha \setminus \{0\}$. Suppose $1 \notin f \cdot \mathcal{P}_n$ and let M denote the $n \times n$ matrix with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$. Then the unique $p_n \in \mathcal{P}_n$ satisfying*

$$\|p_n f - 1\|_\alpha = \text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$$

is given by

$$p_n(z) = p_n(0) \left(1 + \sum_{k=1}^{n} \frac{\det M^{(k)}}{\det M} z^k \right), \qquad (2.2)$$

where $M^{(k)}$ denotes the $n \times n$ matrix obtained from M by replacing the kth column of M by the column with entries $\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \le j \le n$.

Proof. As mentioned before, if p_n is the optimal approximant of order n to f and $p_n(0) \neq 0$, then the optimal approximant of order n to $1/f_n$ is $[p_n(0)]^{-1} p_n$, where $f_n = p_n(0)f$. If $[p_n(0)]^{-1} p_n(z) = 1 + \sum_{k=1}^{n} c_k^* z^k$, then $(c_1^*, \ldots, c_n^*) \in \mathbb{C}^n$ is the unique solution to the system in (2.1) because

$$\langle (z^k f_n)', (z^j f_n)' \rangle_{L^2(\mu_\alpha)} = |p_n(0)|^2 \langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$$

for $0 \le k \le n$ and $1 \le j \le n$. It follows now that the $n \times n$ matrix M with entries $\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)}$ has non-zero determinant and thus

$$c_k^* = \frac{\det M^{(k)}}{\det M}, \quad 1 \le k \le n,$$

by Cramer's rule, where $M^{(k)}$ denotes the $n \times n$ matrix obtained from M by replacing the kth column of M by the column with entries $\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)}$, $1 \le j \le n$. Hence p_n is given by (2.2). $\square$

If f is a polynomial, then the computation of the determinants appearing in (2.2) can be simplified in view of the following proposition.

Proposition 2.2. *Suppose f is a polynomial of degree t. Then the matrix M in Theorem 2.1 is banded and has bandwidth at most $2t + 1$.*

Proof. The orthogonality of z^l and z^m for $l \neq m$ (under the $L^2(\mu_\alpha)$ inner product) implies that the (j, k)-entry of M equals 0 if the degree of $(z^k f)'$ is strictly less than $j - 1$ or if the degree of $(z^j f)'$ is strictly less than $k - 1$; that is, $k + t - 1 < j - 1$ or $j + t - 1 < k - 1$. Therefore, the only entries of M that do not necessarily vanish are the ones whose indices j and k satisfy $-t \le j - k \le t$. Thus, M is banded and has bandwidth at most $2t + 1$. $\square$

2.2. **An explicit example of optimal approximants.** Now, we calculate explicitly optimal approximants to $1/f$, where $f(z) = 1 - z$. Even though f is a low order polynomial, this example is already interesting because f is cyclic in D_α for $\alpha \le 1$, even though it is not invertible for any $\alpha \ge -1$.

We begin with some general computations. Let $\beta = 1 - \alpha$. Then

$$\|z^m\|^2_{L^2(\mu_\alpha)} = \int_0^1 u^{1-\alpha} (1 - u)^m \, du = \prod_{\ell=1}^{m} \frac{\ell}{\ell + \beta}$$

holds for any non-negative integer m. Therefore, if $f(z) = \sum_{i=0}^{t} a_i z^i$, we have[1]

$$\langle (z^k f)', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \sum_{i=0}^{t} \sum_{\ell=0}^{t} a_i \bar{a}_\ell (i+k)(\ell+j) \langle z^{i+k-1}, z^{\ell+j-1} \rangle_{L^2(\mu_\alpha)}$$

$$= \sum_{i=0}^{t} a_i \bar{a}_{i+k-j}(i+k)^2 \| z^{i+k-1} \|_{L^2(\mu_\alpha)}^2$$

$$= \sum_{i=0}^{t} a_i \bar{a}_{i+k-j}(i+k) \prod_{\ell=1}^{i+k} \frac{\ell}{\ell+\beta} \qquad (2.3)$$

because z^l and z^m are orthogonal for $l \neq m$ under the $L^2(\mu_\alpha)$ inner product.

We simplify notation by calling, for $k \in \mathbb{N}$,

$$\Lambda_\beta(k) = k \prod_{\ell=1}^{k} \frac{\ell}{\ell+\beta}.$$

Since $a_0 = 1$ and $a_1 = -1$, it follows from (2.3) that

$$\langle (z^k f)', (z^{k-1} f)' \rangle_{L^2(\mu_\alpha)} = -\Lambda_\beta(k),$$

$$\langle (z^k f)', (z^k f)' \rangle_{L^2(\mu_\alpha)} = \Lambda_\beta(k) + \Lambda_\beta(k+1),$$

$$\langle (z^k f)', (z^{k+1} f)' \rangle_{L^2(\mu_\alpha)} = -\Lambda_\beta(k+1), \quad \text{and}$$

$$\langle -f', (z^j f)' \rangle_{L^2(\mu_\alpha)} = \begin{cases} \Lambda_\beta(1) & \text{if } j = 1 \\ 0 & \text{if } j \geq 2. \end{cases}$$

Thus, in view of (2.1), the coefficients of p_n satisfy the sytem of equations

$$c_1 [\Lambda_\beta(1) + \Lambda_\beta(2)] - c_2 [\Lambda_\beta(2)] = \Lambda_\beta(1)$$

$$-c_{j-1} [\Lambda_\beta(j)] + c_j [\Lambda_\beta(j) + \Lambda_\beta(j+1)] - c_{j+1} [\Lambda_\beta(j+1)] = 0$$

$$-c_{n-1} [\Lambda_\beta(n)] + c_n [\Lambda_\beta(n) + \Lambda_\beta(n+1)] = 0$$

or, interpreting $c_{n+1} = 0$, equivalently, for all $2 \leq j \leq n+1$:

$$\Lambda_\beta(j)(c_j - c_{j-1}) = \Lambda_\beta(1)(c_1 - 1).$$

For fixed k, $2 \leq k \leq n+1$, by a repeated use of the previous identity, we obtain the following:

$$c_k = \left[\Lambda_\beta(1) \sum_{j=1}^{k} \frac{1}{\Lambda_\beta(j)} \right] (c_1 - 1) + 1 \qquad (2.4)$$

In particular, we have

$$c_1 - 1 = -\Lambda_\beta(n+1) c_n$$

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[1]Under the usual convention that $a_i = 0$ for any integer $i < 0$ or $i > t$.

and so, we can recover the value of c_1,

$$c_1 - 1 = -\frac{1}{\Lambda_\beta(1) \sum_{j=1}^{n+1} \frac{1}{\Lambda_\beta(j)}}.$$

Finally, we obtain the explicit solution, which can be expressed as follows, for $1 \le k \le n$:

$$c_k = \left[\sum_{j=k+1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell}\right) \right] \left[\sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell}\right) \right]^{-1}. \qquad (2.5)$$

