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Titre: *Investigation dans des systèmes abstraits avec entrées et sorties comme fonctions partielles de temps*

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Résumé

Cette thèse est consacrée à l'investigation de propriétés de systèmes où les entrées et sorties sont des fonctions partielles sur le domaine temporel. Dans nos travaux, des systèmes de ce genre sont mappés vers des abstractions appelées "blocs". La notion de bloc peut être considérée comme une extension spécifique des notions de systèmes avec entrées et sorties qui ont été étudiés, en plusieurs variantes, en théorie des systèmes. Les aspects essentiels des blocs sont leurs non-déterminisme; partialité des entrées - sorties; et le domaine temps-réel.

Les résultats originaux suivants ont été établis dans cette thèse:

(1) Les notions de non-anticipation faible et forte considérées dans les travaux de la théorie des systèmes de T. Windeknecht, M. Mesarovic, Y. Takahara pour différentes classes de systèmes ont été comparées et étendues aux blocs.

(2) Un théorème de représentation de blocs fortement non-anticipatifs a été prouvé. Il a été montré que de tels blocs peuvent être représentés par une classe de systèmes abstraits dynamiques appelés Systèmes Markoviens Non-déterministes Complets (NCMS). Ces derniers s'appuient sur la notion de système de solution introduit dans la Théorie des Processus de O. Hájek.

(3) Des critères généraux pour l'existence de couples d'entrées - sorties totaux de blocs fortement non-anticipatifs et l'existence de sorties totales pour des entrées totales d'un bloc fortement non-anticipatif.

Les résultats obtenus sont utiles pour la formalisation et l'analyse de langages de spécification basés sur des diagrammes de blocs, ainsi que pour des langages de développement pour des systèmes cyber-physiques et des systèmes de traitement de données temps-réel.

Mots-clés : système abstrait, temps continu, fonction partielle, representation, causalité, système cyber-physique, trajectoire globale, théorie de systèmes.

Discipline : Informatique

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Abstract

The thesis is devoted to investigation of properties of systems with inputs and outputs as partial functions on the real time domain. In our work systems of this kind are mapped to abstractions called blocks. The notion of a block can be considered as a specific extension of the notions of a system with inputs and outputs which were studied in various variants of mathematical systems theory. The main aspects of blocks are nondeterminism, partiality of inputs/outputs, real time domain.

The following novel results concerning blocks were obtained in the thesis:

(1) Weak and strong notions of nonanticipation considered in the works on mathematical systems theory by T. Windeknecht, M. Mesarovic, Y. Takahara for different classes of systems were extended to blocks and compared.

(2) A representation theorem for strongly nonanticipative blocks was proved. It was shown that such blocks can be represented using an introduced class of abstract dynamical systems called Nondeterministic Complete Markovian Systems (NCMS) which is based on the notion of a solution system introduced in the Theory of Processes by O. Hájek.

(3) General criteria for the existence of total input-output pairs of a strongly nonanticipative block and the existence of a total output for a given total input of a strongly nonanticipative block.

The obtained results are useful in formalization and analysis of block diagram-based specification and development languages for cyber-physical systems and real-time information processing systems.

Keywords : abstract system, continuous time, partial function, representation, causality, cyber-physical system, global trajectory, mathematical systems theory.

Scientific field : Informatics

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$\mathbb{N}$	the set of natural numbers $\{1,2,3,...\}$
$\mathbb{N}_0$	the set of non-negative integer numbers, i.e. $\mathbb{N}\cup\{0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of non-negative real numbers, i.e. $[0,+\infty)$
$[a,b],[a,b),(a,b],(a,b)$	bounded intervals in $\mathbb{R}$, if $a,b\in\mathbb{R}$, $a\le b$
$\sup A$	the least upper bound of A, if $A\subseteq\mathbb{R}$, $A\neq\varnothing$ (assuming $\sup A=+\infty$, if A is unbounded from above)
$\min A$	the least element of A, if $A\subseteq\mathbb{R}$ and A has the least element (otherwise, $\min A$ is undefined)
$\max A$	the greatest element of A, if $A\subseteq\mathbb{R}$ and A has the greatest element (otherwise, $\max A$ is undefined)
$\inf A$	the greatest lower bound of A, if $A\subseteq\mathbb{R}$, $A\neq\varnothing$ (assuming $\inf A=-\infty$, if A is unbounded from below)
$f:A\to B$	A total function from A to B
$f:A\tilde{\to}B$	A partial function from A to B
$f\mid_X$	restriction of a function f onto a set X
2^A	the power set of a set A
B^A	the set of all total functions from A to B
${}^AB$	the set of all partial functions from A to B
$f(x)\downarrow$	A function f is defined on the argument x
$f(x)\downarrow=y$	A function f is defined on x and $f(x)=y$ holds
$f(x)\uparrow$	A function f is undefined on the argument x
$dom(f)$	the domain of a function, i.e. $\{x\mid f(x)\downarrow\}$
$range(f)$	the range of a function, i.e. $\{y\mid\exists x\, f(x)\downarrow\wedge y=f(x)\}$

$dom(R)$	the domain of a binary relation, $\{x \mid \exists y\, (x,y) \in R\}$
$range(R)$	the range of a binary relation, $\{y \mid \exists x\, (x,y) \in R\}$
$f(x) \cong g(x)$	strong equality, i.e. $f(x)\downarrow$ if and only if $g(x)\downarrow$, and $f(x)\downarrow$ implies $f(x) = g(x)$
$f \circ g$	functional composition: $(f \circ g)(x) \cong f(g(x))$
$\lim_{\tau \to t-} f(\tau)$	left limit at $t \in \mathbb{R}$
$\lim_{\tau \to t+} f(\tau)$	right limit at $t \in \mathbb{R}$
$t \mapsto f(t,x)$	a function obtained from f by fixing the value of a parameter x, i.e. a function g_x such that $g_x(t) = f(t,x)$ for all t, where x is a fixed value
$X \mapsto y$	constant function defined on X which takes the value y, if X is a given set and y is a given value
$Bool$	the set of logical values $\{false, true\}$
$\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$	the logical operations of negation, disjunction, conjunction, implication, and equivalence respectively
T	time domain (coincides with $\mathbb{R}_+$ throughout the thesis)
$\mathcal{T}_0$	$\{\varnothing, T\} \cup \{[0,t) \mid t \in T \setminus \{0\}\} \cup \{[0,t] \mid t \in T\}$
$\mathfrak{T}$	the set of all intervals (connected sets) I in $\mathbb{R}$ such that $I \subseteq T$ and the cardinality of I is greater than one
W	A fixed non-empty set (signal values)
$[v_1 \mapsto w_1, v_2 \mapsto w_2, ...]$	A named set which maps names v_1, v_1, … to values w_1, w_2, … (Section 1.2.2.3)
$[]$	the empty named set (Section 1.2.2.3)
$sb_1 \preceq sb_2$	A signal bunch sb_1 is a prefix of a signal bunch sb_2 (Definition 1.3)
$sb[x]$	A signal which corresponds to a name x of a signal bunch sb (Definition 1.3)

$\perp$	the trivial signal bunch (Definition 1.3)
$\preceq^2$	partial order on pairs of signal bunches (Section 1.3)
$In(B)$	the sets of input names of a block B (Definition 1.4)
$Out(B)$	the set of output names of a block B (Definition 1.4)
$Op(B)$	the operation of a block B (Definition 1.4)
$IO(B)$	the I/O relation of a block B (Definition 1.6)
$IDS(B)$	the input data space of a block B (Definition 1.6)
$ODS(B)$	the output data space of a block B (Definition 1.6)
$B_1 \trianglelefteq B_2$	B_1 is a sub-block of a block B_2 (Definition 1.7)
$s_1 \sqsubseteq s_2$	s_1 is a subtrajectory of s_2 (Definition 2.2)
$s_1 \sqsubset s_2$	s_1 is a proper subtrajectory of s_2 (Definition 2.2)
$s_1 \doteq_A s_2$	functions s_1, s_2 coincide on a set A (Definition 2.5)
$s_1 \doteq_{t-} s_2$	functions s_1, s_2 coincide in a left neighborhood of t (Definition 2.5)
$s_1 \doteq_{t+} s_2$	functions s_1, s_2 coincide in a right neighborhood of t (Definition 2.5)
$ST(Q)$	the set of all pairs (s,t), where $s : A \to Q$ for some $A \in \mathfrak{T}$ and $t \in A$ (Section 2.3)
$LR(Q)$	the set of all pairs (l,r), where $l : ST(Q) \to Bool$ is a left-local predicate and $r : ST(Q) \to Bool$ is a right-local predicate (Section 2.3)
NCMS	Nondeterministic Complete Markovian System (Definition 2.4)
I/O NCMS	Input-output NCMS (Definition 2.9)
CPR	Closed under Proper Restrictions (Definition 2.1)
$In(\Sigma)$	the set of input names an I/O NCMS Σ (Section 2.5)
$IState(\Sigma)$	the set of internal states of an I/O NCMS Σ (Section 2.5)

$Out(\Sigma)$	the set of output names an I/O NCMS Σ (Section 2.5)
$in(q)$	the first component of a state q of an I/O NCMS, i.e. $in(q) = d_{in}$, if $q = (d_{in}, x, d_{out})$ (Section 2.5)
$istate(q)$	the second component of a state q of an I/O NCMS, i.e. $istate(q) = x$, if $q = (d_{in}, x, d_{out})$ (Section 2.5)
$out(q)$	the third component of a state q of an I/O NCMS, i.e. $out(q) = d_{out}$, if $q = (d_{in}, x, d_{out})$ (Section 2.5)
□	end of a proof or example

INTRODUCTION

Relevance of the topic of research. An abstract view of a computing system as a transformation of data, a function, or an input-output relation is rather common in computer science. In fact, this view is rooted in foundations of computing and is notable in the works of A. Turing and A. Church.

Nevertheless, a large amount of computing systems used today act not as pure data transformers, but as agents interacting with physical processes. Such systems are now frequently called cyber-physical systems [9, 62]. Examples include autonomous automotive systems, robotics, medical devices, energy conservation systems, etc. [104].

As was stressed in [61], an important aspect that cyber-physical systems must take into account is the passage of (physical) time. The actions of such systems must be properly timed. Besides, the computational aspect of a system must be understood and modeled in a close relation with physical processes. However, this is not taken into account when a system is viewed as an input-output relation on data.

One way to resolve this issue is to consider a system as an input-output relation on time-varying quantities (signals). A view of this kind is extensively used in signal processing and control theory [86, 64], but the kinds of mathematical models of systems usually considered in these fields (e.g. difference or differential equations [86]) do not provide high-level abstractions of processes that take place in cyber-physical systems [9]. In contrast, modeling and specification languages like Simulink [102], AADL [25], SysML [41] and others which have applications in the domain of cyber-physical systems employ high-level abstractions to deal with complexity of large systems.

High-level mathematical models that take into account the aspect of time can be found in the mathematical systems theory. During the second half of the XX century a large number of works that dealt with a mathematical theory of systems were published by L. Zadeh [117, 119, 118], R. Kalman [55], M. Arbib [6, 89], G. Klir [56], W. Wymore [115, 116], M. Mesarovic, [73, 74], B.P. Zeigler [121], A.I. Kuhtenko [59], N.P. Buslenko [16], V.M. Matrosov [71], and others [40, 87, 47, 111, 90, 66, 114, 44]. Many of these works were inspired and influenced by the General Systems Theory by L. Bertalanffy [107, 22, 108], Cybernetics introduced by N. Wiener [110], information theory introduced by C. Shannon [101], circuit theory in electrical engineering, automata theory, control theory. A historical account on the mutual influence between these fields is given in [99, 56]. In particular, the approach developed by M. Mesarovic and Y. Takahara [74] is based on a formalization of a system as a relation on objects. Other approaches such as those developed by M. Arbib [6, 89], W. Wymore [115], B.P. Zeigler [121] resulted from unification of the theory of systems described by differential equations and the automata theory.

Most of the mentioned works introduce a certain kind of abstraction of a system as an input-output relation on time-varying quantities (e.g. a general time system [74, Section 2.5], an external behavior of a dynamical system [55, Section 1.1], an oriented abstract object [119, Chapter 1, Paragraph 4], an I/O observational frame [121, Section 5.3]) and consider such a relation as a mathematical representation of the system's observable behavior. The most basic example is the definition of a Mesarovic time system [74] as a binary relation $S \subseteq I \times O$, where I and O are sets of input and output functions on a time domain T ($I \subseteq A^T$, $O \subseteq B^T$).

However, one aspect that is not sufficiently investigated in works on mathematical systems theory with regard to time systems is partiality of input and output signals as functions of time. For example, in a Mesarovic time system inputs and outputs are always total functions of time. In other theories, where analogous models are considered [119, 118, 121], partial inputs and outputs are allowed, but an additional assumption about the equality of the domains of the corresponding inputs and outputs is usually made.

However, the aspect of partiality of inputs and outputs becomes important, when a high-level input-output model of a real-world system is considered as an abstraction of a lower level mathematical model of this system.

Various concrete mathematical models of systems (e.g. those described by differential equations, hybrid automata [33], etc.) admit a situation, when the inputs of a system (e.g. input control signals), if there are any, are defined on the entire time domain, but the system's behavior (a solution of an equation, execution, etc.) and its outputs are not defined on the entire time domain. This can indicate a real phenomenon (e.g. termination or destruction of a real system) or inadequacy of a mathematical model [10].

