Université de Toulouse

THÈSE

En vue de l'obtention du DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par :
Université Toulouse III Paul Sabatier (UT3 Paul Sabatier)

Discipline ou spécialité :
Mathématiques fondamentales

Présentée et soutenue par
BUI VAN BINH
le : 10 Juin 2014

Titre :
Complex and p-adic Selberg integrals and Dyson-Macdonald's identities

---

École doctorale :
Mathématiques Informatique Télécommunications (MITT)

Unité de recherche :
L'Institut de Mathématiques de Toulouse - UMR 5219

Directeur de thèse :
Vadim SCHECHTMAN, Université Toulouse III

Rapporteurs :
Michael FINKELBERG, State University Higher School of Economics, Moscow
Vassily GORBOUNOV, University of Aberdeen, Scotland

Membres du jury :
Vassily GORBOUNOV, University of Aberdeen
Oleg OGIEVETSKY, Université Aix-Marseille
Serge VLADUTS, Université Aix-Marseille
Jean-Pierre RAMIS, Université Toulouse III
Jacques SAULOY, Université Toulouse III
Vadim SCHECHTMAN, Université Toulouse III

Remerciements

Je tiens tout d'arbord à remercier le directeur de cette thèse, Prof. Vadim Schechtman. Je lui serai pour toujours reconnaissant.

C'est pour moi un honneur que Prof. Michael Finkelberg, Prof. Vassily Gorbounov, Prof. Oleg Ogievetsky, Prof. Serge Vladuts, Prof. Jean-Pierre Ramis, Mcf. Jacques Sauloy aient voulu participer à mon jury. Je tiens à leur exprimer toute ma reconnaissance.

Je remercie le gouvernement du Vietnam, en particulière le Ministère de l'Éducation et de la Formation du Vietnam qui m'a donné une bourse du programme 322 pour que je fasse ma thèse. Je remercie le gouvernement Français qui m'a donné la possibilité d'avoir une couverture sociale.

Je tiens à saluer ceux qui m'ont supporté au "bureau 103-1R2", Cyrille Corpet, Guillaume Couffignal, Eric Sageloli .

Ma pensée va également aux professeurs, collègues et amis du Laboratoire Émile Picard. Merci à Cyrille Corpet pour m'avoir aidé à corriger le français de ma thèse.

Je n'oublie pas non plus les Vietnamiens et tous ceux qui ont partagé ma vie et avec qui j'ai passé de bons moments à Toulouse.

J'ai une pensée particulière pour ma femme Tran Thi Hue, ma fille Bui Hong Minh et les membres de ma famille qui ont été d'une aide inestimable par leur soutien et leur confiance.

Bui Van Binh
Toulouse, Juin 2014

Introduction

Cette thèse présente l'étude d'intégrales en lien avec la Théorie Conforme des Champs (CFT) d'une part, et les représentations de groupes de Lie réels, complexes et p-adiques (ainsi que leurs déformations quantiques) et les formes automorphes d'autre part. Le sujet de la CFT, traduit en langage mathématique, et la théorie de représentation de certaines algèbres de Lie de dimension infinie, comme les algèbres de Virasoro ou de Kac-Moody $\hat{\mathfrak{g}}$, $\mathfrak{g}$ est une algèbre de Lie semi-simple (ou réductive) complexe ou réelle. Plus précisément, on étudie les structures tensorielles sur les catégories de représentations appropriées de $\hat{\mathfrak{g}}$ (ou sur les groupes de Kac-Moody $\hat{G}$ correspondant); les éléments de ces représentations sont appelés *champs* et leur produit tensoriel est appelé *fusion* par les physiciens.

La structure de base que l'on obtient, une *algèbre d'opérateur*, est un analogue en dimension infinie de l'algèbre de Frobenius - une algèbre commutative munie d'un produit scalaire invariant. Les constantes structurelles d'une telle algèbre peuvent être exprimées par des intégrales multiples. Les physiciens ont découvert que dans certains cas (en lien avec $G = SU(2)$ par exemple), ces intégrales sont explicitement calculables en termes de quotients de produits de la fonction Gamma d'Euler. Il se trouve que certains de ces intégrales ont déjà été calculées par A. Selberg en 1944; elles généralisent la formule d'Euler classique pour la fonction Beta. La raison de cette calculabilité est discuté plus loin.

Dans le présent ouvrage, nous étudions des intégrales triples en lien avec les constantes structurelles du "modèle de Liouville" de la CFT; ce modèle se réduit à la théorie de représentation du groupe de Kac-Moody correspondant au groupe de Lie $G_{\mathbb{C}} = SL_2(\mathbb{C})$, et de certains groupes semblables. En général, comme il a été découvert par les frères Zamolodchikovs (cf. [ZZ]), les constantes structurelles pour ce modèle admettent une forme compliquée (bien qu'explicite) dont l'expression en termes d'intégrales n'est pas connue (et est un problème intéressant). Cependant, dans certains cas limite, la partie de Kac-Moody disparait et les constantes strucutrelles peuvent être décrites par une intégrale triple sur $G_{\mathbb{C}}$ lui-même, cf. [ZZ]. Après quelques transformations, elle prend la forme

$$\int_{\mathbb{C}^3} (1+|x_1|^2)^{-2\sigma_1}(1+|x_2|^2)^{-2\sigma_2}(1+|x_3|^2)^{-2\sigma_3}$$
$$|x_1-x_2|^{-2-2\nu_3}|x_2-x_3|^{-2-2\nu_1}|x_3-x_1|^{-2-2\nu_2}dx_1dx_2dx_3 =$$
$$= \pi^3 \frac{\Gamma(\sigma_1+\sigma_2+\sigma_3-1)\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(-\nu_3)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)} \tag{1}$$

Ici

$$\nu_1 = \sigma_1 - \sigma_2 - \sigma_3,\ \nu_2 = \sigma_2 - \sigma_3 - \sigma_1,\ \nu_3 = \sigma_3 - \sigma_1 - \sigma_2$$

Il est à remarquer que des intégrales semblables, liées au groupe $G_{\mathbb{R}} = SL_2(\mathbb{R})$, apparaissent dans un développement par Bernstein et Reznikov, apparemment sans lien avec ceci, cherchant à estimer les coefficients de Fourier de formes de Maass, cf. [BR].

Ceci laisse à penser qu'il est raisonnable de considérer aussi les intégrales p-adiques correspondantes (reliées au groupe $G_p = SL_2(\mathbb{Q}_p)$).

Ces remarques sont un point de départ pour cette thèse. Décrivons maintenant ses principaux résultats. Dans le deuxième Chapitre, nous définissons et calculons des analogues p-adiques et q-déformés aux intégrales de Zamolodchikovs-Bernstein-Reznikov (ZZBR). Nous fournissons aussi deux preuves de (1) (cette formule est donnée sans preuve dans [ZZ]). La forme que prend nécessairement l'intégrale ZZBR p-adique peut être extraite de la version p-adique de la formule de Gindikin-Karpelevich par Langlands, cf. [La].

Ensuite, on remarque que dans le cas d'exposants réels, l'intégrale ZZBR devient (après un changement de variables) une identité de Dyson-Macdonald à terme constant (mais pas réductible à une identité de Macdonald standard). La q-déformation des identités de Macdonald est connue. Ceci permet de supposer une version q-déformée de l'intégrale ZZBR comme suit.

Supposons que q soit un nombre réel, $0 < q < 1$. Pour tout $x, a \in \mathbb{C}$ on définit

$$(x;q)_a = \frac{(x;q)_\infty}{(xq^a;q)_\infty}$$

On définit aussi de façon usuelle la fonction q-Gamma

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}$$

Théorème. *Pour* $\Re(a_i) > 0,\ 1 \le i \le 3$

$$\frac{1}{(2\pi i)^3}\int_{T^3}\prod_{1\le i<j\le 3}(y_i/y_j;q)_{a_{ij}}(qy_j/y_i;q)_{a_{ij}}\frac{dy_1}{y_1}\frac{dy_2}{y_2}\frac{dy_3}{y_3} =$$

$$\frac{\Gamma_q(a_1+a_2+a_3+1)\prod_{i=1}^3\Gamma_q(2a_i+1)}{\prod_{i=1}^3\Gamma_q(a_i+1)\prod_{1\le i<j\le 3}\Gamma_q(a_i+a_j+1)} \qquad (2)$$

Ici, T^3 désigne le tore

$$T^3 = \{(y_1,y_2,y_3) \in \mathbb{C}^3 |\ |y_i| = 1,\ 1 \le i \le 3\}$$

Pour une preuve de ceci, voir Chap 2, Thm. 2.1.6

Toutes ces intérgales fournissent des exemples de cas expressibles en termes de quotients de produitsde valeurs de différentes fonctions Gamma (usuelle, p-adique ou encore q déformée). On remarque empiriquement qu'une telle calculabilité est liée à un phénomène de multiplicité 1. À savoir, nous avons un espace de dimension 1 (un group de cohomologie) muni de deux bases naturelles, et notre intégrale (une période) est leur coefficient de proportionnalité.

Un exemple archetype d'un tel espace de dimension 1 est $\mathrm{Hom}_{\mathfrak{g}}(V_a, V_b \otimes V_c)$ où $\mathfrak{g} = \mathfrak{sl}(2)$ et V_a, V_b, V_c sont des $\mathfrak{g}$-modules irreductibles de dimension finie. Dans le troisième Chapitre de la thèse, nous définissons les représentations de séries principales du groupe quantique $U_q\mathfrak{g} := U_q\mathfrak{sl}_2(\mathbb{R})$ et prouvons un théorème de multiplicité 1 pour celles-ci sous la forme suivante:

Théorème. V_i, $1 \leq i \leq 3$ *étant des représentations de séries principales, il existe une unique fonctionnelle, à une constante multiplicative près,* $U_q\mathfrak{g}$*-invariante.*

$$V_1 \otimes V_2 \otimes V_3 \longrightarrow \mathbb{C}$$

Pour une preuve, voir Chap 3, Thm 3.2.5.

On peut remarquer qu'une interprétation de l'intégrale (2) en tant que période liée au groupe quantique $U_q\mathfrak{g}$ reste un problème ouvert intéressant.

Le premier Chapitre de la thèse est de nature introductive. On y rappelle certaines formules et résultats connus en lien avec l'intégrale de Selberg, les identités de Macdonald, ainsi que la preuve de certaines formules moins standard pour les versions complexes et p-adiques de la fonction Beta d'Euler.

Les résultats principaux de cette thèse ont été publiés dans [BS1] - [BS3].

Introduction

The thesis is devoted to the study of some integrals related to the two-dimensional Conformal Field Theory (CFT) on the one hand, and to the representations of real, complex and p-adic Lie groups (and their quantum deformations) and automorphic forms on the other. The subject of CFT, translated into the mathematical language is the representation theory of certain infinite dimensional Lie algebras, like Virasoro algebra and Kac-Moody algebras $\hat{\mathfrak{g}}$, $\mathfrak{g}$ being a semisimple (or reductive) complex or real Lie algebra. More specifically, one studies the tensor structures on the categories of suitable representations of $\hat{\mathfrak{g}}$ (or the corresponding Kac-Moody Lie groups $\hat{G}$); the elements of these representations are called *fields*, and their tensor product is called *fusion* by physicists.

The basic structure which one obtains, an *operator algebra*, is an infinite-dimensional analogue of a Frobenius algebra — a commutative algebra equipped with an invariant scalar product. The structure constants of such an algebra may be expressed by certain multiple integrals. The physicists have discovered that in some cases (related to $G = SU(2)$ for example) these integrals are explicitly computable in terms of ratios of products of the Euler Gamma function. In fact, some of the integrals of this kind have been computed already in 1944 by A.Selberg; they generalize the classical Euler formula for the Beta function. The reason of this computability is discussed later.

In the present work we study triple integrals related to the structure constants of the "Liouville model" of the CFT; this model boils down to the representation theory of the Kac-Moody group corresponding to the Lie group $G_{\mathbb{C}} = SL_2(\mathbb{C})$, and its relatives. In general, as was discovered by brothers Zamolodchikovs (cf. [ZZ]), the structure constants for this model admit a complicated (though explicit) form whose expression in terms of integrals is not known (and is an interesting problem). However, in some limiting cases the Kac-Moody part disappears and the structure constants can be written down as triple integrals over the group $G_{\mathbb{C}}$ itself, cf. [ZZ]. After some transformations it takes the form

$$\int_{\mathbb{C}^3} (1+|x_1|^2)^{-2\sigma_1}(1+|x_2|^2)^{-2\sigma_2}(1+|x_3|^2)^{-2\sigma_3}$$

$$|x_1-x_2|^{-2-2\nu_3}|x_2-x_3|^{-2-2\nu_1}|x_3-x_1|^{-2-2\nu_2}dx_1dx_2dx_3 =$$

$$\pi^3\frac{\Gamma(\sigma_1+\sigma_2+\sigma_3-1)\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(-\nu_3)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)} \tag{1}$$

Here

$$\nu_1 = \sigma_1 - \sigma_2 - \sigma_3,\ \nu_2 = \sigma_2 - \sigma_3 - \sigma_1,\ \nu_3 = \sigma_3 - \sigma_1 - \sigma_2$$

It is remarkable that similar integrals connected with the group $G_{\mathbb{R}} = SL_2(\mathbb{R})$ appeared in a seemingly unrelated development by Bernstein and Reznikov devoted to the estimation of Fourier coefficients of Maass forms, cf. [BR]. This suggests we should consider the corresponding p-adic integrals (related to the group $G_p = SL_2(\mathbb{Q}_p)$) as well. My thesis develops these ideas.

Let us pass to the description of its main results. In the Second Chapter we define and compute p-adic and q-deformed analogs of Zamolodchikovs-Bernstein-Reznikov's (ZZBR) integrals. We also provide two proofs of (1) (this formula was given in [ZZ] without proof). The necessary form of the p-adic ZZBR integrals may be extracted from the Langland's p-adic version of the Gindikin - Karpelevich formula, cf. [La].

Next, one notes that in the case of integer exponents the real ZZBR integral becomes (after a change of variables) a constant term Dyson-Macdonald identity (but not reducible to a standard Macdonald identity). The q-deformation of the Macdonald identities is known. This allowed me to correctly guess a q-deformed version of ZZBR integral as follows.

Suppose that q is a real number, $0 < q < 1$. For any $x, a \in \mathbb{C}$ we define

$$(x;q)_a = \frac{(x;q)_\infty}{(xq^a;q)_\infty}$$

Define the q-Gamma function as usually by

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}$$

Theorem. *For* $\Re(a_i) > 0,\ 1 \le i \le 3$

$$\frac{1}{(2\pi i)^3}\int_{T^3}\prod_{1\le i<j\le 3}(y_i/y_j;q)_{a_{ij}}(qy_j/y_i;q)_{a_{ij}}\frac{dy_1}{y_1}\frac{dy_2}{y_2}\frac{dy_3}{y_3} =$$

$$\frac{\Gamma_q(a_1+a_2+a_3+1)\prod_{i=1}^3\Gamma_q(2a_i+1)}{\prod_{i=1}^3\Gamma_q(a_i+1)\prod_{1\le i<j\le 3}\Gamma_q(a_i+a_j+1)} \tag{2}$$

Here T^3 denotes the torus

$$T^3 = \{(y_1,y_2,y_3)\in\mathbb{C}^3 |\ |y_i| = 1,\ 1\le i\le 3\}$$

For a proof, see Chapter 2, Thm. 2.1.6

All these integrals provide the examples of a wide class expressible in terms of quotients of products of the values of various Gamma functions (the usual ones, or p-adic, or q-Gamma functions). An empirical fact is that such computability is connected with multiplicity one phenomenon. Namely, we have a one dimensional space (a cohomology group) with two natural bases in it, and our integral (a period) is their proportionality coefficient.

An archetypical example of such one-dimensional space is $\mathrm{Hom}_{\mathfrak{g}}(V_a, V_b\otimes V_c)$ where $\mathfrak{g} = \mathfrak{sl}(2)$ and V_a, V_b, V_c are finite-dimensional irreducible $\mathfrak{g}$-modules. In the Third Chapter of the thesis we define the principal series representations of the quantum group $U_q\mathfrak{g} := U_q\mathfrak{sl}_2(\mathbb{R})$ and prove a multiplicity one theorem for them in the following form:

Theorem. $V_i,\ 1\le i\le 3$, *being principal series representations, there exists one, up to a multiplicative constant,* $U_q\mathfrak{g}$*-invariant functional*

$$V_1\otimes V_2\otimes V_3 \longrightarrow \mathbb{C}$$

For a proof see Chapter 3, Thm. 3.2.5

Note that an intepretation of the integral (2) as a period related to the quantum group $U_q\mathfrak{g}$ remains an interesting open problem.

The First Chapter of the dissertation is of introductory nature. We recall there some known formulas and results related to the Selberg integrals and Macdonald identities, and prove some less standard formulas for complex and p-adic versions of the Euler Beta function.

The main results of the thesis have been published in [BS1] - [BS3].

Contents

CHAPTER 1

Generalities, elementary calculations

1.1. Selberg's integral

1.1.1 The Euler functions. Let us recall the definitions

1. Gamma function

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt, \quad Re(s) > 0$$

2. Beta function

$$B(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx, \quad Re(s), Re(t) > 0$$

The following relation between these functions

Theorem.

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

1.1.2. Real and complex Selberg integrals. Let us mention two important generalizations of the theorem :

Real Selberg integral

$$\mathcal{I}_{\mathbb{R}}(n;\alpha,\beta,\rho) = \int_{[0,1]^n} \prod_{j=1}^n x_j^\alpha (1-x_j)^\beta \prod_{i<j} |x_i - x_j|^\rho dx_1 ... dx_n =$$

$$= \prod_{j=1}^n \frac{\Gamma(1+j\rho/2)\Gamma(\alpha+1+(j-1)\rho/2)\Gamma(\beta+1+(j-1)\rho/2)}{\Gamma(1+\rho/2)\Gamma(\alpha+\beta+2+(n+j-2)\rho/2)}$$

where

$$\mathrm{Re}(\alpha+\beta) < -1 < \mathrm{Re}(\alpha), \mathrm{Re}(\beta)$$

$$(n-1)\mathrm{Re}(\rho) + \mathrm{Re}(\alpha+\beta) < -1$$

The proof can be found in chapter 17, [M].

The complex case ,

$$\mathcal{I}_{\mathbb{C}}(n;\alpha,\beta,\rho)=\int_{\mathbb{C}^n}\prod_{j=1}^{n}|z_j|^{\alpha}|1-z_j|^{\beta}\prod_{k<j}|z_k-z_j|^{\rho}(i/2)^n dz_1 d\bar{z}_1 ... dz_n d\bar{z}_n$$

$$=\pi^n\prod_{j=1}^{n}\frac{\gamma(1+j\rho/2)\gamma(\alpha+1+(j-1)\rho/2)\gamma(\beta+1+(j-1)\rho/2)}{\gamma(1+\rho/2)\gamma(\alpha+\beta+2+(n+j-2)\rho/2)}$$

$$=S(n;\alpha,\beta,\rho).\mathcal{I}_{\mathbb{R}}(n;\alpha,\beta,\rho)^2$$

where $\gamma(x)=\Gamma(x)/\Gamma(1-x)$,

$$S(n;\alpha,\beta,\rho)=\prod_{j=1}^{n}\frac{\sin(j\rho\pi/2)\sin(\pi(\alpha+(j-1)\rho/2))\sin(\pi(\beta+(j-1)\rho/2)}{\sin(\rho\pi/2)\sin(\pi(\alpha+\beta+(n+j-2)\rho/2))}$$

We refer the reader to [Ao] or [DF] for details.

Example. $n=1$

$$\int_{\mathbb{C}}|z|^{2a}|z-1|^{2b}\frac{i}{2}dzd\bar{z}=\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi(a+b)}\left(\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right)^2$$

One has an elementary proof for this in Appendix.

1.2. The p-adic case

1.2.1. Definition. We set

$$\Gamma_{\mathbb{Q}_p}(s):=\int_{\mathbb{Q}_p^{\times}}1_{\mathbb{Z}_p}|x|_p^{s-1}d_p x$$

where $d_p x$ is the Haar measure on $\mathbb{Q}_p$.

Similarly, the Beta function on $\mathbb{Q}_p$ is defined by :

$$B_{\mathbb{Q}_p}(s,t)=\int_{\mathbb{Q}_p^{\times}}|x|_p^{s-1}|1-x|_p^{t-1}d_p x$$

The following is a p-adic analogue of Theorem in 1.1.1

1.2.2. Theorem.

$$B_{\mathbb{Q}_p}(s,t)=\frac{\gamma_{\mathbb{Q}_p}(s)\gamma_{\mathbb{Q}_p}(t)}{\gamma_{\mathbb{Q}_p}(s+t)}$$

where $\gamma_{\mathbb{Q}_p}(s)=\Gamma_{\mathbb{Q}_p}(s)/\Gamma_{\mathbb{Q}_p}(1-s)$

Proof. Straighforward calculations show that

$$\Gamma_{\mathbb{Q}_p}(s) = \frac{1-p^{-1}}{1-p^{-s}}$$

$$B_{\mathbb{Q}_p}(s,t) = \frac{(1-p^{s-1})(1-p^{t-1})(1-p^{-s-t})}{(1-p^{-s})(1-p^{-t})(1-p^{s+t-1})}$$

This proves the theorem. □

For the genaral case (i.e. $n \geq 2$), a p-adic version of 1.1.2. on $\mathbb{Q}_p$ is no longer true. One of the possible reasons is the definition of Gamma, Beta is not as good as the real case. So, let us introduce other versions of these functions (see details in Chapter 2, 2.1.7).

Write

$$\phi_p(x) = \max\{|x|_p, 1\}$$

Here are some results

1.2.3. Theorem.

$$(i) I_1 = \int_{\mathbb{Q}_p} |\phi(x)|_p^a d_p x = \frac{1-p^a}{1-p^{a+1}}$$

$$(ii) I_2 = \int_{\mathbb{Q}_p^2} |\phi(x)|_p^a |\phi(y)|_p^b |x-y|_p^c d_p x d_p y$$

$$= \frac{p^c(1-p)(1-p^a)(1-p^b)(1-p^{a+b+2c+2})}{(1-p^{c+1})(1-p^{a+c+1})(1-p^{b+c+1})(1-p^{a+b+c+2})}$$

$$(iii) I_3 = \int_{\mathbb{Q}_p^3} |\phi(x)|_p^a |\phi(y)|_p^b |\phi(z)|_p^c |x-y|_p^\gamma |y-z|^\alpha |z-x|^\beta d_p x d_p y d_p z$$

$$= c \cdot \frac{\Gamma(\sigma_1+\sigma_2+\sigma_3-1)\Gamma(\sigma_1+\sigma_2-\sigma_3)\Gamma(\sigma_2+\sigma_3-\sigma_1)\Gamma(\sigma_3+\sigma_1-\sigma_2)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)}$$

where $c = 1 + p^{-1}, a = -2\sigma_1, b = -2\sigma_2, c = -2\sigma_3,$

$\alpha = -1-\sigma_1+\sigma_2+\sigma_3, \beta = -1-\sigma_2+\sigma_1+\sigma_3, \gamma = -1-\sigma_3+\sigma_2+\sigma_1$ *and* $\Gamma(s)$ *stands for* $\Gamma_{\mathbb{Q}_p}(s)$.