Alternatively, in the case of the Dirichlet space, we can compute the coefficients c_k, $1 \le k \le n$, using determinants as follows. For $n \in \mathbb{N}$, let $M_n = M$ and $M_n^{(k)} = M^{(k)}$ be the $n \times n$ matrices corresponding to f as in Theorem 2.1. By Proposition 2.2, the matrix M_n is tridiagonal and so it suffices to compute the coefficients above and below each entry of its main diagonal. The coefficients in the jth column of M_n are given by

$$\langle (z^{j+\ell} f)', (z^j f)' \rangle_{L^2} = a_0 \bar{a}_\ell (j + \ell) + a_1 \bar{a}_{1+\ell}(j + \ell + 1)$$

where $\ell = -1, 0, 1$. Since $a_0 = 1$ and $a_1 = -1$, we obtain

$$\langle (z^{j-1} f)', (z^j f)' \rangle_{L^2} = -j,$$
$$\langle (z^j f)', (z^j f)' \rangle_{L^2} = 2j + 1,$$
$$\langle (z^{j+1} f)', (z^j f)' \rangle_{L^2} = -(j + 1),$$
$$\text{and } \langle -f', (z^j f)' \rangle_{L^2} = \bar{a}_{1-j}.$$

Consequently,

$$M_1 = 3, \qquad M_1^{(1)} = 1,$$

$$M_2 = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}, \qquad M_2^{(1)} = \begin{pmatrix} 1 & -2 \\ 0 & 5 \end{pmatrix}, \qquad M_2^{(2)} = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 7 \end{pmatrix}, \qquad M_3^{(1)} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 5 & -3 \\ 0 & -3 & 7 \end{pmatrix}, \ldots$$

$$M_4 = \begin{pmatrix} 3 & -2 & 0 & 0 \\ -2 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 9 \end{pmatrix}, \qquad M_4^{(1)} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & -3 & 7 & -4 \\ 0 & 0 & -4 & 9 \end{pmatrix}, \ldots$$

Thus, the optimal approximants to f of orders 1, 2, 3, and 4 are

$$p_1(z) = p_1(0) \left(1 + \frac{1}{3}z \right),$$

$$p_2(z) = p_2(0) \left(1 + \frac{5}{11}z + \frac{2}{11}z^2 \right),$$

$$p_3(z) = p_3(0) \left(1 + \frac{13}{25}z + \frac{7}{25}z^2 + \frac{3}{25}z^3 \right), \quad \text{and}$$

$$p_4(z) = p_4(0) \left(1 + \frac{77}{137}z + \frac{47}{137}z^2 + \frac{27}{137}z^3 + \frac{12}{137}z^4 \right).$$

What we have shown is that, for any integer n, the optimal approximant for the Dirichlet space is an example of a generalized Riesz mean polynomial: more specifically, defining $H_n = \sum_{j=1}^{n} \frac{1}{j}$ and $H_0 = 0$,

$$p_n(z) = p_n(0) \left(\sum_{k=0}^{n} \left(1 - \frac{H_k}{H_{n+1}} \right) z^k \right).$$

Moreover, for the Hardy space, the optimal approximant is a modified Cesàro mean polynomial,

$$p_n(z) = p_n(0) \left(\sum_{k=0}^{n} \left(1 - \frac{k + H_k}{n + 1 + H_{n+1}} \right) z^k \right),$$

and for the Bergman space, the optimal approximants are

$$p_n(z) = p_n(0) \left(1 + \sum_{k=1}^{n} \left(1 - \frac{k(k+7) + 4H_k}{(n+1)(n+8) + 4H_{n+1}} \right) z^k \right).$$

We will return to these polynomials in Section 3.

3. Rate of decay of the optimal norms

In this section, we obtain estimates for $\text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$ as $n \to \infty$, $f \in D_\alpha$. It turns out that the example of $f(z) = 1 - z$ in the previous section is a model example for the rate of decay of $\text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$. We first examine the rate of decay for this function, then establish such estimates when f is a polynomial whose zeros are simple and lie in $\mathbb{C} \setminus \mathbb{D}$, and then extend our results to arbitrary polynomials. We conclude with estimates on functions that admit an analytic continuation to the closed unit disk yet have at least one zero on the circle.

To simplify notation, define the auxiliary function φ_α on $[0, \infty)$ to be

$$\varphi_\alpha(s) = \begin{cases} s^{1-\alpha}, & \text{if } \alpha < 1 \\ \log^+(s), & \text{if } \alpha = 1. \end{cases}$$

Lemma 3.1. *If $f(z) = \zeta - z$, for $\zeta \in \mathbb{T}$, then $\text{dist}_{D_\alpha}^2(1, f \cdot \mathcal{P}_n)$ is comparable to $\varphi_\alpha^{-1}(n+1)$ for all sufficiently large n.*

Proof. First notice that, for any polynomial p and $\zeta \in \mathbb{T}$, the polynomial $q(z) = \zeta p(\zeta z)$ satisfies $\|p(z)(\zeta - z) - 1\|_\alpha = \|q(z)(1 - z) - 1\|_\alpha$ because rotation by ζ is an isometry in D_α. Therefore, it is enough to consider the case when $\zeta = 1$, i.e. $f(z) = 1 - z$.

Now, recall that by (2.5), if $f(z) = 1 - z$, the optimal approximant of order n to $1/f$ is

$$p_n(z) = p_n(0) \sum_{k=0}^{n} c_k z^k,$$

where

$$c_k = \left[\sum_{j=k+1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell} \right) \right] \left[\sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell} \right) \right]^{-1}, \quad 0 \le k \le n,$$

and $\beta = 1 - \alpha$. We claim that $\|p_n f - 1\|_\alpha^2$ is comparable to $\varphi_\alpha^{-1}(n + 1)$ for all sufficiently large n.

First of all, notice that

$$p_n(z)f(z) - 1 = p_n(0) - 1 + p_n(0) \left[\sum_{k=1}^{n} (c_k - c_{k-1}) z^k - c_n z^{n+1} \right].$$

To simplify notation, define for $1 \le k \le n$

$$a_k = c_k - c_{k-1} = - \left[\frac{1}{k} \prod_{\ell=2}^{k} \left(1 + \frac{\beta}{\ell} \right) \right] \left[\sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell} \right) \right]^{-1}$$

and $a_{n+1} = -c_n$. Then

$$\sum_{k=1}^{n} k^\alpha |a_k|^2 = \left[\sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell} \right) \right]^{-2} \sum_{k=1}^{n} k^{\alpha-2} \left[\prod_{\ell=2}^{k} \left(1 + \frac{\beta}{\ell} \right) \right]^{2}. \quad (3.1)$$

Recalling that $2^{-1} x \le \log(1 + x) \le x$ holds for all $x \in [0, 1]$, we see that

$$\prod_{\ell=2}^{k} \left(1 + \frac{\beta}{\ell} \right) = \exp \left[\sum_{\ell=2}^{k} \log \left(1 + \frac{\beta}{\ell} \right) \right]$$

is comparable to

$$\exp \left[\beta \sum_{\ell=2}^{k} \frac{1}{\ell} \right],$$

and so comparable to k^β, when $1 \le k \le n + 1$. Thus, the sum in (3.1) and

$$(n + 1)^\alpha |a_{n+1}|^2 = (n + 1)^\alpha \left[\frac{1}{n + 1} \prod_{\ell=2}^{n+1} \left(1 + \frac{\beta}{\ell} \right) \right]^{2} \left[\sum_{j=1}^{n+1} \frac{1}{j} \prod_{\ell=2}^{j} \left(1 + \frac{\beta}{\ell} \right) \right]^{-2}$$

are comparable to

$$\left[\sum_{j=1}^{n+1}\frac{1}{j^\alpha}\right]^{-2}\sum_{k=1}^n\frac{1}{k^\alpha}\quad\text{and}\quad\frac{1}{(n+1)^\alpha}\left[\sum_{j=1}^{n+1}\frac{1}{j^\alpha}\right]^{-2},$$

respectively. Since $\sum_{j=1}^n j^{-\alpha}$ is comparable to $\varphi_\alpha(n+1)$, the sum

$$\sum_{k=1}^{n+1}k^\alpha|a_k|^2$$

is comparable to $\varphi_\alpha^{-1}(n+1)$ when $n\geq 2$. This proves the lemma. $\qquad\square$