An example of such a situation is the phenomenon of a finite time blow-up in differential equations [10, 31]. It is characterized by the unbounded growth of the value of one or several system variables during a bounded time interval. This can be illustrated by a (non-zero) solution $x(t) = 1/(c-t)$, $c = const$ of the differential equation $\frac{d}{dt}x(t) = x^2(t)$, for which $|x(t)| \to +\infty$, when $t \to c$. A survey of the respective results and applications can be found in [35, 10, 31, 63, 21, 13, 53].

Another kind of a situation when a mathematical model does not define a system's behavior on the entire time domain is a Zeno behavior [1, 122, 4, 103] of a hybrid (discrete-continuous) system [33, 42]. In this case, a hybrid system performs an infinite sequence of discrete steps during a bounded total time, but each step takes a non-zero time. A simple example in which this behavior arises is a hybrid automaton [42] which models a bouncing ball [122].

It should be noted that in either case, the problems of detection of finite time blow-ups or Zeno behaviors in a mathematical model, their physical interpretation, and if necessary, adjustment of a model to avoid such behaviors are non-trivial. For this reason, generally, one cannot assume that any available and useful model of a real-world system would be free of such behaviors.

This dictates that an input-output abstraction of a real system which is based on concrete mathematical models of this system must take into account the possibility of partial input and outputs.

The arguments mentioned above show that a study of abstract system models which take into account partiality of inputs and outputs as functions of time is an important topic of theoretical research.

Connection of the work with scientific programs, plans, topics. The work is a part of the scientific research conducted at the department of Theory and Technology of Programming of the Faculty of Cybernetics of Taras Shevchenko National University of Kyiv devoted to the following fundamental and applied themes: "Development of constructive mathematical formalisms for intelligent decision support systems, knowledge processing, and standardization of modern DBMS and CASE tools" (№ 0106U005856, 2006-2010), "Formal specifications and methods of development of reliable software systems" (№ 0111U007052, 2011-2015).

The work was supported in part by the project Verisync (ANR-10-BLAN-0310) of Institut de Recherche en Informatique de Toulouse (IRIT), France, devoted to development of methods for ensuring safety and reliability of embedded software.

Aim and objectives of the thesis. The *aim* of the work is formalization and analysis of systems that admit inputs and outputs which are partial functions of time. The main *objectives* of the research are listed below.

1) Give a definition of an abstract system which admits partial inputs and outputs.

2) Give an adequate definition of the notion of causality (nonanticipation) for abstract systems with partial inputs and outputs. Informally, this property means

that the current output values of a system do not depend on the future values of the inputs [86, 112, 28, 72].

3) Find a relation between causality (nonanticipation) and the existence of a representation in the form of a dynamical system for abstract systems with partial inputs and outputs. A connection between the existence of a state-space (dynamical system) representation of a system with inputs an outputs and nonanticipation was studied in the works on mathematical systems theory [112, 74, 56]. For example, in the theory by M. Mesarovic and Y. Takahara [74], a time system is causal if and only if it has a state space representation [74, Proposition 2.8]. The aim is to establish an analogous result for the systems considered in this work.

4) Obtain criteria that allow one to determine the existence of pairs of the corresponding total inputs and total outputs (total input-output pairs) and the existence of a total output for a given total input for abstract systems with partial inputs and outputs (here “total” means “defined on the entire time domain”).

The object of the research is a class of abstract systems with inputs and outputs which are partial functions on the real time domain.

The subjects of the research are the aspects of causality (nonanticipation), representation, and the existence of total input-output pairs of abstract systems with inputs and outputs which are partial functions on the real time domain.

*Research methods***.** The research is based on methodological principles of the composition-nominative approach [84] which aims to construct a hierarchy of program and system models of various abstraction levels and generality. This approach is a development of the compositional programming by V.N. Red’ko [92, 91] of Kyiv school of cybernetics which was inspired by the principle of composition by G. Frege and investigations of A.A. Lyapunov, Yu.I. Yanov, A.P. Ershov, V.M. Glushkov and others.

Scientific novelty of the obtained results. The following novel results were obtained in the thesis.

1) A new class of abstract systems with partially defined inputs and outputs called *blocks* was introduced. Basic properties of the systems of this class were studied.

2) Weak and strong notions of nonanticipation considered in [112, 74] were *improved*. These notions were extended to blocks and compared.

3) *For the first time* a representation theorem for strongly nonanticipative blocks was proved. It was shown that such blocks can be represented using an introduced class of abstract dynamical systems called initial Nondeterministic Complete Markovian Systems (NCMS) which is based on the notion of a solution system by O. Hájek [37].

4) *For the first time* general criteria for the existence of total input-output pairs of a strongly nonanticipative block (i.e. input-output pairs (i,o) such that both i and o are total functions of time) and the existence of a total output for a given total input of a strongly nonanticipative block were proved.

5) *For the first time* a general criterion for the existence of global trajectories of NCMS was obtained. This criterion expresses the existence of global trajectories in terms of conditions of the existence of locally defined trajectories of NCMS.

Theoretical and practical significance of the obtained results. The work is theoretical. The obtained results can be used for constructing high-level abstractions of cyber-physical, real-time information processing and other similar systems or their components.

The results of the thesis were used in the course “Formal methods of program development” at the Faculty of Cybernetics of Taras Shevchenko National University of Kyiv.

Personal contribution of the applicant. All results present in this thesis were obtained personally by the applicant. In the works published in co-authorship:

– in the article [17] the following sections belong to the applicant: “3. Uncertain Markov processes”, “4. Systems with uncertain structure”;

– in the article [81] the following sections belong to the applicant: "2. Nominative data", "5. Nominative equivalence", "7. Nominative stability of programs of $SCION_A$";
– in the article [83] the following sections belong to the applicant: "1. Nominative and complex-named data", "2. Properties of complex-named data", "3. Monotonicity of the operations on complex-named data";
– in the article [52] the following sections belong to the applicant: "2. Motivating example", "3 Possibility theory and Markov-like processes", "4. Simple systems with uncertain switching".

Approbation of the results of the thesis. The main results of this work were presented at the following scientific conferences and workshops:

1) XVI All-Ukrainian Scientific Conference "Modern problems in applied mathematics and informatics", October 8-9, 2009, Lviv, Ukraine.

2) The 6th International Conference "Theoretical and Applied Aspects of Program Systems Development", December 8-10, 2009, Kyiv, Ukraine.

3) International Scientific Conference "Simulation-2010", May 12-14, 2010, Kyiv, Ukraine.

4) The 7th International Scientific and Practical Conference on Programming UkrPROG'2010, May 25-27, 2010, Kiev, Ukraine.

5) International Scientific Conference on Computer Science and Engineering (CSE'2010), September 20-22, 2010, Košice, Slovakia.

6) The Third International Conference "Nonlinear Dynamics – 2010", September, 21-24, 2010, Kharkiv, Ukraine.

7) XV International Conference "Dynamical system modeling and stability investigation" (DSMSI-2011), May 24–27, 2011, Kyiv, Ukraine.

8) The 8th International Conference "Theoretical and Applied Aspects of Program Systems Development", September 19-23, 2011, Kyiv, Ukraine.

9) International Scientific Conference INFORMATICS'2011, November 16-18, 2011, Rožňava, Slovakia.

10) International Workshop on Algebraic, Logical, and Algorithmic Methods of System Modeling, Specification and Verification (SMSV), June 6-10, 2012, Kherson, Ukraine.

11) The 9th International Conference "Theoretical and Applied Aspects of Program Systems Development", December 3-7, 2012, Kyiv, Ukraine.

12) The 10th International Conference "Theoretical and Applied Aspects of Program Systems Development", May 25 – June 2, 2013, Yalta, Ukraine.

13) The 2nd International Workshop on Algebraic, Logical, and Algorithmic Methods of System Modeling, Specification and Verification (SMSV 2013), June 19-22, 2013, Kherson, Ukraine.

Publications. The main results of the thesis were reflected in 20 scientific publications, including 6 papers in Ukrainian professional scientific journals, 1 paper in a foreign (non-Ukrainian) scientific journal which is included in international academic journal databases (Inspec, Google Scholar, and others), 1 chapter in a book which belongs to a series included in international academic journal databases (Scopus, Google Scholar, Mathematical Reviews, and others), and 12 publications in conference proceedings.

Structure and size of the thesis. The thesis consists of an introduction, three chapters, conclusion, and a list of references. The main part of the thesis consists of 149 pages. The total size of the thesis is 172 pages, including 12 figures. The list of references contains 122 items (13 pages in total).

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CHAPTER 1
ABSTRACT SYSTEMS WITH PARTIAL INPUTS AND OUTPUTS

1.1 Overview

As we have noted in the introduction, a large amount of computing systems used today act as agents interacting with physical processes. They are now frequently called cyber-physical systems [9, 62]. Modeling and specification of such systems requires taking into account the passage of (physical) time, so they cannot be viewed as pure data transformations or pure input-output relations on data.

Let us give some quotes from the Cyber-Physical Systems (CPS) concept map [20] by S.S. Sunder of NIST (USA), E.A. Lee of UC Berkeley (USA) and others:

"CPS integrates the dynamics of the physical processes with those of the software and networking, providing abstractions and modeling, design, and analysis techniques for the integrated whole." [20]

"Classical models of computation in computer science, rooted in Turing-Church theories for non-concurrent systems, and in nondeterministic transition systems and process algebras for concurrent systems, do not handle temporal dynamics well." [20]

"A key CPS challenge is to conjoin the engineering abstractions for continuous dynamics (such as differential equations) with computer science abstractions (such as algorithms)." [20]

Besides, the following research needs in CPS are outlined in [9]: Abstraction and Architectures, Distributed Computations and Networked Control, and Verification and Validation. With regard to the first aspect (Abstraction and Architectures) it is stated that

“Innovative approaches to abstraction and architectures that enable seamless integration of control, communication, and computation must be developed for rapid design and deployment of CPS.” [9].

The mentioned challenges imply the importance of development of adequate system models of various levels of abstraction and generality with the emphasis on the temporal behavior of a system (which should not be restricted to a purely discrete or purely continuous evolution).

It should be noted that the essential role of mathematical modeling in systems engineering was already recognized in early works in that field [34, 39, 115, 16].

Although not aimed specifically at solving the mentioned challenges, many concrete models that combine a discrete and continuous behavior in some way were and are studied in control theory, theory of differential equations, and computer science, e.g. variable structure systems [11, 23, 106], impulsive differential equations [96, 60], differential equations with discontinuous right hand sides [26], switched systems [65], hybrid control systems [113, 105, 79, 14, 33], hybrid automata [46, 2, 42], phase transition systems [69], hybrid reactive modules [3], hybrid I/O automata [68]. It is reasonable to assume that on some level of abstraction models of these kinds would be useful in the context of CPS.

A more general treatment of models that can combine discrete and continuous behavior (including studies of model hierarchies) can be found in many variants of mathematical systems theory [119, 6, 55, 74, 89, 121, 87, 88, 16, 71, 90, 66, 114, 44]. With regard to the way of modeling system’s behavior, these variants of mathematical systems theory can be roughly classified into those which on the most abstract level consider a system as a “black box” which interacts with the environment and those which on the most abstract level describe the behavior of a system using the notion of state.

We consider the approaches of the former kind preferable. For example, consider the Architecture Analysis and Design Language (AADL) standardized by the Society of Automotive Engineers (SAE) [25, 24, 45] which is applicable to the domains related to cyber-physical systems. The following quote gives a general description of this langauge [24, p. 13]:

"The language employs formal modeling concepts for the description and analysis of application system architectures in terms of distinct components and their interactions. It includes abstractions of software, computational hardware, and system components for (a) specifying and analyzing real-time embedded and high dependability systems, complex systems of systems, and specialized performance capability systems and (b) mapping of software onto computational hardware elements."

The core AADL language concepts include a Component Type which defines interface elements and externally observable attributes of a component and a Component Implementation which defines a component's internal structure in terms of interconnections of subcomponents, subprogram call sequences, etc. One component type can have several corresponding implementations. A system is also viewed as a kind of (composite) component. A component type can be considered as a high-level ("black box") model and a component implementation as its refinement.

The description given above supports a view that approaches which on the abstract level consider a system as a "black box" are preferable (this still allows one to take into account the internal organization of a system on lower levels).

Let us consider several variants of mathematical systems theory of this kind.

– System theory by L. Zadeh [119, 118]. In this theory an *oriented abstract object* is defined as a family $\{R_{[t_0,t_1]}\}$ of sets indexed by segments of time $[t_0,t_1]$, where each set $R_{[t_0,t_1]}$ consists of pairs (u,y) (called input-output pairs) of functions of time u, y (called segments) defined on a common domain $[t_0,t_1]$. The family must satisfy a consistency condition: if a pair (u,y) belongs to $R_{[t_0,t_1]}$, then any pair $(u|_{[\tau_0,\tau_1]}, y|_{[\tau_0,\tau_1]})$ with $[\tau_0,\tau_1]\subseteq[t_0,t_1]$ also belongs to some member of this family. A system is defined as a combination of abstract objects which can be represented as a block diagram. Links between abstract objects mean equality constraints. Although it is noted that in general an input-output pair does not need to be extendable to (be a restriction of) some input-output pair defined on T (i.e. globally in time), this case does not receive much attention. Instead, a special class of oriented abstract objects (oriented objects) is introduced in which for each pair (u, y) there is an input-output pair (u_T, y_T) such that u, y are restrictions of u_T, y_T respectively and both u_T and y_T are defined on T. This class is then considered. In particular, a kind of state-state representation is introduced (following an informal principle that a state at a certain time is an information that is needed to determine the future behavior of a system [118]), the problems of identification and input-output analysis are considered, and further subclasses (linear systems) are studied.