Proof. (i) By definition

$$I_1 = \int_{\mathbb{Z}_p} 1 d_p x + \int_{\mathbb{Q}_p/\mathbb{Z}_p} |x|_p^a d_p x = 1 + \sum_{n=-\infty}^{-1} \int_{X_n} |x|_p^a d_p x$$

where $X_n = \{x \in \mathbb{Q}_p | x = a_n p^n + ... + a_0 + a_1 p + ..., a_i \in \{0, 1, ..., p-1\} \forall i, a_n \neq 0\}$.

Since $d_p(X_n) = p^{-n}(1-p^{-1})$ and $|x_p| = p^{-n}$ on X_n

$$I_1 = 1 + \sum_{n=-\infty}^{-1} p^{-na} d_p(X_n) = 1 + \sum_{n=-\infty}^{-1} (1-p^{-1}) p^{-n(a+1)}$$

$$= 1 + (1-p^{-1}) \frac{p^{a+1}}{1-p^{a+1}} = \frac{1-p^a}{1-p^{a+1}}.$$

ii) See Thm. 2.2.4 in Chapter 2.
iii) The detailed calculations are included in the Appendix. In fact, we obtain

$$I_3 = (1-p^{-1})^2(\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}})\frac{1}{1-p^{-(\alpha+\beta+\gamma+2)}} \tag{1}$$

$$+(1-p^{-1})(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+2)}} \tag{2}$$

$$+(1-p^{-1})^2\sum \frac{1}{1-p^{-(\alpha+1)}}\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}} \tag{3}$$

$$+(1-p^{-1})(p-2)p^{-1}\sum \frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}} \tag{4}$$

$$+(1-p^{-1})^2\sum \left(\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}+\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}}\right)\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}} \tag{5}$$

$$+(1-p^{-1})^2\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}} \tag{6}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{7}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\sum \frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{8}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\sum \frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{9}$$

$$+(1-p^{-1})(p-2)(p-3)p^{-2}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{10}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{11}$$

$$+(1-p^{-1})^3\sum \left(\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}+\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}}\right)\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\times$$

$$\times\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{12}$$

$$+(1-p^{-1})^3\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{13}$$

$$+(1-p^{-1})^3\sum \frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{14}$$

$$+(1-p^{-1})^3\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{15}$$

Let

$$a=-2\sigma_1, b=-2\sigma_2, c=-2\sigma_3,$$
$$\alpha=-1-\sigma_1+\sigma_2+\sigma_3, \beta=-1-\sigma_2+\sigma_3+\sigma_1, \gamma=-1-\sigma_3+\sigma_1+\sigma_2$$

then

$$a+b+c+\alpha+\beta+\gamma+3=-(\sigma_1+\sigma_2+\sigma_3), -(\alpha+\beta+\gamma+2)=1-(\sigma_1+\sigma_2+\sigma_3)$$
$$\alpha+1=-\sigma_1+\sigma_2+\sigma_3, \beta+1=-\sigma_2+\sigma_3+\sigma_1, \gamma+1=-\sigma_3+\sigma_1+\sigma_2$$
$$a+\beta+\gamma+1=b+\alpha+\gamma+1=c+\alpha+\beta+1=-1$$
$$a+b+\alpha+\beta+\gamma+2=-1-\sigma_1-\sigma_2+\sigma_3$$
$$b+c+\alpha+\beta+\gamma+2=-1-\sigma_2-\sigma_3+\sigma_1$$
$$c+a+\alpha+\beta+\gamma+2=-1-\sigma_3-\sigma_1+\sigma_2$$

We deduce that

$$(4)+(5)+(6)=\sum(1-p^{-1})\frac{p^{-1-\sigma_2-\sigma_3+\sigma_1}}{1-p^{-1-\sigma_2-\sigma_3+\sigma_1}}$$
$$+\sum(1-p^{-1})^2\frac{p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}\frac{p^{-1-\sigma_2-\sigma_3+\sigma_1}}{1-p^{-1-\sigma_2-\sigma_3+\sigma_1}}$$
$$=(1-p^{-1})p^{-1}\sum\frac{p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}$$
$$(9)+(10)=(1-p^{-1})(p-2)p^{-1}\frac{p^{-(\sigma_1+\sigma_2+\sigma_3)}}{1-p^{-(\sigma_1+\sigma_2+\sigma_3)}}$$
$$(7)+(8)+(12)+(13)+(14)=(1-p^{-1})^2\frac{p^{-(\sigma_1+\sigma_2+\sigma_3)}}{1-p^{-(\sigma_1+\sigma_2+\sigma_3)}}\sum\frac{p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}$$

Thus we get

$$I_3=(1-p^{-1})^2\sum\frac{p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}\cdot\frac{1}{1-p^{1-(\sigma_1+\sigma_2+\sigma_3)}}$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{1}{1-p^{1-(\sigma_1+\sigma_2+\sigma_3)}}$$
$$+(1-p^{-1})p^{-1}\sum\frac{1+p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{p^{-(\sigma_1+\sigma_2+\sigma_3)}}{1-p^{1-(\sigma_1+\sigma_2+\sigma_3)}}$$
$$+(1-p^{-1})^2\frac{p^{-(\sigma_1+\sigma_2+\sigma_3)}}{1-p^{1-(\sigma_1+\sigma_2+\sigma_3)}}\sum\frac{p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}$$
$$=(1-p^{-1})^2\sum\frac{p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}\cdot\frac{1+p^{-(\sigma_1+\sigma_2+\sigma_3)}}{1-p^{1-(\sigma_1+\sigma_2+\sigma_3)}}$$
$$+(1-p^{-1})p^{-1}\sum\frac{1+p^{\sigma_1-\sigma_2-\sigma_3}}{1-p^{\sigma_1-\sigma_2-\sigma_3}}$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{1+p^{-(\sigma_1+\sigma_2+\sigma_3)}}{1-p^{1-(\sigma_1+\sigma_2+\sigma_3)}}$$

Set

$$x=\sigma_1-\sigma_2-\sigma_3, y=\sigma_2-\sigma_1-\sigma_3, z=\sigma_3-\sigma_2-\sigma_1$$

then

$$x+y=-2\sigma_3, y+z=-2\sigma_1, x+z=-2\sigma_2$$

$$I_3 = (1-p^{-1})\Bigg((1-p^{-1})\frac{1+p^{x+y+z}}{1-p^{x+y+z+1}}\sum\frac{p^x}{1-p^x}$$
$$+(p-2)p^{-1}\frac{1+p^{x+y+z}}{1-p^{x+y+z+1}}+p^{-1}\sum\frac{1+p^x}{1-p^x}\Bigg)$$

We have

$$(1-p^{-1})\frac{1+p^{x+y+z}}{1-p^{x+y+z+1}}\sum\frac{p^x}{1-p^x}+(p-2)p^{-1}\frac{1+p^{x+y+z}}{1-p^{x+y+z+1}}+p^{-1}\sum\frac{1+p^x}{1-p^x}$$
$$=\frac{1-\sum p^{x+y}+\sum p^{2x+y+z}-p^{2x+2y+2z}+}{(1-p^{x+y+z+1})\prod(1-p^x)}$$
$$+\frac{p^{-1}-\sum p^{x+y-1}+\sum p^{2x+y+z-1}-p^{2x+2y+2z-1}}{(1-p^{x+y+z+1})\prod(1-p^x)}$$
$$=\frac{(1+p^{-1})\prod(1-p^{x+y})}{(1-p^{x+y+z+1})\prod(1-p^x)}$$

Therefore

$$I_3=\frac{(1+p^{-1})(1-p^{-1})(1-p^{-2\sigma_1})(1-p^{-2\sigma_2})(1-p^{-2\sigma_3})}{(1-p^{\sigma_1-\sigma_2-\sigma_3})(1-p^{\sigma_2-\sigma_1-\sigma_3})(1-p^{\sigma_3-\sigma_2-\sigma_1})(1-p^{1-(\sigma_1+\sigma_2+\sigma_3)})}$$
$$=(1+p^{-1})\times$$
$$\frac{\Gamma_p(\sigma_1+\sigma_2-\sigma_3)\Gamma_p(\sigma_2+\sigma_3-\sigma_1)\Gamma_p(\sigma_3+\sigma_1-\sigma_2)\Gamma_p(\sigma_1+\sigma_2+\sigma_3-1)}{\Gamma_p(2\sigma_1)\Gamma_p(2\sigma_2)\Gamma_p(2\sigma_3)}$$

and the proof is complete. □

1.3. Dyson-Macdonald identities

1.3.1. Dyson identity(or Dyson's conjecture) for the root system A_n looks as follows :

$$\mathrm{CT}\prod_{1\le i<j\le n}(1-x_i/x_j)^a(1-x_j/x_i)^a=\frac{\Gamma(na+1)}{\Gamma(a+1)^n}$$

where "CT" means *constant term.*

The conjecture above is given by F. Dyson in 1962, [Dy] .

If $a\in\mathbb{N}$ then it becomes

$$\mathrm{CT}\prod_{1\le i<j\le n}(1-x_i/x_j)^a(1-x_j/x_i)^a=\frac{(na)!}{(a!)^n}$$

This identity was proved by J.Gunson and K.Wilson, who showed in more generally that

$$\mathrm{CT}\prod_{i\neq j}(1-x_i/x_j)^{a_i}=\frac{(a_1+...+a_n)!}{a_1!...a_n!}$$

where $a_1,...,a_n$ are nonnegative integers.

Andrews (1975) found a q-analog of Dyson's conjecture, stating that the constant term of

$$\prod_{1\le i<j\le n}\left(\frac{x_i}{x_j};q\right)_{a_i}\left(\frac{qx_j}{x_i};q\right)_{a_j}$$

is

$$\frac{(q;q)_{a_1+\cdots+a_n}}{(q;q)_{a_1}\cdots(q;q)_{a_n}}$$

where $(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)=(1-q)(1-aq)...(1-aq^{n-1})$. This conjecture becomes Dyson's conjecture when $q=1$, and was proved by Zeilberger and Bressoud (1985).

Moreover, Macdonald (1982,[Ma]) proved the following generalization. If R is a reduced root system, e^α denotes the formal exponential corresponding to $\alpha\in R$, then for any nonnegative integer k, we have:

1.3.2. Macdonald identity

$$\text{CT}\prod_{\alpha\in R}(1-e^\alpha)^k=\prod_{i=1}^{l}\binom{kd_i}{k}$$

He also extended the same formula with "q-analogues" in the following theorem

1.3.3. Theorem.(Macdonald's conjecture) . *Let k be a positive integer, R_+ be set of positive roots in a reduced root system R then*

$$\text{CT}\prod_{\alpha\in R_+}\prod_{i=1}^{k}(1-q^{i-1}e^{-\alpha})(1-q^ie^\alpha)=\prod_{i=1}^{l}\begin{bmatrix}kd_i\\k\end{bmatrix}$$

where $\begin{bmatrix}n\\r\end{bmatrix}$ *is the "q-binomical coefficient"*

$$\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-r-1})}{(1-q)(1-q^2)...(1-q^r)}$$

□

In **1.3.2** and **1.3.3** one can replace k by k_α, which is a multiplicity function such that $k_{w\alpha}=k_\alpha$, for all $w\in W$(i.e. Weyl group). With this assumption, the first proof for **1.3.2** was found by E. Opdam [O]. And there are so many interesting results we will not mention here. Finally, **Macdonald's conjectures** were proved in full generality by (Cherednik 1995, [CI]) using doubly affine Hecke algebras. In Chapter 2, we shall prove a conjecture for **q-analogues** in the case $n=3$ (correspond to the root system A_2) and k_α are distinct different.

The Dyson-Macdonald identities concern with some problems in the theory of Jacobi polynomial and Hecke algebra.

CHAPTER 2

Zamolodchikov's integral

2.1. Introduction

2.1.1. The following remarkable integral

$$I_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3)=\int_{\mathbb{C}^3}(1+|x_1|^2)^{-2\sigma_1}(1+|x_2|^2)^{-2\sigma_2}(1+|x_3|^2)^{-2\sigma_3}$$

$$|x_1-x_2|^{-2-2\nu_3}|x_2-x_3|^{-2-2\nu_1}|x_3-x_1|^{-2-2\nu_2}dx_1dx_2dx_3 \qquad (2.1.1.1)$$

has appeared in [ZZ] in connection with the Liouville model of the conformal field theory. Here $\sigma_i\in\mathbb{C}$,

$$\nu_1=\sigma_1-\sigma_2-\sigma_3,\ \nu_2=\sigma_2-\sigma_3-\sigma_1,\ \nu_3=\sigma_3-\sigma_1-\sigma_2$$

et dx denotes the standard Haar measure on $\mathbb{C}$.

Set

$$\tilde{I}_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3)=\int_{\mathbb{C}^2}(1+|x_1|^2)^{-2\sigma_1}(1+|x_2|^2)^{-2\sigma_2}|x_1-x_2|^{-2-2\nu_3}dx_1dx_2 \qquad (2.1.1.2)$$

To compute (2.1.1.1) the authors of [ZZ] first note that

$$I_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3)=\pi\tilde{I}_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) \qquad (2.1.1.3)$$

This may be proven using an $SU(2)$-symmetry (cf. 2.5.1 below). So (2.1.1.1) and (2.1.1.2) converge for $\Re\sigma_1,\Re\sigma_2,\Re\nu_3$ sufficiently large.

Then the authors give (without proof) the value of $\tilde{I}_{\mathbb{C}}$ and hence that of $I_{\mathbb{C}}$:

$$I_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3)=\pi^3\frac{\Gamma(\sigma_1+\sigma_2+\sigma_3-1)\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(-\nu_3)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)} \qquad (2.1.1.4)$$

A proof (somewhat artificial) of (2.1.1.4) may be found in [HMW]. We propose another proof in 2.5.3 below.

2.1.2. In this note we take the study of the *real*, *q-deformed* and *p-adic* versions of (2.1.1.1).

A real version of (2.1.1.1) is the integral (2.1.2.1) below, cf. 2.6. It has appeared in [BR] in connection with a study of periods of automorphic triple products:

$$I_{\mathbb{R}}(\sigma_1,\sigma_2,\sigma_3) := \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\prod_{i=1}^{3}|\sin(\theta_i-\theta_{i+1})|^{(\nu_{i+2}-1)/2}d\theta_1 d\theta_2 d\theta_3 =$$

$$= \frac{\Gamma((\nu_1+1)/4)\Gamma((\nu_2+1)/4)\Gamma((\nu_3+1)/4)\Gamma((\sum_i \nu_i+1)/4)}{\Gamma(1/2)^3\Gamma((1-\sigma_1)/2)\Gamma((1-\sigma_2)/2)\Gamma((1-\sigma_3)/2)} \tag{2.1.2.1}$$

The index i under the integral is understood modulo 3. The authors of [BR] provide an elegant proof of (2.1.2.1) using Gaussian integrals.

In this note we propose and calculate a q-deformation of this integral (see Thm. 2.1.6 and 2.4 below); in the limit $q \to 1$ this gives (2.1.2.1).

Set

$$a_i = \frac{\nu_i - 1}{4}$$

Then (2.1.2.1) rewrites as

$$J(a_1,a_2,a_3) := \frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\prod_{i=1}^{3}|\sin(\theta_i-\theta_{i+1})|^{2a_{i+2}}d\theta_1 d\theta_2 d\theta_3 =$$

$$= \frac{\Gamma(a_1+1/2)\Gamma(a_2+1/2)\Gamma(a_3+1/2)\Gamma(\sum a_i+1)}{\Gamma(1/2)^3\Gamma(a_1+a_2+1)\Gamma(a_2+a_3+1)\Gamma(a_3+a_1+1)}$$

$$= \frac{\prod_i \Gamma(2a_i+1)\Gamma(\sum a_i+1)}{4^{\sum a_i}\prod_i\Gamma(a_i+1)\prod_{i<j}\Gamma(a_i+a_j+1)} \tag{2.1.2.2}$$

where we have used the duplication formula

$$\Gamma(2a+1) = 2^{2a}\pi^{-1/2}\Gamma(a+1/2)\Gamma(a+1)$$

Let us suppose that a_i are positive integers. After a change of variables $y_j = e^{2i\theta_j}$ it is easy to see that (2.1.2.2) is equivalent to

$$\mathrm{CT}\prod_{1\le i<j\le 3}(1-y_i/y_j)^{a_{ij}}(1-y_j/y_i)^{a_{ij}}$$

$$= \frac{(a_1+a_2+a_3)!\prod_{i=1}^{3}(2a_i)!}{\prod_{i=1}^{3}a_i!\prod_{1\le i<j\le 3}(a_i+a_j)!} \tag{2.1.2.3}$$

Here $a_{ij} := a_k$ where $\{k\} = \{1,2,3\}\setminus\{i,j\}$ and CT means "constant term".

If $a_1 = a_2 = a_3 = a$, this gets into

$$\mathrm{CT}\prod_{1\le i\ne j\le 3}(1-y_i/y_j)^a = \frac{\Gamma(3a+1)}{\Gamma(a+1)^3} \tag{2.1.2.4}$$

which is the classical Dyson formula for the root system A_2, cf. [Dy], (142).

In 2.3 below we give an independent proof of (2.1.2.3).

2.1.3. The reformulation (2.1.2.3) allows us to write down a q-deformation of it. It looks as follows. For a natural a denote as usual

$$(x;q)_a = \prod_{i=0}^{a-1}(1-xq^i) = \frac{(x;q)_\infty}{(xq^a;q)_\infty}$$

where

$$(x;q)_\infty = \prod_{i=0}^{\infty}(1-xq^i) \tag{2.1.3.1}$$

Here q is a formal variable.

Denote

$$[a]_q^! = \frac{(q;q)_a}{(1-q)^a}$$

2.1.4. Theorem. *Let a_1, a_2, a_3 be positive integers. Then*

$$\mathrm{CT} \prod_{1\le i<j\le 3} (y_i/y_j;q)_{a_{ij}}(qy_j/y_i;q)_{a_{ij}}$$

$$= \frac{[a_1+a_2+a_3]_q^! \prod_{i=1}^3 [2a_i]_q^!}{\prod_{i=1}^3 [a_i]_q^! \prod_{1\le i<j\le 3}[a_i+a_j]_q^!} \tag{2.1.4.1}$$

For a proof see 2.4 below.

If $a_1 = a_2 = a_3$ this becomes the (proven) Macdonald's q-constant term conjecture for the root system A_2.

In fact, (2.1.4.1) is in turn a particular case of the following beautiful formula due to W.Morris.

Theorem (W.Morris). *Let a_1, a_2, a_3 be natural and $\sigma \in \Sigma_3$ be an arbitrary permutation. Then*

$$\mathrm{CT} \prod_{1\le i<j\le 3} (y_i/y_j;q)_{a_{ij}}(qy_j/y_i;q)_{a_{\sigma(i)\sigma(j)}}$$

$$= \frac{[a_1+a_2+a_3]_q^! \prod_{i=1}^3 [a_i+a_{\sigma(i)}]_q^!}{\prod_{i=1}^3 [a_i]_q^! \prod_{1\le i<j\le 3}[a_i+a_j]_q^!} \tag{2.1.4.2}$$

This is the case of the A_2-*isolated labeling* of Morris' conjecture, cf. [Mo], 4.3. The proof of (2.1.4.2) is contained in *op. cit.* 5.12. The formula (2.1.4.1) is the case $\sigma =$ the identity permutation of (2.1.4.2). Our proof of (2.1.4.1) is less involved than Morris's proof of the general case.

2.1.5. Let us generalize Thm. 2.1.4 to the case of complex a_i. To this end we suppose that q is a real number, $0 < q < 1$.

For any $x, a \in \mathbb{C}$ we define

$$(x;q)_a = \frac{(x;q)_\infty}{(xq^a;q)_\infty}$$

Define as usually

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}$$

We denote by T^3 the torus

$$T^3 = \{(y_1, y_2, y_3) \in \mathbb{C}^3 |\ |y_i| = 1,\ 1 \le i \le 3\}$$

2.1.6. Theorem. *For* $\Re(a_i) > 0,\ 1 \le i \le 3$

$$\frac{1}{(2\pi i)^3} \int_{T^3} \prod_{1\le i<j\le 3} (y_i/y_j; q)_{a_{ij}} (qy_j/y_i; q)_{a_{ij}} \frac{dy_1}{y_1}\frac{dy_2}{y_2}\frac{dy_3}{y_3} =$$

$$\frac{\Gamma_q(a_1 + a_2 + a_3 + 1) \prod_{i=1}^3 \Gamma_q(2a_i + 1)}{\prod_{i=1}^3 \Gamma_q(a_i + 1) \prod_{1\le i<j\le 3} \Gamma_q(a_i + a_j + 1)} \tag{2.1.6.1}$$

For a proof see 2.4.6 below. Passing to the limit $q \to 1$ gives (2.1.2.1).

2.1.7. Let us describe a p-adic version of (2.1.1.1).

Let p be a prime number; consider the field $\mathbb{Q}_p$ of rational p-adic numbers. Let d_px denote the Haar measure on $\mathbb{Q}_p$ normalized by the condition

$$\int_{\mathbb{Z}_p} d_px = 1$$

Let

$$|.|_p :\ \mathbb{Q}_p^\times \longrightarrow \mathbb{R}_{>0}^\times$$

be the standard p adic norm, $|p|_p = p^{-1}$; we set $|0|_p = 0$.

We have $d_p(ax) = |a|_p d_px$, so $|a|_p$ is a p-adic analog of $|z|^2, z \in \mathbb{C}$.

Define a function $\psi_p(x), x \in \mathbb{Q}_p$, by

$$\psi_p(x) = \max\{|x|_p, 1\} \tag{2.1.7.1}$$

This is an analog of $|z|^2 + 1, z \in \mathbb{C}$, see 2.5 below.

Set

$$\Gamma_{\mathbb{Q}_p}(\sigma) = \frac{1 - p^{-1}}{1 - p^{-\sigma}},\ \sigma \in \mathbb{C}$$

2.1.8. The following integrals are p-adic analogs of (2.1.1.1):

$$I_{\mathbb{Q}_p}(\sigma_1, \sigma_2, \sigma_3) = \int_{\mathbb{Q}_p^3} \psi_p(x_1)^{-2\sigma_1} \psi_p(x_2)^{-2\sigma_2} \psi_p(x_3)^{-2\sigma_3}$$

$$|x_1 - x_2|_p^{-1-\nu_3} |x_2 - x_3|_p^{-1-\nu_1} |x_3 - x_1|_p^{-1-\nu_2} d_px_1 d_px_2 d_px_3 \tag{2.1.8.1}$$

and of (2.1.1.2):

$$\tilde{I}_{\mathbb{Q}_p}(\sigma_1, \sigma_2, \sigma_3) = \int_{\mathbb{C}^2} \psi_p(x_1)^{-2\sigma_1} \psi_p(x_2)^{-2\sigma_2} |x_1 - x_2|_p^{-1-\nu_3} d_px_1 d_px_2 \tag{2.1.8.2}$$

2.1.9. Theorem. (i)

$$I_{\mathbb{Q}_p}(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{\Gamma_p(2)} \tilde{I}_{\mathbb{Q}_p}(\sigma_1, \sigma_2, \sigma_3) \tag{2.1.9.1}$$

(ii)

$$\tilde{I}_{\mathbb{Q}_p}(\sigma_1, \sigma_2, \sigma_3) = \frac{\Gamma_{\mathbb{Q}_p}(\sigma_1 + \sigma_2 + \sigma_3 - 1)\Gamma_{\mathbb{Q}_p}(-\nu_1)\Gamma_{\mathbb{Q}_p}(-\nu_2)\Gamma_{\mathbb{Q}_p}(-\nu_3)}{\Gamma_{\mathbb{Q}_p}(2\sigma_1)\Gamma_{\mathbb{Q}_p}(2\sigma_2)\Gamma_{\mathbb{Q}_p}(2\sigma_3)} \tag{2.1.9.2}$$

For a proof, see 2.2. below.