Let us now examine the rate of decay of optimal norms for polynomials whose zeros are simple and lie in $\mathbb{C}\backslash\mathbb{D}$. To begin, let us introduce some notation. Let $A(\mathbb{T})$ denote the *Wiener algebra*, that is, $A(\mathbb{T})$ consists of functions f, defined on $\mathbb{T}$, whose Fourier coefficients are absolutely summable, and is equipped with the norm

$$\|f\|_{A(\mathbb{T})}=\sum_{k=-\infty}^\infty|a_k|.$$

The positive Wiener algebra consists of analytic functions whose Fourier coefficients satisfy $\sum_{k=0}^\infty|a_k|<\infty$; in particular, these functions belong to H^∞, the space of bounded analytic functions in $\mathbb{D}$, and $\|f\|_{H^\infty}\leq\|f\|_{A(\mathbb{T})}$ holds for all f in the positive Wiener algebra, where $\|f\|_{H^\infty}=\sup\{|f(z)|\colon z\in\mathbb{D}\}$.

Proposition 3.2. *Let $\alpha\leq 1$, $t\in\mathbb{N}$ and $f\in\mathcal{P}_t$. If the zeros of f are simple and lie in $\mathbb{C}\backslash\mathbb{D}$, then for each $n>t$ there is $p_n\in\mathcal{P}_n$ such that $(p_nf)(0)=1$,*

$$\|p_nf-1\|_\alpha^2\leq C\varphi_\alpha^{-1}(n+1)\tag{3.2}$$

holds for some constant C that depends on f and α but not on n, and such that the sequence $\{p_nf\}_{n>t}$ is bounded in $A(\mathbb{T})-norm$.

Proof. Suppose f has simple zeros $z_1,\ldots,z_t\in\mathbb{C}\setminus\mathbb{D}$. Then there are constants $d_1,\ldots,d_t$ such that

$$\frac{1}{f(z)}=\sum_{j=1}^t\frac{d_j}{z_j-z}=\sum_{k=0}^\infty\left(\sum_{j=1}^t\frac{d_j}{z_j^{k+1}}\right)z^k.$$

Define $b_k=\sum_{j=1}^t d_jz_j^{-(k+1)}$ for $k\geq 0$. It follows that the sequence $\{b_k\}_{k=0}^\infty$ is bounded in modulus by $\sum_{j=1}^t|d_j|$, and the Taylor series representations of f and $1/f$ centered at the origin are of the form

$$f(z)=\sum_{k=0}^t a_kz^k\quad\text{and}\quad\frac{1}{f(z)}=\sum_{k=0}^\infty b_kz^k,$$

for some $a_0, \ldots, a_t \in \mathbb{C}$. Set $a_k = 0$ for $k > t$. Consequently,

$$\sum_{j=0}^{k} b_j a_{k-j} = 0 \quad \text{for } k \in \mathbb{N} \backslash \{0\}. \tag{3.3}$$

Consider the polynomial $p_n(z) = \sum_{k=0}^{n} c_k z^k$ with coefficients

$$c_0 = a_0^{-1} \quad \text{and} \quad c_k = \left(1 - \frac{\varphi_\alpha(k)}{\varphi_\alpha(n+1)}\right) b_k \text{ for } 1 \le k \le n.$$

For convenience of notation, let $c_k = 0$ if $k > n$. Evidently, $(p_n f)(0) = 1$. Let us prove (3.2). To estimate $\|p_n f - 1\|_\alpha^2$, we consider separately the norms of

$$\mathbf{mp} = \sum_{k=t+1}^{n+t} \left(\sum_{i=0}^{k} c_i a_{k-i}\right) z^k, \quad \text{and}$$

$$\mathbf{sp} = \sum_{k=1}^{t} \left(\sum_{i=0}^{k} c_i a_{k-i}\right) z^k,$$

and note that

$$\|p_n f - 1\|_\alpha^2 = \|\mathbf{mp}\|_\alpha^2 + \|\mathbf{sp}\|_\alpha^2 \tag{3.4}$$

and

$$\sum_{i=0}^{k} c_i a_{k-i} = \frac{-1}{\varphi_\alpha(n+1)} \sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i} \tag{3.5}$$

by (3.3). To estimate the norm of $\mathbf{mp}$, we need the following result.

Lemma 3.3 (Control Lemma). *Under the assumptions of Proposition 3.2, if $k > t$, there is a constant $C = C(\alpha, f)$ such that*

$$\left|\sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i}\right| \le \frac{C}{(k+1)^\alpha}.$$

We finish the proof of Proposition 3.2 before proving the Control Lemma. By (3.5) and the Control Lemma 3.3,

$$\|\mathbf{mp}\|_\alpha^2 = \sum_{k=t+1}^{n+t} \left|\sum_{i=0}^{k} c_i a_{k-i}\right|^2 (k+1)^\alpha$$

$$= \frac{1}{\varphi_\alpha^2(n+1)} \sum_{k=t+1}^{n+t} \left|\sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i}\right|^2 (k+1)^\alpha$$

$$\le \frac{C_1}{\varphi_\alpha^2(n+1)} \sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha}$$

for some constant $C_1 = C_1(\alpha, f)$. It follows now from the estimates

$$\sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha} \leq \begin{cases} n(n+t+1)^{-\alpha} & \text{if } \alpha \leq 0, \\ (1-\alpha)^{-1}[(n+t+1)^{1-\alpha} - (t+1)^{1-\alpha}] & \text{if } 0 < \alpha < 1, \\ \log(n+t+1) - \log(t+1) & \text{if } \alpha = 1 \end{cases}$$

and the elementary inequalities

$$\begin{aligned} (n+t+1)^{-\alpha} &\leq 2^{-\alpha}(n+1)^{-\alpha} & \text{for } \alpha \leq 0, \\ (n+t+1)^{1-\alpha} &\leq 2^{1-\alpha}(n+1)^{1-\alpha} & \text{for } \alpha > 0, \text{ and} \\ \log(n+t+1) - \log(t+1) &\leq \log(n+1), \end{aligned}$$

that there is a constant $C_2 = C_2(\alpha, f)$ such that

$$\sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha} \leq C_2 \varphi_\alpha(n+1), \tag{3.6}$$

and so

$$\|\mathbf{mp}\|_\alpha^2 \leq \frac{C_1 C_2}{\varphi_\alpha(n+1)}. \tag{3.7}$$

Next, we estimate the norm of $\mathbf{sp}$. Recalling (3.5), we see that

$$\|\mathbf{sp}\|_\alpha^2 = \frac{1}{\varphi_\alpha^2(n+1)} \sum_{k=1}^{t} \left| \sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i} \right|^2 (k+1)^\alpha.$$

By the Triangle inequality and since φ is increasing, if $1 \leq k \leq t$, then

$$\left| \sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i} \right| \leq \|b\|_{\ell^\infty} \|a\|_{\ell^\infty} (t+1)\varphi_\alpha(t), \tag{3.8}$$

where $a = \{a_k\}_{k=0}^\infty$ and $b = \{b_k\}_{k=0}^\infty$. Thus,

$$\|\mathbf{sp}\|_\alpha^2 \leq \frac{1}{\varphi_\alpha^2(n+1)} \|b\|_{\ell^\infty}^2 \|a\|_{\ell^\infty}^2 (t+1)^2 \varphi_\alpha^2(t) \sum_{k=1}^{t} (k+1)^\alpha,$$

and so

$$\|\mathbf{sp}\|_\alpha^2 \leq \frac{C_3}{\varphi_\alpha(n+1)} \tag{3.9}$$

as $\varphi_\alpha(t) \leq \varphi_\alpha(n+1)$. Hence, (3.2) follows from (3.4), (3.7) and (3.9).