– Abstract systems theory by M. Mesarovic and Y. Takahara [73, 74]. A system is defined on the abstract level as a relation on sets $S \subseteq V_1 \times V_2 \times \ldots \times V_n$ (meaning a relation among objects). As a special case, an input/output system ("terminal system") is obtained by partitioning $\{V_1,\ldots,V_n\}$ into inputs (causes) and outputs (effects). A special case of an input/output system, a time system, is defined as a relation $S \subseteq X \times Y$, where $X \subseteq A^T$ and $Y \subseteq B^T$ are called time objects and their elements are called abstract time functions (total functions of time). Besides other classes of systems, the class of time systems receives much attention [74, Chapter 5]. In particular, the topics of state-space representation, causality, feedback are considered.

– Systems theory by B.P. Zeigler [120, 121]. In this theory a hierarchy of system specifications is defined (it is noted in [121] that the levels of this hierarchy are close to epistemological levels defined by G. Klir [56] with the difference that Zeigler makes emphasis on time and dynamics). The initial level 0 (observational frame) of this hierarchy corresponds to knowledge of how to stimulate a system with inputs, which variables to measure, and how to observe them. The level 1 (I/O behavior) corresponds to knowledge of a set of time-indexed collections of input and output data (input/output pairs of a system, pairs of input and output trajectories). Subsequent levels include knowledge of the state and structure of a system. On the initial level a system is formalized as an I/O observation frame $IO = (T, X, Y)$, where T is a time domain (time base) and X, Y are input and output value sets. On the level 1 (I/O relation observation) a system is formalized as a tuple $IORO = (T, X, \Omega, Y, R)$, where T, X, Y are defined as in I/O observation frame, Ω is a set of allowable input segments, i.e. functions defined on a time interval which take values in X, and R is an I/O relation consisting of pairs of input segments and output segments (function from a time interval to Y) such that $dom(\omega) = dom(\rho)$ for all $(\omega, \rho) \in R$.

– **Behavioral approach to systems theory by J.C. Willems [111, 90].** A mathematical model is defined as a pair $(\mathbb{U}, \mathcal{B})$, where $\mathbb{U}$ is a set of outcomes and $\mathcal{B} \subseteq \mathbb{U}$ is a behavior. Informally, a model defines a subset of possible outcomes of a set of all outcomes. A (dynamical) system is defined as a triple $(\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is a time domain, $\mathbb{W}$ is a signal space, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is a behavior. Thus the behavior of system is a set of trajectories which have a common domain. An I/O dynamical system is defined as a tuple $(\mathbb{T}, \mathbb{U}, \mathbb{Y}, \mathcal{B})$, where $\mathbb{T}$ is a time domain, $\mathbb{U}$ and $\mathbb{Y}$ are input and output signal spaces, and $\mathcal{B} \subseteq (\mathbb{U} \times \mathbb{Y})^{\mathbb{T}}$ is a behavior. Thus the behavior is a set of total functions of time which determine input-output value pairs. Additional constraints are imposed that guarantee that the input is "free", i.e. is not restricted by the system, and that for any given input signal, any two corresponding output signals which have a common prefix (till some time t) coincide.

Some works in the field of computer science provide abstract models close to the models described above. Examples are given below.

– **An approach to functional specification of real-time and hybrid systems proposed in [79]**. In this work a notion of a stream processing function is defined as a function $f : (M_1^{\mathbb{R}_+})^m \to (M_2^{\mathbb{R}_+})^n$, where M_1, M_2 are sets of input and output values, i.e. a mapping from tuples of total signals to tuples of total signals. It is considered as a functional specification that describes the behavior of a component/system. The problems of composition and feedback are studied.

– **An approach to modeling timed concurrent systems proposed in [67, 72].** A signal is considered as a partial function $s : T \tilde{\rightarrow} V$ from a time domain to a set of values, or as a set of pairs (t,v), $t \in T$, $v \in V$ (called events) which is a graph of a partial function. The set of all such signals is denoted as $S(T,V)$ and is equipped with a set-valued distance-like function. A (partial) function F from $S(T,V)$ to $S(T,V)$ is considered as a model of a component/system. The question of feedback composition of such functions is studied for a special subclass of causal functions (strictly contracting functions [72]).

We see that the approaches described above provide abstract input-output models of a system. These models can capture a temporal behavior of a system and do not restrict it to purely discrete or purely continuous evolutions. At the same time, we see an aspect that is not sufficiently investigated. Namely, the case when the components of an input-output pair are partial functions of time which do not necessarily have equal domains. Among the approaches described above, this case is explicitly considered in [67, 72], but only for a special subclass of deterministic (functional) systems.

We will investigate not necessarily deterministic abstract systems with partial inputs and outputs (as functions of time) in this thesis. We will define a class of such systems which we call *blocks*. A block can be seen as a generalization of the notion of a Mesarovic time systems [74]. It maps a collection of input signals (*input signal bunch*) to one or more collections of output signals (*output signal bunches*). Then we will study the main aspects of blocks such as nonanticipation, representation, and existence of total input-output pairs.

We use the term block, because the notion of a system is already very overloaded and in order to stress that regardless of the way of its actual specification, a block is viewed abstractly as a black box which receives input signals and produces output signals.

1.2 Preliminaries

In this section we describe general methodological aspects of our work and general aspects of the mathematical framework which we use.

1.2.1 Methodological aspects

In the thesis we use some principles of the composition-nominative approach to program and system formalization [84]. The main principles of this approach are:

- *Development principle* (from abstract to concrete): notions should be introduced as a process of their development that starts from abstract understanding and proceeds to more concrete considerations.
- *Principle of priority of semantics over syntax*: program or system semantic and syntactical aspects should be first studied separately, then in their integrity in which semantic aspects prevail over syntactical ones.
- *Compositionality principle*: programs or systems can be constructed from simpler programs or systems with the help of special operations, called compositions, which form a kernel of semantics structures.
- *Nominativity principle*: nominative (naming) relations are the basic ones in constructing data.

In accordance with the Development principle, we start our study with an abstract view of an input-output system in Chapter 1, and later in Chapter 2 and Chapter 3 we consider such systems on a more concrete level. In accordance with the Principle of priority of semantics over syntax, we focus on semantic properties of blocks, although we define blocks in a way that admits development of the syntactic aspect. We use the principles of Compositionality and Nominativity in Chapter 1 when we define block compositions and represent input and output data of blocks as named sets [84, 82, 81].

1.2.2 Mathematical aspects

1.2.2.1 Binary relations and functions

We consider a binary relation as a subset of the Cartesian product of two sets and do not distinguish formally the notions of a binary relation R and the graph of R. We do not distinguish formally the notion of a function and a functional binary relation. However, generally, we do not apply set membership notation to functions ($(x, y) \in f$) and instead use a functional notation $(y = f(x))$.

The notation $f : A \to B$ (or $f : A \tilde{\to} B$) indicates that f is defined on a set A (or subset of A) and takes values in B. When we write that a function $f : A \tilde{\to} B$ is total or surjective, we mean that f is total on the set A specifically ($f(x)$ is defined for all $x \in A$) or, respectively, is onto B (for each $y \in B$ there exists $x \in dom(f)$ such that $f(x) = y$).

We will write $f(x)\downarrow$ to indicate that a function f is defined on a given argument x, and $f(x)\downarrow= y$ to indicate that f is defined on x and takes the value y on x. To indicate that f is undefined on x, we will write $f(x)\uparrow$. We will write $f(x) \cong g(x)$ to indicate that $f(x)\downarrow$ if and only if $g(x)\downarrow$, and $f(x)\downarrow$ implies $f(x) = g(x)$.

1.2.2.2 Multi-valued functions

A multi-valued function (multifunction) [84] associates one or more resulting values with each argument value.

Definition 1.1. A (total) multi-valued function from a set A to a set B (denoted as $f : A \to^{tm} B$) is a function $f : A \to 2^B \setminus \{\varnothing\}$.

An inclusion $y \in f(x)$ means that y is one of possible values of f on x.

1.2.2.3 Named sets

We will use the notion of a *named set* [84] to formalize an assignment of values to variable names.

Definition 1.2. A named set is a partial function $d : V \tilde{\to} W$ from a non-empty set of names V to a set of values W.

A named set can be considered as a special case of more general notions of *nominative data* and *complex-named data* [84, 82, 81, 83, 80, 48, 49, 75, 50] which reflect hierarchical data organizations. Operations on such data were described in [82]. We will use a special notation for the set of named sets: ${}^{V}W$ denotes the set of all named sets $d : V \tilde{\to} W$ (this only emphasises that V is a set of names).

An expression of the form $[n_1 \mapsto a_1, n_2 \mapsto a_2, ...]$ (where $n_1, n_2, ...$ are distinct names) denotes a named set d such that $dom(d) = \{n_1, n_2, ...\}$ and $d(n_i) = a_i$.

The unique named set with empty domain is called the *empty named set* and is denoted as $[]$.

For any named sets d_1, d_2 we write $d_1 \subseteq d_2$ (*named set inclusion*), if (the graph of) d_1 is a subset of (the graph of) d_2.

We give a special meaning to the operations of union $\cup$, intersection $\cap$ and difference $\setminus$ of named sets: if d_1, d_2 are named sets and the union of (graphs of) d_1 and d_2 is a named set d, then $d_1 \cup d_2 = d$. Otherwise (i.e. the union of graphs of d_1, d_2 is not functional), $d_1 \cup d_2$ is undefined.

The union of more than two named sets and the intersection of named sets are defined similarly.

1.2.2.4 Axiom of choice

We assume the axiom of choice [43] throughout the thesis and use it or equivalent statements (Zorn's lemma [43]) without special mentioning.

1.3 An abstract block

Informally, a block is an abstract model of a system which receives *input signals* and produces *output signals* (Fig. 1.1). The input signals can be thought of as certain time-varying characteristics (attributes) of the external environment of the system which are relevant for (the operation of) this system. Each instance of an input signal has a certain time domain on which it is defined (present).

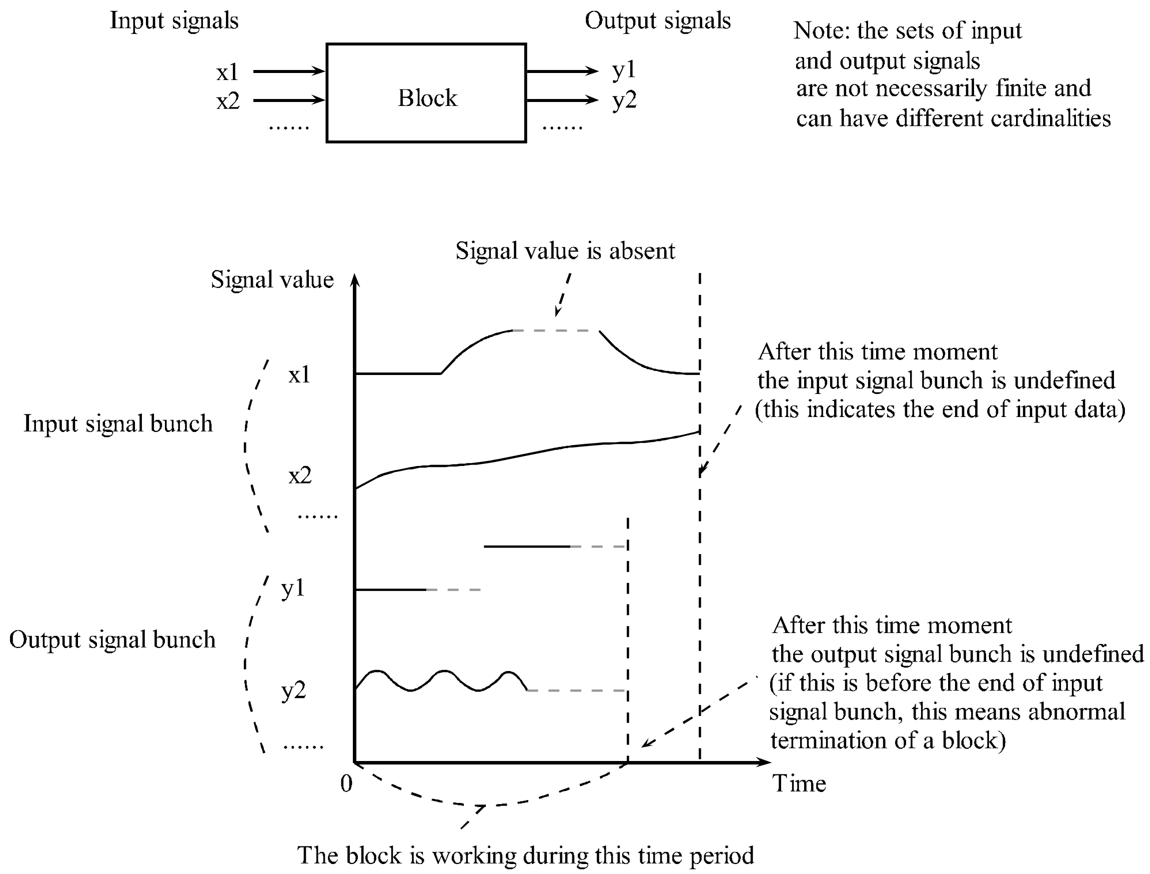

Fig. 1.1. An illustration of a block with input signals x_1, x_2, ... and output signals y_1, y_2, ... The plot displays example evolutions of input and output signals. Solid lines represent (present) signal values. Dashed horizontal segments indicate signal absence. Dashed vertical lines indicate the right boundaries of the domains of signal bunches.