2.1.10. Let $G = PGL(2)$. The integral $I_{\mathbb{C}}$ (resp. $I_{\mathbb{R}}, I_{\mathbb{Q}_p}$) is related to invariant functionals on triple products $V_1 \otimes V_2 \otimes V_3$ where V_i are irreducible $G(K)$-representations of the principal series, with $K = \mathbb{C}$ (resp. $\mathbb{R}$ or $\mathbb{Q}_p$), cf. [BR] for the real case and [BS2] for the complex case. So its q-deformation (2.1.6.1) should be related to the same objects connected with the quantum group $U_q\mathfrak{g}(\mathbb{R})$ where $\mathfrak{g} = Lie\ G$.

2.2. The p-adic case

2.2.1. Notation. Let

$$v_p:\ \mathbb{Q}_p \longrightarrow \mathbb{Z} \cup \{\infty\}$$

denote the usual p-adic valuation, i.e. $v_p(x) = n$ if $x \in p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$, $v_p(0) = \infty$.

For $n, m \in \mathbb{Z}$ we denote

$$A_{\le n} = \{x \in \mathbb{Q}_p |\ v_p(x) \le n\},\ A_{\ge n} = \{x \in \mathbb{Q}_p |\ v_p(x) \ge n\},$$

$$A_{[n,m]} = A_{\ge n} \cap A_{\le m},\ A_n = A_{[n,n]}$$

We also set

$$\Gamma_{\mathbb{Q}_p}(\infty) := \lim_{a\to\infty} \Gamma_{\mathbb{Q}_p}(a) = 1 - p^{-1}$$

2.2.2. A p-adic hypergeometric function. Define

$$F_{\mathbb{Q}_p}(a, c; y) = \int_{\mathbb{Q}_p} \psi_p(x)^a |x - y|_p^c d_p x,$$

$a, c \in \mathbb{C}; y \in \mathbb{Q}_p$.

2.2.3. Lemma. (i) *If $v_p(y) \ge 0$ then*

$$F_{\mathbb{Q}_p}(a, c; y) = \Gamma_{\mathbb{Q}_p}(c + 1) - \Gamma_{\mathbb{Q}_p}(a + c + 1)$$

(ii) *If $v_p(y) = n < 0$ then*

$$F_{\mathbb{Q}_p}(a, c; y) = p^{-n(a+c+1)}\Gamma_{\mathbb{Q}_p}(c + 1) - p^{-n(a+c+1)}\Gamma_{\mathbb{Q}_p}(a + c + 1)$$

$$+\frac{p^{-nc}\Gamma_{\mathbb{Q}_p}(\infty)}{\Gamma_{\mathbb{Q}_p}(n(a + 1) + 1)} - \frac{p^{-nc}\Gamma_{\mathbb{Q}_p}(\infty)\Gamma_{\mathbb{Q}_p}(a + 1)}{\Gamma_{\mathbb{Q}_p}(n(a + 1))}$$

Proof. Let us denote for brevity

$$f(a,c;x,y) = \psi_p(x)^a|x-y|_p^c$$

(i) Let $v_p(y) = n \geq 0$. Decompose $\mathbb{Q}_p$ into the following areas:

$$\mathbb{Q}_p = A_{<0} \cup A_{[0,n-1]} \cup A_n \cup A_{\geq n}$$

Then

$$\int_{A_{<0}} f d_p x = -\Gamma_{\mathbb{Q}_p}(a+c+1),$$

$$\int_{A_{[0,n-1]}} f d_p x = (1-p^{-n(c+1)})\Gamma_{\mathbb{Q}_p}(c+1),$$

$$\int_{A_{>n}} f d_p x = p^{-n(c+1)-1}$$

To evaluate $\int_{A_n} f(x,y)d_px$, we decompose A_n into two areas depending on $y \in A_n$: $A_n = A'_n(y) \cup A''_n(y)$ where

$$A'_n(y) = \{x \in A_n |\ v_p(x-y) = n\},\ A''_n(y) = \{x \in A_n |\ v_p(x-y) > n\} \tag{2.2.3.1}$$

Then

$$\int_{A'_n(y)} f d_p x = (p-2)p^{-n(c+1)-1}$$

and

$$\int_{A''_n(y)} f d_p x = p^{-(n+1)(c+1)}\Gamma_{\mathbb{Q}_p}(c+1),$$

so that

$$\int_{A_n} f d_p x = (p-2)p^{-n(c+1)-1} + p^{-(n+1)(c+1)}\Gamma_{\mathbb{Q}_p}(c+1).$$

Adding up, we get (i).

(ii) is proved in a similar manner. Let $v_p(y) = n < 0$. We decompose

$$\mathbb{Q}_p = A_{<n} \cup A_n \cup A_{[-n+1,-1]} \cup A_{\geq 0},$$

and for $y \in A_n$

$$A_n = A'_n(y) \cup A''_n(y)$$

as in (2.2.3.1). Then

$$\int_{A_{<n}} f d_p x = -p^{-n(a+c+1)}\Gamma_{\mathbb{Q}_p}(a+c+1),$$

$$\int_{A_{[-n+1,-1]}} f d_p x = (1-p^{-1})p^{-nc} \sum_{m=n+1}^{-1} p^{-m(a+1)},$$

$$\int_{A_{\geq 0}} f d_p x = p^{-nc},$$

$$\int_{A'_n(y)} f d_p x = (p-2)p^{-n(a+c+1)-1}$$

and

$$\int_{A''_n(y)} f d_p x = p^{-(n+1)(c+1)} \Gamma_{\mathbb{Q}_p}(c+1)$$

Adding up, we get (ii). □

2.2.4. Theorem.

$$J(a,b,c) := \int\int_{\mathbb{Q}_p^2} \psi_p(x)^a \psi_p(y)^b |x-y|_p^c d_p x d_p y =$$

$$= \frac{\Gamma_{\mathbb{Q}_p}(c+1)\Gamma_{\mathbb{Q}_p}(-a-c-1)\Gamma_{\mathbb{Q}_p}(-b-c-1)\Gamma_{\mathbb{Q}_p}(-a-b-c-2)}{\Gamma_{\mathbb{Q}_p}(-a)\Gamma_{\mathbb{Q}_p}(-b)\Gamma_{\mathbb{Q}_p}(-a-b-2c-2)} \tag{2.2.4.1}$$

Proof. By definition

$$J(a,b,c) = \int_{\mathbb{Q}_p} \psi_p(y)^b F_{\mathbb{Q}_p}(a,c;y) d_p y$$

Using Lemma 2.2.3 we readily compute this integral and arrive at (2.2.4.1). □

This theorem is equivalent to (2.1.9.2).

2.2.5. Proof of (2.1.9.1). We shall use the same method as in the complex case, cf. 2.5 below.

Let

$$K = SL_2(\mathbb{Z}_p) \subset G = SL_2(\mathbb{Q}_p)$$

If $v \in \mathbb{Q}_p^2$ and $g \in K$ then

$$|v|_p = |gv|_p \tag{2.2.5.1}$$

where

$$|(a,b)|_p = \max\{|a|_p, |b|_p\}$$

For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ z \in \mathbb{Q}_p$$

set

$$g \cdot z = \frac{az+b}{cz+d}$$

It follows:

$$\psi_p(g \cdot x) = \frac{\psi_p(x)}{|cx+d|_p},\ g \in K \tag{2.2.5.2}$$

We have also for $g \in G$

$$g \cdot x - g \cdot y = \frac{x-y}{(cx+d)(cy+d)} \tag{2.2.5.3}$$

and

$$d_p(g \cdot z) = \frac{d_p z}{|cz+d|_p^2}, \tag{2.2.5.4}$$

cf. [GGPS], Ch. II, §3, no. 1.

We have

$$I_{\mathbb{Q}_p}(\sigma_1,\sigma_2,\sigma_3) = \int_{\mathbb{Q}_p} \psi_p(x_3)^{-2\sigma_3}\Bigg(\int_{\mathbb{Q}_p^2} \psi_p(x_1)^{-2\sigma_1}\psi_p(x_2)^{-2\sigma_2}$$

$$|x_1-x_2|_p^{-1-\nu_3}|x_2-x_3|^{-1-\nu_1}|x_3-x_1|^{-1-\nu_2}d_px_1d_px_2\Bigg)d_px_3 \tag{2.2.5.5}$$

Given $y\in\mathbb{Q}_p$, set $a(y)=y$ if $y\in\mathbb{Z}_p$ and $a(y)=y^{-1}$ if $y\notin\mathbb{Z}_p$; so $a(y)\in\mathbb{Z}_p$ in any case.

Define a matrix $k(y)\in K$ by:

(i) if $y\in\mathbb{Z}_p$ then

$$k(y)=\begin{pmatrix}1 & -a(y)\\ 0 & 1\end{pmatrix}$$

(ii) if $y\in\mathbb{Q}_p\setminus\mathbb{Z}_p$ then

$$k(y)=\begin{pmatrix}a(y) & -1\\ 1 & 0\end{pmatrix}$$

In the internal integral in (2.2.5.5) let us make a change of variables

$$x_i = k(x_3)^{-1}\cdot y_i,\ \ i=1,2$$

Using (2.2.5.2) - (2.2.5.4) we get

$$I_{\mathbb{Q}_p}(\sigma_1,\sigma_2,\sigma_3)=\int_{\mathbb{Q}_p}\frac{d_px_3}{\psi_p(x_3)^2}\cdot I'_{\mathbb{Q}_p}(\sigma_1,\sigma_2,\sigma_3)$$

where

$$\cdot I'_{\mathbb{Q}_p}(\sigma_1,\sigma_2,\sigma_3)=$$

$$\int\int_{\mathbb{Q}_p^2}|y_1|_p^{-1-\nu_2}|y_2|_p^{-1-\nu_1}\psi_p(y_1)^{-2\sigma_1}\psi_p(y_2)^{-2\sigma_2}|y_1-y_2|_p^{-1-\nu_3}d_py_1d_py_2$$

After one more substitution $y_i\mapsto y_i^{-1},\ i=1,2,$

$$I'_{\mathbb{Q}_p}(\sigma_1,\sigma_2,\sigma_3)=\tilde{I}_{\mathbb{Q}_p}(\sigma_1,\sigma_2,\sigma_3)$$

(note that $d_p(y^{-1})=d_py/|y|_p^2$).

Finally we conclude by the following easily proved p-adic version of (2.5.1.6):

2.2.6. Lemma.

$$\int_{\mathbb{Q}_p}\frac{d_px}{\psi_p(x)^2}=1+p^{-1}$$

□

2.3. The real case

2.3.1. Theorem. *If $a, b, c \in \mathbb{N}$ then*

$$\mathrm{CT}(1-y_1/y_2)^c(1-y_2/y_1)^c(1-y_1/y_3)^b(1-y_3/y_1)^b(1-y_2/y_3)^a(1-y_3/y_2)^a$$

$$= \frac{(2a)!(2b)!(2c)!(a+b+c)!}{a!b!c!(a+b)!(a+c)!(b+c)!}$$

2.3.2. Lemma. (A.C.Dixon's identity).

$$\sum_{n=-\infty}^{\infty} (-1)^n \binom{a+b}{a+n}\binom{b+c}{b+n}\binom{a+c}{c+n} = \frac{(a+b+c)!}{a!b!c!}$$

(We set $\binom{a}{b} = 0$ for $b < 0$.)
See [K], 1.2.6, Exercice 62 and Answer to Ex. 62, p. 490 (one finds also interesting references there). □

2.3.3. Proof of 2.3.1. The Laurent polynomial on the left hand side is

$$f(y_1, y_2, y_3) =$$

$$= \sum_{i,j,k} (-1)^{a+b+c-i-j-k} \binom{2c}{i}\binom{2a}{j}\binom{2b}{k} y_1^{b-c+i-k} y_2^{c-a+j-i} y_3^{a-b+k-j}$$

whence the constant term corresponds to the values

$$b - c + i - k = c - a + j - i = a - b + k - j = 0$$

Set $n = c - i$; then $n = b - k = a - j$ as well, so

$$\mathrm{CT}\, f(y_1, y_2, y_3) = \sum_{n=0}^{\infty} (-1)^{-3n} \binom{2a}{a+n}\binom{2b}{b+n}\binom{2c}{c+n} =$$

$$\frac{(2a)!(2b)!(2c)!}{(a+b)!(a+c)!(b+c)!} \sum_{n=0}^{\infty} (-1)^n \binom{a+b}{a+n}\binom{b+c}{b+n}\binom{a+c}{c+n},$$

and the application of 2.3.2 finishes the proof. □

2.4. The real q-deformed case

2.4.1. Notation:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]^!_q}{[b]^!_q [a-b]^!_q}$$

If $a, b \in \mathbb{Z}, a \geq 0, b < 0$ we set

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = 0$$

Set

$$u = x_2/x_1,\ v = x_3/x_2,\ w = x_1/qx_3;$$

$$a = a_3,\ b = a_1,\ c = a_2.$$

We are interested in the constant term of

$$F_q(u, v, w) = (qu; q)_a (u^{-1}; q)_a (qv; q)_b (v^{-1}; q)_b (qw; q)_b (w^{-1}; q)_b \tag{2.4.1.1}$$

where $uvw = q^{-1}$.

2.4.2. Lemma . (K.Kadell).

$$(qx; q)_b (x^{-1}; q)_a = \sum_{i=-a}^{b} q^{i(i+1)/2} \begin{bmatrix} a+b \\ a+i \end{bmatrix}_q (-x)^i$$

See [Ka], (3.31). □

2.4.3. Lemma. (q-Dixon identity). *On has two equivalent formulas:*
(i)

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \begin{bmatrix} a+b \\ a+n \end{bmatrix}_q \begin{bmatrix} b+c \\ b+n \end{bmatrix}_q \begin{bmatrix} c+a \\ c+n \end{bmatrix}_q =$$

$$= \frac{[a+b+c]^!_q}{[a]^!_q [b]^!_q [c]^!_q} \tag{2.4.3.1}$$

(ii)

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \begin{bmatrix} 2a \\ a+n \end{bmatrix}_q \begin{bmatrix} 2b \\ b+n \end{bmatrix}_q \begin{bmatrix} 2c \\ c+n \end{bmatrix}_q =$$

$$= \frac{[2a]^!_q [2b]^!_q [2c]^!_q [a+b+c]^!_q}{[a]^!_q [b]^!_q [c]^!_q [a+b]^!_q [b+c]^!_q [a+c]^!_q} \tag{2.4.3.2}$$

Cf. [K], answer to Exercice 1.2.6, [C]. □

2.4.4. Now we can prove (2.1.4.1). Replace in the product (2.4.1.1) the double products like $(qu;q)_a(u^{-1};q)_a$ using 2.4.2. In the resulting expression the constant term will be the sum of coefficients at $u^iv^iv^i$ divided by q^i. Thus

$$\mathrm{CT}\,F_q(u,v,w) = \sum_{n=-\infty}^{\infty} (-1)^n q^{-n} q^{3n(n+1)/2} \begin{bmatrix} 2a \\ a+n \end{bmatrix}_q \begin{bmatrix} 2b \\ b+n \end{bmatrix}_q \begin{bmatrix} 2c \\ c+n \end{bmatrix}_q$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \begin{bmatrix} 2a \\ a+n \end{bmatrix}_q \begin{bmatrix} 2b \\ b+n \end{bmatrix}_q \begin{bmatrix} 2c \\ c+n \end{bmatrix}_q$$

$$= \frac{[2a]_q^![2b]_q^![2c]_q^![a+b+c]_q^!}{[a]_q^![b]_q^![c]_q^![a+b]_q^![b+c]_q^![a+c]_q^!}$$

by (2.4.3.2). This finishes the proof of (2.1.4.1). □

Let us prove Thm. 2.1.6. We shall use an idea going back to Hardy, cf. [B], 5.5; [S]; [M], 17.2.

First (2.1.6.1) is true if all $a_i \in \mathbb{N}$ — this is Thm. 2.1.4. Now we shall use

2.4.5. Lemma. *Let $f(z)$ be a function holomorphic and bounded for $\Re z \geq 0$ such that $f(z) = 0$ for $z \in \mathbb{N}$. Then $f(z) \equiv 0$.*

This is a particular case of **Carlsson's theorem**, cf. [B], 5.3; [T], 5.8.1. □

2.4.6. Set

$$\tilde{\Gamma}_q(a) = \frac{\prod_{i=1}^{\infty}(1-q^i)}{\prod_{i=0}^{\infty}(1-q^{a+i})},$$

so that

$$\Gamma_q(a) = (1-q)^{1-a}\tilde{\Gamma}_q(a);$$

Recall that $0 < q < 1$.

We have

$$1 - |b|q^{\Re s} \leq |1 - bq^s| \leq 1 + |b|q^{\Re s},\ \Re s \geq 0,$$

and

$$1 + t \leq e^t,\ t \geq 0.$$

It follows:

$$|\prod_{i=0}^{\infty}(1-q^{a+i})| \leq \prod_{i=0}^{\infty}(1+q^{\Re a+i}) \leq$$

$$\prod_{i=0}^{\infty} e^{q^{\Re a+i}} = e^{\sum_{i=0}^{\infty} q^{\Re a+i}} = e^{q^{\Re a}/(1-q)} \leq e^{1/(1-q)}$$

for $\Re a \geq 0$.

On the other hand

$$|\prod_{i=0}^{\infty}(1-q^{a+i})| \geq \prod_{i=0}^{\infty}(1-q^{\Re a+i}) \geq \prod_{i=0}^{\infty}(1-q^{a_0+i})$$

for $\Re a \geq a_0 > 0$.

Fix $a_0 > 0$. It follows that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \leq \tilde{\Gamma}_q(a) \leq C_2 \qquad (2.4.6.1)$$

for all a, $\Re a \geq a_0$.

Consider the right hand side of (2.1.6.1)

$$f(a_1,a_2,a_3) := \frac{\Gamma_q(a_1+a_2+a_3+1)\prod_{i=1}^3 \Gamma_q(2a_i+1)}{\prod_{i=1}^3 \Gamma_q(a_i+1)\prod_{1\leq i<j\leq 3}\Gamma_q(a_i+a_j+1)} =$$

$$\frac{\tilde{\Gamma}_q(a_1+a_2+a_3+1)\prod_{i=1}^3 \tilde{\Gamma}_q(2a_i+1)}{\prod_{i=1}^3 \tilde{\Gamma}_q(a_i+1)\prod_{1\leq i<j\leq 3}\tilde{\Gamma}_q(a_i+a_j+1)}$$

It follows from (2.4.6.1) that there a constant $C_3 > 0$ such that

$$|f(a_1,a_2,a_3)| \leq C_3$$

for all a_1, a_2, a_3 with the real part $\geq a_0$.

In the same manner we prove that if $g(a_1,a_2,a_3;x_1,x_2,x_3)$ is the expression under the integral from the left hand side of (2.1.6.1), there exist a constant $C_4 > 0$ such that

$$|g(a_1,a_2,a_3)| \leq C_4$$

for all a_1, a_2, a_3 with the real part $\geq a_0$ and $(x_1,x_2,x_3) \in T^3$; thus

$$h(a_1,a_2,a_3) = \frac{1}{(2\pi^3)}\left|\int_{T^3} g(a_1,a_2,a_3;x_1,x_2,x_3)dx_1dx_2dx_3\right|$$

is also bounded by a constant not depending on a_i.

By Thm. 2.1.4 we know that $h(a_1,a_2,a_3) = f(a_1,a_2,a_3)$ if all $a_i \in \mathbb{N}$. Now applying (3 times) 2.4.5 we conclude that this is true for all a_i with $\Re a_i \geq a_0$. This proves Thm. 2.1.6. □

2.5. The complex case

2.5.1. Proof of . (2.1.1.3). We give some details because we use exactly the same argument in the p-adic case, cf. 2.5.

Let

$$K = SU(2) = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} \mid a,b\in\mathbb{C},\ |a|^2+|b|^2 = 1\} \subset$$

$$\subset G = SL_2(\mathbb{C}) \qquad (2.5.1.1)$$

If $v \in \mathbb{C}^2$ and $g \in K$ then

$$|v| = |gv| \qquad (2.5.1.2)$$

where

$$|(a,b)|^2 = |a|^2 + |b|^2$$

For

$$g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G,\ z \in \mathbb{C}$$

set

$$g \cdot z = \frac{az+b}{cz+d}$$

It follows:

$$1+|g \cdot x|^2 = \frac{1+|x|^2}{|cx+d|^2},\ g \in K \tag{2.5.1.3}$$

We have also for $g \in G$

$$g \cdot x - g \cdot y = \frac{x-y}{(cx+d)(cy+d)} \tag{2.5.1.4}$$

and

$$d(g \cdot z) = \frac{dz}{|cz+d|^4} \tag{2.5.1.5}$$

(recall that in real coordinates $z = x+iy$ we have $dz = dxdy$).

Using this, let us evaluate

$$I_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) =$$

$$\int_{\mathbb{C}} dx_3(1+|x_3|^2)^{-2\sigma_3}\left(\int_{\mathbb{C}^2}\prod_{i=1}^{2}(1+|x_i|^2)^{-2\sigma_i}\prod_{i=1}^{3}|x_i-x_{i+1}|^{-2-2\nu_{i+2}}dx_1dx_2\right)$$

In the internal integral let us make a change of variables $x_i = k^{-1}\cdot y_i,\ i=1,2$ with $k \in K$ as in (2.5.1.1) with

$$a = \frac{1}{(1+|x_3|^2)^{1/2}},\ b = -\frac{x_3}{(1+|x_3|^2)^{1/2}},$$

so $y_3 = k \cdot x_3 = 0$.

We get

$$I_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) = \int_{\mathbb{C}}\frac{dx_3}{(1+|x_3|^2)^2}\cdot I'_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3)$$

where

$$I'_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) =$$

$$\int_{\mathbb{C}^2}|y_1|^{-2-2\nu_2}|y_2|^{-2-2\nu_1}(1+|y_1|^2)^{-2\sigma_1}(1+|y_2|^2)^{-2\sigma_2}|y_1-y_2|^{-2-2\nu_3}dy_1dy_2$$

After one more substitution $y_i \mapsto y_i^{-1},\ i=1,2,$

$$I'_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) = \tilde{I}_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3)$$

(note that $d(y^{-1}) = dy/|y|^4$).

Passing to polar coordinates we get

$$\int_{\mathbb{C}}\frac{dx}{(1+|x|^2)^2} = \pi, \tag{2.5.1.6}$$

cf. Lemma 2.2.6.

Thus

$$I_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) = \pi\tilde{I}_{\mathbb{C}}(\sigma_1,\sigma_2,\sigma_3) \tag{2.5.1.7}$$

which establishes (2.1.1.3). □

2.5.2. To establish (2.1.1.4) it remains to compute the integral (2.1.1.2). Introduce new parameters $a_i = -1 - \nu_i,\ i = 1,2,3$. In terms of them (2.1.1.2) becomes

$$I_{\mathbb{C};2}(a_1,a_2,a_3) := \int_{\mathbb{C}^2} (1+|z_1|^2)^{-2-a_2-a_3}(1+|z_2|^2)^{-2-a_1-a_3}|z_1-z_2|^{2a_3}dz_1dz_2.$$

In view of 2.5.1, (2.1.1.4) is equivalent to

2.5.3. Theorem. *For $\Re a_1, \Re a_2, \Re a_3$ sufficiently large*

$$I_{\mathbb{C};2}(a_1,a_2,a_3) = = \pi^2 \frac{\Gamma(a_1+a_2+a_3+2)\Gamma(a_1+1)\Gamma(a_2+1)\Gamma(a_3+1)}{\Gamma(a_1+a_2+2)\Gamma(a_1+a_3+2)\Gamma(a_2+a_3+2)} \tag{2.5.3.1}$$

Proof. Let us make a change of variables $z_k = r_k e^{i\phi_k}$. If $z_k = x_k + iy_k$ then $dz_k = dx_kdy_k = r_kd\phi_k$.