Finally, we show that the sequence $\{p_n f\}_{n>t}$ is bounded in $A(\mathbb{T})$. Notice that, for $1 \leq k \leq t$, (3.5) and (3.8) imply

$$\left| \sum_{i=0}^{k} c_i a_{k-i} \right| \leq \|b\|_{\ell^\infty} \|a\|_{\ell^\infty} (t+1) \tag{3.10}$$

because $\varphi_\alpha(t) \leq \varphi_\alpha(n+1)$. On the other hand, for $t < k \leq n+t$, (3.5) and the Control Lemma 3.3 imply

$$\left| \sum_{i=0}^{k} c_i a_{k-i} \right| \leq \frac{C}{(k+1)^\alpha} \varphi_\alpha^{-1}(n+1) \tag{3.11}$$

for some constant $C = C(\alpha, f)$. Therefore, by (3.10), (3.11), and (3.6),

$$\|p_n f\|_{A(\mathbb{T})} = \sum_{k=1}^{n+t} \left| \sum_{i=0}^{k} c_i a_{k-i} \right|$$

$$\leq \sum_{k=1}^{t} \|b\|_{\ell^\infty} \|a\|_{\ell^\infty} (t+1) + \frac{C}{\varphi_\alpha(n+1)} \sum_{k=t+1}^{n+t} \frac{1}{(k+1)^\alpha}$$

$$\leq \|b\|_{\ell^\infty} \|a\|_{\ell^\infty} (t+1)t + C \cdot C_2$$

and so $\{p_n f\}_{n>t}$ is bounded in $A(\mathbb{T})$. This completes the proof. $\qquad\square$

We now proceed to prove Lemma 3.3.

Proof of Control Lemma 3.3. For $k - t \leq s \leq k$,

$$\varphi_\alpha'(s) \leq \begin{cases} (1-\alpha)k^{-\alpha} & \text{if } \alpha < 0 \\ (1-\alpha)(k-t)^{-\alpha} & \text{if } 0 \leq \alpha < 1 \\ (k-t)^{-1} & \text{if } \alpha = 1. \end{cases} \qquad (3.12)$$

Thus, the Mean Value Theorem, (3.12), and the inequality

$$(k-t)^{-\alpha} \leq (t+2)^\alpha (k+1)^{-\alpha} \text{ for } \alpha \geq 0 \text{ and } k \geq t+1,$$

imply that there is a constant $C = C(\alpha, t)$ such that

$$\varphi_\alpha(k) - \varphi_\alpha(i) \leq C(k-i)(k+1)^{-\alpha} \text{ for } k \geq i.$$

Recalling (3.3) and that $a_i = 0$ for $i > t$, we obtain

$$\left| \sum_{i=0}^{k} \varphi_\alpha(i) b_i a_{k-i} \right| \leq \left| \sum_{i=0}^{k} [\varphi_\alpha(k) - \varphi_\alpha(i)] b_i a_{k-i} \right|$$

$$\leq \sum_{i=k-t}^{k} [\varphi_\alpha(k) - \varphi_\alpha(i)] \cdot |b_i a_{k-i}|$$

$$\leq \|a\|_{\ell^\infty} \|b\|_{\ell^\infty} C (k+1)^{-\alpha} \sum_{i=k-t}^{k} (k-i),$$

where $a = \{a_i\}_{i=0}^{\infty}$ and $b = \{b_i\}_{i=0}^{\infty}$. Hence, the conclusion holds with constant $\|a\|_{\ell^\infty} \|b\|_{\ell^\infty} C t(t+1)/2$. $\qquad\square$

It seems natural to ask whether the proof of Theorem 3.2 can be extended to polynomials f whose zeros are not necessarily simple. However, even in the simple case of $f(z) = (1-z)^2$, the coefficients of the Taylor series representation centered at the origin of $1/f$ are not bounded; consequently, the proof of Proposition 3.2 cannot be extended directly because the boundedness of these coefficients is needed. Nevertheless, if f is an arbitrary polynomial, we can obtain an estimate for $\text{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_n)$. Moreover, using Lemma 3.1, we will be able to show this rate of decay is sharp.

Theorem 3.4. *Let $\alpha \leq 1$. If f is a polynomial whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that*

$$\operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m) \leq C \varphi_\alpha^{-1}(m+1) \tag{3.13}$$

holds for all sufficiently large m. Moreover, this estimate is sharp in the sense that if such a polynomial f has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\tilde{C}\varphi_\alpha^{-1}(m+1) \leq \operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m).$$

Proof. Suppose f has factorization

$$f(z) = K \prod_{k=1}^{s} (z - z_k)^{r_k}$$

with $r_1, \ldots, r_s \in \mathbb{N}$, $z_1, \ldots, z_s \in \mathbb{C} \setminus \mathbb{D}$ are distinct, and $K \in \mathbb{C} \setminus \{0\}$. Define

$$g(z) = \prod_{k=1}^{s} (z - z_k) \quad \text{and} \quad h(z) = K^{-1} \prod_{k=1}^{s} (z - z_k)^{\gamma - r_k},$$

where $\gamma = \max\{r_1, \ldots, r_s\}$, and let d equal the degree of h. Then $fh = g^\gamma$,

$$\operatorname{dist}_{D_\alpha}(1, f \cdot \mathcal{P}_{n+d}) \leq \operatorname{dist}_{D_\alpha}(1, fh \cdot \mathcal{P}_n) \text{ for } n \in \mathbb{N}, \tag{3.14}$$

and the zeros of g are simple and lie in $\mathbb{C} \setminus \mathbb{D}$.

By Proposition 3.2, for $n > s$, we can choose $q_n \in \mathcal{P}_n$ such that $(q_n g)(0) = 1$ and

$$\|q_n g - 1\|_\alpha^2 \leq C_1 \varphi_\alpha^{-1}(n+1) \tag{3.15}$$

holds for some $C_1 = C_1(\alpha, g)$, and such that the sequence $\{q_n g\}_{n>s}$ is bounded in $A(\mathbb{T})$.