An *input signal bunch*, or simply an *input* of the block, can be thought of as a collection of instances of input signals of the system. Each input signal bunch i has an associated domain of the existence ($dom(i)$) which is a superset of the union of the domains of signals contained in i. The domain of an input signal bunch can be thought of as a time span of the existence of the external environment of the system.

The output signals can be considered as effects (results) of the system's operation. An *output signal bunch*, or simply an *output* of the block, can be thought of as a collection of instances of output signals of the system. The output signals have domains of definition (presence) and each output signal bunch o has an associated domain of the existence ($dom(o)$) which is a superset of the union of the domains of signals contained in o. The domain of an output signal bunch can be thought of as a time span during which the system operates.

It is assumed that for an output signal bunch o which corresponds to a given input signal bunch i the inclusion $dom(o) \subseteq dom(i)$ holds (i.e. the system does not operate when the environment does not exist). However, in the general case, the presence of a given input signal at a given time does not imply the presence of a certain output signal at the same or any other time moment.

A block may operate nondeterministically, i.e. for one input signal bunch it may choose an output signal bunch from a set of possible variants. But for any input signal bunch there exists at least one corresponding output signal bunch (although the values of all signals in it may be absent at all times, which means that the block does not produce any output values).

Normally, a block processes the whole input signal bunch, and does or does not produce output values. However, in the general case, a block may not process the whole input signal bunch and may terminate at some time moment before its end. This is interpreted as an abnormal termination.

Let us give formal definitions. Let $T = \mathbb{R}_+$ denote a time scale. We will use the same time scale T throughout the thesis. We assume that T is equipped with the topology induced by the standard topology on $\mathbb{R}$ [77], i.e. the open sets in T have the form $T \cap \bigcup_{j \in J} I_j$, where $(I_j)_{j \in J}$ is an indexed family of open intervals in $\mathbb{R}$. Let us define the following class of sets:

$$\mathcal{T}_0 = \{\varnothing, T\} \cup \{[0,t) \mid t \in T \setminus \{0\}\} \cup \{[0,t] \mid t \in T\},$$

i.e. $\mathcal{T}_0$ is the set of all (bounded or unbounded) intervals with the left end 0 together with the empty set. Obviously, $\mathcal{T}_0$ is closed under arbitrary unions and intersections, and thus is a complete lattice of sets [93].

Let W be a fixed non-empty set of values.

Definition 1.3.

1) A signal is a partial function from T to W ($f : T \tilde{\to} W$).
2) A V-signal bunch (where V is a set of names) is a function $sb : T \tilde{\to} {}^V W$ such that $dom(sb) \in \mathcal{T}_0$. The set of all V-signal bunches is denoted as $Sb(V, W)$.
3) A signal bunch is a V-signal bunch for some V.
4) A signal bunch sb is called trivial, if $dom(sb) = \varnothing$, and is called total, if $dom(sb) = T$. The trivial signal bunch is denoted as $\perp$.
5) For a V-signal bunch sb, a signal corresponding to a name $x \in V$ is a function $sb[x] : dom(sb) \tilde{\to} W$ such that $sb[x](t) \cong (sb(t))(x)$ for all t.
6) A signal bunch sb_1 is a prefix of a signal bunch sb_2 (denoted as $sb_1 \preceq sb_2$), if $sb_1 = sb_2|_A$ for some $A \in \mathcal{T}_0$.

Note that a signal is not considered as a special case of a signal bunch.

Lemma 1.1. If $sb_1 = sb_2|_A$ for some signal bunches sb_1, sb_2 and $A \in \mathcal{T}_0$, then either $A = dom(sb_1)$, or $sb_1 = sb_2$.

Proof. Assume $sb_1 = sb_2|_A$. Then $dom(sb_1) = dom(sb_2) \cap A$. Because $dom(sb_2)$ and A belong to $\mathcal{T}_0$, they are comparable with respect to inclusion $\subseteq$, so we have either $dom(sb_1) = A$, or $dom(sb_1) = dom(sb_2)$. In the latter case we have $sb_1 = sb_2$, because $sb_1 = sb_2|_A$. □

Lemma 1.2. $\preceq$ is a partial order on V-signal bunches.

Proof. Reflexivity of $\preceq$ follows from the fact that $sb = sb|_{dom(sb)}$ and $dom(sb) \in \mathcal{T}_0$ for any V-signal bunch sb.

If $sb_1 \preceq sb_2$ and $sb_2 \preceq sb_1$, then $sb_1 = sb_2|_A$ and $sb_2 = sb_1|_{A'}$ for some $A, A' \in \mathcal{T}_0$, whence $dom(sb_1) \subseteq dom(sb_2)$ and $dom(sb_2) \subseteq dom(sb_1)$. Then $dom(sb_1) = dom(sb_2)$. Moreover, $dom(sb_2) = dom(sb_1) \subseteq A$, because $sb_1 = sb_2|_A$. Then $sb_1 = sb_2$. Thus $\preceq$ is antisymmetric.

If $sb_1 \preceq sb_2$, $sb_2 \preceq sb_3$, then $sb_1 = sb_2 |_A$, $sb_2 = sb_3 |_{A'}$ for some $A, A' \in \mathcal{T}_0$. Then $sb_1 \preceq sb_3$, because $sb_1 = sb_3 |_{A \cap A'}$ and $A \cap A' \in \mathcal{T}_0$. Thus $\preceq$ is transitive. □

Later we will need a generalized version of the prefix relation $\preceq$ for pairs of signal bunches.

For any signal bunches sb_1, sb_2, sb_1', sb_2' let denote $(sb_1, sb_2) \preceq^2 (sb_1', sb_2')$, if there exists $A \in \mathcal{T}_0$ such that $sb_1 = sb_1' |_A$ and $sb_2 = sb_2' |_A$.

It is easy to see that $\preceq^2$ is a partial order on pairs of V-signal bunches (note that this is not a product order [93]). The notation $\preceq^2$ is not to be confused with the composition of a binary relation with itself.

Now let us give the definition of a block. A block has a syntactic aspect (e.g. a description in some specification language) and a semantic aspect – a partial multi-valued function on signal bunches.

Definition 1.4.

1) A block is an object B (syntactic aspect) together with an associated set of input names $In(B)$, a set of output names $Out(B)$, and a total multi-valued function $Op(B): Sb(In(B), W) \to^{tm} Sb(Out(B), W)$ (operation, semantic aspect) such that the membership $o \in Op(B)(i)$ implies $dom(o) \subseteq dom(i)$.
2) Two blocks B_1, B_2 are semantically identical, if $In(B_1) = In(B_2)$, $Out(B_1) = Out(B_2)$, and $Op(B_1) = Op(B_2)$.

A membership $o \in Op(B)(i)$ means that o is a possible output of a block B on the input i. For each input signal bunch i there exists some output signal bunch o. The domain of o is a subset of the domain of i. A situation when o becomes undefined at some time t, but i is still defined at t we interpret as an error during the operation of the block B (the block cannot resume its operation after t).

If there is only one possible output signal bunch for each input signal bunch, we call a block deterministic.

Definition 1.5. A block B is deterministic, if $Op(B)(i)$ is a singleton set for each $In(B)$-signal bunch i.

Definition 1.6.

1) An input/output (I/O) pair of a block B is a pair of signal bunches (i,o) such that $o \in Op(B)(i)$. In such a pair i is called the input signal bunch and o is called the output signal bunch.
2) The I/O relation of a block B is the set of all I/O pairs of B, i.e. is the graph of the multifunction $Op(B)$. The I/O relation of B is denoted as $IO(B)$.
3) The input data space of a block B is the set $IDS(B) = {}^{In(B)}W$ and the output data space of B is the set $ODS(B) = {}^{Out(B)}W$.

As the inclusions $(i,o) \in IO(B)$ and $o \in Op(B)(i)$ are equivalent, we will use the one which is more convenient in a given context.

From Definition 1.6 we have that if $(i,o) \in IO(B)$, then i takes values in $IDS(B)$ and o takes values in $ODS(B)$.

Definition 1.7. A block B is a sub-block of a block B' (denoted as $B \trianglelefteq B'$), if $In(B) = In(B')$, $Out(B) = Out(B')$, and $IO(B) \subseteq IO(B')$.

Obviously, the sub-block relation $\trianglelefteq$ on blocks is reflexive and transitive, so it is a preorder. It can be interpreted as a kind of refinement of models, in particular, if a block B is considered as a model of a real system and B' is considered as a specification of requirements to the behavior of this system, the relation $B \trianglelefteq B'$ can be interpreted as a statement that the system satisfies the specification.

Definition 1.8. An I/O pair (i,o) of a block B is called

1) trivial, if $(i,o) = (\perp, \perp)$;
2) non-trivial, if $(i,o) \neq (\perp, \perp)$;
3) normal, $dom(i) = dom(o)$;
4) total, if $dom(i) = dom(o) = T$;

5) abnormal, if $dom(o) \subset dom(i)$.

From Definition 1.4 we immediately have that the trivial pair $(\bot, \bot)$ is an I/O pair of any block. This pair means that a block does not output any value when it has no available input. A normal I/O pair corresponds to the case when a block operates normally and processes the whole available input. An abnormal I/O pair corresponds to the case when a block terminates before the end of the available input signal bunch, which we interpret as an error during its operation.

The output signal bunch of an abnormal I/O pair is in some sense non-continuable. Formally, this is expressed by the following lemma.

Lemma 1.3. Let (i,o) be an abnormal I/O pair of some block and i', o' be signal bunches such that $(i,o) \preceq^2 (i',o')$. Then $o = o'$.

Proof. Assume that $(i,o) \preceq^2 (i',o')$. Then $i = i'|_A$ and $o = o'|_A$ for some $A \in \mathcal{T}_0$. By Lemma 1.1, either $A = dom(o)$, or $o = o'$. In the latter case the proposition holds, so consider the former case, i.e. $A = dom(o)$. We have $dom(i) \subseteq A = dom(o)$, because $i = i'|_A$. This contradicts the assumption that (i,o) is abnormal, because $dom(o) \subset dom(i)$. Thus $o = o'$. □

1.4 Composition of blocks

The usual ways in which input-output systems like blocks can be combined include the sequential and parallel composition. Other ways of combining such systems are also possible (e.g. a composition involving a feedback [74, 118]), but we do not consider them in the thesis. Formally, we define the compositions of blocks as follows.

Definition 1.9. If B_1, B_2 are blocks such that $Out(B_1) = In(B_2)$, then a block B is called a *sequential composition* of B_1 and B_2 (Fig. 1.2), if $In(B) = In(B_1)$, $Out(B) = Out(B_2)$, and $Op(B)(i) = \bigcup_{o \in Op(B_1)(i)} Op(B_2)(o)$.

Definition 1.10. If $(B_j)_{j\in J}$ is an indexed family of blocks, where $J \neq \varnothing$ is a set of indices, such that $In(B_j) \cap In(B_{j'}) = \varnothing$ and $Out(B_j) \cap Out(B_{j'}) = \varnothing$ for all $j, j' \in J$ such that $j \neq j'$, then a block B is called a *direct product* (or a *parallel composition with independent inputs*) of $(B_j)_{j\in J}$ (Fig. 1.3), if $In(B) = \bigcup_{j\in J} In(B_j)$, $Out(B) = \bigcup_{j\in J} Out(B_j)$, and $Op(B)(i)$ is the set of all $o \in Sb(Out(B), W)$ such that there exists an indexed family $(o_j)_{j\in J}$ such that

1) $o_j \in Op(B_j)(f_j \circ i)$ for all $j \in J$, where for each $j \in J$ $f_j : {}^{In(B)}W \to {}^{In(B_j)}W$ is a function such that $f_j(d) = d|_{In(B_j)}$ for all $d \in {}^{In(B)}W$;

2) $dom(o) = \bigcap_{j\in J} dom(o_j)$;

3) $o(t) = \bigcup_{j\in J} o_j(t)$ for each $t \in \bigcap_{j\in J} dom(o_j)$, where $\bigcup$ is the union of named sets (note that $\bigcup_{j\in J} o_j(t)$ is defined, because the sets $Out(B_j)$, $j \in J$ are disjoint).

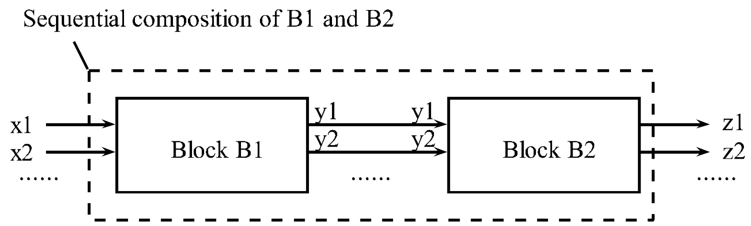

Fig. 1.2. An illustration of a sequential composition of blocks.