So we get:

$$I_{\mathbb{C};3}(a_1,a_2,a_3) = \int_{\mathbb{R}^2_+} (1+r_1^2)^{-2-a_2-a_3}(1+r_2^2)^{-2-a_1-a_3} r_1 r_2 \left(\int_{[0,2\pi]^2} |r_1e^{i\phi_1} - r_2e^{i\phi_2}|^{2a_3} d\phi_1 d\phi_2 \right) dr_1 dr_2.$$

We have

$$|r_1e^{i\phi_1} - r_2e^{i\phi_2}|^2 = (r_1e^{i\phi_1} - r_2e^{i\phi_2})(r_1e^{-i\phi_1} - r_2e^{-i\phi_2}) = (r_1 - r_2e^{i(\phi_2-\phi_1)})(r_1 - r_2e^{-i(\phi_2-\phi_1)})$$

Suppose that $a_3 \in \mathbb{N}$. Note that

$$\frac{1}{(2\pi)^2}\int_{[0,2\pi]^2} (r_1 - r_2e^{i(\phi_2-\phi_1)})^a(r_1 - r_2e^{-i(\phi_2-\phi_1)})^a d\phi_1 d\phi_2 = \mathrm{CT}_z(r_1 - r_2z_1/z_2)^a(r_1 - r_2z_2/z_1)^a$$

Let us introduce a polynomial

$$\phi_a(r_1,r_2) := \mathrm{CT}_z(r_1 - r_2z_1/z_2)^a(r_1 - r_2z_2/z_1)^a = \sum_{i=0}^{a}\binom{a}{i}^2 r_1^{2i}r_2^{2a-2i}$$

Note that

$$\phi_a(1,1) = \binom{2a}{a}$$

(this is the simplest case of the Dyson identity).

We get

$$I_{\mathbb{C};2}(a_1,a_2,a_3) = = \sum_{i=0}^{a_3}\binom{a_3}{i}^2 \int_{\mathbb{R}^2_+} (1+r_1^2)^{-2-a_2-a_3}(1+r_2^2)^{-2-a_1-a_3} r_1^{1+2i} r_2^{1+2a_3-2i} dr_1 dr_2$$

We have

$$I(a,b) := \int_0^\infty (1+r^2)^b r^{1+2a} dr =$$

$(u = r^2)$

$$\frac{1}{2}\int_0^\infty (1+u)^b u^a du =$$

$(u = v/(1-v))$

$$\frac{1}{2}\int_0^1 (1-v)^{-b-a-2} v^a dv = \frac{1}{2}B(a+1, -a-b-1) =$$

$$= \frac{1}{2} \cdot \frac{\Gamma(a+1)\Gamma(-a-b-1)}{\Gamma(-b)}$$

It follows:

$$\int_{\mathbb{R}_+^2} (1+r_1^2)^{-2-a_2-a_3}(1+r_2^2)^{-2-a_1-a_3} r_1^{1+2i} r_2^{1+2a_3-2i} dr_1 dr_2 =$$

$$= \frac{1}{4} \cdot \frac{\Gamma(i+1)\Gamma(1-i+a_2+a_3)\Gamma(a_3-i+1)\Gamma(1+i+a_1)}{\Gamma(a_2+a_3+2)\Gamma(a_1+a_3+2)}$$

So

$$I_{\mathbb{C};2}(a_1,a_2,a_3) = \frac{\pi^2\Gamma(a_3+1)^2}{\Gamma(a_2+a_3+2)\Gamma(a_1+a_3+2)}$$

$$\sum_{i=0}^{a_3} \frac{\Gamma(1-i+a_2+a_3)\Gamma(1+i+a_1)}{\Gamma(i+1)\Gamma(a_3-i+1)}$$

Note that if $a_2, a_3 \in \mathbb{N}$,

$$\frac{\Gamma(1-i+a_2+a_3)\Gamma(1+i+a_1)}{\Gamma(i+1)\Gamma(a_3-i+1)} =$$

$$= \Gamma(a_1+1)\Gamma(a_2+1)\binom{a_1+i}{a_1}\binom{a_2+a_3-i}{a_2}$$

Thus

$$I_{\mathbb{C};2}(a_1,a_2,a_3) = \frac{\pi^2\Gamma(a_3+1)^2\Gamma(a_1+1)\Gamma(a_2+1)}{\Gamma(a_2+a_3+2)\Gamma(a_1+a_3+2)}\times$$

$$\times \sum_{i=0}^{a_3} \binom{a_1+i}{a_1}\binom{a_2+a_3-i}{a_2} \qquad (2.5.3.2)$$

Next we shall use an elementary

2.5.4. Lemma. *For positive integers a_1, a_2, a_3,*

$$\sum_{i=0}^{a_3} \binom{a_1+i}{a_1}\binom{a_2+a_3-i}{a_2} = \binom{a_1+a_2+a_3+1}{a_3}$$

Proof. The case $a_2 = 0$ is a consequence of the identity

$$\binom{a+1}{b+1} = \binom{a}{b+1} + \binom{a}{b}.$$

Suppose that $a_2 \geq 1$ and make the induction on a_3. The case $a_3 = 0$ is clear. Suppose we have proven the assertion for $a_3 \leq n$. We have:

$$A(n+1) := \sum_{i=0}^{n+1} \binom{a_1+i}{a_1}\binom{a_2+n+1-i}{a_2} =$$

$$\sum_{i=0}^{n} \binom{a_1+i}{a_1}\binom{a_2+n+1-i}{a_2} + \binom{a_1+n+1}{a_1} \qquad (*)$$

But

$$\binom{a_2+n+1-i}{a_2} = \binom{a_2+n-i}{a_2} + \binom{a_2+n-i}{a_2-1}$$

Inserting this into (*) and using the induction hypothesis one finishes the proof. □

Combining (2.5.3.2) with the above lemma we get the assertion of Thm. 2.5.3 for natural a_i.

On the other hand, one can verify that both sides of (2.5.3.1) are bounded when for one i $\Re a_i \to \infty$ and the other two a_j's are fixed. So by Carlsson's theorem (cf. Lemma 2.5.4), the identity (2.5.3.1) holds true for all a_i with $\Re a_i$ sufficiently large (so that the integral converges). □

Another proof of (2.1.1.4), along the lines of [BR], the interested reader may find in [BS2].

2.6. Why the real integral is analogous to the complex one

2.6.1. Consider a change of variables

$$x = \tan\alpha,\ y = \tan\beta,\ z = \tan\gamma$$

We have

$$\sin(\alpha-\beta) = (\tan\alpha - \tan\beta)\cos\alpha\cos\beta,$$
$$1 + x^2 = \cos^{-2}\alpha$$

and

$$d\alpha = \frac{dx}{1+x^2}$$

2.6.2. Consider the integral (2.1.2.1)

$$\mathcal{I}(a,b,c) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\sin(\alpha-\beta)|^{2c}|\sin(\alpha-\gamma)|^{2b}|\sin(\beta-\gamma)|^{2a}d\alpha d\beta d\gamma.$$

The function under the integral is π-periodic with respect to each argument. It follows that

$$\mathcal{I}(a,b,c) =$$

$$= 8\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} |\sin(\alpha-\beta)|^{2c}|\sin(\alpha-\gamma)|^{2b}|\sin(\beta-\gamma)|^{2a}d\alpha d\beta d\gamma$$

$$= 8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+x^2)^{-(b+c+1)}(1+y^2)^{-(a+c+1)}(1+z^2)^{-(a+b+1)}\times$$

$$\times|x-y|^{2c}|x-z|^{2b}|y-z|^{2a}dxdydz$$

We see that this integral is similar to (2.1.1.1). □

2.7. Invariant functionals and Zamolodchikov's integral

2.7.1. Recall. Consider a triple complex integral

$$I(\sigma_1,\sigma_2,\sigma_3) = \int_{\mathbb{C}^3}(1+|z_1|^2)^{-2\sigma_1}(1+|z_2|^2)^{-2\sigma_2}(1+|z_3|^2)^{-2\sigma_3}$$

$$|z_1-z_2|^{2\nu_3-2}|z_2-z_3|^{2\nu_1-2}|z_3-z_1|^{2\nu_2-2}dz_1dz_2dz_3. \quad (2.7.1.1)$$

It has appeared in a remarkable paper [ZZ] in connection with the Liouville model of the Conformal field theory. Here $\sigma_i \in \mathbb{C}$,

$$\nu_i = \sigma_{i+1}+\sigma_{i+2}-\sigma_i,\ i \mod 3, \quad (2.7.1.2)$$

$dz := dxdy,\ z = x+iy$, stands for the standard Lebesgue measure on $\mathbb{C}$. The integral converges if $\Re\sigma_i > 1/2,\ \Re\nu_i > 0,\ i = 1,2,3$. The aim of this section is to prove the following assertion.

2.7.2. Theorem.

$$I(\sigma_1,\sigma_2,\sigma_3) = \pi^3\frac{\Gamma(\nu_1+\nu_2+\nu_3-1)\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)}{\Gamma(2\sigma_1)\Gamma(2\sigma_2)\Gamma(2\sigma_3)} \quad (2.7.2.1)$$

This formula is given without proof in [ZZ]. A proof of Theorem 2.7.2 has appeared in [HMW]; it uses some complicated change of variables. A different proof which uses Macdonald type constant term identities may be found in [BS2]. In this section we provide still another proof of (2.7.2.1) which uses an elegant method of Bernstein and Reznikov who computed a similar real integral, cf. [BR]. Their procedure is based on some multiplicity one considerations which allow one to reduce the computation of

(2.7.1.1) to a computation of a Gaussian integral, cf. Thm 2.7.8 below; this last integral is of independent interest.

Notation. For a smooth variety Y we denote by $C(Y)$ the space of C^∞ functions $f : Y \longrightarrow \mathbb{C}$.

Principal series. Let $G := SL_2(\mathbb{C}) \supset K := SU(2)$; $X = \mathbb{C}^2 \setminus \{0\}$. Let $C(X)$ denote the space of smooth functions $f : X \longrightarrow \mathbb{C}$; for $\lambda \in \mathbb{C}$ let $C_\lambda(X) \subset C(X)$ be the subspace of f such that $f(ax) = |a|^{2\lambda} f(x)$ for all $a \in \mathbb{C}^*$. The group G is acting on $C_\lambda(X)$ by the rule $(gf)(x) = f(xg),\ g \in G$; we denote this representation by V_λ.

Line realization, cf. [GGPS]. The map $f \mapsto f(x, 1)$ induces an isomorphism $\phi : V_\lambda \xrightarrow{\sim} V'_\lambda$ where V'_λ is a subspace of $C(\mathbb{C})$ depending on λ.

The induced action of G on V'_λ is given by

$(gf)(x) = |bx + d|^{2\lambda} f((ax + c)/(bx + d)),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

2.7.3. Lemma. *The dimension of the space $Hom_K(V_\lambda, \mathbb{C})$ of continuous K-invariant maps $V_\lambda \longrightarrow \mathbb{C}$ is equal to* 1.

Proof. Let $\ell \in Hom_K(V_\lambda, \mathbb{C})$. Choose a Haar measure on K such that $\int_K dk = 1$. Since ℓ is continuous and K invariant, for all $f \in V_\lambda$ $\ell(f) = \ell(\bar{f})$ where $\bar{f}(x) = \int_K f(kx)dk$. But $\bar{f}$ is homogeneous and K-invariant, and such a function is unique up to a multiplicative constant. □

2.7.4. Lemma. (i) *The functional $\ell'_\lambda :\ V'_{\lambda-2} \longrightarrow \mathbb{C}$ given by*

$$\ell'_\lambda(f) = \int_{\mathbb{C}} (1 + |z|^2)^{-\lambda} f(z) dz$$

is K-invariant.The integral converges for all $\lambda \in \mathbb{C},\ f \in V'_{\lambda-2}$. ℓ'_λ is G-invariant iff $\lambda = 0$.

(ii) *The functional $\ell_\lambda :\ V_{\lambda-2} \longrightarrow \mathbb{C}$ given by*

$$\ell_\lambda(f) = \frac{1}{\pi} \int_{\mathbb{C}^2} f(z_1, z_2) e^{-|z_1|^2 - |z_2|^2} dz_1 dz_2$$

is K-invariant. The integral converges for $\Re\lambda > 0$. □

By Lemma 2.7.3, $\ell_\lambda = c_\lambda \ell'_\lambda \circ \phi$ for some $c_\lambda \in \mathbb{C}$.

2.7.5. Lemma. $c_\lambda = \Gamma(\lambda)\pi^{-1}$. □

More generally, given n complex numbers $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ we define a G-module

$$V_{\bar{\lambda}} = \{f \in C(X^n) | f(a_1 x_1, \ldots, a_n x_n) = \prod_{i=1}^{n} |a_i|^{2\lambda_i} f(x_1, \ldots, x_n)\}$$

with the diagonal action of G.

The map $f \mapsto f((x_1, 1), \ldots, (x_n, 1))$ induces an isomorphism of G-modules

$$\phi :\ V_{\bar{\lambda}} \xrightarrow{\sim} V'_{\bar{\lambda}} \subset C(\mathbb{C}^n)$$

with the action of G on $V'_{\bar\lambda}$ given by

$$(gf)(x_1,\dots,x_n)=\prod_{i=1}^n|bx_i+d|^{-2\lambda_i}f\left(\frac{ax_1+c}{bx_1+d},\dots,\frac{ax_n+c}{bx_n+d}\right),\ g=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

We have K-invariant functionals $\ell'_{\bar\lambda}:\ V'_{\lambda_1-2,\dots,\lambda_n-2}\longrightarrow\mathbb{C}$,

$$\ell'_{\bar\lambda}(f)=\int_{\mathbb{C}^n}\prod_{i=1}^n(1+|z_i|^2)^{-\lambda_i}f(z_1,\dots,z_n)dz_1\dots dz_n,$$

(the integral converges for all λ_i)

and $\ell_{\bar\lambda}:\ V_{\lambda_1-2,\dots,\lambda_n-2}\longrightarrow\mathbb{C}$,

$$\ell_{\bar\lambda}(f)=\int_{(\mathbb{C}^2)^n}f(x_1,y_1,\dots,x_n,y_n)e^{-\sum_{i=1}^n(|x_i|^2+|y_i|^2)}dx_1dy_1\dots dx_ndy_n$$

(the integral converges if $\Re\lambda_i>0$ for all i).

2.7.6. Corollary.

$$\ell_{\bar\lambda}=\pi^{-n}\prod_{i=1}^n\Gamma(\lambda_i)\cdot\ell'_{\bar\lambda}\circ\phi.$$

Indeed, the factorisable functions $\prod f_i(x_i),\ f_i\in V_{\lambda_i}$ are dense in $V_{\lambda_1,\dots,\lambda_n}$, and we apply the Fubini theorem.

For a function $f:\ Y\longrightarrow\mathbb{C}$ where $Y\subset\mathbb{C}^n$ with the complement of measure zero we denote

$$\mathcal{G}(f)=\frac{1}{\pi^{n/2}}\int_{\mathbb{C}^n}f(z_1,\dots,z_n)e^{-\sum_{i=1}^n|z_i|^2}dz_1\dots dz_n$$

(when the integral converges). The following proposition is a complex analog of properties of real Gaussian integrals from [BR].

2.7.7. Proposition. (i) *Set* $r(z)=(\sum_{i=1}^n|z_i|^2)^{1/2}$. *Then* $\mathcal{G}(r^s)=\Gamma(s/2+n)/\Gamma(n)$.
(ii) *If* $h(z)=\sum c_iz_i$ *then* $\mathcal{G}(|h|^s)=\Gamma(s/2+1)||h||^s$ *where* $||h||=(\sum_{i=1}^n|c_i|^2)^{1/2}$.
(iii) *Consider the space* $\mathbb{C}^4=(\mathbb{C}^2)^2$ *and the determinant*

$$d(w_1,w_2)=z_{11}z_{22}-z_{12}z_{21},\ w_i=(z_{1i},z_{2i}).$$

Then $\mathcal{G}(|d|^s)=\Gamma(s/2+1)\Gamma(s/2+2)$. □

Now consider the space $L=(\mathbb{C}^2)^3$ whose elements we shall denote $(w_1,w_2,w_3),\ w_i\in\mathbb{C}^2$. Consider the following function on L:

$$K_{\nu_1,\nu_2,\nu_3}(w_1,w_2,w_3)=|d(w_1,w_2)|^{2\nu_3-2}|d(w_1,w_3)|^{2\nu_2-2}|d(w_2,w_3)|^{2\nu_1-2}$$

2.7.8. Theorem.

$$\mathcal{G}(K_{\nu_1,\nu_2,\nu_3})=\Gamma(\nu_1+\nu_2+\nu_3-1)\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3).$$

The proof goes along the same lines as in [BR], where a similar real Gaussian integral is calculated. □

Now we easily deduce Theorem 2.7.2. The function K_{ν_1,ν_2,ν_3} belongs to the space $V_{2\sigma_1-2,2\sigma_2-2,2\sigma_3-2}$ where σ_i are defined from (2.7.1.2). We have

$$\mathcal{G}(K_{\nu_1,\nu_2,\nu_3}) = \ell_{2\sigma_1,2\sigma_2,2\sigma_3}(K_{\nu_1,\nu_2,\nu_3}).$$

On the other hand, by Corollary 2.7.6

$$I(\sigma_1,\sigma_2,\sigma_3) = \ell'_{2\sigma_1,2\sigma_2,2\sigma_3}(K'_{\nu_1,\nu_2,\nu_3}) = \pi^3 \prod_{i=1}^{3} \Gamma(2\sigma_i)^{-1} \ell_{2\sigma_1,2\sigma_2,2\sigma_3}(K_{\nu_1,\nu_2,\nu_3})$$

where $K'_{\nu_1,\nu_2,\nu_3} = \phi(K_{\nu_1,\nu_2,\nu_3})$. This, together with Theorem 2.7.8, implies Theorem 2.7.2. □

The functional $\ell : V_{-2\sigma_1,-2\sigma_2,-2\sigma_3} \longrightarrow \mathbb{C}$,

$$\ell(f) = \int_{\mathbb{C}^3} f K'_{\nu_1,\nu_2,\nu_3} dz_1 dz_2 dz_3$$

is G-invariant. Similarly to the real case treated in [BR], the space of such G-invariant functionals is one-dimensional, cf. [L]. This is a "methaphysical reason", why the integral (2.7.1.1) (the value of this functional on some spherical vector) is computable in terms of Gamma-functions.

We are grateful to A.Reznikov for discussions and for the notes by A.Yomdin about a p-adic analogue of the Bernstein - Reznikov method which were very useful for us.

CHAPTER 3

Invariant triple functionals over $U_q\mathfrak{sl}_2$

Introduction

Before describing the content of this chapter let us discuss some motivation and questions behind it.

The fact that an irreducible finite dimensional representation $V(\lambda_1)$ of highest weight λ_1 of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ occurs with multiplicity at most 1 in a tensor product $V(\lambda_2) \otimes V(\lambda_3)$ is easy and classical. Since these representations are isomorphic to their duals, the same thing may be expressed by saying that the dimension of the space of $\mathfrak{g}$-invariant functionals

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3), \mathbb{C}) \leq 1 \tag{0.1}$$

The multiplicity one statements like (0.1) hold true as well if $V(\lambda_i)$ are irreducible infinite dimensional representations of real, complex and p-adic Lie group or Lie algebra close to GL_2 (their proof being usually more difficult).

As an example, such an statement for the group $G = PGL_2(\mathbb{R})$ and the representation of the principal series is applied in [BR]. In that case a representation $V(\lambda)$ may be realized (before Hilbert completion) in the space of smooth functions on the unit circle $f : S^1 \longrightarrow \mathbb{C}$ and the tensor product $V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)$ - in the space of functions of three variables $f : (S^1)^3 \longrightarrow \mathbb{C}$. An explicit linear functional

$$\ell_{\lambda_1,\lambda_2,\lambda_3} : V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \longrightarrow \mathbb{C}$$

may be defined in the form of an integral

$$\ell_{\lambda_1,\lambda_2,\lambda_3}(f) = \int_{(S^1)^3} f(\theta_1, \theta_2, \theta_3) \mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3}(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2 d\theta_3 \tag{0.2}$$

against some naturally defined G-invariant kernel $\mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3}$ cf. [BR], 5.1.1, [Ok], (0.10), (0.12). On the other hand our triple product contains a distinguished spherical (i.e $PO(2)^3$-invariant) vector $v_{\lambda_1,\lambda_2,\lambda_3}$, the constant function 1.

The value

$$\ell_{\lambda_1,\lambda_2,\lambda_3}(v_{\lambda_1,\lambda_2,\lambda_3}) = \int_{(S^1)^3} \mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3}(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2 d\theta_3 \tag{0.3}$$

is equal to certain quotient of products of Gamma values. Its asymptotics with respect to λ_i (which follows from Stirling formula) is one of the ingredients used in [BR] for an estimation of Fourier coefficients of automorphic triple products.

In Chapter 2 we have calculated the integrals similar to (0.3) corresponding to complex and p-adic groups $PGL_2(\mathbb{C}), PGL_2(\mathbb{Q}_p)$, and also an analogous *q-deformed* integral which has the form

$$\int_{(S^1)^3} \mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3;q}(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3 \tag{0.4}$$

where $\mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3;q}$ is a certain q-deformation of the kernel $\mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3}$. These integrals are expressed in term of the complex, p-adic and q-deformed versions of Gamma-functions respectively. One could expect that it is possible to find representations $V_q(\lambda)$ of the q-deformed algebra $U_q\mathfrak{gl}_2$ in the space of function on S^1 so that the q-deformed kernel $\mathfrak{K}_{\lambda_1,\lambda_2,\lambda_3;q}$ will be a $U_q\mathfrak{gl}_2$-invariant element of the triple product $V_q(\lambda_1)\otimes V_q(\lambda_2)\otimes V_q(\lambda_3)$.

We do not pursue this direction here, but we prove some multiplicity one statement like (0.1) over the quantum group. Our starting point was a theorem of Hung Yean Loke [L] who proves in particular that (0.1) holds true if $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{R})$ and $V(\lambda_i)$ are irreducible representations of the (infinitesimal) principal series defined by Jacquet - Langlands, cf. [JL], Ch. I, §5. The space of such a representation is much smaller than the spaces of smooth functions above, it is rather a "discrete analog" of it, and the structures that appear are quite similar.