Let $d\mu_\alpha(z) = (1 - |z|^2)^{1-\alpha}\, dA(z)$. Recalling that $\|p\|_\alpha^2$ is comparable to $|p(0)|^2 + D_\alpha(p) = |p(0)|^2 + \|p'\|^2_{L^2(\mu_\alpha)}$ for all $p \in D_\alpha$, we obtain

$$\begin{aligned}
\|q_n^\gamma g^\gamma - 1\|_\alpha^2 &\leq C_2 \|(q_n^\gamma g^\gamma)'\|^2_{L^2(\mu_\alpha)} \\
&= C_2 \|(q_n g)^{\gamma-1}\gamma(q_n' g + q_n g')\|^2_{L^2(\mu_\alpha)} \\
&\leq C_2 \gamma^2 \|q_n g\|_{H^\infty}^{2\gamma-2} \|q_n' g + q_n g'\|^2_{L^2(\mu_\alpha)} \\
&\leq C_3 \gamma^2 \|q_n g\|_{H^\infty}^{2\gamma-2} \|q_n g - 1\|_\alpha^2 \\
&\leq C_3 \gamma^2 \|q_n g\|_{A(\mathbb{T})}^{2\gamma-2} \|q_n g - 1\|_\alpha^2 \tag{3.16}
\end{aligned}$$

for some constants $C_2 = C_2(\alpha)$ and $C_3 = C_3(\alpha)$, as $(q_n g)(0) = 1$. Therefore, (3.15) and (3.16) imply that there is a constant $C_4 = C_4(\alpha, \gamma, g)$ such that

$$\operatorname{dist}^2_{D_\alpha}(1, g^\gamma \cdot \mathcal{P}_{n\gamma}) \leq C_4 \varphi_\alpha^{-1}(n+1)$$

because $q_n^\gamma \in \mathcal{P}_{n\gamma}$ and $\{q_n g\}_{n>s}$ is bounded in $A(\mathbb{T})$. Thus, by (3.14),

$$\operatorname{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_{n\gamma+d}) \leq C_4 \varphi_\alpha^{-1}(n+1) \text{ when } n > s. \tag{3.17}$$

Let $m > d + (s+1)\gamma$. Then there exists an integer a and an $n \in \mathbb{N}$ such that $0 \le a < \gamma$ and $m - d = n\gamma + a$. In particular, $n > s$ and

$$\text{dist}_{D_\alpha}^2(1, f \cdot \mathcal{P}_m) \le C_4 \varphi_\alpha^{-1}(n+1) \tag{3.18}$$

follows from (3.17) as $m \ge n\gamma + d$. Finally, the elementary inequalities

$$(1 + n\gamma + d + a) \le (\gamma + d)(1 + n) \quad \text{and} \quad (1 + n\gamma + d + a) \le (1+n)^{2\gamma + d}$$

valid for all $n \in \mathbb{N}$ imply the existence of a constant $C_5 = C_5(\alpha, \gamma, d)$ such that

$$\varphi_\alpha(m+1) \le C_5 \varphi_\alpha(n+1). \tag{3.19}$$

Hence, (3.13) holds for $m > d + (s+1)\gamma$ by (3.18) and (3.19).

Let us now show that the inequality is sharp. If f is any polynomial with zeros outside $\mathbb{D}$ that has at least one zero on $\mathbb{T}$, then $f(z) = h(z)(\zeta - z)$ for some polynomial h of degree say d. Then for any polynomial p_m of degree at most m,

$$\|p_m(z)h(z)(\zeta - z) - 1\|_\alpha^2 \ge \text{dist}_{D_\alpha}^2(1, (\zeta - z) \cdot \mathcal{P}_{m+d}).$$

By Lemma 3.1, there exists a constant $C_1 = C_1(\alpha)$ such that

$$\text{dist}_{D_\alpha}^2(1, (\zeta - z) \cdot \mathcal{P}_{m+d}) \ge C_1 \varphi_\alpha^{-1}(m + d + 1).$$

Now, in a manner similar to (3.19), we can choose a constant $C_2 = C_2(\alpha, d)$ such that

$$\varphi_\alpha^{-1}(m + d + 1) \ge C_2 \varphi_\alpha^{-1}(m + 1).$$

Finally, letting $\tilde{C} = C_1 C_2$ and noting that the polynomial p_m was arbitrary, we obtain the desired result that

$$\text{dist}_{D_\alpha}^2(1, f \cdot \mathcal{P}_m) \ge \tilde{C} \varphi_\alpha^{-1}(m + 1). \qquad \square$$

In fact, the rates in Theorem 3.4 hold for more general functions f, namely functions that have an analytic continuation to the closed unit disk. Since such functions can be factored as $f(z) = h(z)g(z)$, where h is a polynomial with a finite number of zeros on the circle and g is a function analytic in the closed disk with no zeros there, the estimates in Theorem 3.4 hold for h. Moreover, we can obtain estimates on g that will allow us to give upper bounds on the product $h(z)g(z)$. The estimates needed for g are contained in the following lemma.

Lemma 3.5. *Let $\alpha \le 1$ and let g be analytic in the closed disk. If $T_n(g)$ is the Taylor polynomial of degree n of g, then*

$$\|g - T_n(g)\|_\alpha^2 = o\left(\varphi_\alpha^{-1}(n+1)\right),$$

and there exists a constant $C = C(\alpha)$ such that

$$\|T_n(g)\|_{M(D_\alpha)} \le C.$$

Proof. Suppose $g(z) = \sum_{k=0}^{\infty} d_k z^k$ is convergent in the closed unit disk. Then there exist constants $R > 1$ and $C_1 > 0$ such that $|d_k| \leq C_1 R^{-k}$. Therefore

$$\|g - T_n(g)\|_{\alpha}^2 = \sum_{k=n+1}^{\infty} (k+1)^{\alpha} |d_k|^2$$

$$\leq C_1 R^{-2n} \sum_{j=1}^{\infty} (j + n + 1)^{\alpha} R^{-2j}$$

$$\leq C_1 R^{-2n} (n+1)^{\alpha} C_2,$$

where $C_2 = C_2(\alpha, R) = \sum_{j=1}^{\infty} (j+1)^{\alpha} R^{-2j}$ if $\alpha \geq 0$ and $C_2 = \sum_{j=1}^{\infty} R^{-2j}$ if $\alpha < 0$. In either case, $C_2 < \infty$ and is independent of n. Therefore, we have that for all $\alpha \leq 1$,

$$\|g - T_n(g)\|_{\alpha}^2 \leq C_1 C_2 R^{-2n} (n+1)^{\alpha}.$$

Noting that R^{-2n} decays exponentially as $n \to \infty$ while φ_{α}^{-1} decays at a polynomial or logarithmic rate, we obtain that

$$\|g - T_n(g)\|_{\alpha}^2 = o\left(\varphi_{\alpha}^{-1}(n+1)\right).$$

The same type of argument can be used to show that the Taylor polynomials $T_n(g)$ have uniformly bounded multiplier norms. Indeed, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in D_{\alpha}$, then in a manner similar to that above, using the exponential decay of the coefficients d_k of g, one can easily show that for every integer k,

$$\|d_k z^k \cdot f\|_{\alpha} \leq R^{-k} C_1 C_3 \|f\|_{\alpha},$$

where $C_3 = C_3(k, \alpha) = (k+1)^{\alpha/2}$ if $\alpha \geq 0$, otherwise, $C_3 = 1$. Therefore,

$$\|T_n(g) \cdot f\|_{\alpha} \leq \sum_{k=0}^{n} \|d_k z^k \cdot f\|_{\alpha} \leq \left(\sum_{k=0}^{n} C_1 C_3 R^{-k}\right) \|f\|_{\alpha}.$$

Since the series $C = \sum_{k=0}^{\infty} C_1 C_3 R^{-k}$ converges, we obtain

$$\|T_n(g)\|_{M(D_{\alpha})} \leq C,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.6. *Let $\alpha \leq 1$. If f is a function admitting an analytic continuation to the closed unit disk and whose zeros lie in $\mathbb{C} \setminus \mathbb{D}$, then there exists a constant $C = C(\alpha, f)$ such that*

$$\mathrm{dist}_{D_{\alpha}}^2(1, f \cdot \mathcal{P}_m) \leq C \varphi_{\alpha}^{-1}(m+1)$$

holds for all sufficiently large m. Moreover, this estimate is sharp in the sense that if such a function f has at least one zero on $\mathbb{T}$, then there exists a constant $\tilde{C} = \tilde{C}(\alpha, f)$ such that

$$\tilde{C} \varphi_{\alpha}^{-1}(m+1) \leq \mathrm{dist}_{D_{\alpha}}^2(1, f \cdot \mathcal{P}_m).$$

Proof. Let us first examine the upper bound. Without loss of generality, f is not identically 0, and therefore can only have a finite number of zeros on the unit circle $\mathbb{T}$. Write $f(z) = h(z)g(z)$, where h is the polynomial formed from the zeros of f that lie on $\mathbb{T}$, and g is analytic in the closed disk with no zeros there. Therefore, $1/g$ is also analytic in the closed unit disk (and obviously has no zeros there), and hence Lemma 3.5 applies to $1/g$. Notice also that g and g' are bounded in the disk, and therefore g is a multiplier for D_α.