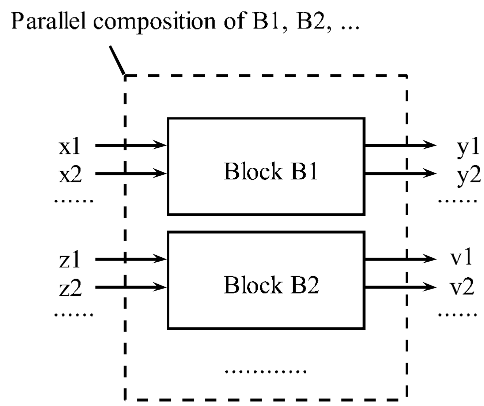

Fig. 1.3. An illustration of a direct product of blocks.

Proposition 1.1. Let B_1, B_2 be blocks such that $Out(B_1) = In(B_2)$. Then

1) a sequential composition of B_1 and B_2 exists;

2) if each of B and B' is a sequential composition of B_1 and B_2, then B and B' are semantically identical.

Proof. 1) For each $i \in Sb(In(B_1), W)$ denote $O(i) = \bigcup_{o \in Op(B_1)(i)} Op(B_2)(o)$.

Let $i \in Sb(In(B_1), W)$. Then $Op(B_1)(i) \neq \varnothing$, so there exists some $o' \in Op(B_1)(i)$, and because $Op(B_2)(o') \neq \varnothing$, we have $O(i) \neq \varnothing$.

Moreover, if $o \in O(i)$, then $o \in Op(B_2)(o')$ for some $o' \in Op(B_1)(i)$, so $dom(o) \subseteq dom(o') \subseteq dom(i)$. Thus $dom(o) \subseteq dom(i)$ for all $o \in O(i)$.

Let B be the triple $(In(B_1), Out(B_2), (O(i))_{i \in Sb(In(B_1), W)})$. Let us associate with B the sets $In(B) = In(B_1)$, $Out(B) = Out(B_2)$, and a function $Op(B): Sb(In(B), W) \rightarrow^{tm} Sb(Out(B), W)$ such that $Op(B)(i) = O(i)$ for all i. Then B is a block and is a sequential composition of B_1 and B_2 by Definition 1.9.

2) Follows immediately from Definition 1.9. □

Proposition 1.2. Let $(B_j)_{j \in J}$ be an indexed family of blocks, where $J \neq \varnothing$ is a set of indices, such that $In(B_j) \cap In(B_{j'}) = \varnothing$ and $Out(B_j) \cap Out(B_{j'}) = \varnothing$ for all $j, j' \in J$ such that $j \neq j'$. Then

1) a direct product of $(B_j)_{j \in J}$ exists;

2) if each of B and B' is a direct product of $(B_j)_{j \in J}$, then B and B' are semantically identical.

Proof. 1) Denote $IN = \bigcup_{j \in J} In(B_j)$, $OUT = \bigcup_{j \in J} Out(B_j)$. For each $j \in J$ $f_j : {}^{IN}W \rightarrow {}^{In(B_j)}W$ is a function such that $f_j(d) = d|_{In(B_j)}$ for all $d \in {}^{IN}W$.

For each $i \in Sb(IN, W)$ let $O(i)$ be the set of all $o \in Sb(OUT, W)$ such that there exists an indexed family $(o_j)_{j \in J}$ such that $o_j \in Op(B_j)(f_j \circ i)$ for all $j \in J$, $dom(o) = \bigcap_{j \in J} dom(o_j)$, and $o(t) = \bigcup_{j \in J} o_j(t)$ for each $t \in \bigcap_{j \in J} dom(o_j)$.

Let $i \in Sb(IN,W)$. Then for each $j \in J$, $f_j \circ i \in Sb(In(B_j),W)$ and there exists some $o_j \in Op(B_j)(f_j \circ i)$. Let $A = \bigcap_{j \in J} dom(o_j)$ and $o : A \to^{OUT} W$ be a function such that $o(t) = \bigcup_{j \in J} o_j(t)$ for all $t \in A$. Then $A \in \mathcal{T}_0$, so $o \in Sb(OUT,W)$. Thus $o \in O(i)$. So we have $O(i) \neq \varnothing$ for all $i \in Sb(IN,W)$.

Moreover, for each $o \in O(i)$ we have $dom(o) = \bigcap_{j \in J} dom(o_j)$ for some $(o_j)_{j \in J}$ such that $o_j \in Op(B_j)(f_j \circ i)$ for all $j \in J$, so because $J \neq \varnothing$, we have $dom(o) \subseteq \bigcap_{j \in J} dom(f_j \circ i) = dom(i)$.

Let B be the triple $(IN, OUT, (O(i))_{i \in Sb(IN,W)})$. Let us associate with B the sets $In(B) = IN$ and $Out(B) = OUT$, and $Op(B) : Sb(In(B),W) \to^{tm} Sb(Out(B),W)$ such that $Op(B)(i) = O(i)$ for all i. Then B is a block and is a direct product of $(B_j)_{j \in J}$ by Definition 1.10.

2) Follows immediately from Definition 1.10. □

1.5 Causality in input-output systems

In the case of input-output systems, causality (or nonanticipation) basically means the output does not depend on future values of the input. This notion frequently appears in mathematical systems theory [112, 74, 28, 66] and signal processing [86]. Systems that work in real (physical) time satisfy this condition. However, the details of a formal definition for different classes of systems vary.

In signal processing, electrical engineering, control theory the following definition is frequently used [86, 64]: if a system maps signals x_1, x_2 to signals y_1, y_2 and $x_1(t) = x_2(t)$ for all $t \leq t_0$, then $y_1(t) = y_2(t)$ for all $t \leq t_0$. It is presupposed that a system is deterministic. We can reformulate it for blocks as follows.

Definition 1.11. A deterministic block B is causal, if for each $i_1, i_2 \in Sb(In(B), W)$ and $t \in T$, if $i_1|_{[0,t]} = i_2|_{[0,t]}$, $o_1 \in Op(B)(i_1)$, and $o_2 \in Op(B)(i_2)$, then $o_1|_{[0,t]} = o_2|_{[0,t]}$.

The following lemma shows that in this definition one can consider signal bunches which coincide not only on a segment of the form $[0,t]$, but on any $A \in \mathcal{T}_0$.

Lemma 1.4. A deterministic block B is causal if and only if for all signal bunches i_1, i_2, o_1, o_2 and $A \in \mathcal{T}_0$ such that $o_1 \in Op(B)(i_1)$, $o_2 \in Op(B)(i_2)$, the equality $i_1|_A = i_2|_A$ implies $o_1|_A = o_2|_A$.

Proof. The "if" part of the statement follows immediately from Definition 1.11, because $[0,t] \in \mathcal{T}_0$. Consider the "only if" part of the statement. Assume that B is deterministic. Let $A \in \mathcal{T}_0$ and i_1, i_2, o_1, o_2 be signal bunches such that $o_1 \in Op(B)(i_1)$, $o_2 \in Op(B)(i_2)$, and $i_1|_A = i_2|_A$. If $A = [0,t]$ for some $t \in T$, then $o_1|_A = o_2|_A$ by Definition 1.11. Otherwise, $A = \bigcup_{0 \le t < \sup A} [0,t]$, whence $o_1|_A = o_2|_A$, because $o_1|_{[0,t]} = o_2|_{[0,t]}$ for each $t \in T$ by Definition 1.11. □

The condition for causality is illustrated in Fig. 1.4.

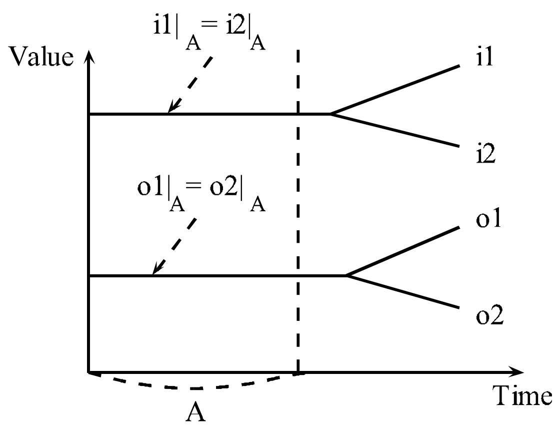

Fig. 1.4. An illustration of Lemma 1.4.
The graphs of signal bunches are depicted as solid lines.

Some works in the domain of mathematical systems theory [112, 74, 28, 66] extend this notion of causality to the nondeterministic case. However, there is no unified approach to an extension of such kind.

For example, in the work [112] a notion of a *non-anticipatory* system of the form $S \subseteq A^{\overline{T}} \times B^{\overline{T}}$, where $\overline{T}$ is a time domain (in [112] $\overline{T}$ is denoted as T, but we changed this symbol here to avoid a conflict with our notation). When $\overline{T} = (t_0, +\infty)$ with the standard ordering, where $t_0 \in \mathbb{R}$, this notion can be described as follows [112, Definition 2.4]:

- If S is a functional relation, S is called *non-anticipatory*, if for all $t \in \overline{T}$ and $x, x' \in dom(S)$, if $x|_{(t_0,t]} = x'|_{(t_0,t]}$, then $S(x)(t) = S(x')(t)$.
- Otherwise, S is called *non-anticipatory*, if there exists a set $F \subseteq range(S)^{dom(S)}$ of functional non-anticipatory systems such that $S = \bigcup\{(x, y) \mid (\exists f)\, f \in F \wedge y = f(x)\}$.

In the theory [74], a time system $S \subseteq X \times Y$, where X and Y are sets of (total) functions on a time domain is called *causal* [74, Chapter 3, Definition 2.2], if it has a causal initial response function, which means [74, Chapter 3, Definition 1.1 and Definition 2.1] a function $\rho_0 : C \times X \to Y$, where C is an arbitrary set, such that $(x, y) \in S$ if and only if $\exists c\, (y = \rho_0(c, x))$, and for any $x, x' \in X$ and c, t, if $x|_{\{t':t' \le t\}} = x'|_{\{t':t' \le t\}}$, then $\rho_0(c, x)|_{\{t':t' \le t\}} = \rho_0(c, x')|_{\{t':t' \le t\}}$. The idea here is essentially similar to the definition given in [112]. In the same work a number of related notions are defined. In particular, a notion of a *pre-causal system* $S \subseteq X \times Y$ is defined as follows [74, Chapter 3, Definition 2.4]: for any $x, x' \in X$ and t, if $x|_{\{t':t' \le t\}} = x'|_{\{t':t' \le t\}}$, then $S(x)|_{\{t':t' \le t\}} = S(x')|_{\{t':t' \le t\}}$, where $S(x) = \{y \mid (x, y) \in S\}$ and $S(x)|_A$ means $\{y|_A \mid y \in S(x)\}$. It it shown that the notions of a pre-causal and causal system are equivalent on a special class of time systems (the class of output-complete systems [74, Chapter 3, Definition 2.5 and Proposition 2.1]). Other notions defined in [74] include *strongly causal*, *past-determined*, and *strongly past-*

determined systems [74, Chapter 3]. They are shown to be stronger than the notion of a causal system.

In the theory [66] a notion of a *precausal* system is defined in the same way as the notion of a pre-causal system is defined in [74], but is used for a special class of linear time systems [66, p. 276].

In [28] the authors define another notion of a *causal* or *non-anticipatory* system. They consider a function $F:\Omega \to 2^{\Sigma}$ (i-o function), where Ω and Σ are sets of (total) input and output functions on a time domain $Time$ which is assumed to be $\mathbb{N}_0$, and define [28, Definition 7] that F is *causal* (or *non-anticipatory*), if for any f, g such that $f(t)=g(t)$ for all $t \le k$, $F(f)|_k = F(g)|_k$, where $F(f)|_k$ is the set of restrictions of $F(f)$ on time moments $t \le k$. The idea here is essentially the same as in pre-causal systems in the sense of [74].

Considering the definitions mentioned above, we can distinguish two recurring ideas: a non-anticipatory system in the sense of [112] (or causal system in the sense of [74]) and a pre-causal time system in the sense of [74]. We will apply both ideas to blocks. To avoid clash with terminology used in different works, we will introduce the notions of a *strongly nonanticipative* and *weakly nonanticipative* block on the basis of ideas of a non-anticipatory system in the sense of [112] and pre-causal system in the sense of [74].

Definition 1.12. A block B is strongly nonanticipative, if for each $(i,o) \in IO(B)$ there exists a deterministic causal sub-block $B' \trianglelefteq B$ such that $(i,o) \in IO(B')$.

Definition 1.13. A block B is weakly nonanticipative, if for each $A \in \mathcal{T}_0$ and $i_1, i_2 \in Sb(In(B), W)$, if $i_1|_A = i_2|_A$, then

$$\{o|_A \mid o \in Op(B)(i_1)\} = \{o|_A \mid o \in Op(B)(i_2)\}.$$

These notions can be considered as adaptations of the notions of causality/nonanticipation which were considered in [74] and [112] for certain classes of systems with total inputs and outputs to blocks.

1.6 Deterministic causal, weakly nonanticipative, and strongly nonanticipative blocks

Let us compare the introduced notions of nonanticipation. Firstly, note that the notions of a weakly and strongly nonanticipative block indeed can be considered as generalizations of the notion of a deterministic causal block (Definition 1.11).

Lemma 1.5. Let B be a deterministic block. Then:

1) B is causal if and only if B is weakly nonanticipative.
2) B is causal if and only if B is strongly nonanticipative.