A base $\{e_q, q \in Q\}$ of $V(\lambda)$ is enumerated by a set Q which may be identified with coroot lattice of $\mathfrak{g}$ (or with $\mathbb{Z}$). Thus elements of $V(\lambda)$ are finite linear combinations

$$\sum a(q)e_q$$

which we can consider as functions $a : Q \to \mathbb{C}$ which are compactly supported, i.e. all but finitely many $a(q)$ are zeros. The Lie algebra $\mathfrak{g}$ acts on these functions by difference operators (depending on $\lambda \in \mathbb{C}$) of order ≤ 1 (to avoid the confusion, $V(\lambda)$ is not a highest weight module). Thus, elements of a triple product $V(\lambda_1)\otimes V(\lambda_2)\otimes V(\lambda_3)$ are compactly supported functions $a : Q^3 \to \mathbb{C}$. Similarly, a trilinear functional

$$\ell : V(\lambda_1)\otimes V(\lambda_2)\otimes V(\lambda_3) \longrightarrow \mathbb{C}$$

is uniquely determined by its action on the basis elements. If we denote

$$\mathcal{K}(q_1,q_2,q_3) = \ell(e_{q_1}\otimes e_{q_2}\otimes e_{q_3}),$$

we get a function $\mathcal{K} : Q^3 \longrightarrow \mathbb{C}$ (an arbitrary, not necessarily compactly supported one). The value of ℓ is given by

$$\ell(a) = \sum_{(q_1,q_2,q_3)\in Q^3} a(q_1,q_2,q_3)\mathcal{K}(q_1,q_2,q_3);$$

this formula is similar to (0.2).

The functional is $\mathfrak{g}$-invariant iff the corresponding $\mathcal{K}$ satisfies a simple system of difference equations. The result of [L] says that the space of such functions $\mathcal{K}$ is one-dimensional. It would be interesting to find a nice explicit formula for a solution.

In **3.2** of the present chapter we define principal series representations over the quantum group $U_q\mathfrak{sl}_2$ which are q-deformations of the Jacquet - Langlands modules. Then we define natural intertwining ("reflection") operators between them (cf. **3.2.4**) and finally prove for them an analog of (0.1), cf. Thm. **3.2.5** for the precise formulation; this is our main result. The proof is a q-deformation of the argument from [L].

In **3.1** we recall the definitions from [JL] and the original argument of [L] and present some comments on it, cf. 3.1.6, in the spirit of I.M.Gelfand's philosophy considering the Clebsch - Gordan coefficients as discrete orthogonal polynomials, cf. [NSU].

We thank F.Malikov who drew our attention to a very interesting paper [FM].

3.1. Invariant triple functionals over $\mathfrak{sl}_2$

3.1.1. Principal series. First we recall the classical definition of the principal series following Jacquet - Langlands. Another definition of these modules may be found in [FM].

Let $\mathfrak{g} = \mathfrak{sl}_2$ and E, F, H be the standard base of $\mathfrak{g}$.

Let $s \in \mathbb{C}$, $\epsilon \in \{0, 1\}$. Following [JL], §5 and [L] 2.2 consider the following representation $M(s, \epsilon)$ of $\mathfrak{g}$ (for a motivation of the definition see 3.1.10 below.)

The underlying vector space of $M(s, \epsilon)$ has a $\mathbb{C}$-base $\{v_n\}_{n\in\epsilon+2\mathbb{Z}}$. We denote $M_n = \mathbb{C} \cdot v_n$, so $M(s, \epsilon) = \oplus_{n\in\mathbb{Z}} M_n$ where we set $M_n = 0$ if $n \notin \epsilon + 2\mathbb{Z}$.

The action of $\mathfrak{g}$ is given by

$$Hv_n = nv_n, \tag{3.1.1.1}$$

$$Ev_n = \frac{1}{2}(s + n + 1)v_{n+2}, Fv_n = \frac{1}{2}(s - n + 1)v_{n-2} \tag{3.1.1.2}$$

Thus,

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$$

3.1.2. Lemma. *If $s - \epsilon \notin 2\mathbb{Z} + 1$ then $M(s, \epsilon)$ is an irreducible $\mathfrak{g}$-module.*

Proof. Let $W \in M := M(s, \epsilon)$ be a $\mathfrak{g}$-submodule, $W \neq 0$. Since W is H-invariant, $W = \oplus_n W \cap M_n$. Thus there exists $x \in W \cap M_n, x \neq 0$. Due to the hypothesis $F^m x \neq 0$ and $E^m x \neq 0$ for all $m \in \mathbb{Z}$, whence $W = M$. □

3.1.3. The reflection operator. Cf. [JL], between 5.11 and 5.12. Consider two modules $M(\pm s, \epsilon)$. A linear map

$$f : M(s, \epsilon) \longrightarrow M(-s, \epsilon)$$

is $\mathfrak{g}$-equivariant iff it respects the gradings (since it commutes with H), say $f(v_n) = f_n v'_n$, and the numbers f_n satisfy two relations

$$(s + n + 1)f_{n+2} = (-s + n + 1)f_n, \tag{3.1.3.1}$$

(commutation with E) and

$$(s-n+1)f_{n-2} = (-s-n+1)f_n \tag{3.1.3.2}$$

(commutation with F). In fact these equations are equivalent: for example (3.1.3.2) is the same as (3.1.3.1) with n replaced by $n-2$, multiplied by -1.

These relations are satisfied if

$$f_n = \frac{\Gamma((-s+n+1)/2)}{\Gamma((s+n+1)/2)},$$

We shall denote the corresponding intertwining operator by

$$R(s) : M(s,\epsilon) \xrightarrow{\sim} M(-s,\epsilon)$$

In fact, these are the only possible intertwiners between different modules of principal series.

One has

$$R(-s)R(s) = Id_{M(s)}.$$

3.1.4. Theorem, cf. [L], Thm 1.2 (1). *Consider three $\mathfrak{g}$-modules $M^i = M(s_i,\epsilon_i), i = 1,2,3$. Suppose that $s_i - \epsilon_i \notin 1+2\mathbb{Z}$. There exists a unique, up to a multiplicative constant, function*

$$f : M := M^1 \otimes M^2 \otimes M^3 \longrightarrow \mathbb{C}$$

such that

$$f(Xm) = 0, X \in \mathfrak{g}, m \in M \tag{3.1.4.1}$$

and

$$f(\omega m) = f(m) \tag{3.1.4.2}$$

where $\omega : M \xrightarrow{\sim} M$ is an automorphism defined by

$$\omega(v_n \otimes v_m \otimes v_k) = v_{-n} \otimes v_{-m} \otimes v_{-k}$$

3.1.5. Proof . (sketch). The condition (3.1.4.1) for $X = H$ implies that $f(v_n \otimes v_m \otimes v_k) = 0$ unless $n+m+k=0$.

Let us denote

$$a(n,m) = f(v_n \otimes v_m \otimes v_{-n-m})$$

The conditions (3.1.4.1) and (3.1.4.2) are equivalent to a system of 3 equations on the function $a(n,m)$:

$$a(n,m) = a(-n,-m), \tag{3.1.5.0}$$

$$(s_1+n+1)a(n+2,m)+(s_2+m+1)a(n,m+2)+(s_3-n-m-1)a(n,m) = 0 \tag{3.1.5.1}$$

and

$$(s_1-n+1)a(n-2,m)+(s_2-m+1)a(n,m-2)+(s_3+n+m-1)a(n,m) = 0 \tag{3.1.5.2}$$

One has to show that these equations admit a unique, up to scalar, solution.

By considering the "bonbon" configuration

$$B = \{(n,m),(n-2,m),(n,m+2),(n-2,m+2),(n+2,m),(n,m-2),$$
$$(n+2,m+2)\}$$

one shows that (3.1.5.1-2) imply an equation

$$(s_1 - n + 1)(s_2 + m + 1)a(n-2, m+2) - (s_3^2 - s_1^2 - s_2^2 - 2nm + 1)a(n,m)$$
$$+(s_1 + n + 1)(s_2 - m + 1)a(n+2, m-2) = 0 \qquad (3.1.5.3)$$

After that it is almost evident that a solution of (3.1.5.1-2) is uniquely defined by its two values on a diagonal, like $a(n,m), a(n-2,m+2)$. The parity condition (3.1.5.0) implies that the space of solutions has dimension ≤ 1.

The non-trivial part is a proof of the *existence* of a solution. It is a direct computation. Cf. the argument for the q-deformed case in **3.2** below. □

3.1.6. Difference equations on the root lattice of type A_2.

Let X denote the lattice $\{(n_1, n_2) \in \mathbb{Z}^2 | \; n_i - \epsilon_i \in 2\mathbb{Z}\}$. (Note that initially it comes in the above proof as a lattice

$$\{(n_1, n_2, n_3) \in \mathbb{Z}^3 | \quad n_i - \epsilon_i \in 2\mathbb{Z}, \quad \sum n_i = 0\}$$

and resembles the root lattice of the root system of type A_2)

Consider the space of maps of sets $Y = \{a : X \longrightarrow \mathbb{C}\}$; Y is a complex vector space. Define two linear operators $L_\pm \in \mathrm{End}\, Y$ by

$$L_+ a(n,m) = (p+n)a(n+2,m) + (s+n)a(n,m+2) + (r-n-m)a(n,m), \quad (3.1.6.1a)$$
$$L_- a(n,m) = (p-n)a(n-2,m) + (s-n)a(n,m-2) + (r+n+m)a(n,m), \quad (3.1.6.1b)$$

where $p = s_1 + 1$, $s = s_2 + 1$, $r = s_3 - 1$.

One can rewrite the equations (3.1.5.1-2) in the form

$$L_+ a = 0, L_- a = 0 \qquad (3.1.6.2)$$

3.1.7. Lemma. $[L_+, L_-] = 2(L_+ - L_-)$. □

It follows that L_+ and L_- span a 2-dimensional Lie algebra isomorphic to a Borel subalgebra of $\mathfrak{sl}_2$.

Following [NSU], Ch. II, §1, introduce the forward and backward difference ("discrete derivatives") operators acting on functions $f(n)$ of an integer argument:

$$\Delta f(n) = f(n+2) - f(n), \quad \nabla f(n) = f(n) - f(n-2)$$

These operators give rise to "discrete partial derivatives" acting on the space of functions of two variables $a(n,m)$ as above. We denote by subscripts $_n$ or $_m$ the operators acting on the first (resp. second) argument, for example

$$\nabla_n a(n,m) = a(n,m) - a(n-2,m),$$

etc. Then the equations (3.1.6.2) rewrite as follows:

$$((n+p)\nabla_n + (m+s)\nabla_m)a = -(p+s+r)a \qquad (3.1.6.3a)$$
$$((n-p)\nabla_n + (m-s)\nabla_m)a = -(p+s+r)a \qquad (3.1.6.3b)$$

These equations are similar to [NSU], Ch. IV, §2, (30).

Let us fix k and consider the functions $b(n) := a(n, k-n)$. The equation (3.1.5.3) is a second order equation satisfied by these functions that may be written as

$$\{(n+p)(n+s-k)\nabla\Delta + 2(pn + sn - pk)\nabla - r(r-2)\}b = 0 \qquad (3.1.6.4)$$

It is a "difference equation of hypergeometric type" in the terminology of [NSU], Ch. II, §1. Their solutions can be called "Hahn functions".

3.1.8. Analogous differential equations. It is instructive to consider the continuous analogs of the previous operators.

Let us consider the following operators on the space $\mathcal{Y}$ of differentiable functions $a(x, y) : \mathbb{R}^2 \longrightarrow \mathbb{C}$ which is a continuous analog of the space Y :

$$\mathfrak{L}_+ = (p + x)\partial_x + (s + y)\partial_y + p + s + r$$

$$\mathfrak{L}_- = (-p + x)\partial_x + (-s + y)\partial_y + p + s + r$$

3.1.9. Lemma. $[\mathfrak{L}_+, \mathfrak{L}_-] = \mathfrak{L}_+ - \mathfrak{L}_-$. □

The analog of (3.1.6.4) is a hypergeometric equation

$$(x + p)(x + q - k)b''(x) + 2((p + s)x - pk)b'(x) - r(r - 2)b(x) = 0 \qquad (3.1.9.1)$$

where $b(x) = a(x, k - x)$.

3.1.10. Motivation: Jacquet-Langlands principal series over $GL_2(\mathbb{R})$ Cf. [GL], Ch. I, §5. Recall that a *quasicharacter* of the group $\mathbb{R}^\star$ is a continuous homomorphism $\mu : \mathbb{R}^\star \longrightarrow \mathbb{C}^\star$. All such homomorphisms have the form

$$\mu(x) = \mu_{s,m}(x) = |x|^s (x/|x|)^m$$

where $s \in \mathbb{C}$, $m \in \{0, 1\}$. Let $\mu_i = \mu_{s_i,m_i}$, $i = 1, 2$, be two such quasicharacters. Let $B'(\mu_1, \mu_2)$ denote the space of C^∞-functions $f : G := GL_2(\mathbb{R}) \longrightarrow \mathbb{C}$ such that

$$f\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} g \right) = \mu_1(a)\mu_2(b)(|a/b|)^{1/2} f(g)$$

for all $g \in G, a, b \in \mathbb{R}^\star, c \in \mathbb{R}$. G acts on $B'(\mu_1, \mu_2)$ in the obvious way.

Set $s = s_1 - s_2$ and $m = |m_1 - m_2|$. For any $n \in \mathbb{Z}$ such that $n - m \in 2\mathbb{Z}$ define a function $\phi_n \in B'(\mu_1, \mu_2)$ by

$$\phi_n\left(\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} k(\theta) \right) = \mu_1(a)\mu_2(b)(|a/b|)^{1/2} e^{in\theta}$$

where

$$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K := O(2) \in G$$

Let $B(\mu_1, \mu_2) \subset B'(\mu_1, \mu_2)$ be the (dense) subspace generated by all ϕ_n.

Let us describe explicitely the induced action of $\mathfrak{B} = Lie(G)_\mathbb{R} \otimes \mathbb{C}$ on $B(\mu_1, \mu_2)$. Following [L], consider a matrix $A = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ so that $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ (cf. [Ba], (3.5)).

Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$A^{-1}XA = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} =: Y',$$

$$A^{-1}YA = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} =: X',$$
$$A^{-1}(-iH)A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = k(\pi/2)$$

or more generally

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = Ak(\theta)A^{-1}$$

Thus if $K' = AKA^{-1}$ then $Lie(K')_{\mathbb{C}} = \mathbb{C}.H$.

The action of G on $B'(\mu_1, \mu_2)$ induces an action of $\mathfrak{B}$ on $B(\mu_1, \mu_2)$ which looks as follows:

$$2X'\phi_n = (s+n+1)\phi_{n+2}, \quad 2Y'\phi_n = (s-n+1)\phi_{n-2}, \tag{3.1.10.1}$$

cf. [JL], Lemma 5.6.

The space $B(\mu_1, \mu_2)$ is a $(\mathfrak{B}, K)-module$, which means that it is a $\mathfrak{B}$-module and a K-module and the action of $\mathfrak{k} := LieK$ induced from $\mathfrak{B}$ coincides with the one induced from K.

3.2. A q-deformation

3.2.1. Category $\mathcal{C}_q$ and tensor product. Cf. [Lu]. Let q be a complex number different from 0 and not a root of unity. We fix a value of $\log q$ and for any $s \in \mathbb{C}$ define $q^s := e^{s\log q}$.

Let $U_q = U_q\mathfrak{sl}_2$ denote the $\mathbb{C}$-algebra generated by E, F, K, K^{-1} subject to relations

$$KK^{-1} = 1,$$
$$KE = q^2EK, \quad KF = q^{-2}FK,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}},$$

cf. [Lu], 3.1.1.

Introduce a comultiplication $\Delta : U_q \longrightarrow U_q \otimes U_q$ as a unique algebra homomorphism such that

$$\Delta(K) = K \otimes K$$
$$\Delta(E) = E \otimes 1 + K \otimes E$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

cf. [Lu], Lemma 3.1.4.

Let $\mathcal{C}_q$ denote the category of $\mathbb{Z}$-graded U_q-modules $M = \oplus_{i\in\mathbb{Z}} M_i$ such that

$$Kx = q^i x, x \in M_i.$$

The comutliplication Δ above makes $\mathcal{C}_q$ a tensor category.

In particular if M^i, $i = 1, 2, 3$, are objects of $\mathcal{C}_q$ then their tensor product $M = M^1 \otimes M^2 \otimes M^3$ is defined as a vector space; it is the tensor product of vector spaces underlying M^i. The action of U_q is given by

$$K(x \otimes y \otimes z) = Kx \otimes Ky \otimes Kz,$$

$$E(x \otimes y \otimes z) = Ex \otimes y \otimes z + Kx \otimes Ey \otimes z + Kx \otimes Ky \otimes Ez,$$

$$F(x \otimes y \otimes z) = Fx \otimes K^{-1}y \otimes K^{-1}z + x \otimes Fy \otimes K^{-1}z + x \otimes y \otimes Fz.$$

3.2.2. Infinitesimal principal series. Set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$q \in \mathbb{R}_{>0}$, $s \in \mathbb{C}$. Thus

$$\lim_{q \to 1} [n]_q = n.$$

Let $s \in \mathbb{C}$, $\epsilon \in \{0, 1\}$. Define an object $M_q(s, \epsilon) \in \mathcal{C}_q$ as follows. As a $\mathbb{Z}$-graded vector space $M_q(s, \epsilon) = \oplus M_i$ where $M_i = \mathbb{C} \cdot v_i$ if $i \in \epsilon + 2\mathbb{Z}$ and 0 otherwise.

An action of the operators E, F are given by

$$Ev_n = [(s + n + 1)/2]_q v_{n+2}, \quad Fv_n = [(s - n + 1)/2]_q v_{n-2}.$$

One checks that

$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}} = \frac{K - K^{-1}}{q - q^{-1}}$$

where

$$Kv_n = q^n v_n,$$

so $M_q(s, \epsilon)$ is an U_q-module.

3.2.3. Lemma. *If $s - \epsilon \notin 2\mathbb{Z} + 1$ then $M_q(s, \epsilon)$ is an irreducible U_q-module.*

The proof is the same as in the non-deformed case (see Lemma 3.1.2). □

3.2.4. The reflection operator. As in 3.1.3, let us construct an intertwining operator

$$R_q(s) : M_q(s, \epsilon) \xrightarrow{\sim} M_q(-s, \epsilon).$$

Suppose that

$$R_q(s)v_n = r_n v_n$$

for some $r_n \in \mathbb{C}$. As in *loc. cit.*, $R_q(s)$ is U_q-equivariant iff the numbers r_n satisfy the equation

$$r_{n+2} = \frac{[(-s + n + 1)/2]_q}{[(s + n + 1)/2]_q} \cdot r_n \qquad (3.2.4.1)$$

Suppose we have found a function $\phi(x)$, $x \in \mathbb{C}$, satisfying a functional equation

$$\phi(x + 1) = [x]_q \phi(x). \qquad (3.2.4.2)$$

Then

$$r_n = \frac{\phi((-s+n+1)/2)}{\phi((s+n+1)/2)}$$

satisfies (3.2.4.1).

Suppose that $|q| < 1$. In that case consider the q-Gamma function defined by a convergent infinite product

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

cf. [GR]. It satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$$

It follows that a function

$$\phi(x) = q^{a(x)}\Gamma_{q^2}(x)$$

satisfies (3.2.4.2) if $a(x)$ satisfies

$$a(x+1) - a(x) = 1 - x,$$

for example

$$a(x) = -\frac{x^2}{2} + \frac{3x}{2}$$

Thus, *if we set*

$$\phi(x) = q^{-(x^2-3x)/2}\Gamma_{q^2}(x),$$

the operator $R_q(s)$ defined by

$$R_q(s)v_n = \frac{\phi((-s+n+1)/2)}{\phi((s+n+1)/2)}v_n$$

is an isomorphism $R_q(s) : M_q(s,\epsilon) \xrightarrow{\sim} M_q(-s,\epsilon)$ in $\mathcal{C}_q$.

It possesses the unitarity property

$$R_q(-s)R_q(s) = Id_{M(s)}.$$

If $|q| = 1$ then a solution to the functional equation (3.2.4.2) may be given in terms of the Shintani-Kurokawa *double sine function* (aka Ruijsenaars hyperbolic Gamma function), cf. [NU], Prop. 3.3, [R], Appendix A. This function is a sort of a "modular double" of Γ_q.

3.2.5. Theorem. *Let $M^i = M_q(s_i,\epsilon_i) \in \mathcal{C}_q$ be 3 objects as above such that $s_i - \epsilon_i \notin 2\mathbb{Z}+1$, $i = 1,2,3$.*

There exists a unique, up to a scalar multiple, function

$$f : M := M^1 \otimes M^2 \otimes M^3 \longrightarrow \mathbb{C} \tag{3.2.5.1}$$

such that

$$f(Xm) = 0, X \in \{E,F\}, m \in M, \tag{3.2.5.2}$$

$$f(Km) = f(m),$$
$$f(\omega m) = f(m) \tag{3.2.5.3}$$

where $\omega : M \xrightarrow{\sim} M$ *is an automorphism defined by*

$$\omega(v_n \otimes v_m \otimes v_k) = v_{-n} \otimes v_{-m} \otimes v_{-k}$$

3.2.6. Proof(beginning). The argument below is a straightforward generalization of the argument from [L], §2. The condition $f(Km) = f(m)$ implies that $f(v_n \otimes v_m \otimes v_k) = 0$ unless $n + m + k = 0$. Let us denote

$$a_q(n, m) := f(v_n \otimes v_m \otimes v_{-n-m-2}).$$

The condition $f(\omega m) = f(m)$ gives

$$a_q(n, m) = a_q(-n, -m) \tag{3.2.6.0}$$

Since

$$f(E(v_n \otimes v_m \otimes v_{-n-m-2})) = 0,$$

we get

$$[(s_1 + n + 1)/2]_q a_q(n + 2, m) + q^n[(s_2 + m + 1)/2]_q a_q(n, m + 2)+$$
$$+q^{n+m}[(s_3 - n - m - 1)/2]_q a_q(n, m) = 0 \tag{3.2.6.1}$$

or

$$[(s_3 - n - m - 1)/2]_q a_q(n, m) = -q^{-n-m}[(s_1 + n + 1)/2]_q a_q(n + 2, m)$$
$$-q^{-m}[(s_2 + m + 1)/2]_q a_q(n, m + 2) \tag{3.2.6.1$'$}$$

Similarly,

$$f(F(v_n \otimes v_m \otimes v_{-n-m+2})) = 0$$

implies

$$q^{n-2}[(s_1 - n + 1)/2]_q a_q(n - 2, m) + q^{n+m-2}[(s_2 - m + 1)/2]_q a_q(n, m - 2)$$
$$+[(s_3 + n + m - 1)/2]_q a_q(n, m) = 0 \tag{3.2.6.2}$$

or

$$[(s_3 + n + m - 1)/2]_q a_q(n, m) = -q^{n-2}[(s_1 - n + 1)/2]_q a_q(n - 2, m)$$
$$-q^{n+m-2}[(s_2 - m + 1)/2]_q a_q(n, m - 2) \tag{3.2.6.2$'$}$$

It follows from (3.2.6.1)$'$:

$$[(s_3 - n - m + 1)/2]_q a_q(n - 2, m) = -q^{-n-m+2}[(s_1 + n - 1)/2]_q a_q(n, m)$$
$$-q^{-m}[(s_2 + m + 1)/2]_q a_q(n - 2, m + 2) \tag{3.2.6.3}$$

and

$$[(s_3 - n - m + 1)/2]_q a_q(n, m - 2) = -q^{-n-m+2}[(s_1 + n + 1)/2]_q a_q(n + 2, m - 2)$$
$$-q^{-m+2}[(s_2 + m - 1)/2]_q a_q(n, m) \tag{3.2.6.4}$$

(One could write (3.2.6.3) = $(3.2.6.1)'_{n-2,m}$ and (3.2.6.4) = $(3.2.6.1)'_{n,m-2}$)

Sustitute (3.2.6.3) and (3.2.6.4) into (3.2.6.2)′:

$$\Big([(s_3-n-m+1)]_q[(s_3+n+m-1)/2]_q - q^{-m}[(s_1+n-1)/2]_q[(s_1-n+1)/2]_q -$$

$$-q^n[(s_2-m+1)/2]_q[(s_2+m-1)/2]_q\Big)a_q(n,m)$$

$$= q^{n-m-2}[(s_1-n+1)/2]_q[(s_2+m+1)/2]_q a_q(n-2,m+2)+$$
$$+[(s_2-m+1)/2]_q[(s_1+n+1)/2]_q a_q(n+2,m-2) \tag{3.2.6.5}$$

This is a q-deformed (3.1.5.3). □

Now comes the main point.