Now, for $m \in \mathbb{N}$, let q_m be the optimal approximant of order m to $1/h$ and define $p_m = q_m T_m(1/g)$. By the Triangle Inequality,

$$\|p_m f - 1\|_\alpha \leq \|T_m(1/g)g(q_m h - 1)\|_\alpha + \|T_m(1/g)g - 1\|_\alpha.$$

Since we know that g is a multiplier for D_α, that the q_m are optimal for h, and that $T_m(1/g)$ are uniformly bounded in multiplier norm by Lemma 3.5, we see that the square of the first summand on the right-hand side is dominated by a constant times $\varphi_\alpha(m+1)$, for some constant independent of m. On the other hand, by the second part of Lemma 3.5, the square of the second summand is $o(\varphi_\alpha(m+1))$, and thus is negligible by comparison. Therefore,

$$\mathrm{dist}_{D_\alpha}^2(1, f \cdot \mathcal{P}_m) \leq C\varphi_\alpha^{-1}(m+1)$$

for some constant $C = C(\alpha, f)$, as desired.

Let us now address the lower bound for such functions f. Notice first that if the lower bound holds for functions of the form $(\zeta - z)g(z)$, where g is analytic and without zeros in the closed unit disk, then the conclusion holds for f. Moreover, as in the proof of Lemma 3.1, it is enough to consider $\zeta = 1$. Therefore, we write $f(z) = h(z)g(z)$, where $h(z) = 1 - z$ and g as above. Again, since g is analytic and has no zeros in the closed disk, note that both g and $1/g$ are multipliers for D_α. Therefore, if p_m is any polynomial of degree less than or equal to m,

$$\|p_m f - 1\|_\alpha \leq \|g\|_{M(D_\alpha)}\|p_m h - 1/g\|_\alpha \leq \|g\|_{M(D_\alpha)}\|1/g\|_{M(D_\alpha)}\|p_m f - 1\|_\alpha.$$

Now, let's choose p_m to be the optimal approximants of degree less than or equal to m to $1/f$. Then by the above discussion, we can assume $p_m h - 1/g \to 0$ in D_α, and in particular, the norms $\|p_m h\|_\alpha$ are bounded. We thus obtain

$$\begin{aligned}\|p_m f - 1\|_\alpha &= \|p_m h(g - T_m(g) + T_m(g)) - 1\|_\alpha \\ &\geq \|p_m h T_m(g) - 1\|_\alpha - \|p_m h(g - T_m(g))\|_\alpha\end{aligned}$$

Now, by Lemma 3.1, $\|p_m h T_m(g) - 1\|_\alpha^2$ is greater than or equal to a constant times $\varphi_\alpha^{-1}(2m+1)$, which in turn is comparable to $\varphi_\alpha^{-1}(m+1)$. On the other hand,

$$\|p_m h(g - T_m(g))\|_\alpha \leq \|p_m h\|_\alpha\|g - T_m(g)\|_{M(D_\alpha)},$$

so by Lemma 3.5 and since the norms of $\|p_m h\|_\alpha$ are bounded, this term decays at an exponential rate. Therefore, there exist constants C_1 and C_2

such that

$$\text{dist}^2_{D_\alpha}(1, f \cdot \mathcal{P}_m) = \|p_m f - 1\|^2_\alpha \geq C_1 \|p_m h T_m(g) - 1\|_\alpha \geq C_2 \varphi^{-1}_\alpha(m+1),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.7. The methods used in the proofs of Theorems 3.4 and 3.6 yield an independent proof of the upper bound for the optimal norm in the Dirichlet space (the case $\alpha = 1$), valid for a class of functions with the property that the Fourier coefficients of f and of $1/f$ exhibit simultaneously rapid decay. More specifically, if $\{a_j\}$ denotes the sequence of Taylor coefficients of a function $f \in D$ and $\{b_i\}$ denotes the coefficients of $1/f$, we say that f is a *strongly invertible function* if f has no zeros in $\mathbb{D}$ and, if for all j and k, we have $|a_j| \leq C(j+1)^{-3}$, and $|b_k| \leq C(k+1)^{-1}$, for some constant C. For example, one can show that if f is strongly invertible, then $1/f$ is in the Dirichlet space. (In fact, much more is true; $1/f \in D_2$.) That is, strongly invertible implies invertible in the Dirichlet space, and such functions are known to be cyclic (see [4], p. 274), and are therefore of interest. By defining polynomials analogous to those at the end of Section 2, namely,

$$P_n(z) = \sum_{k=0}^n \left(1 - \frac{H_k}{H_{n+1}}\right) b_k z^k,$$

one can use the stronger condition on the decay of the coefficients of $1/f$ to prove a version of the Control Lemma 3.3 with the coefficients H_k and then one can obtain the conclusion of Theorem 3.6 for these strongly invertible functions. In particular, it can be shown that the following holds.

Proposition 3.8. *Let f be a strongly invertible function, $\gamma \in \mathbb{N}$ and $g = f^\gamma$. Then there exist polynomials q_n of degree at most n for which $\|q_n g - 1\|^2_D \leq C/\log(n+2)$.*

It would be natural to investigate whether these Riesz-type polynomials provide close to optimal approximants for more general functions, in particular functions of the form $f_\beta(z) = (1-z)^\beta$, when $\beta < 1$. Another interesting question would be whether the rate of decay that we have observed for functions admitting an analytic continuation to the closed disk holds for other functions that vanish precisely on the same set.

4. LOGARITHMIC CONDITIONS

It is well-known that if f is invertible in the Hardy or Dirichlet space, then f is cyclic in that space. In addition, it is easy to see that if both f and $1/f$ are in D_α and f is bounded then $\log f \in D_\alpha$, but the converse does not hold. The condition that $\log f \in D_\alpha$ can be thought of as an intermediate between $f \in D_\alpha$ and $1/f \in D_\alpha$. Indeed, $\log f \in D_\alpha$ is equivalent to f'/f being a $D_{\alpha-2}$ function. On the other hand, $f \in D_\alpha$ if and only if $f' \in D_{\alpha-2}$, while $1/f \in D_\alpha$ if and only if $f'/f^2 \in D_{\alpha-2}$. We therefore want to study the following question:

Problem 4.1. Is any function $f \in D_\alpha$, with logarithm $q = \log f \in D_\alpha$, cyclic in D_α?

In several cases the statement is true: If $\alpha > 1$ or $\alpha = 0$, and $f \in D_\alpha$ with its logarithm $q = \log f \in D_\alpha$, then f is cyclic in D_α. Indeed, for $\alpha > 1$, $\log f \in D_\alpha$ implies $1/f \in H^\infty$, which is equivalent to the cyclicity of f (see p. 274 of [4]). For $\alpha = 0$, it is easy to see that if $\log f \in H^1$, then $\log |f(0)| = (1/2\pi) \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$, and therefore f is outer, that is, cyclic in H^2. Moreover, the logarithmic condition implies the following interpolation result, valid for all $\alpha < 2$.