Proof. The item 1 follows immediately from Lemma 1.4 and Definition 1.13, while the item 2 follows from the fact that $IO(B) \neq \varnothing$ and that $B' \trianglelefteq B$ if and only if $B' = B$ for a deterministic block B. □

By Definition 1.12, informally, the operation of a strongly nonanticipative block B can be interpreted as a two-step process:

1) before receiving input signals, the block B chooses an arbitrary deterministic causal sub-block $B' \trianglelefteq B$ (one can call this a response strategy);
2) the block B' receives the input signals of B and produces the corresponding output signals (response) which become the output signals of B.

Intuitively, it is clear that in this scheme at any time the block B does not need a knowledge of the future of its input signals in order produce the corresponding output signals.

Let us prove the following (alternative) characterization of weakly nonanticipative blocks which does not rely on comparison of sets of signal bunches.

Theorem 1.1. A block B is weakly nonanticipative if and only if the following conditions are satisfied:

1) if $(i,o) \in IO(B)$ and $(i',o') \preceq^2 (i,o)$, then $(i',o') \in IO(B)$;
2) if $o \in Op(B)(i)$ and $i \preceq i'$, then $(i,o) \preceq^2 (i',o')$ for some $o' \in Op(B)(i')$.

Proof.

Let us prove the "if" part of the theorem. Assume that the conditions 1 and 2 of the theorem are satisfied. Assume that $A \in \mathcal{T}_0$, $i_1, i_2 \in Sb(In(B), W)$, and $i_1|_A = i_2|_A$. Let $o \in Op(B)(i_1)$. Then from the condition 1 we have $o|_A \in Op(B)(i_1|_A)$, because $(i_1|_A, o|_A) \preceq^2 (i_1, o)$. Moreover, $i_1|_A \preceq i_2$, because $i_1|_A = i_2|_A$. Thus $(i_1|_A, o|_A) \preceq^2 (i_2, o')$ for some $o' \in Op(B)(i_2)$ by the condition 2.

If $(i_1|_A, o|_A)$ is an abnormal I/O pair of B, then $o|_A = o'$ by Lemma 1.3 and $o|_A = o'|_A$, whence $o|_A \in \{o''|_A \mid o'' \in Op(B)(i_2)\}$.

Now consider the case when $(i_1|_A, o|_A)$ is a normal I/O pair of B. Because $\{A, dom(o)\} \subseteq \mathcal{T}_0$, only the following two cases are possible:

- $A \subseteq dom(o)$. We have $o|_A = o'|_{A'}$ for some $A' \in \mathcal{T}_0$. By Lemma 1.1, either $o|_A = o'$, or $A' = dom(o|_A) = A$. In both cases, $o|_A = o'|_A$, whence $o|_A \in \{o''|_A \mid o'' \in Op(B)(i_2)\}$.
- $A \supset dom(o)$. Then $o|_A = o$ and $dom(i_1|_A) = dom(i_1) \cap A = dom(o)$. Then $dom(i_1) = dom(o)$, because $dom(i_1), A \in \mathcal{T}_0$. Then $i_2|_A = i_1|_A = i_1$. Because $A \neq dom(i_1)$, we have $i_1 = i_2$ by Lemma 1.1. Then $o|_A \in \{o''|_A \mid o'' \in Op(B)(i_2)\}$.

Thus we have proved that for any $A \in \mathcal{T}_0$, $i_1, i_2 \in Sb(In(B), W)$ such that $i_1|_A = i_2|_A$, if $o \in Op(B)(i_1)$, then $o|_A \in \{o''|_A \mid o'' \in Op(B)(i_2)\}$. This immediately implies that B is weakly nonanticipative by Definition 1.13.

Now let us prove the "only if" part of the theorem.

Assume that B is weakly nonanticipative.

Assume that $(i, o) \in IO(B)$ and $(i', o') \preceq^2 (i, o)$. Then $i' = i|_A$ and $o' = o|_A$ for some $A \in \mathcal{T}_0$. Then $i'|_A = (i|_A)|_A = i|_A$, whence

$$o' = o|_A \in \{o''|_A \mid o'' \in Op(B)(i)\} = \{o''|_A \mid o'' \in Op(B)(i')\}$$

by Definition 1.13. Then $o' = o''|_A$ for some $o'' \in Op(B)(i')$. Moreover, $dom(o'') \subseteq dom(i') \subseteq A$. Thus $o' = o''$ and $(i',o') \in IO(B)$.

Assume that $o \in Op(B)(i)$ and $i \preceq i'$. Then $i = i'|_A$ for some $A \in \mathcal{T}_0$. Then $i|_A = (i'|_A)|_A = i'|_A$, whence

$$o|_A \in \{o''|_A \mid o'' \in Op(B)(i)\} = \{o''|_A \mid o'' \in Op(B)(i')\}$$

by Definition 1.13. Then $o|_A = o'|_A$ for some $o' \in Op(B)(i')$. Moveover, $dom(o) \subseteq dom(i) \subseteq A$, whence $o = o|_A = o'|_A$. Thus $(i,o) \preceq^2 (i',o')$. □

The conditions of Theorem 1.1 are illustrated in Fig. 1.5 and Fig. 1.6 below.

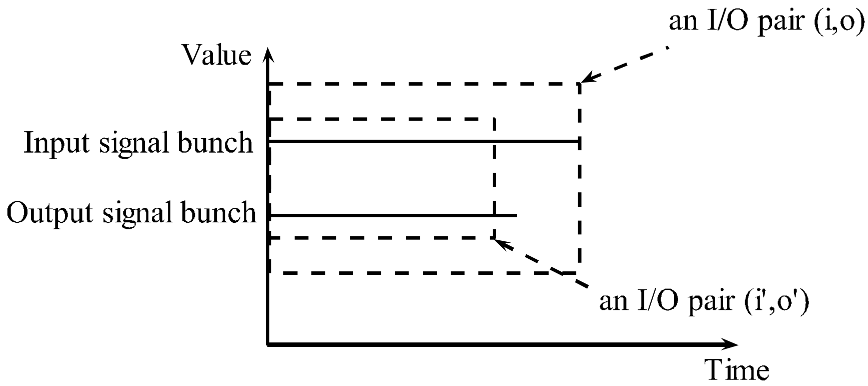

Fig. 1.5. An illustration of the condition 1 of Theorem 1.1.

Dashed rectangles enclose I/O pairs.

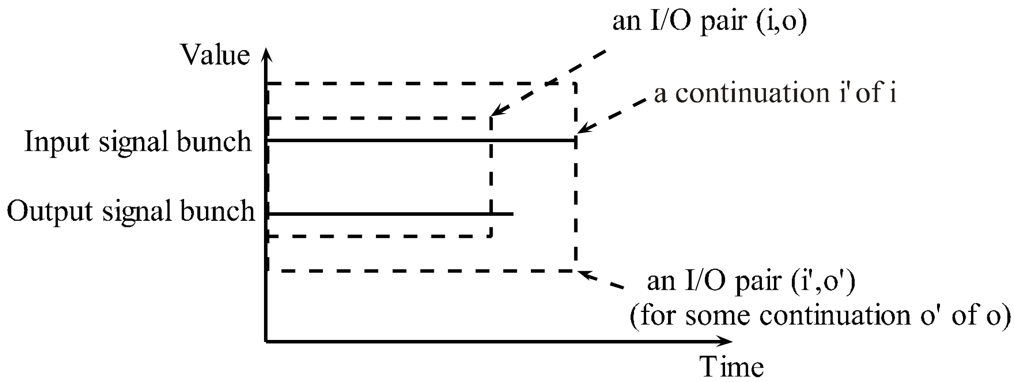

Fig. 1.6. An illustration of the condition 2 of Theorem 1.1.

Dashed rectangles enclose I/O pairs.

This theorem has the following corollary.

Lemma 1.6. Let $\mathcal{B}$ be a non-empty set of weakly nonanticipative blocks and B be a block such that $IO(B)=\bigcup_{B'\in\mathcal{B}} IO(B')$. Then B is weakly nonanticipative.

Proof. Let us check that the condition 1 of Theorem 1.1 holds for B. Let $(i,o)\in IO(B)$ and $(i',o')\preceq^2(i,o)$. Then $(i,o)\in IO(B')$ for some $B'\in\mathcal{B}$. Then $(i',o')\in IO(B')$ by Theorem 1.1 for the block B', because B' is weakly nonanticipative. Then $(i',o')\in IO(B)$.

Let us check that the condition 2 of Theorem 1.1 holds for B. Let $o\in Op(B)(i)$ and $i\preceq i'$. Then $(i,o)\in IO(B')$ for some $B'\in\mathcal{B}$. Then $(i,o)\preceq^2(i',o')$ for some $o'\in Op(B')(i')$ by Theorem 1.1 for the block B', because B' is weakly nonanticipative. Then $(i',o')\in IO(B')\subseteq IO(B)$ and $o'\in Op(B)(i')$.

We conclude that B is weakly nonanticipative by Theorem 1.1. □

Lemma 1.7. Let B be a weakly nonanticipative block and $(i,o)\in IO(B)$. Then there exists a weakly nonanticipative sub-block $B_0\trianglelefteq B$ such that $Op(B_0)(i)=\{o\}$.

Proof. Assume that B is weakly nonanticipative and $(i,o)\in IO(B)$. For each $i'\in Sb(In(B),W)$ let us define a set

$$O(i')=\{o'\in Op(B)(i')\mid \forall A\in\mathcal{T}_0\ (i'|_A=i|_A\Rightarrow o'|_A=o|_A)\}.$$

Let us show that $O(i')\neq\varnothing$ for all i'. Let $i'\in Sb(In(B),W)$ and $A^*=\bigcup\{A\in\mathcal{T}_0\mid i'|_A=i|_A\}$. Then $A^*\in\mathcal{T}_0$ and $i'|_{A^*}=i|_{A^*}$. Then from Definition 1.13 it follows that $o|_{A^*}\in\{o''|_{A^*}\mid o''\in Op(B)(i')\}$. Then there is some $o'\in Op(B)(i')$ such that $o'|_{A^*}=o|_{A^*}$. Then for each $A\in\mathcal{T}_0$, if $i'|_A=i|_A$, then $A\subseteq A^*$, whence $o'|_A=o|_A$. Thus $o'\in O(i')$ and $O(i')\neq\varnothing$.

Obviously, $O(i')\subseteq Op(B)(i')$ for each i', and because $O(i')\neq\varnothing$ for all i', we conclude that O is an operation of some sub-block of B, i.e. there is a sub-block $B_0 \trianglelefteq B$ with $Op(B_0)(i') = O(i')$ for all i'. Moreover, $Op(B_0)(i) = O(i) = \{o\}$ by the definition of O.

Let us show that B_0 is weakly nonanticipative. Indeed, let $i_1, i_2 \in Sb(In(B_0), W) = Sb(In(B), W)$, $A \in \mathcal{T}_0$, and $i_1|_A = i_2|_A$. For $j = 1,2$ let us denote

$$U_j = \{o'|_A \mid o' \in Op(B)(i_j)\}$$

$$V_j = \{o'|_A \mid o' \in Sb(Out(B), W), \forall A' \in \mathcal{T}_0\ (i_j|_{A'} = i|_{A'} \Rightarrow o'|_{A'} = o|_{A'})\}.$$

Then for $j = 1,2$ we have

$$\{o'|_A \mid o' \in Op(B_0)(i_j)\} = \{o'|_A \mid o' \in O(i_j)\} =$$

$$= \{o'|_A \mid o' \in Op(B)(i_j) \wedge \forall A' \in \mathcal{T}_0\ (i_j|_{A'} = i|_{A'} \Rightarrow o'|_{A'} = o|_{A'})\} =$$

$$= U_j \cap V_j.$$

We have $U_1 = U_2$ by Definition 1.13, because B is weakly nonanticipative.

Now let us check that $V_1 \subseteq V_2$. Assume that $o'|_A \in V_1$ for some o' such that $i_1|_{A'} = i|_{A'} \Rightarrow o'|_{A'} = o|_{A'}$ for each $A' \in \mathcal{T}_0$. Two cases are possible: $i_1|_A = i|_A$ and $i|_A \neq i|_A$. Consider the first case ($i_1|_A = i|_A$). Let o'' be an arbitrary element of $O(i_2)$ (which exists as we have shown above). Then $i_2|_{A'} = i|_{A'} \Rightarrow o''|_{A'} = o|_{A'}$ for each $A' \in \mathcal{T}_0$, whence $o''|_A \in V_2$. Then $o''|_A = o|_A = o'|_A$, because $i_2|_A = i_1|_A = i|_A$. Thus $o'|_A \in V_2$.

Consider the second case ($i_1|_A \neq i_2|_A$). Then $i_2|_A = i_1|_A \neq i|_A$ and the equality $i_2|_{A'} = i|_{A'}$ implies $A' \subset A$. Then $i_2|_{A'} = i|_{A'} \Rightarrow i_1|_{A'} = i|_{A'} \Rightarrow o'|_{A'} = o|_{A'}$ for each $A' \in \mathcal{T}_0$. Thus $o'|_A \in V_2$ by the definition of V_2.

So in both cases, $o'|_A \in V_2$, and because o' is arbitrary, we conclude that $V_1 \subseteq V_2$. By exchanging indices 1, 2 in the proof above we can show that $V_2 \subseteq V_1$. Thus $V_1 = V_2$. Then we have

$$\{o'|_A \mid o' \in Op(B_0)(i_1)\} = U_1 \cap V_1 = U_2 \cap V_2 = \{o'|_A \mid o' \in Op(B_0)(i_2)\}.$$

Thus B_0 is weakly nonanticipative by Definition 1.13. □

Although Definition 1.13 can seem to be a natural generalization of the notion of a causal deterministic block, it has certain more or less counter-intuitive consequences which we will describe below.