3.2.7. Lemma. *$N \in \mathbb{Z}$ be such that $N \equiv \epsilon_1 + \epsilon_2$ (mod 2). Suppose we are given $a_q(n,m)$ for $n+m = N$ and they satisfy* (3.2.6.5). *Using* (3.2.6.2) *let us define $a_q(n,m)$ for $n+m = N+2k (k \geq 1)$ inductively.*

Then $a_q(n,m)$ satisfies (3.2.6.1) *for $n + m \geq N$.*

Proof. We will prove the lemma by induction on $n + m$.

By induction we assume that (3.2.6.1) is satisfied for all $n + m \leq N - 2$. Hence $a_q(n,m)$ also satisfies (3.2.6.5) for all $n + m \leq N - 2$.

Let $n+m = N-2$, we want to prove (3.2.6.1) where $a_q(n+2,m)$ and $a_q(n,m+2)$ are defined from (3.2.6.2)′ :

$$ta_q(n+2,m) =$$
$$= -q^n[(s_1-n-1)/2]_q a_q(n,m) - q^{n+m}[(s_2-m+1)/2]_q a_q(n+2,m-2) \tag{3.2.7.1}$$

$$ta_q(n,m+2) =$$
$$= -q^{n-2}[(s_1-n+1)/2]_q a_q(n-2,m+2) - q^{n+m}[(s_2-m-1)/2]_q a_q(n,m) \tag{3.2.7.2}$$

where $t = [(s_3+n+m+1)/2]_q \neq 0$ by assumption.

We put (3.2.7.1) and (3.2.7.2) into the right hand side of (3.2.6.1)′

$$-q^{-n-m}[(s_1+n+1)/2]_q a_q(n+2,m) - q^{-m}[(s_2+m+1)/2]_q a_q[n,m+2] =$$

$$= -q^{-n-m}[(s_1+n+1)/2]_q t^{-1}\times$$

$$\Big(-q^n[(s1-n-1)/2]_q a_q(n,m) - q^{n+m}[(s_2-m+1)/2]_q a_q(n+2,m-2)\Big)-$$

$$-q^{-m}[(s_2+m+1)/2]_q t^{-1}\times$$

$$\Big(-q^{n-2}[(s_1-n+1)/2]_q a_q(n-2,m+2) - q^{n+m}[(s_2-m-1)/2]_q a_q(n,m)\Big)$$

$$= t^{-1}\Big(q^{-m}[(s_1+n+1)/2]_q[(s_1-n-1)/2]_q a_q(n,m)+$$
$$+[(s_1+n+1)/2]_q[(s_2-m+1)/2]_q a_q(n+2,m-2)+$$
$$+q^{n-m-2}[(s_2+m+1)/2]_q[(s_1-n+1)/2]_q a_q(n-2,m+2)+$$
$$+q^n[(s_2+m+1)/2]_q[(s_2-m-1)/2]_q a_q(n,m)\Big)$$

(we substitute (3.2.6.5) for the second and third terms)

$$= t^{-1}\Big(q^{-m}[(s_1+n+1)/2]_q[(s_1-n-1)/2]_q+[(s_3-n-m+1)/2]_q[(s_3+n+m-1)/2]_q-$$
$$-q^{-m}[(s_1+n-1)/2]_q[(s_1-n+1)/2]_q - q^n[(s_2-m+1)/2]_q[(s_2+m-1)/2]_q+$$
$$+q^n[(s_2+m+1)/2]_q[(s_2-m-1)/2]_q\Big)a_q(n,m)$$

$$= t^{-1}\Big(q^{s_3}+q^{-s_3}-q^{m+n-1}-q^{-m-n+1}-q^{-m+n+1}-q^{-m-n-1}+q^{-m+n-1}+q^{-m-n+1}-$$
$$-q^{m+n+1}-q^{-m+n-1}+q^{m+n-1}+q^{-m+n+1}\Big)a_q(n,m)$$
$$= t^{-1}\Big(q^{s_3}+q^{-s_3}-q^{-m-n-1}-q^{m+n+1}\Big)a_q(n,m)$$
$$= t^{-1}[(s_3+n+m+1)/2]_q[(s_3-n-m-1)/2]_q a_q(n,m)$$
$$= [(s_3-n-m-1)/2]_q a_q(n,m)$$

But this is exactly (3.2.6.1)$'$! This proves the lemma. □

3.2.8. End of the proof of Thm. 3.2.5. By (3.2.6.5) and equality $a_q(2,-2) = a_q(-2,2)$ we have

$$\Big([(s_3+1)/2]_q[(s_3-1)/2]_q-$$
$$[(s_1+1)/2]_q[(s_1-1)/2]_q - [(s_2+1)/2]_q[(s_2-1)/2]_q\Big)a_q(0,0) =$$
$$= 2[(s_1+1)/2]_q[(s_2+1)/2]_q a_q(2,-2) \tag{3.2.8.1}$$

Let us construct a solution a_q of equations (3.2.6.0) - (3.2.6.2) as follows.

(i) If $\epsilon_1 = 0$, we start from an arbitrary value of $a_q(0,0)$ and define $a_q(2,-2)$ by (3.2.8.1).

(ii) If $\epsilon_1 = 1$, we start from an arbitrary value of $a_q(1,-1)$ and set $a_q(-1,1) = a_q(1,-1)$.

Using (3.2.6.5), repeatedly, we determine $a_q(n,-n)$ for all positive n. Using (3.2.6.0), we determine $a_q(n,-n)$ for all $n = 0$. Applying (3.2.4.2) inductively one defines $a_q(n,m)$ for all $n+m > 0$. Finally (3.2.6.0) gives $a_q(n,m)$ for $n+m < 0$.

From the construction, $a_q(n,m)$ satisfies (3.2.6.0) and (3.2.6.2) if $n+m > 0$ and (3.2.4.1) if $n+m < 0$. Lemma 3.2.7 shows that (3.2.6.1) is satisfied when $n+m = 0$. This proves the existence of a_q.

Since $a_q(n,m)$ is completely determined by its value at $a_q(0,0)$ or $a_q(1,-1)$, the dimension of the space of solutions of the system (3.2.6.0) - (3.2.6.2) is equal to 1. This completes the proof of Thm. 3.2.5. □

Appendix

1. Proof of Example. at page 16

Lemma 1. Feynman's integral for n=1

$$\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}(a_1t+a_2(1-t))^{-(\alpha_1+\alpha_2)}dt = B(\alpha_1,\alpha_2)a_1^{-\alpha_1}a_2^{-\alpha_2}.$$

Proof.

It is sufficient to show that

$$a_1a_2\int_0^1 (a_1t)^{\alpha_1-1}(a_2(1-t))^{\alpha_2-1}(a_1t+a_2(1-t))^{-(\alpha_1+\alpha_2)}dt = B(\alpha_1,\alpha_2)(1)$$

Set

$$X = \frac{a_1t}{a_1t+a_2(1-t)} \Rightarrow dX = \frac{a_1a_2}{(a_1t+a_2(1-t))^2}dt$$

then the left hand sight of (1) is

$$\int_0^1 X^{\alpha_1-1}(1-X)^{\alpha_2-1}dX$$

and the lemma is proved.

Lemma 2.

$$\int_R (y^2+a)^\lambda dy = a^{\lambda+\frac{1}{2}}B(-\lambda-\frac{1}{2},\frac{1}{2}), \quad (a \geq 0)$$

Proof .

We have

$$\int_R (y^2+a)^\lambda dy = a^\lambda\int_R ((\frac{y}{\sqrt{a}})^2+1)^\lambda = a^{\lambda+\frac{1}{2}}\int_R (Y^2+1)^\lambda dY$$

$$= a^{\lambda+\frac{1}{2}}\int_R (s+1)^\lambda s^{-\frac{1}{2}}ds = a^{\lambda+\frac{1}{2}}\int_0^1 t^{-\lambda-\frac{3}{2}}(1-t)^{-\frac{1}{2}}dt$$

$$= a^{\lambda+\frac{1}{2}}B(-\lambda-\frac{1}{2},\frac{1}{2})$$

where

$$y = \sqrt{a}.Y, s = Y^2, t = \frac{1}{s+1}.$$

Now, we turn to the example.

By definition $z = x + iy$,

$$I = \int_C |z|^{2a}|z-1|^{2b}\frac{i}{2}dzd\bar{z} = \int_{R^2}(x^2+y^2)^a((x-1)^2+y^2)^b dxdy$$

Setting $\hat{x} = x + \frac{1}{2}$, one has

$$I = \int_{R^2}[(\hat{x}+\frac{1}{2})^2+y^2]^a\cdot[(\hat{x}-\frac{1}{2})^2+y^2]^b d\hat{x}dy$$

$$= \frac{1}{4^{a+b}} \int_{R^2} [(2\hat{x}+1)^2 + 4y^2]^a \cdot [(2\hat{x}-1)^2 + 4y^2]^b d\hat{x}dy.$$

Let us denote $X = 2\hat{x}, Y = 2y$, it follows that

$$I = \frac{1}{4^{a+b+1}} \int_{R^2} ((X+1)^2 + Y^2)^a ((X-1)^2 + Y^2)^b dXdY$$

$$= \frac{1}{4^{a+b+1}} \int_{R^2} ((x+1)^2 + y^2)^a ((x-1)^2 + y^2)^b dxdy$$

$$= \frac{1}{4^{a+b+1}} \int_{R^2} (y^2 + x^2 + 1 + 2x)^a (y^2 + x^2 + 1 - 2x)^b dxdy$$

$$= \frac{1}{4^{a+b+1}} \cdot J$$

By Lemma 1 with

$$\alpha_1 = -a, \alpha_2 = -b, a_1 = y^2 + x^2 + 1 + 2x, a_2 = y^2 + x^2 + 1 - 2x,$$

we obtain

$$J = \frac{1}{B(-a,-b)} \int_0^1 t^{-a-1}(1-t)^{-b-1} dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [y^2 + x^2 + 1 + 2(2t-1)x]^{a+b} dxdy.$$

By Lemma 2

$$\int_{-\infty}^{\infty} [y^2 + x^2 + 1 + 2(2t-1)x]^{a+b} dy = (x^2 + 1 + 2(2t-1)x)^{a+b+\frac{1}{2}} B(-a-b-\frac{1}{2}, \frac{1}{2}),$$

We thus get

$$J = \frac{B(-a-b-\frac{1}{2}, \frac{1}{2})}{B(-a,-b)} \int_0^1 t^{-a-1}(1-t)^{-b-1} dt \int_R (x^2 + 1 + 2(2t-1)x)^{a+b+\frac{1}{2}} dx.$$

Consider

$$J_1 = \int_0^1 t^{-a-1}(1-t)^{-b-1} dt \int_R (x^2 + 1 + 2(2t-1)x)^{a+b+\frac{1}{2}} dx$$

$$= \int_0^1 t^{-a-1}(1-t)^{-b-1} dt \int_R [(x + 2t - 1)^2 + 4t(1-t)]^{a+b+\frac{1}{2}} dx.$$

Set $X = x + 2t - 1$,

$$J_1 = \int_0^1 t^{-a-1}(1-t)^{-b-1} dt \int_R [X^2 + 4t(1-t)]^{a+b+\frac{1}{2}} dX.$$

Using Lemma 2 again we have

$$\int_R [X^2 + 4t(1-t)]^{a+b+\frac{1}{2}} dX = [4t(1-t)]^{a+b+1} B(-a-b-1, \frac{1}{2}),$$

Since

$$J_1 = \int_0^1 t^{-a-1}(1-t)^{-b-1} [4t(1-t)]^{a+b+1} B(-a-b-1, \frac{1}{2}) dt$$

$$= 4^{a+b+1} B(-a-b-1, \frac{1}{2}) \int_0^1 t^b (1-t)^a dt$$

$$= 4^{a+b+1}B(-a-b-1,\frac{1}{2})B(a+1,b+1)$$

we deduce that

$$I = \frac{1}{4^{a+b+1}}\cdot J = \frac{1}{4^{a+b+1}}\cdot\frac{B(-a-b-\frac{1}{2},\frac{1}{2})}{B(-a,-b)}J_1$$

$$= \frac{B(-a-b-\frac{1}{2},\frac{1}{2})B(-a-b-1,\frac{1}{2})}{B(-a,-b)}\cdot B(a+1,b+1)$$

$$= \frac{\frac{\Gamma(-a-b-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(-a-b)}\times\frac{\Gamma(-a-b-1)\Gamma(\frac{1}{2})}{\Gamma(-a-b-\frac{1}{2})}}{\frac{\Gamma(-a)\Gamma(-b)}{\Gamma(-a-b)}}\cdot B(a+1,b+1)$$

$$= \frac{\pi\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)}\cdot B(a+1,b+1)$$

On the other hand

$$\Gamma(-a)\Gamma(-b) = \frac{-\pi}{sin(\pi a)\Gamma(a+1)}\times\frac{-\pi}{sin(\pi b)\Gamma(b+1)}$$

$$= \frac{\pi^2}{sin(\pi a)sin(\pi b)\Gamma(a+1)\Gamma(b+1)}$$

$$\Gamma(-a-b-1) = \frac{\pi}{sin((-a-b-1)\pi)\Gamma(a+b+2)}$$

$$= \frac{\pi}{sin((a+b)\pi)\Gamma(a+b+2)}.$$

Finally

$$I = \frac{sin(\pi a)sin(\pi b)\Gamma(a+1)\Gamma(b+1)}{sin((a+b)\pi)\Gamma(a+b+2)}B(a+1,b+1)$$

$$= \frac{sin(\pi a)sin(\pi b)}{sin((a+b)\pi)}(\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)})^2.$$

The proof is complete. □

2. Calculations of Thm. 1.2.3. We can rewrite J as

$$J = \int_{\mathbb{Z}_p}|z-y|^{\alpha}|z-x|^{\beta}dz + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|z|_p^c|z-y|^{\alpha}|z-x|^{\beta}dz$$

$$= J_1 + J_2,$$

We first compute

$$J_1 = \int_{\mathbb{Z}_p}|z-y|^{\alpha}|z-x|^{\beta}dz = \sum_{m=0}^{+\infty}\int_{C_m}|z-y|^{\alpha}|z-x|^{\beta}dz$$

Set $f = |z-y|_p^{\alpha}|z-x|_p^{\beta}$

Case 1. $x\in A_k, y\in B_l, \quad k,l<0$

If $z \in C_m, C_m = \{z \in \mathbb{Z}_p | z = c_m p^m + c_{m+1}p^{m+1} + ...\}, m \geq 0$ then $z - x \in A_k, z - y \in B_l$. Therefore

$$J_1 = \int_{\mathbb{Z}_p} p^{-l\alpha} p^{-k\beta} dz = p^{-(l\alpha+k\beta)} \mu(\mathbb{Z}_p) = p^{-(l\alpha+k\beta)}$$

Case 2. $x \in A_k, y \in B_l, k \geq 0, l < 0$
Hence $z - y \in B_l, |z - y|_p = p^{-l}$,

$$J_1 = \sum_{m=0}^{+\infty} \int_{C_m} p^{-l\alpha} |z - x|^{\beta} dz =$$

$$= p^{-l\alpha}\Big((1 - p^{-1}) \sum_{m=0}^{k-1} p^{-m(\beta+1)} + (p - 2)p^{-1} p^{-k(\beta+1)} +$$

$$+(1 - p^{-1})p^{-k(\beta+1)} \frac{p^{-(\beta+1)}}{1 - p^{-(\beta+1)}} + p^{-k(\beta+1)} p^{-1}\Big)$$

$$= p^{-l\alpha}(1 - p^{-1}) \frac{1}{1 - p^{-(\beta+1)}}$$

Case 3. $x \in A_k, y \in B_l, k < 0, l \geq 0$
then $z - x \in A_k, |z - x|_p = p^{-k}$

$$J_1 = \sum_{m=0}^{+\infty} \int_{C_m} p^{-k\beta} |z - y|^{\alpha} dz = p^{-k\beta}(1 - p^{-1}) \frac{1}{1 - p^{-(\alpha+1)}}$$

Case 4. $x \in A_k, y \in B_l, k, l \geq 0$
$\star$ Suppose that $\underline{k > l}$.

$$J_1 = \sum_{m=0}^{+\infty} \int_{C_m} f dz$$

$$= \sum_{m=0}^{l-1} \int_{C_m} f dz + \int_{C_l} f dz + \sum_{m=l+1}^{k-1} \int_{C_m} f dz + \int_{C_k} f dz + \sum_{m=k+1}^{+\infty} \int_{C_m} f dz$$

*If $m < l < k$ then $z - y \in C_m, z - x \in C_m$, thus

$$\sum_{m=0}^{l-1} \int_{C_m} f dz = \sum_{m=0}^{l-1} p^{-m\alpha} p^{-m\beta} p^{-m} (1 - p^{-1}) =$$

$$= (1 - p^{-1}) \frac{1 - p^{-l(\alpha+\beta+1)}}{1 - p^{-(\alpha+\beta+1)}}$$

*If $m = l$ then

$$\int_{C_l} f dz = p^{-l\beta} \int_{C_l} |z - y|_p^{\alpha} dz$$

$$= (p - 2)p^{-1} p^{-l(\alpha+\beta+1)} + (1 - p^{-1})p^{-l(\alpha+\beta+1)} \frac{p^{-(\alpha+1)}}{1 - p^{-(\alpha+1)}}$$

*If $l < m < k$ then $z - x \in C_m, z - y \in B_l$

$$\sum_{m=l+1}^{k-1} \int_{C_m} f dz = \sum_{m=l+1}^{k-1} p^{-l\alpha} p^{-m\beta} p^{-m}(1 - p^{-1})$$

$$= (1 - p^{-1}) p^{-l(\alpha+\beta+1)} \frac{p^{-(\beta+1)}}{1 - p^{-(\beta+1)}} - (1 - p^{-1}) p^{-l\alpha} \frac{p^{-k(\beta+1)}}{1 - p^{-(\beta+1)}}$$

*If $m = k, z - y \in B_l$ then

$$\int_{C_k} f dz = p^{-l\alpha} \int_{C_k} |z - x|_p^{\beta} dz$$

$$= (p - 2) p^{-1} p^{-l\alpha} p^{-k(\beta+1)} + (1 - p^{-1}) p^{-l\alpha} p^{-k(\beta+1)} \frac{p^{-(\beta+1)}}{1 - p^{-(\beta+1)}}$$

*If $m > k, z - y \in B_l, z - x \in A_k$ then

$$\sum_{m=k+1}^{+\infty} \int_{C_m} f dz = \sum_{m=k+1}^{+\infty} p^{-l\alpha} p^{-k\beta} p^{-m}(1 - p^{-1}) = p^{-l\alpha} p^{-k\beta} p^{-(k+1)}$$

⋆ In the case $\underline{k < l}$,

$$J_1 = \sum_{m=0}^{+\infty} \int_{C_m} f dz$$

$$= \sum_{m=0}^{k-1} \int_{C_m} f dz + \int_{C_k} f dz + \sum_{m=k+1}^{l-1} \int_{C_m} f dz + \int_{C_l} f dz + \sum_{m=l+1}^{+\infty} \int_{C_m} f dz$$

Similarly, we obtain

*

$$\sum_{m=0}^{k-1} \int_{C_m} f dz = (1 - p^{-1}) \frac{1 - p^{-k(\alpha+\beta+1)}}{1 - p^{-(\alpha+\beta+1)}}$$

*

$$\int_{C_k} f dz = (p - 2) p^{-1} p^{-k(\alpha+\beta+1)} + (1 - p^{-1}) p^{-k(\alpha+\beta+1)} \frac{p^{-(\alpha+1)}}{1 - p^{-(\alpha+1)}}$$

*

$$\sum_{m=k+1}^{l-1} \int_{C_m} f dz = (1 - p^{-1}) p^{-k(\alpha+\beta+1)} \frac{p^{-(\alpha+1)}}{1 - p^{-(\alpha+1)}} - (1 - p^{-1}) p^{-k\beta} \frac{p^{-l(\alpha+1)}}{1 - p^{-(\alpha+1)}}$$

*

$$\int_{C_l} f dz = (p - 2) p^{-1} p^{-k\beta} p^{-l(\alpha+1)} + (1 - p^{-1}) p^{-k\beta} p^{-l(\alpha+1)} \frac{p^{-(\alpha+1)}}{1 - p^{-(\alpha+1)}}$$

*

$$\sum_{m=l+1}^{+\infty} \int_{C_m} f dz = (1 - p^{-1}) p^{-l\alpha} p^{-k\beta} p^{-(l+1)}$$

⋆ Now, let us consider $\underline{k = l}$

$$J_1 = \sum_{m=0}^{+\infty} \int_{C_m} f dz$$
$$= \sum_{m=0}^{k-1} \int_{C_m} f dz + \int_{C_k} f dz + \sum_{m=k+1}^{+\infty} \int_{C_m} f dz$$

where

*

$$\sum_{m=0}^{k-1} \int_{C_m} f dz = (1 - p^{-1}) \frac{1 - p^{-k(\alpha+\beta+1)}}{1 - p^{-(\alpha+\beta+1)}}$$

*

$$\sum_{m=k+1}^{+\infty} \int_{C_m} f dz = \sum_{m=k+1}^{+\infty} p^{-k\alpha} p^{-k\beta} p^{-m} (1 - p^{-1}) = (1 - p^{-1}) p^{-k(\alpha+\beta)} p^{-(k+1)}$$

*If $m = k$ then

$$\int_{C_k} f dz = \int_{C_k, x, y \in C_k} |z - y|_p^{\alpha} |z - x|_p^{\beta} dz$$

Assume that $x = a_k p^k + a_{k+1} p^{k+1} + ...$

and $\quad y = b_k p^k + b_{k+1} p^{k+1} + ...$

and $\quad z = c_k p^k + c_{k+1} p^{k+1} + ..., a_k, b_k, c_k \neq 0$

• If $a_k = b_k$ then

$$\int_{C_k} f dz = \int_{C_k, c_k = a_k = b_k} |z - y|_p^{\alpha} |z - x|_p^{\beta} dz + \int_{C_k, c_k \neq a_k = b_k} |z - y|_p^{\alpha} |z - x|_p^{\beta} dz$$

It is easily seen that

$$\int_{C_k, c_k \neq a_k = b_k} |z - y|_p^{\alpha} |z - x|_p^{\beta} dz = p^{-k\alpha} p^{-k\beta} (p - 2) p^{-(k+1)}$$

It remains to calculate

$$K = \int_{C_k, c_k = a_k = b_k} |z - y|_p^{\alpha} |z - x|_p^{\beta} dz$$

Let s denote the first index such that $a_{k+s} \neq b_{k+s}, s \geq 1$, we thus get

$$a_{k+i} = b_{k+i}, 1 \leq i \leq s - 1$$

This implies that

$$K = \int_{C_k, c_k = a_k = b_k} f dz =$$
$$= \left(\sum_{i=1}^{s-1} p^{-(k+i)\alpha} p^{-(k+i)\beta} p^{-(k+i)} (1 - p^{-1}) \right) +$$
$$+ p^{-(k+s)\alpha} (1 - p^{-1}) \frac{p^{-(k+s+1)(\beta+1)}}{1 - p^{-(\beta+1)}} +$$
$$+ p^{-(k+s)\beta} (1 - p^{-1}) \frac{p^{-(k+s+1)(\alpha+1)}}{1 - p^{-(\alpha+1)}} + p^{-(k+s)(\alpha+\beta+1)} (p - 2) p^{-1}$$