Lemma 4.2. Suppose $f \in D_\alpha$ and $\log f \in D_\alpha$. Then, for any $\tau \in (0, 1]$, we have
$$D_\alpha(f^\tau) \leq \tau^2 \left(D_\alpha(f) + D_\alpha(\log f) \right),$$
and consequently, $f^\tau \in D_\alpha$.

Proof. It suffices to establish the bound on $D_\alpha(f^\tau)$. To this end, we write

$$
\begin{aligned}
D_\alpha(f^\tau) &= \int_{\mathbb{D}} |(f^\tau)'(z)|^2 d\mu_\alpha(z) = \tau^2 \int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} \right|^2 |f(z)|^{2\tau} d\mu_\alpha(z) \\
&= \tau^2 \int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} \right|^2 |f(z)|^{2\tau} \chi_{\{z \in \mathbb{D}: |f(z)| < 1\}} d\mu_\alpha(z) \\
&\quad + \tau^2 \int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} \right|^2 |f(z)|^{2\tau} \chi_{\{z \in \mathbb{D}: |f(z)| \geq 1\}} d\mu_\alpha(z) \\
&\leq \tau^2 \int_{\mathbb{D}} \left| \frac{f'(z)}{f(z)} \right|^2 \chi_{\{z \in \mathbb{D}: |f(z)| < 1\}} d\mu_\alpha(z) \\
&\quad + \tau^2 \int_{\mathbb{D}} |f'(z)|^2 \chi_{\{z \in \mathbb{D}: |f(z)| \geq 1\}} d\mu_\alpha(z),
\end{aligned}
$$

and the resulting integrals can be bounded in terms of $D_\alpha(f)$ and $D_\alpha(\log f)$, as claimed. $\square$

This lemma allows us to show that for a function f in the Dirichlet space D, corresponding to the case $\alpha = 1$, the condition $\log f \in D$ does imply the cyclicity of f. The proof relies on the following theorem due to Richter and Sundberg (see [11, Theorem 4.3] and let μ be Lebesgue measure).

Theorem 4.3 (Richter and Sundberg, 1992)**.** *If $f \in D$ is an outer function, and if $\tau > 0$ is such that $f^\tau \in D$, then $[f] = [f^\tau]$.*

In [11], Richter and Sundberg applied this theorem by showing that if f is univalent and non-vanishing, then $f^\tau \in D$, and hence is cyclic. In what follows, we do not require univalence.

Theorem 4.4. *Suppose $f \in D$ and $\log f \in D$. Then f is cyclic in the Dirichlet space.*

Proof. As discussed above, the logarithmic condition $\log f \in D$ implies that f is outer. Next, by Lemma 4.2, $f^\tau \in D$ for all $\tau > 0$, and so $[f] = [f^\tau]$ for each τ. Since the Lemma also implies $f^\tau \to 1$ in D as $\tau \to 0$, we have $[f] = [1]$, and the assertion follows. $\square$

The following is the main result for the remaining cases $\alpha < 0$ and $0 < \alpha < 1$.

Theorem 4.5. *Let $f \in H^\infty$ and $q = \log f \in D_\alpha$. Suppose there is a sequence of polynomials $\{q_n\}$ that approach q in D_α norm with*

$$2 \sup_{z \in \mathbb{D}} \mathrm{Re}(q(z) - q_n(z)) + \log(\|q - q_n\|_\alpha^2) \le C$$

for some constant $C > 0$. Then f is cyclic in D_α.

Remark 4.6. An immediate consequence of Theorem 4.5 is that if $q = \log f$ can be approximated in D_α by polynomials $\{q_n\}$ with $\sup_{z \in \mathbb{D}} \mathrm{Re}(q(z) - q_n(z)) < C$, then f is cyclic. Brown and Cohn proved (see [3, Theorem B]) that for any closed set of logarithmic capacity zero $E \subset \partial\mathbb{D}$, there exists a cyclic function f in D such that $\mathcal{Z}(f^*) = E$. The functions they build satisfy this hypothesis on q_n, and therefore, these assumptions are always satisfied by at least one cyclic function, for any potential cyclic function zero set.

Proof of Theorem 4.5. We can assume $\alpha \le 1$, because otherwise the statement is immediate. As discussed earlier in this section, the function f is in D_α. Now, for any sequence of polynomials p_n, by the triangle inequality

$$\|p_n f - 1\|_\alpha \le \|p_n f - e^{-q_n} f\|_\alpha + \|e^{-q_n} f - 1\|_\alpha. \tag{4.1}$$

The first summand on the right hand side can be bounded by

$$\|(p_n - e^{-q_n})f\|_\alpha \le \|p_n - e^{-q_n}\|_{M(D_\alpha)} \|f\|_\alpha.$$

Moreover, for $\alpha \le 1$, the multiplier norm of a function is controlled by the H^∞ norm of its derivative. Hence, a good choice of approximating polynomials is to select $\{p_n\}$ so that $p_n(0) = e^{-q_n(0)}$ and $\|p_n' + q_n' e^{-q_n}\|_{H^\infty} \le 1/n$, which is possible by Weierstrass' Theorem. With this choice, the first summand on the right hand side in (4.1) approaches 0 as $n \to \infty$.

Note that these polynomials p_n converge pointwise to $1/f$, and therefore, to prove the cyclicity of f, it is sufficient to show that the norms of $p_n f - 1$ stay bounded. So what remains to show is that, as n goes to infinity, $\|e^{-q_n} f - 1\|_\alpha^2$ is uniformly bounded. To evaluate this expression for large n, we use the norm in terms of the derivative:

$$\|e^{-q_n} f - 1\|_\alpha^2 \approx \| - q_n' e^{-q_n} f + e^{-q_n} f'\|_{\alpha-2}^2 + |e^{-q_n(0)} f(0) - 1|^2.$$

The last term tends to 0 since q_n approaches q pointwise.

In the first summand on the right hand side, taking out a common factor, we see that

$$
\begin{aligned}
\| -q_n' e^{-q_n} f + e^{-q_n} f' \|_{\alpha-2}^2 \;\; &\leq \;\; \| e^{q-q_n} \|_{H^\infty}^2 \left\| \frac{f'}{f} - q_n' \right\|_{\alpha-2}^2 \\
&\leq \;\; C e^{2 \sup \operatorname{Re}(q-q_n)} \| q - q_n \|_\alpha^2
\end{aligned}
$$

for some constant C. Given our assumptions on q_n, the right hand side is less than a constant. This concludes the proof. $\qquad \square$

It would be interesting to determine whether the required approximation property of the polynomials q_n in Theorem 4.5 is a consequence of the other hypotheses.

5. Asymptotic zero distributions for approximating polynomials

In this paper we have primarily been interested in functions $f \in D_\alpha$ that are cyclic and have $f^*(\zeta) = 0$ for at least one $\zeta \in \mathbb{T}$. Prime examples of such a function are

$$
f_\beta(z) = (1 - z)^\beta, \quad \beta \in [0, \infty),
$$

which we have examined closely in this paper for β a natural number.