Example 1.1 (f-limit block). Assume that $W=\mathbb{R}$. We will call a block B an f-limit block, where $f:\mathbb{R}\to\mathbb{R}$ is a function, if $In(B)=\{x\}$ and $Out(B)=\{y\}$ for some names x,y, and for each $i\in Sb(In(B),\mathbb{R})$, $Op(B)(i)$ is defined as follows:

- if $dom(i[x])=T$ and $\lim_{t\to+\infty} i[x](t)$ exists and finite, then $Op(B)(i)$ is the set of all $\{y\}$-signal bunches o such that $dom(o)=dom(o[y])=T$ and $\lim\limits_{t\to+\infty} o[y](t)=f\left(\lim\limits_{t\to+\infty} i[x](t)\right)$;
- otherwise, $Op(B)(i)$ is the set of all $\{y\}$-signal bunches o such that $dom(o)=dom(o[y])=\bigcup\{A\in\mathcal{T}_0 \mid A\subseteq dom(i[x])\}$.

Obviously, in this definition $Op(B)(i)\neq\varnothing$ (because $\mathcal{T}_0$ is closed under unions) and $dom(o)\subseteq dom(i)$ for each $o\in Op(B)(i)$. This implies that an f-limit block exists for each $f:\mathbb{R}\to\mathbb{R}$. □

Informally, for a given real-valued input signal ($i[x]$) of infinite duration ($dom(i[x])=T$) which converges to some finite limit L as $t\to+\infty$, an f-limit block produces an output signal ($o[y]$) which converges to the value $f(L)$. If the input signal does not converge or has a bounded duration, the block outputs an arbitrary signal while the value of the input signal is defined.

The following proposition shows that f-limit blocks are weakly nonanticipative.

Proposition 1.3. Let B be an f-limit block for some $f:\mathbb{R}\to\mathbb{R}$. Then B is weakly nonanticipative.

Proof. Assume that $In(B)=\{x\}$ and $Out(B)=\{y\}$.

Let us check the condition 1 of Theorem 1.1 for B. Let $(i,o)\in IO(B)$ and $(i',o')\preceq^2(i,o)$. Then $i'=i|_{A'}$ and $o'=o|_{A'}$ for some $A'\in\mathcal{T}_0$.

If $dom(i'[x])=T$, then $A'=T$, whence $i'=i$ and $o'=o$ and $(i',o')\in IO(B)$.

Consider the case when $dom(i'[x]) \neq T$. The definition of an f-limit block implies that $dom(o) = dom(o[y]) = \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i[x])\}$. Then

$$dom(o') = dom(o) \cap A' = dom(o[y]) \cap A' = dom((o|_{A'})[y]) = dom(o'[y]).$$

Moreover,

$$dom(o') = dom(o) \cap A' = \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i[x])\} \cap A' =$$

$$= \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i[x]) \cap A'\},$$

because $\mathcal{T}_0$ is closed under intersections. We have $dom(i[x]) \cap A' = dom((i|_{A'})[x]) = dom(i'[x])$. From this and the equalities given above, $dom(o') = dom(o'[y]) = \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i'[x])\}$. Then $(i',o') \in IO(B)$, because $dom(i'[x]) \neq T$.

Thus the condition 1 of Theorem 1.1 holds for B.

Let us check the condition 2 of Theorem 1.1 for B. Assume that $o \in Op(B)(i)$ and $i \preceq i'$, where $i' \in Sb(\{x\}, \mathbb{R})$.

Note that we have $dom(o) = dom(o[y]) = \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i[x])\}$ from the definition of an f-limit block. Consider the following cases.

a) $dom(o) = T$. In this case, $dom(i) = T$ and $i' = i$. Then there exists $o' \in Op(B)(i')$ such that $(i,o) \preceq^2 (i',o')$ (one can choose $i' = i, o' = o$).

b) $dom(o) \neq T$, $dom(i'[x]) = T$, and a value $L = \lim_{t \to +\infty} i'[x](t)$ exists and finite. Let $o' \in Sb(\{y\}, \mathbb{R})$ be a signal bunch such that $dom(o') = T$, $o'(t) = o(t)$, if $t \in dom(o)$, and $o'(t) = [y \mapsto f(L) + 2^{-t}]$, if $t \in T \setminus dom(o)$. Then $dom(o'[y]) = dom(o[y]) \cup (T \setminus dom(o)) = T$. Moreover, $dom(o)$ is a bounded subset of T, because $dom(o) \neq T$, whence $\lim_{t \to +\infty} o'[y](t) = f(L)$ and thus $o' \in Op(B)(i')$ by the definition of an f-limit block. Because $dom(i'[x]) = T$, we have $dom(i[x]) = dom(i)$. Then because $o \in Op(B)(i)$, the definition of an f-limit block implies that $dom(o) = dom(i)$. Then $(i,o) \preceq^2 (i',o')$.

c) Either $dom(i'[x]) \neq T$, or $\lim_{t\to+\infty} i'[x](t)$ does not exist, or is infinite. Let us define $A'' = \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i'[x])\} \in \mathcal{T}_0$. Because $i \preceq i'$, we have $dom(i[x]) \subseteq dom(i'[x])$, whence $dom(o) \subseteq A''$. Let $o' \in Sb(\{y\}, \mathbb{R})$ be a signal bunch such that $dom(o') = A''$, $o'(t) = o(t)$, if $t \in dom(o)$, and $o'(t) = [y \mapsto 0]$, if $t \in A'' \setminus dom(o)$ (o' is correctly defined, because $dom(o) \subseteq A''$ and $A'' \in \mathcal{T}_0$). Then $dom(o') = dom(o'[y]) = A''$, because $dom(o) = dom(o[y])$. Then from the definition of f-limit block and of A'' it follows that $o' \in Op(B)(i')$. If $dom(o) = dom(i)$, then $o'|_{dom(o)} = o$ and $i'|_{dom(o)} = i$, whence $(i,o) \preceq^2 (i',o')$. Now let us assume that $dom(o) \subset dom(i)$. Then $dom(i[x]) \subset dom(i)$, because $o \in Op(B)(i)$. We have $i'|_{dom(i)} = i$, whence $dom(i[x]) = dom((i'|_{dom(i)})[x]) = dom(i'[x]) \cap dom(i)$. Then for each $A \in \mathcal{T}_0$ such that $A \subseteq dom(i'[x])$ we have $A \cap dom(i) \subseteq dom(i[x]) \subset dom(i)$, whence $A \subset dom(i)$ (because $A, dom(i) \in \mathcal{T}_0$), and thus $A \subseteq dom(i[x])$. Then $dom(o') = A'' \subseteq dom(o)$. This implies that $o' = o$. Then $i'|_{dom(i)} = i$ and $o'|_{dom(i)} = o$, whence $(i,o) \preceq^2 (i',o')$.

In all cases a)-c) there exists $o' \in Op(B)(i')$ such that $(i,o) \preceq^2 (i',o')$. Thus the condition 2 of Theorem 1.1 is satisfied. We conclude that B is weakly nonanticipative by Theorem 1.1. □

When a function f is discontinuous, this result can be seen as rather counter-intuitive, at least if weak nonanticipation is understood as a formalization of the idea that at any time a block does not know the future values of the input signals and cannot use them to determine the current output value.

For example, if f is the signum function (i.e. $f(0) = 0$, $f(x) = 1$, if $x > 0$, and $f(x) = -1$, if $x < 0$), then an f-limit block B outputs a signal $o[y]$ which converges to 1 (when $t \to +\infty$), when the input signal $i[x]$ converges to a positive number (when $t \to +\infty$). Moreover, it outputs a signal $o[y]$ which converges to 0, when the input signal $i[x]$ converges to 0. Then, intuitively, for each time t, the knowledge of $i[x]|_{[0,t]}$ (i.e. a prefix of the input signal till t) does not give B a useful information to distinguish between the cases $\lim_{t\to+\infty} i[x]=0$ and $\lim_{t\to+\infty} i[x]>0$, but the block still manages to output a signal which converges to different values in each of these cases.

The following proposition clarifies the characteristics of an f-limit block for a discontinuous f.

Proposition 1.4. Let $f:\mathbb{R}\to\mathbb{R}$ be a function and B be an f-limit block. Then B has a deterministic causal sub-block if and only if f is continuous.

Proof. Assume that $In(B)=\{x\}$ and $Out(B)=\{y\}$.

Let us prove the "if" part of the proposition.

Assume that f is continuous. Let B' be a block such that $In(B')=\{x\}$, $Out(B')=\{y\}$, and for each $i\in Sb(\{x\},\mathbb{R})$, $Op(B')(i)$ is defined as follows: $Op(B)(i)=\{o\}$, where o is the (unique) $\{y\}$-signal bunch such that $dom(o)=\bigcup\{A\in\mathcal{T}_0 \mid A\subseteq dom(i[x])\}$ and $o(t)=[y\mapsto f(i[x](t))]$ for each $t\in dom(o)$. Obviously, B' satisfies the definition of a block and is deterministic.

Let us check that B' is causal. Let $t\in T$, $o_1\in Op(B)(i_1)$, $o_2\in Op(B)(i_2)$, $i_1|_{[0,t]}=i_2|_{[0,t]}$. Then

$$dom(i_1[x])\cap[0,t]=dom((i_1|_{[0,t]})[x])=dom((i_2|_{[0,t]})[x])=dom(i_2[x])\cap[0,t].$$

Then the following holds:

$$dom(o_1|_{[0,t]})=dom(o_1)\cap[0,t]=\bigcup\{A\in\mathcal{T}_0 \mid A\subseteq dom(i_1[x])\}\cap[0,t]=$$
$$=\bigcup\{A\in\mathcal{T}_0 \mid A\subseteq dom(i_1[x])\cap[0,t]\}=$$
$$=\bigcup\{A\in\mathcal{T}_0 \mid A\subseteq dom(i_2[x])\cap[0,t]\}=$$
$$=\bigcup\{A\in\mathcal{T}_0 \mid A\subseteq dom(i_2[x])\}\cap[0,t]=dom(o_2)\cap[0,t]=dom(o_2|_{[0,t]}).$$

Then for each $t\in dom(o_1|_{[0,t]})\subseteq dom(i_1|_{[0,t]})=dom(i_2|_{[0,t]})$, we have

$$o_1(t)=[y\mapsto f(i_1[x](t))]=[y\mapsto f(i_2[x](t))]=o_2(t).$$

Thus $o_1|_{[0,t]}=o_2|_{[0,t]}$ and B is causal.

Let us show that $B' \trianglelefteq B$. Let $i \in Sb(In(B'),\mathbb{R})$ and $o \in Op(B')(i)$. If $dom(i[x]) = T$ and $\lim_{t\to+\infty} i[x](t)$ exists and is finite, then $dom(o) = dom(o[y]) = T$ and $o[y](t) = f(i[x](t))$ for each $t \in T$, whence

$$\lim_{t\to+\infty} o[y](t) = f\left(\lim_{t\to+\infty} i[x](t)\right)$$

by continuity of f. Otherwise,

$$dom(o) = dom(o[y]) = \bigcup\{A \in \mathcal{T}_0 \mid A \subseteq dom(i[x])\}.$$

Thus $o \in Op(B)(i)$, because B is an f-limit block. Then $B' \trianglelefteq B$. Thus B' is a deterministic causal sub-block of B.

Let us prove the "only if" part of the proposition.

Assume that B has a deterministic causal sub-block B'. Let $a \in \mathbb{R}$ and $a_k \in \mathbb{R}$, $k = 1,2,...$ be a sequence such that $\lim_{k\to\infty} a_k = a$.

Let us show that $\lim_{k\to\infty} f(a_k) = f(a)$.

Let us define sequences $i_k \in Sb(\{y\},\mathbb{R})$, $o_k \in Sb(\{y\},\mathbb{R})$, and $t_k \in T$, $k = 1,2,...$ by induction as follows.

Let $i_1(t) = [x \mapsto a_1]$ for all $t \in T$, o_1 be a unique member of $Op(B')(i_1)$, and $t_1 = 0$. If $i_1, i_2, ..., i_k$ are already defined, let $i_{k+1}(t) = i_k(t)$, if $t \in [0, t_k]$ and $i_{k+1}(t) = [x \mapsto a_{k+1}]$, if $t \in T \setminus [0, t_k]$. Let o_{k+1} be a unique member of $Op(B')(i_{k+1})$. Because B' is a sub-block of an f-limit block, $dom(o_{k+1}) = dom(o_{k+1}[y]) = T$ and

$$\lim_{t\to+\infty} o_{k+1}[y](t) = f(\lim_{t\to+\infty} i_{k+1}[x](t)) = f(a_{k+1}).$$

Then let

$$t_{k+1} = 1 + \max\{t_k, \inf\{\tau \in T \mid$$

$$\sup\{|o_{k+1}[y](t) - f(a_{k+1})| \mid t \geq \tau\} \leq \frac{1}{k+1}\}\}.$$

We have defined sequences i_k, o_k, t_k for $k = 1,2,...$. The sequence t_k, $k = 1,2,...$ is a strictly increasing and unbounded from above and $t_1 = 0$.