• If $a_k \neq b_k$ then

$$\int_{C_k} f dz = \int_{C_k, c_k = a_k \neq b_k} f dz + \int_{C_k, c_k = b_k \neq a_k} f dz + \int_{C_k, c_k \neq \{a_k, b_k\}} f dz$$

$$= p^{-k\alpha}(1-p^{-1})\frac{p^{-(k+1)(\beta+1)}}{1-p^{-(\beta+1)}} + p^{-k\beta}(1-p^{-1})\frac{p^{-(k+1)(\alpha+1)}}{1-p^{-(\alpha+1)}} +$$

$$+(p-3)p^{-1}p^{-k(\alpha+\beta+1)}$$

Similar calculations apply to J_2

$$J_2 = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |z|_p^c |z-y|_p^\alpha |z-x|_p^\beta dz = \sum_{m=-\infty}^{-1} \int_{C_m} p^{-mc} f dz$$

Case $\bar{1}$ **.** $x \in A_k, y \in B_l, k, l \geq 0$

$$J_2 = \sum_{m=-\infty}^{-1} \int_{C_m} p^{-mc} p^{-m\alpha} p^{-m\beta} p^{-m}(1-p^{-1}) = \frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}(1-p^{-1})$$

Case $\bar{2}$ **.** $x \in A_k, y \in B_l, k \geq 0, l < 0$

$$J_2 = \sum_{m=-\infty}^{-1} \int_{C_m} p^{-mc} p^{-m\beta} |z-y|_p^\alpha dz =$$

$$= \sum_{m=-\infty}^{l-1} \int_{C_m} p^{-m(c+\beta)} |z-y|_p^\alpha dz + \int_{C_l} p^{-l(c+\beta)} |z-y|_p^\alpha dz +$$

$$+ \sum_{m=l+1}^{-1} \int_{C_m} p^{-m(c+\beta)} |z-y|_p^\alpha dz$$

$$= \sum_{m=-\infty}^{l-1} p^{-m(c+\beta)} p^{-m\alpha} p^{-m}(1-p^{-1}) + \int_{C_l} p^{-l(c+\beta)} |z-y|_p^\alpha dz +$$

$$+ \sum_{m=l+1}^{-1} p^{-m(c+\beta)} p^{-l\alpha} p^{-m}(1-p^{-1})$$

$$= (1-p^{-1})\frac{p^{(1-l)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}$$

$$+p^{-l(c+\beta)}\left[(p-2)p^{-1}p^{-l(\alpha+1)} + (1-p^{-1})\frac{p^{-(l+1)(\alpha+1)}}{1-p^{-(\alpha+1)}}\right] +$$

$$+(1-p^{-1})p^{-l\alpha}p^{c+\beta+1}\frac{1-p^{(-l-1)(c+\beta+1)}}{1-p^{c+\beta+1}}$$

Case $\bar{3}$. $x \in A_k, y \in B_l, k < 0, l \geq 0$

$$J_2 = \sum_{m=-\infty}^{-1} \int_{C_m} p^{-mc} p^{-m\alpha} |z-x|_p^{\beta} dz =$$

$$= (1-p^{-1}) \frac{p^{(1-k)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}} +$$

$$+ p^{-k(c+\alpha)} \left[(p-2)p^{-1}p^{-k(\beta+1)} + (1-p^{-1}) \frac{p^{-(k+1)(\beta+1)}}{1-p^{-(\beta+1)}} \right] +$$

$$+ (1-p^{-1})p^{-k\beta}p^{c+\alpha+1} \frac{1-p^{(-k-1)(c+\alpha+1)}}{1-p^{c+\alpha+1}}$$

Case $\bar{4}$. $x \in A_k, y \in B_l, k, l < 0$

⋆ $\underline{k > l}$

$$J_2 = \sum_{m=-\infty}^{l-1} p^{-mc} f dz + \int_{C_l} p^{-lc} f dz + \sum_{m=l+1}^{k-1} \int_{C_m} p^{-mc} f dz$$

$$+ \int_{C_k} p^{-kc} f dz + \sum_{m=k+1}^{-1} \int_{C_m} p^{-mc} f dz$$

where

*

$$\sum_{m=-\infty}^{l-1} p^{-mc} f dz = (1-p^{-1}) \frac{p^{(1-l)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}$$

*

$$\int_{C_l} p^{-lc} f dz = (p-2)p^{-1}p^{-l(\alpha+\beta+c+1)} + (1-p^{-1})p^{-l(\alpha+\beta+c+1)} \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}$$

*

$$\sum_{m=l+1}^{k-1} \int_{C_m} p^{-mc} f dz =$$

$$= (1-p^{-1})p^{-l(\alpha+\beta+c+1)} \frac{p^{-(c+\beta+1)}}{1-p^{-(c+\beta+1)}} - (1-p^{-1})p^{-l\alpha} \frac{p^{-k(c+\beta+1)}}{1-p^{-(c+\beta+1)}}$$

*

$$\int_{C_k} p^{-kc} f dz = (p-2)p^{-1}p^{-l\alpha}p^{-k(c+\beta+1)} + (1-p^{-1})p^{-l\alpha}p^{-k(c+\beta+1)} \frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}$$

*

$$\sum_{m=k+1}^{-1} \int_{C_m} p^{-mc} f dz = (1-p^{-1})p^{-(k\beta+l\alpha)}p^{c+1} . \frac{1-p^{(-k-1)(c+1)}}{1-p^{c+1}}$$

⋆ $\underline{k < l}$ we have

$$J_2 = \sum_{m=-\infty}^{k-1} p^{-mc} f dz + \int_{C_k} p^{-kc} f dz + \sum_{m=k+1}^{l-1} \int_{C_m} p^{-mc} f dz + \int_{C_l} p^{-lc} f dz +$$

$$+\sum_{m=l+1}^{-1}\int_{C_m} p^{-mc} f dz$$

*

$$\sum_{m=-\infty}^{k-1} p^{-mc} f dz = (1-p^{-1})\frac{p^{(1-k)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}$$

*

$$\int_{C_k} p^{-kc} f dz = (p-2)p^{-1}p^{-k(\alpha+\beta+c+1)} + (1-p^{-1})p^{-k(\alpha+\beta+c+1)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}$$

*

$$\sum_{m=k+1}^{l-1}\int_{C_m} p^{-mc} f dz =$$

$$= (1-p^{-1})p^{-k(\alpha+\beta+c+1)}\frac{p^{-(c+\alpha+1)}}{1-p^{-(c+\alpha+1)}} - (1-p^{-1})p^{-k\beta}\frac{p^{-l(c+\alpha+1)}}{1-p^{-(c+\alpha+1)}}$$

*

$$\int_{C_l} p^{-lc} f dz = (p-2)p^{-1}p^{-k\beta}p^{-l(c+\alpha+1)} + (1-p^{-1})p^{-k\beta}p^{-l(c+\alpha+1)}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}$$

*

$$\sum_{m=l+1}^{-1}\int_{C_m} p^{-mc} f dz = (1-p^{-1})p^{-(k\beta+l\alpha)}p^{c+1}.\frac{1-p^{(-l-1)(c+1)}}{1-p^{c+1}}$$

⋆ $\underline{k=l}$

$$J_1 = \sum_{m=-\infty}^{-1}\int_{C_m} p^{-mc} f dz$$

$$= \sum_{m=-\infty}^{k-1}\int_{C_m} p^{-mc} f dz + \int_{C_k} p^{-kc} f dz + \sum_{m=k+1}^{-1}\int_{C_m} p^{-mc} f dz$$

*

$$\sum_{m=-\infty}^{k-1}\int_{C_m} p^{-mc} f dz = (1-p^{-1})\frac{1-p^{(1-k)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}$$

*

$$\sum_{m=k+1}^{-1}\int_{C_m} p^{-mc} f dz = \sum_{m=k+1}^{-1} p^{-mc}p^{-k\alpha}p^{-k\beta}p^{-m}(1-p^{-1}) =$$

$$= (1-p^{-1})p^{-k(\alpha+\beta)}p^{c+1}.\frac{1-p^{(-k-1)(c+1)}}{1-p^{c+1}}$$

* Let us compute

$$\int_{C_k} p^{-kc} f dz$$

In the same manner we can see that

•$a_k = b_k$

$$\int_{C_k, c_k \neq a_k = b_k} p^{-kc}|z-y|_p^{\alpha}|z-x|_p^{\beta}dz = p^{-kc}p^{-k\alpha}p^{-k\beta}(p-2)p^{-(k+1)}$$

$$\int_{C_k, c_k = a_k = b_k} p^{-kc}|z-y|_p^{\alpha}|z-x|_p^{\beta}dz =$$

$$= p^{-kc}\Big(\sum_{i=1}^{s-1} p^{-(k+i)\alpha}p^{-(k+i)\beta}p^{-(k+i)}(1-p^{-1})\Big)+$$

$$+p^{-kc}p^{-(k+s)\alpha}(1-p^{-1})\frac{p^{-(k+s+1)(\beta+1)}}{1-p^{-(\beta+1)}}+$$

$$+p^{-kc}p^{-(k+s)\beta}(1-p^{-1})\frac{p^{-(k+s+1)(\alpha+1)}}{1-p^{-(\alpha+1)}}+$$

$$+p^{-kc}p^{-(k+s)(\alpha+\beta+1)}(p-2)p^{-1}$$

•$a_k \neq b_k$

$$\int_{C_k} p^{-kc} f dz = \int_{C_k, c_k = a_k \neq b_k} p^{-kc} f dz + \int_{C_k, c_k = b_k \neq a_k} p^{-kc} f dz$$

$$+\int_{C_k, c_k \neq \{a_k, b_k\}} p^{-kc} f dz$$

$$= p^{-kc}p^{-k\alpha}(1-p^{-1})\frac{p^{-(k+1)(\beta+1)}}{1-p^{-(\beta+1)}}$$

$$+p^{-kc}p^{-k\beta}(1-p^{-1})\frac{p^{-(k+1)(\alpha+1)}}{1-p^{-(\alpha+1)}} + (p-3)p^{-1}p^{-k(\alpha+\beta+c+1)}$$

Now,we shall calculate I_3

$$I_3 = \int_{\mathbb{Q}_p^3} |\phi(x)|_p^a|\phi(y)|_p^b|\phi(z)|_p^c|x-y|_p^{\gamma}|y-z|^{\alpha}|z-x|^{\beta}dxdydz$$

$$= \int_{\mathbb{Q}_p^2} |\phi(x)|_p^a|\phi(y)|_p^b|x-y|_p^{\gamma}\int_{\mathbb{Z}_p} f dz + \int_{\mathbb{Q}_p^2} |\phi(x)|_p^a|\phi(y)|_p^b|x-y|_p^{\gamma}\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |z|_p^c f dz$$

$$= X_1 \qquad + \qquad X_2$$

In fact

$$X_1 = \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |\phi(x)|_p^a|\phi(y)|_p^b|x-y|_p^{\gamma}dx\int_{\mathbb{Z}_p} f dz+$$

$$+\int_{\mathbb{Z}_p} dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |\phi(x)|_p^a|\phi(y)|_p^b|x-y|_p^{\gamma}dx\int_{\mathbb{Z}_p} f dz+$$

$$+\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dy \int_{\mathbb{Z}_p} |\phi(x)|_p^a|\phi(y)|_p^b|x-y|_p^{\gamma}dx\int_{\mathbb{Z}_p} f dz+$$

$$+\int_{\mathbb{Z}_p} dy \int_{\mathbb{Z}_p} |\phi(x)|_p^a|\phi(y)|_p^b|x-y|_p^{\gamma}dx\int_{\mathbb{Z}_p} f dz$$

$$= X_{11} + X_{12} + X_{13} + X_{14}$$

• It is easily to check that

$$X_{11} = \sum_{l=-\infty}^{-1} \int_{B_l} |y|_p^b dy \sum_{k=-\infty}^{-1} \int_{A_k} |x|_p^a |x-y|_p^\gamma dx (p^{-k\beta} p^{-l\alpha})$$

$$= \sum_{l=-\infty}^{-1} \int_{B_l} |y|_p^b p^{-l\alpha} dy \Big(\sum_{k=-\infty}^{-1} \int_{A_k} p^{-ka} p^{-k\beta} |x-y|_p^\gamma dx \Big)$$

Since

$$A = \sum_{k=-\infty}^{-1} \int_{A_k} p^{-ka} p^{-k\beta} |x-y|_p^\gamma dx = (1-p^{-1}) \frac{p^{(1-l)(a+\beta+\gamma+1)}}{1-p^{(a+\beta+\gamma+1}} +$$

$$+p^{-l(a+\beta)} \left[(p-2)p^{-1} p^{-l(\gamma+1)} + (1-p^{-1}) \frac{p^{-(l+1)(\gamma+1)}}{1-p^{-(\gamma+1)}} \right] +$$

$$+(1-p^{-1}) p^{-l\gamma} \frac{p^{a+\beta+1}}{1-p^{a+\beta+1}} - (1-p^{-1}) p^{-l\gamma} \frac{p^{-l(a+\beta+1)}}{1-p^{a+\beta+1}}$$

it follows that

$$X_{11} = \sum_{l=-\infty}^{-1} \int_{B_l} |y|_p^b . p^{-l\alpha} . A dy = \sum_{l=-\infty}^{-1} p^{-l(b+\alpha+1)} (1-p^{-1}) A$$

$$= (1-p^{-1})^2 \frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}} . \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})(p-2)p^{-1} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2 \frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2 \frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

• We also have

$$X_{12} = \int_{\mathbb{Z}_p} 1 dy \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} |x|_p^a |x-y|_p^\gamma dx \int_{\mathbb{Z}_p} f dz \quad \text{(see} \quad \textbf{Case3}.)$$

$$= (1-p^{-1})^2 \frac{p^{a+\beta+\gamma+1}}{(1-p^{-(\alpha+1)})(1-p^{a+\beta+\gamma+1})}$$

• Similarly,

$$X_{13} = \int_{\mathbb{Q}_p \setminus \mathbb{Z}_p} dy \int_{\mathbb{Z}_p} |\phi(x)|_p^a |\phi(y)|_p^b |x-y|_p^\gamma dx \int_{\mathbb{Z}_p} f dz$$

$$= (1-p^{-1})^2 \frac{p^{b+\alpha+\gamma+1}}{(1-p^{-(\beta+1)})(1-p^{b+\alpha+\gamma+1})}$$

• For computing X_{14}, let us rewrite

$$X_{14} = \sum_{k,l,k>l\geq 0} \int_{A_k} dx \int_{B_l} |x-y|_p^\gamma dy \int_{\mathbb{Z}_p} f dz$$

$$+\sum_{k,l,l>k\geq 0}\int_{A_k}dx\int_{B_l}|x-y|_p^{\gamma}dy\int_{\mathbb{Z}_p}fdz$$
$$+\sum_{k=0}^{+\infty}\int_{A_k}dx\int_{B_k}|x-y|_p^{\gamma}dy\int_{\mathbb{Z}_p}fdz$$
$$=X_{141}+X_{142}+X_{143}$$

We get

$$X_{141}=\sum_{k,l,k>l\geq 0}\int_{A_k}dx\int_{B_l}|x-y|_p^{\gamma}dy\int_{\mathbb{Z}_p}fdz$$
$$=\sum_{l=0}^{+\infty}\int_{B_l}dy(\sum_{k=l+1}^{+\infty}\int_{A_k}|x-y|_p^{\gamma}dx)\int_{\mathbb{Z}_p}fdz$$
$$=\sum_{l=0}^{+\infty}\int_{B_l}dy(\sum_{k=l+1}^{+\infty}\int_{A_k}|x-y|_p^{\gamma}Hdx)$$

where

$$H=(1-p^{-1})\frac{1-p^{-l(\alpha+\beta+1)}}{1-p^{-(\alpha+\beta+1)}}+(p-2)p^{-1}p^{-l(\alpha+\beta+1)}$$
$$+(1-p^{-1})p^{-l(\alpha+\beta+1)}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}+(1-p^{-1})p^{-l(\alpha+\beta+1)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}$$
$$-(1-p^{-1})p^{-l\alpha}\frac{p^{-k(\beta+1)}}{1-p^{-(\beta+1)}}+(p-2)p^{-1}p^{-l\alpha}p^{-k(\beta+1)}$$
$$+(1-p^{-1})p^{-l\alpha}p^{-k(\beta+1)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}+p^{-1}p^{-l\alpha}p^{-k\beta}p^{-k}$$

Clearly,

$$\sum_{k=l+1}^{+\infty}\int_{A_k}|x-y|_p^{\gamma}Hdx=\sum_{k=l+1}^{+\infty}p^{-l\gamma}p^{-k}(1-p^{-1})H$$
$$=(1-p^{-1})p^{-1}\frac{p^{-l(\gamma+1)}}{1-p^{-(\alpha+\beta+1)}}-(1-p^{-1})p^{-1}\frac{p^{-l(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+1)}}$$
$$+(p-2)p^{-2}p^{-l(\alpha+\beta+\gamma+2)}+(1-p^{-1})p^{-1}p^{-l(\alpha+\beta+\gamma+2)}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}$$
$$+(1-p^{-1})p^{-1}p^{-l(\alpha+\beta+\gamma+2)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}-$$
$$-(1-p^{-1})^2\frac{p^{-(\beta+2)}}{(1-p^{-(\beta+1)})(1-p^{-(\beta+2)})}p^{-l(\alpha+\beta+\gamma+2)}+$$
$$+(1-p^{-1})^2p^{-l(\alpha+\beta+\gamma+2)}\frac{p^{-(\beta+2)}}{1-p^{-(\beta+2)}}$$
$$+(1-p^{-1})^2p^{-l(\alpha+\beta+\gamma+2)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{-(\beta+2)}}{1-p^{-(\beta+2)}}$$

Therefore

$$X_{141} = \sum_{l=0}^{+\infty} \int_{B_l} dy (\sum_{k=l+1}^{+\infty} \int_{A_k} |x-y|_p^\gamma H dx)$$

$$= \sum_{l=0}^{+\infty} p^{-l}(1-p^{-1})(\sum_{k=l+1}^{+\infty} \int_{A_k} |x-y|_p^\gamma H dx)$$

$$= (1-p^{-1})^2 p^{-1} \frac{p^{-(\gamma+2)}}{(1-p^{-(\gamma+2)})(1-p^{-(\alpha+\beta+\gamma+3)})}$$

$$+(p-2)(1-p^{-1})p^{-2} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

$$+(1-p^{-1})^2 p^{-1} \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

$$+(1-p^{-1})^2 p^{-1} \frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

By the same method as above, we have

$$X_{142} = \sum_{k,l,l>k\geq 0} \int_{A_k} dx \int_{B_l} |x-y|_p^\gamma dy \int_{\mathbb{Z}_p} f dz$$

$$= (1-p^{-1})^2 p^{-1} \frac{p^{-(\gamma+2)}}{(1-p^{-(\gamma+2)})(1-p^{-(\alpha+\beta+\gamma+3)})}$$

$$+(p-2)(1-p^{-1})p^{-2} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

$$+(1-p^{-1})^2 p^{-1} \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

$$+(1-p^{-1})^2 p^{-1} \frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

Consider

$$X_{143} = \sum_{k=0}^{+\infty} \int_{A_k} dx \int_{B_k} |x-y|_p^\gamma dy \int_{\mathbb{Z}_p} f dz$$

To compute this, we set

$$X_{143} = \sum_{k=0}^{+\infty} \int_{A_k} dx \int_{B_k} |x-y|_p^\gamma dy (\sum_{m=0}^{k-1} \int_{C_m} f dz)$$

$$+ \sum_{k=0}^{+\infty} \int_{A_k} dx \int_{B_k} |x-y|_p^\gamma dy (\int_{C_k} f dz)$$

$$+ \sum_{k=0}^{+\infty} \int_{A_k} dx \int_{B_k} |x-y|_p^\gamma dy (\sum_{m=k+1}^{+\infty} \int_{C_m} f dz)$$

$$= X_{1431} + X_{1432} + X_{1433}$$

- $$X_{1431}=\sum_{k=0}^{+\infty}\int_{A_k}dx\int_{B_k}|x-y|_p^{\gamma}dy(\sum_{m=0}^{k-1}\int_{C_m}fdz)$$
$$=\sum_{k=0}^{+\infty}\int_{B_k}dy\int_{A_k}|x-y|_p^{\gamma}(1-p^{-1})\frac{1-p^{-k(\alpha+\beta+1)}}{1-p^{-(\alpha+\beta+1)}}dx$$
$$=\sum_{k=0}^{+\infty}p^{-k}(1-p^{-1})\left((p-2)p^{-1}p^{-k(\gamma+1)}+(1-p^{-1})p^{-k(\gamma+1)}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\right)\times$$
$$\times(1-p^{-1})\frac{1-p^{-k(\alpha+\beta+1)}}{1-p^{-(\alpha+\beta+1)}}$$
$$+\sum_{k=0}^{+\infty}(1-p^{-1})^3\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}p^{-k(\gamma+2)}\frac{1-p^{-k(\alpha+\beta+1)}}{1-p^{-(\alpha+\beta+1)}}$$
$$=(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\gamma+2)}}{(1-p^{-(\gamma+2)})(1-p^{-(\alpha+\beta+\gamma+3)})}+$$
$$+(1-p^{-1})^3\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{-(\gamma+2)}}{(1-p^{-(\gamma+2)})(1-p^{-(\alpha+\beta+\gamma+3)})}$$

- $$X_{1433}=\sum_{k=0}^{+\infty}\int_{B_k}dy\int_{A_k}|x-y|_p^{\gamma}dx(\sum_{m=k+1}^{+\infty}\int_{C_m}fdz)=$$
$$=\sum_{k=0}^{+\infty}p^{-k}(1-p^{-1})\Bigg((p-2)p^{-1}p^{-k(\gamma+1)}$$
$$+(1-p^{-1})p^{-k(\gamma+1)}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\Bigg)p^{-k(\alpha+\beta+1)}p^{-1}$$
$$+\sum_{k=0}^{+\infty}(1-p^{-1})^2p^{-1}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}p^{-k(\alpha+\beta+\alpha+3)}$$
$$=(1-p^{-1})(p-2)p^{-2}\frac{1}{p^{-(\alpha+\beta+\alpha+3)}}+(1-p^{-1})^2p^{-1}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{1}{p^{-(\alpha+\beta+\alpha+3)}}$$

- $$X_{1432}=\sum_{k=0}^{+\infty}\int_{A_k}dx\int_{B_k}|x-y|_p^{\gamma}dy(\int_{C_k}fdz)$$
$$=\sum_{k=0}^{+\infty}\int_{B_k}dy\int_{A_k,a_k=b_k}|x-y|_p^{\gamma}dx(\int_{C_k}fdz)$$
$$+\sum_{k=0}^{+\infty}\int_{B_k}dy\int_{A_k,a_k\neq b_k}|x-y|_p^{\gamma}dx(\int_{C_k}fdz)$$