Numerical experiments, described below, suggest that a study of the zero sets $\mathcal{Z}(p_n)$ of approximating polynomials may be interesting from the point of view of cyclicity. It seems that the rate at which zeros approach the circle is related to the extent to which the corresponding polynomials furnish approximants in D_α. For instance, we have compared the zero sets associated with the Taylor polynomials of $1/f_\beta$ with those of Riesz-type polynomials,

$$
\mathcal{R}_n \left(\frac{1}{f_\beta} \right) (z) = \sum_{k=0}^n \left(1 - \frac{H_k}{H_{n+1}} \right) b_k z^k, \quad n \geq 1. \tag{5.1}
$$

Intuitively, since $1/f_\beta$ has a pole at $z = 1$, we should expect the approximating polynomials p_n to be "large" in the intersection of disks of the form $B(1, r)$ with the unit disk. On the other hand, the remainder functions $p_n f - 1$ have to tend to zero in norm (and hence pointwise). We note that since $1/f_\beta$ has a pole on $\mathbb{T}$, the Taylor series of $1/f_\beta$ cannot have radius of convergence greater than 1. It therefore follows from Jentzsch's theorem that every point on $\mathbb{T}$ is a limit point of the zeros of the sequence $\{T_n(1/f_\beta)\}_{n=1}^\infty$. We refer the reader to [13] for background material concerning sections of polynomials, and for useful computer code.

We start with the simplest case $f_1(z) = 1 - z$. The zeros of the Taylor polynomials $T_n(1/f)$, the Cesàro polynomials $C_n(1/f)$, and the Riesz polynomials $R_n(1/f)$, for $n = 1, \ldots, 50$, can be found in Figure 5.1. All the zeros of these polynomials are located outside the unit disk, and inside
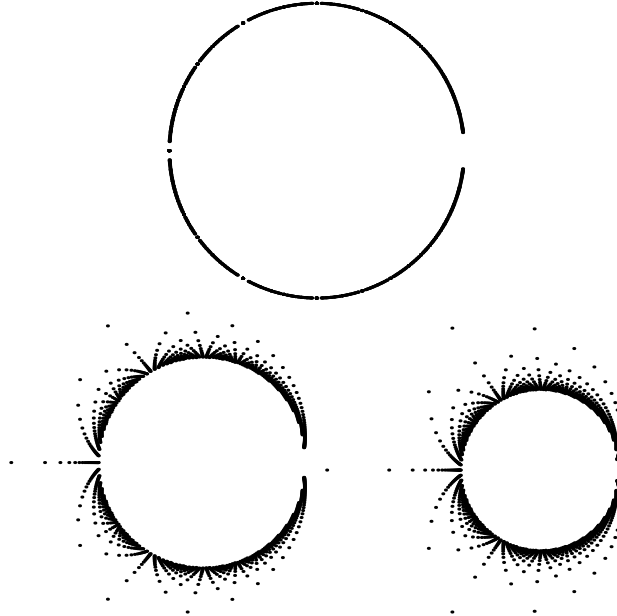
FIGURE 5.1. Left to right: Successive zero sets $\mathcal{Z}(T_n)$, $\mathcal{Z}(C_n)$, and $\mathcal{Z}(R_n)$, for $f_1 = 1 - z$, and $n = 1, \ldots, 50$.

a certain cardioid-like curve. In the case of the Taylor polynomials, the explicit formula

$$T_n(1/f_1)(z) = \frac{1 - z^{n+1}}{1 - z}$$

holds, and so $\mathcal{Z}(T_n)$ simply consists of the n-th roots of unity, minus the point $\zeta = 1$. Replacing Taylor polynomials by Cesàro polynomials has the effect of repelling zeros away from the unit circle, and into the exterior of the disk. This effect is even more pronounced for the Riesz polynomials (5.1), where it appears that convergence of roots to the unit circumference, and the roots of unity in particular, is somewhat slower. Note also the relative absence of zeros close to the pole of $1/f_1$, and the somewhat tangential approach region at $\zeta = 1$.

Next, we turn to a function with two simple zeros on $\mathbb{T}$, namely

$$f = 1 - z + z^2.$$

Plots of zeros of successive approximating polynomials are displayed in Figure 5.2. While $\mathcal{Z}(T_n)$ is more complicated, the general features of Figure 5.1 persist. We again note a relative absence of zeros close to the two poles of $1/f$, and the zeros of the Cesàro and Riesz polynomials are again located in the exterior disk, and seem to tend to $\mathbb{T}$ more slowly. We observe approach regions with vertices at the symmetrically placed poles, and the angle at these vertices seems to decrease as we move from Taylor polynomials through Cesàro polynomials to the polynomials in (5.1).
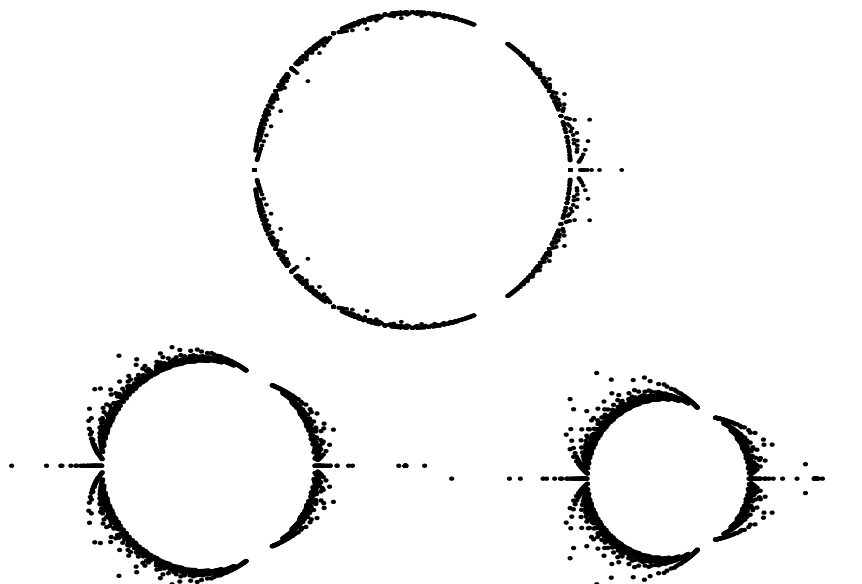
FIGURE 5.2. Left to right: Successive zero sets $\mathcal{Z}(T_n)$, $\mathcal{Z}(C_n)$, and $\mathcal{Z}(R_n)$, for $f_1 = 1 - z + z^2$, and $n = 1, \ldots, 50$.

It seems natural to suspect that locally the picture would be similar for a polynomial with a large number of zeros on the unit circle.

It would be interesting to investigate whether there is a relationship between zeros of approximating polynomials, the region of convergence of the Taylor series of $1/f$, and the cyclicity of f in future work.

Acknowledgments. The authors would like to thank Omar El-Fallah, Håkan Hedenmalm, Dima Khavinson, Artur Nicolau, Boris Shekhtman, and Dragan Vukotić for many helpful conversations during the writing of this paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, 4202 E. FOWLER AVENUE, TAMPA, FL 33620, USA.
E-mail address: `cbenetea@usf.edu`

DEPARTMENT OF MATHEMATICS, FLORIDA GULF COAST UNIVERSITY, 10501 FGCU BOULEVARD SOUTH, FORT MYERS, FL 33965-6565, USA.
E-mail address: `acondori@fgcu.edu`

DEPARTMENT OF MATHEMATICS, BAYLOR UNIVERSITY, ONE BEAR PLACE #97328, WACO, TX 76798-7328, USA.
E-mail address: `Constanze_Liaw@baylor.edu`

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA, SPAIN.
E-mail address: `dseco@mat.uab.cat`

STATISTICAL LABORATORY, CENTRE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK.
E-mail address: `a.sola@statslab.cam.ac.uk`