◇

$$\sum_{k=0}^{+\infty}\int_{B_k}dy\bigg(\int_{A_k,a_k=b_k}|x-y|_p^{\gamma}dx\int_{C_k}fdz\bigg)=$$

$$=\sum_{k=0}^{+\infty}p^{-k}(1-p^{-1})\bigg(\frac{p^{-(k+1)(\gamma+1)}}{1-p^{-(\gamma+1)}}(1-p^{-1})p^{-k(\alpha+\beta+1)}(p-2)p^{-1}\bigg)+$$

$$+\sum_{k=0}^{+\infty}p^{-k}(1-p^{-1})\bigg[(1-p^{-1})^2p^{-k(\alpha+\beta+\gamma+2)}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^2p^{-k(\alpha+\beta+\gamma+2)}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

$$+(1-p^{-1})^2p^{-k(\alpha+\beta+\gamma+2)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

$$+(1-p^{-1})(p-2)p^{-1}p^{-k(\alpha+\beta+\gamma+2)}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}\bigg]$$

$$=(1-p^{-1})^2(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}+$$

$$+(1-p^{-1})^3\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^3\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^3\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

◇

$$\sum_{k=0}^{+\infty}\int_{B_k}dy\int_{A_k,a_k\neq b_k}|x-y|_p^{\gamma}dx(\int_{C_k}fdz)=$$

$$=\sum_{k=0}^{+\infty}p^{-k}(1-p^{-1})((p-2)p^{-1}p^{-k(\gamma+1)})\bigg(p^{-k\alpha}(1-p^{-1})\frac{p^{-(k+1)(\beta+1)}}{1-p^{-(\beta+1)}}+$$

$$+p^{-k\beta}(1-p^{-1})\frac{p^{-(k+1)(\alpha+1)}}{1-p^{-(\alpha+1)}}+(p-3)p^{-1}p^{-k(\alpha+\beta+1)}\bigg)$$

$$=(1-p^{-1})^2(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}+$$

$$+(1-p^{-1})(p-2)(p-3)p^{-2}\frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

Thus we get

$$X_1 = X_{11} + X_{12} + X_{13} + X_{141} + X_{142} + X_{1431} + X_{1432} + X_{1433}$$

$$= (1-p^{-1})^2 \frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}} \cdot \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})(p-2)p^{-1} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2 \frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2 \frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}} +$$

$$+(1-p^{-1})^2 \frac{p^{a+\beta+\gamma+1}}{(1-p^{-(\alpha+1)})(1-p^{a+\beta+\gamma+1})}$$

$$+(1-p^{-1})^2 \frac{p^{b+\alpha+\gamma+1}}{(1-p^{-(\beta+1)})(1-p^{b+\alpha+\gamma+1})} +$$

$$+(1-p^{-1})^2 (\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}) \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

$$+(1-p^{-1})^3 (\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}) \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}} \frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

$$+(1-p^{-1})^2 (p-2)p^{-1} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}} \frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

$$+(1-p^{-1})(p-2)p^{-1} \frac{1}{1-p^{-(\alpha+\beta+\gamma+3)}}$$

It is easily seen that

$$1 + (1-p^{-1}) \frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}} = \frac{1-p^{-(\alpha+\beta+\gamma+3)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

we obtain

$$X_1 = (1-p^{-1})^2 \frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}} \cdot \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})(p-2)p^{-1} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2 \frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2 \frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}} \frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}} +$$

$$+(1-p^{-1})^2 \frac{p^{a+\beta+\gamma+1}}{(1-p^{-(\alpha+1)})(1-p^{a+\beta+\gamma+1})}$$

$$+(1-p^{-1})^2 \frac{p^{b+\alpha+\gamma+1}}{(1-p^{-(\beta+1)})(1-p^{b+\alpha+\gamma+1})} +$$

$$+(1-p^{-1})^2(\sum \frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}})\frac{1}{1-p^{-(\alpha+\beta+\gamma+2)}}$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

Our next goal is to evalute the integral

$$X_2 = \int_{\mathbb{Q}_p^2} |\phi(x)|_p^a |\phi(y)|_p^b |x-y|_p^\gamma \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |z|_p^c f dz$$

Set

$$X_2 = \int_{\mathbb{Z}_p} dx \int_{\mathbb{Z}_p} dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dz + \int_{\mathbb{Z}_p} dx \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dz$$
$$+\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dx \int_{\mathbb{Z}_p} dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dz + \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dx \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} dz$$
$$= X_{21} + X_{22} + X_{23} + X_{24}$$

Apply to this case

- $$X_{21} = \int_{\mathbb{Z}_p} dx \int_{\mathbb{Z}_p} |x-y|_p^\gamma dy \int_{\mathbb{Q}_p\setminus\mathbb{Z}_p} |z|_p^c f dz$$
$$= \int_{\mathbb{Z}_p} dx \int_{\mathbb{Z}_p} |x-y|_p^\gamma dy \frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}(1-p^{-1})$$
$$= (1-p^{-1})^2 \frac{1}{1-p^{-(\gamma+1)}} \frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}$$

- $$X_{22} = \int_{\mathbb{Z}_p} dx \sum_{l=-\infty}^{-1} |y|_p^b |x-y|_p^\gamma dy \Bigg[(1-p^{-1})\frac{p^{(1-l)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}} +$$
$$+p^{-l(c+\beta)}\left((p-2)p^{-1}p^{-l(\alpha+1)} + (1-p^{-1})\frac{p^{-(l+1)(\alpha+1)}}{1-p^{-(\alpha+1)}}\right)+$$
$$+(1-p^{-1})p^{-l\alpha}p^{c+\beta+1}\frac{1-p^{(-l-1)(c+\beta+1)}}{1-p^{c+\beta+1}}\Bigg]$$
$$= \sum_{l=-\infty}^{-1} p^{-lb}p^{-l\gamma}p^{-l}(1-p^{-1})\Bigg[(1-p^{-1})\frac{p^{(1-l)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}} +$$
$$+p^{-l(c+\beta)}\left((p-2)p^{-1}p^{-l(\alpha+1)} + (1-p^{-1})\frac{p^{-(l+1)(\alpha+1)}}{1-p^{-(\alpha+1)}}\right)+$$
$$+(1-p^{-1})p^{-l\alpha}p^{c+\beta+1}\frac{1-p^{(-l-1)(c+\beta+1)}}{1-p^{c+\beta+1}}\Bigg]$$
$$= (1-p^{-1})^2 \frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}} \frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$
$$+(1-p^{-1})^2\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$

- By the same method

$$X_{23}=\int_{\mathbb{Z}_p}dy\sum_{k=-\infty}^{-1}|x|_p^a|x-y|_p^\gamma dx\bigg[(1-p^{-1})\frac{p^{(1-k)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}+$$
$$+p^{-k(c+\alpha)}\bigg((p-2)p^{-1}p^{-k(\beta+1)}+(1-p^{-1})\frac{p^{-(k+1)(\beta+1)}}{1-p^{-(\beta+1)}}\bigg)+$$
$$+(1-p^{-1})p^{-k\beta}p^{c+\alpha+1}\frac{1-p^{(-k-1)(c+\alpha+1)}}{1-p^{c+\alpha+1}}\bigg]$$
$$=(1-p^{-1})^2\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$
$$+(1-p^{-1})^2\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$
$$+(1-p^{-1})^2\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$

-

$$X_{24}=\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|x|_p^a dx\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|y|_p^b|x-y|_p^\gamma dy\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|z|_p^c f dzdz$$
$$=\sum_{k,l,0>k>l}\int_{A_k}|x|_p^a dx\int_{B_l}|y|_p^b|x-y|_p^\gamma dy\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|z|_p^c f dz+$$
$$+\sum_{k,l,0>l>k}\int_{A_k}|x|_p^a dx\int_{B_l}|y|_p^b|x-y|_p^\gamma dy\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|z|_p^c f dz$$
$$+\sum_{k=-\infty}^{-1}\int_{A_k}|x|_p^a dx\int_{B_k}|y|_p^b|x-y|_p^\gamma dy\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}|z|_p^c f dz$$
$$=X_{241}+X_{242}+X_{243}$$

-

$$X_{241}=\sum_{l=-\infty}^{-1}\int_{B_l}|y|_p^b dy\sum_{k=l+1}^{-1}p^{-ka}p^{-l\gamma}p^{-k}(1-p^{-1})\bigg[(1-p^{-1})\frac{p^{(1-l)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}$$
$$+(p-2)p^{-1}p^{-l(\alpha+\beta+c+1)}+(1-p^{-1})p^{-l(\alpha+\beta+c+1)}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}$$
$$+(1-p^{-1})p^{-l(\alpha+\beta+c+1)}\frac{p^{-(c+\beta+1)}}{1-p^{-(c+\beta+1)}}-(1-p^{-1})p^{-l\alpha}\frac{p^{-k(c+\beta+1)}}{1-p^{-(c+\beta+1)}}$$

$$+(p-2)p^{-1}p^{-l\alpha}p^{-k(c+\beta+1)}+(1-p^{-1})p^{-l\alpha}p^{-k(c+\beta+1)}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}$$

$$+(1-p^{-1})p^{-(k\beta+l\alpha)}p^{c+1}.\frac{1-p^{(-k-1)(c+1)}}{1-p^{c+1}}\Big]$$

$$=\sum_{l=-\infty}^{-1}\int_{B_l}|y|_p^b dy\sum_{k=l+1}^{-1}p^{-k(a+1)}p^{-l\gamma}(1-p^{-1}).L \quad (L=[...])$$

And

$$M=\sum_{k=l+1}^{-1}p^{-k(a+1)}p^{-l\gamma}(1-p^{-1}).L$$

$$=(1-p^{-1})^2\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}p^{-l(c+\alpha+\beta+\alpha+1)}\frac{p^{a+1}}{1-p^{a+1}}(1-p^{(-l-1)(a+1)})+$$

$$+(1-p^{-1})(p-2)p^{-1}p^{-l(c+\alpha+\beta+\alpha+1)}\frac{p^{a+1}}{1-p^{a+1}}(1-p^{(-l-1)(a+1)})+$$

$$+(1-p^{-1})^2\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}p^{-l(c+\alpha+\beta+\alpha+1)}\frac{p^{a+1}}{1-p^{a+1}}(1-p^{(-l-1)(a+1)})$$

$$+(1-p^{-1})^2\frac{p^{-(c+\beta+1)}}{1-p^{-(c+\beta+1)}}p^{-l(c+\alpha+\beta+\alpha+1)}\frac{p^{a+1}}{1-p^{a+1}}(1-p^{(-l-1)(a+1)})$$

$$-(1-p^{-1})^2p^{-l(\alpha+\gamma)}\frac{1}{1-p^{-(c+\beta+1)}}\frac{p^{a+c+\beta+2}}{1-p^{a+c+\beta+2}}(1-p^{(-l-1)(a+c+\beta+2)})$$

$$+(1-p^{-1})(p-2)p^{-1}p^{-l(\alpha+\gamma)}\frac{p^{a+c+\beta+2}}{1-p^{a+c+\beta+2}}(1-p^{(-l-1)(a+c+\beta+2)})$$

$$+(1-p^{-1})^2\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}p^{-l(\alpha+\gamma)}\frac{p^{a+c+\beta+2}}{1-p^{a+c+\beta+2}}(1-p^{(-l-1)(a+c+\beta+2)})$$

$$+(1-p^{-1})^2p^{-l(\alpha+\gamma)}\frac{p^{c+1}}{1-p^{c+1}}\frac{p^{a+\beta+1}}{1-p^{a+\beta+1}}(1-p^{(-l-1)(a+\beta+1)})$$

$$-(1-p^{-1})^2p^{-l(\alpha+\gamma)}\frac{1}{1-p^{c+1}}\frac{p^{a+c+\beta+2}}{1-p^{a+c+\beta+2}}(1-p^{(-l-1)(a+c+\beta+2)})$$

This imlies that

$$X_{241}=\sum_{l=-\infty}^{-1}\int_{B_l}|y|_p^b M dy=\sum_{l=-\infty}^{-1}p^{-lb}p^{-l}(1-p^{-1})M$$

$$=(1-p^{-1})^3\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^3\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^3\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

- Similarly, we have

$$X_{242}=(1-p^{-1})^3\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^3\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^3\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^3\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^3\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

- $$X_{243}=\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k}|x|_p^a|x-y|_p^\gamma dx(\sum_{m=-\infty}^{k-1}\int_{C_m}p^{-mc}fdz)+$$
$$+\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k}|x|_p^a|x-y|_p^\gamma(\int_{C_k}p^{-kc}fdz)+$$
$$+\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k}|x|_p^a|x-y|_p^\gamma dx(\sum_{m=k+1}^{-1}\int_{C_m}p^{-mc}fdz)$$
$$=X_{2431}+X_{2432}+X_{2433}$$

- $$X_{2431}=\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k}|x|_p^a|x-y|_p^\gamma dx(\sum_{m=-\infty}^{k-1}\int_{C_m}p^{-mc}fdz)$$
$$=\sum_{k=-\infty}^{-1}\Bigg[p^{-k(b+1+a)}(1-p^{-1})\times\Bigg((p-2)p^{-1}p^{-k(\gamma+1)}$$

$$+(1-p^{-1})p^{-k(\gamma+1)}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\Big)\times(1-p^{-1})\frac{1-p^{(1-k)(\alpha+\beta+c+1)}}{1-p^{\alpha+\beta+c+1}}\Big]$$

$$=(1-p^{-1})^2(p-2)p^{-1}\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

- $$X_{2433}=\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k}|x|_p^a|x-y|_p^\gamma dx(\sum_{m=k+1}^{-1}\int_{C_m}p^{-mc}fdz)$$

$$=\sum_{k=-\infty}^{-1}p^{-k(b+1+a)}(1-p^{-1})\Big((p-2)p^{-1}p^{-k(\gamma+1)}$$

$$+(1-p^{-1})p^{-k(\gamma+1)}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\Big)\times(1-p^{-1})p^{-k(\alpha+\beta)}p^{c+1}.\frac{1-p^{(-k-1)(c+1)}}{1-p^{c+1}}$$

$$=(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

- $$X_{2432}=\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k,a_k\neq b_k}|x|_p^a|x-y|_p^\gamma(\int_{C_k}p^{-kc}fdz)$$

$$+\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k,a_k=b_k}|x|_p^a|x-y|_p^\gamma(\int_{C_k}p^{-kc}fdz)$$

◇

$$\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k,a_k\neq b_k}|x|_p^a|x-y|_p^\gamma(\int_{C_k}p^{-kc}fdz)=$$

$$=\sum_{k=-\infty}^{-1}p^{-k(b+1)}(1-p^{-1})p^{-ka}(p-2)p^{-(k+1)}\Big(p^{-kc}p^{-k\alpha}(1-p^{-1})\frac{p^{-(k+1)(\beta+1)}}{1-p^{-(\beta+1)}}+$$

$$+p^{-kc}p^{-k\beta}(1-p^{-1})\frac{p^{-(k+1)(\alpha+1)}}{1-p^{-(\alpha+1)}}+(p-3)p^{-1}p^{-k(\alpha+\beta+c+1)}\Big)$$

$$=(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

$$+(1-p^{-1})(p-2)(p-3)p^{-2}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

◇

$$\sum_{k=-\infty}^{-1}\int_{B_k}|y|_p^b\int_{A_k,a_k=b_k}|x|_p^a|x-y|_p^\gamma(\int_{C_k}p^{-kc}fdz)$$

$$=\sum_{k=-\infty}^{-1}\int_{B_k}p^{-kb}dy\Big(\int_{A_k,a_k=b_k}p^{-ka}|x-y|_p^\gamma dx\int_{C_k,c_k\neq a_k=b_k}p^{-kc}fdz\Big)+$$

$$+\sum_{k=-\infty}^{-1}\int_{B_k}p^{-kb}dy\Big(\int_{A_k,a_k=b_k}p^{-ka}|x-y|_p^\gamma dx\int_{C_k,c_k=a_k=b_k}p^{-kc}fdz\Big)$$

$$=(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}+$$

$$+(1-p^{-1})^3\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^3\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^3\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

Hence

$$X_2=X_{21}+X_{22}+X_{23}+X_{241}+X_{242}+X_{2431}+X_{2432}+X_{2433}$$

$$=(1-p^{-1})^2\frac{1}{1-p^{-(\gamma+1)}}\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}+$$

$$+(1-p^{-1})^2\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})(p-2)p^{-1}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})(p-2)p^{-1}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^2\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}$$

$$+(1-p^{-1})^3\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+c+\alpha+\beta+\gamma+2}}{1-p^{a+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{a+\beta+\gamma+1}}{1-p^{a+\beta+\gamma+1}}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{\alpha+\beta+c+1}}{1-p^{\alpha+\beta+c+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^3\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{a+b+\alpha+\beta+\gamma+2}}{1-p^{a+b+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$

$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}$$
$$+(1-p^{-1})(p-2)(p-3)p^{-2}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}+$$
$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}+$$
$$+(1-p^{-1})^3\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$
$$+(1-p^{-1})^3\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$
$$+(1-p^{-1})^3\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\beta+1)}}{1-p^{-(\beta+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}+$$
$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}$$

Therefore

$$I_3 = X_1 + X_2$$
$$= (1-p^{-1})^2\left(\sum\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\right)\frac{1}{1-p^{-(\alpha+\beta+\gamma+2)}} \quad (1)$$
$$+(1-p^{-1})(p-2)p^{-1}\frac{1}{1-p^{-(\alpha+\beta+\gamma+2)}} \quad (2)$$
$$+(1-p^{-1})^2\sum\frac{1}{1-p^{-(\alpha+1)}}\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}} \quad (3)$$
$$+(1-p^{-1})(p-2)p^{-1}\sum\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}} \quad (4)$$
$$+(1-p^{-1})^2\sum\left(\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}+\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}}\right)\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}} \quad (5)$$
$$+(1-p^{-1})^2\sum\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}} \quad (6)$$
$$+(1-p^{-1})^2(p-2)p^{-1}\sum\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \quad (7)$$
$$+(1-p^{-1})^2(p-2)p^{-1}\sum\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \quad (8)$$
$$+(1-p^{-1})^2(p-2)p^{-1}\sum\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} (9)$$
$$+(1-p^{-1})(p-2)(p-3)p^{-2}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \quad (10)$$
$$+(1-p^{-1})^2(p-2)p^{-1}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \quad (11)$$

$$+(1-p^{-1})^3\sum\left(\frac{p^{b+\alpha+\gamma+1}}{1-p^{b+\alpha+\gamma+1}}+\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}}\right)\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\times$$

$$\times\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{12}$$

$$+(1-p^{-1})^3\sum\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{b+c+\alpha+\beta+\gamma+2}}{1-p^{b+c+\alpha+\beta+\gamma+2}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{13}$$

$$+(1-p^{-1})^3\sum\frac{p^{-(\gamma+1)}}{1-p^{-(\gamma+1)}}\frac{p^{c+\beta+\alpha+1}}{1-p^{c+\beta+\alpha+1}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{14}$$

$$+(1-p^{-1})^3\sum\frac{p^{-(\alpha+1)}}{1-p^{-(\alpha+1)}}\frac{p^{-(\alpha+\beta+\gamma+2)}}{1-p^{-(\alpha+\beta+\gamma+2)}}\frac{p^{a+b+c+\alpha+\beta+\gamma+3}}{1-p^{a+b+c+\alpha+\beta+\gamma+3}} \tag{15}$$

□

Bibliography

[Ao] K. Aomoto, On the complex Selberg integral, *Quart. J. Math. Oxford Ser.* (2)38 (1987), no. 152, 385 - 399

[B] W.N.Bailey, Generalised hypergeometric series, Cambridge University Press, 1935.

[Ba] V.Bargmann, Irreducible unitary representations of the Lorentz group, *Ann. Math.* 48 (1947), 568 - 640.

[BR] J.Bernstein, A.Reznikov, Estimates of automorphic functions, *Moscow Math. J.*, Vol.4 (2004), 19 - 37.

[BS1] B.V.Binh, V.Schechtman, Remarks on a triple integral, *Moscow Math. J.*, Vol4. 2013, pp585-600, arXiv:1204.2117.

[BS2] B.V.Binh, V.Schechtman, Invariant functionals and Zamolodchikov's integral, *Funct. Analysis and Its Appl.* to appear , arXiv:1302.6046.

[BS3] B.V.Binh, V.Schechtman, Invariant triple functionals over $U_q\mathfrak{sl}_2$, arXiv:1305.0685, Preprint, Toulouse, 2013.

[C] L.Carlitz, Note on a q-identity, *Math. Scand.* **3** (1955), 281 - 282.

[CI] Cherednik. I, Double affine Hecke algebras and Macdonald's conjectures. *Ann. of Math.* (2) 141 (1995), no. 1, 191 - 216

[DF] Vl.S.Dotsenko, V.A.Fateev, Four - point correlation fucntions and the operator algebra in 2D conformal invariant theories with central charge $c \leq 1$, *Nuclear Physics B* 251[FS13] (1985) 691 - 734.

[Dy] F.J.Dyson, Statistical theory of energy levels of complex systems. I, *J. Math. Phys.* **3** (1962), 140 - 156.

[FM] B.Feigin, F.Malikov, Integral intertwining operators and complex powers of differential (q-difference) operators, pp. 15 - 63 in: Unconventional Lie algebras, *Adv. Soviet Math.* 17, Amer. Math. Soc., Providence, RI, 1993.

[GR] G.Gasper, M.Rahman, Basic hypergeometric series.

[GGPS] I.Gelfand, M.Graev, I.Pyatetsky-Shapiro, Representation theory and automorphic forms.

[HMW] D.Harlow, J.Maltz, E.Witten, Analytic continuation of Liouville theory, arXiv:1108.441.

[JL] H.Jacquet, R.Langlands, Automorphic forms on GL(2).

[K] D.E.Knuth, The art of computer programming, vol. 1, 3rd Edition.

[Ka] K.W.J.Kadell, A proof of Andrews' q-Dyson conjecture for $n = 4$, *Trans. AMS*, **290** (1985), 127 - 144.

[L] H.Y.Loke, Trilinear forms of gl2, *Pacific J. Math.*, 197 (2001), 119 - 144.

[La] R.Langlands, Euler products.

[Lu] G.Lusztig, Introduction to quantum groups.

[M] M.L.Mehta, Random matrices, 2nd Edition.

[Ma], Macdonald, I. G. , Some conjectures for root systems, *SIAM Journal on Mathematical Analysis* 13 (6), 988 - 1007.

[Mo] W.G.Morris, II, Constant term identities for finite and affine root systems: conjectures and theorems, PhD Thesis, University of Wisconsin - Madison, 1982.

[NSU] A.Nikiforov, S.Suslov, V.Uvarov, Classical ortothogonal polynomials of a discrete variable.

[NU] M.Nishizawa, K.Ueno, Integral solutions of q-difference equations of hypergeometric type with $|q| = 1$, arXiv:q-alg/9612014.

[O] E. Opdam, Some applications of hypergeometric shift operators, *Inv. Math.* 98 (1989), 1 - 18.

[Ok] A.Oksak, Trilinear Lorentz invariant forms, *Comm. Math. Phys.* 29 (1973), 189 - 217.

[R] S.Ruijsenaars, A generalized hypergeometric function satisfying four analytic difference equations of Askey-Wilson type, *Comm. Math. Phys.* 206 (1999), 639 - 690.

[S] A.Selberg, Bemerkninger om et multiplet integral, *Norsk Matematisk Tidsskrift*, **26** (1944), 71 - 78.

[T] E.Titchmarsh, The theory of functions, Oxford University Press, 1939.

[Ze] D.Zeilberger, A proof of the G_2 case of Macdonald's root system - Dyson conjecture, *SIAM J. Math. Anal.* **18** (1987), 880 - 883.

[ZZ] A.Zamolodchikov, Al.Zamolodchikov, Conformal bootstrap in Liouville field theory, *Nucl. Phys.* **B477** (1996), 577 - 605.