Let i be a $\{x\}$-signal bunch such that $dom(i)=T$, $i(t_1)=i_1(t_1)$, and $i(t)=i_{k+1}(t)$, if $t\in(t_k,t_{k+1}]$, $k\in\mathbb{N}$, and o be a (unique) member of $Op(B')(i)$. We have $i_{k+1}[x](t)=a_{k+1}$ for all $k=1,2,...$ and $t>t_k$. Then $i[x](t)\in\{a_{k+1},a_{k+2},...\}$ for all $k\in\mathbb{N}$ and $t>t_k$. For each $\varepsilon>0$ there exists $k\in\mathbb{N}$ such that $|a_{k'}-a|<\varepsilon$ for all $k'\geq k$, whence $|i[x](t)-a|<\varepsilon$ for all $t>t_k$. Thus $\lim_{t\to+\infty}i[x](t)=a$. Then $dom(o)=dom(o[y])=T$ and $\lim_{t\to+\infty}o[y](t)=f(a)$, because B' is a sub-block of an f-limit block.

On the other hand, $i_{k+1}|_{[0,t_k]}=i_k|_{[0,t_k]}$ for all $k\in\mathbb{N}$. Because t_k is an increasing sequence, we have $i_{k'}|_{[0,t_k]}=i_k|_{[0,t_k]}$ for all k and $k'\geq k$. Besides, $i|_{(t_k,t_{k+1}]}=i_{k+1}|_{(t_k,t_{k+1}]}$ for all $k\in\mathbb{N}$, whence $i|_{(t_k,t_{k+1}]}=i_{k'}|_{(t_k,t_{k+1}]}$ for all $k'\geq k+1$. Also, $i_k(t_1)=i_1(t_1)$ for all $k\in\mathbb{N}$. Then

$$i|_{[0,t_k]}=i|_{\{t_1\}\cup(t_1,t_2]\cup...\cup(t_{k-1},t_k]}=i_k|_{[0,t_k]} \text{ for all } k=2,3,...,$$

whence $o|_{[0,t_k]}=o_k|_{[0,t_k]}$, because B' is causal. Then $o(t_k)=o_k(t_k)$ for all $k=2,3,...$, and from the definition of t_k we have

$$|o[y](t_k)-f(a_k)|=|o_k[y](t_k)-f(a_k)|\leq\frac{1}{k} \text{ for all } k=2,3,....$$

This implies that $\lim_{k\to\infty}f(a_k)=f(a)$, because $\lim_{t\to+\infty}o[y](t)=f(a)$. We conclude that f is sequentially continuous [43] and thus is continuous. □

This proposition implies that for a discontinuous function f, an f-limit block has no deterministic causal sub-block.

Now we can show the following relation between the notions of a weakly and strongly nonanticipative block.

Theorem 1.2 (About strongly nonanticipative block).

1) Each strongly nonanticipative block is weakly nonanticipative.
2) There exists a weakly nonanticipative block which is not strongly nonanticipative.

Proof.

1) Assume that B is strongly nonanticipative. Let $\mathbf{R}$ be the set of all relations $R \subseteq IO(B)$ such that R is an I/O relation of a weakly nonanticipative block. For each $R \in \mathbf{R}$ let us define a block B_R such that $IO(B_R) = R$, $In(B_R) = In(B)$, $Out(B_R) = Out(B)$. Let $\mathcal{B} = \{B_R \mid R \in \mathbf{R}\}$. Then each element of $\mathcal{B}$ is weakly nonanticipative. From Definition 1.12 and Lemma 1.5 we have $IO(B) \subseteq \bigcup \mathbf{R} = \bigcup_{B' \in \mathcal{B}} IO(B')$. On the other hand, $IO(B') \subseteq IO(B)$ for any $B' \in \mathbf{R}$, so $IO(B) = \bigcup_{B' \in \mathcal{B}} IO(B')$. Then B is weakly nonanticipative by Lemma 1.6.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous function and B be an f-limit block. By Proposition 1.3, B is weakly nonanticipative. By Proposition 1.4, B has no causal deterministic sub-blocks. Because $IO(B) \neq \varnothing$, B is not strongly nonanticipative. □

Consider some examples. Firstly, consider an example of a strongly nonanticipative block. Let u, y be names and $W = \mathbb{R}$ (W is the set of signal values).

Example 1.2. Let B be a block such that $In(B) = \{u\}$, $Out(B) = \{y\}$, and for each i, $Op(B)(i) = \{o_1(i), o_2(i)\}$, where $o_1(i), o_2(i) \in Sb(Out(B), W)$ are signal bunches such that $dom(o_1(i)) = dom(o_2(i)) = dom(i)$ and for $j = 1,2$ we have

– $o_j(i)(t) = [y \mapsto j \cdot i[u](t)]$, if $t \in dom(i)$ and $i[u](t) \downarrow$;

– $o_j(i)(t) = []$, if $t \in dom(i)$ and $i[u](t) \uparrow$.

Informally, this means that B is a “gain” block with a slope j which is either 1 or 2 during the whole duration of the block’s operation.

Obviously, B satisfies Definition 1.4, i.e. is indeed a block.

Let us show that B is strongly nonanticipative. For $j = 1,2$ let $B_j \trianglelefteq B$ be a sub-block such that $Op(B_j)(i) = \{o_j(i)\}$ for all $i \in Sb(In(B), W)$ (i.e. B_1 always selects $o_1(i)$ from $Op(B)(i)$ and B_2 always selects $o_2(i)$).

The blocks B_1, B_2 are deterministic. Let us check that they are causal.

Let $j \in \{1,2\}$. Let $i,i' \in Sb(In(B_j),W)$, $\tau \in T$, $i|_{[0,\tau]} = i'|_{[0,\tau]}$, $o \in Op(B_j)(i)$, and $o' \in Op(B)(i')$. Then $o(t) = [y \mapsto j \cdot i[u](t)]$ for all $t \in dom(i[u])$, $o(t) = []$ for all $t \in dom(i) \setminus dom(i[u])$, and $o(t)\uparrow$, for all $t \notin dom(i)$. Similarly, we have $o'(t) = [y \mapsto j \cdot i'[u](t)]$ for $t \in dom(i'[u])$, $o'(t) = []$ for all $t \in dom(i') \setminus dom(i'[u])$, and $o(t)\uparrow$, for all $t \notin dom(i')$. Then $dom(i) \cap [0,\tau] = dom(i') \cap [0,\tau]$ and $i[u]|_{[0,\tau]} = i'[u]|_{[0,\tau]}$, because $i|_{[0,\tau]} = i'|_{[0,\tau]}$. Then we conclude that $o|_{[0,\tau]} = o'|_{[0,\tau]}$. Thus B_j is causal.

Obviously, each I/O pair $(i,o) \in IO(B)$ belongs either to $IO(B_1)$, or to $IO(B_2)$, so B is strongly nonanticipative by Definition 1.12. □

Above we have given an example of a weakly nonanticipative block which is not strongly nonanticipative (f-limit block for a discontinuous f). Now consider an example of a block which is not weakly nonanticipative.

Example 1.3. Let B' be a block such that $In(B') = \{u\}$, $Out(B') = \{y\}$, and the operation is defined as follows:

- $Op(B')(i) = \{o_1\}$, where $dom(o_1) = dom(i)$ and $o_1(t) = [y \mapsto 1]$ for all $t \in dom(i)$, if $dom(i[u]) = T$;
- $Op(B')(i) = \{o_2\}$, where $dom(o_2) = dom(i)$ and $o_2(t) = [y \mapsto 0]$ for all $t \in dom(i)$, otherwise.

Informally, the block B' decides whether its input signal u is total.

It is easy to see that B' indeed satisfies Definition 1.4 (i.e. is a block), but the condition 1 of Theorem 1.1 is not satisfied, because $(i,o) \in IO(B')$, where $i(t) = [u \mapsto 0]$ for all $t \in T$, $o(t) = [y \mapsto 1]$ for all $t \in T$, and $(i|_{[0,1]}, o|_{[0,1]}) \preceq^2 (i,o)$, but $(i|_{[0,1]}, o|_{[0,1]}) \notin IO(B')$. So B' is not weakly nonanticipative.

Informally, the reason is that at each time t the current value of y depends on the entire input signal. □

1.7 Classes of blocks

The classes of blocks that we have introduced are illustrated in Fig. 1.7. Arguably, the notion of a strongly nonanticipative block conforms to the informal idea of nonanticipation as non-dependence of the current output signal values of the block on the future of the input. However, for weakly nonanticipative blocks this is not so clear and is debatable, because of Proposition 1.3 and Proposition 1.4. We will consider the notion of a strongly nonanticipative block as possibly not the most general, but adequate generalization of the notion of a causal block to the nondeterministic setting and investigate such blocks in the next chapters.

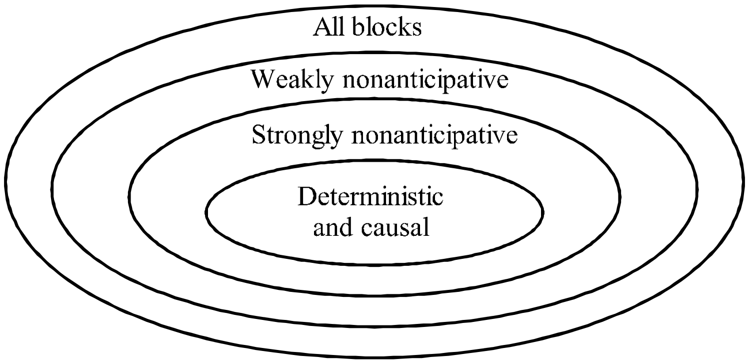

Fig. 1.7. Classes of blocks.

1.8 Conclusions from the chapter

We have introduced the notion of a block as an input-output system which maps an input signal bunch to one or more output signal bunches. Input and output signal bunches are not necessarily total functions of time. We have introduced two notions of nonanticipation for blocks (weakly nonanticipative and strongly nonanticipative blocks) on the basis of similar notions that appear in the literature for different kinds of input-output systems [112, 74, 28, 66] and compared them.

CHAPTER 2
REPRESENTATION OF STRONGLY NONANTICIPATIVE BLOCKS

2.1 Overview

Typically, even in the variants of mathematical systems theory which on the abstract level consider a system as a "black box", e.g. [119, 74, 121, 111], the concept of a system's state is still introduced and a link between black box and state-based models is established.

– **In the work [118] by L. Zadeh** the concept of state is discussed and the following description is given:

"Roughly, a state of a system at any given time is the information needed to determine the behavior of the system from that time on.".

Abstractly, a system is represented by a family of pairs of time functions $\mathfrak{A} = \{(u_{[t_0,t_1]}, y_{[t_0,t_1]})\}, t_0, t_1 \in (-\infty, +\infty)$, where $u_{[t_0,t_1]}$ and $y_{[t_0,t_1]}$ are an input and output defined on a time segment $[t_0, t_1]$ (there may be more than one pair defined on a given time segment). It is assumed that the family is closed under segmentation (CUS), i.e. a restriction of an input-output pair which belongs to $\mathfrak{A}$ and is defined on $[t_0, t_1]$ onto a sub-segment of $[t_0, t_1]$ still belongs to $\mathcal{A}$.

Formally, a state is defined for such a system using the following construction. A *bundle of input-output pairs* is a subset of the set $\mathfrak{A}(t_0)$ of all pairs from $\mathfrak{A}$ defined on a segment of the form $[t_0, t]$, $t \geq t_0$ for a fixed t_0. Members of a chosen indexed family of bundles which satisfies several special conditions (covering, closure under truncation, uniqueness, continuation [118]) are called *aggregates* and their indices (tags) are called *states* of $\mathfrak{A}$ at time t_0. This construction is used to represent a system as an input-output-state relation of the form $y = \overline{A}(x(t_0);u)$, where $x(t_0)$ is an (initial) state, $u \in dom(\mathfrak{A}_{x(t_0)}(t_0))$, $y \in range(\mathfrak{A}_{x(t_0)}(t_0))$, $\mathfrak{A}_{x(t_0)}(t_0)$ is the aggregate corresponding to the index (tag) $x(t_0)$. This equality expresses a functional dependence ($\overline{A}$) of the future output (y) of the system on the current state ($x(t_0)$) and the future input (u). Other questions related to the notion of state are also discussed in [118], such as equivalence of states, state equation, association of states with a system, etc.

– **In the work [74] by M. Mesarovic and Y. Takahara** the following reasons for introducing the concept of state are given:

"(i) A system is, in general, a relation; i.e. the same input can lead to different outputs. The state enables a representation of the system as a function. The idea is that if one knows what state the system is in, he could with assurance ascertain what the output will be. In such a way one regains "predictability" believed to be present if a complete set of observations is available.

(ii) The state enables the determination of a future output solely on the basis of the future input and the state the system is in. In other words, the state enables a "decoupling" of the past from the present and future. The state embodies all past history of the system. Knowing the state supplants knowledge of the past. Apparently, for this role to be meaningful, the notion of past and future must be relevant for the system considered; this leads to the notion of an abstract time system." [74, p. 45]

A notion of a *pre-state space representation* of a time system $S \subseteq X \times Y$ ($X \subseteq A^T$, $Y \subseteq B^T$) is introduced [74, p. 80] as a pair of families of mapping (ϕ, μ), $\phi = \{\phi_{tt'} \mid \phi_{tt'} : C_t \times X_{tt'} \to C_{t'}, t, t' \in T, t' \geq t\}$, $\mu = \{\mu_t \mid \mu_t : C_t \times A \to B, t \in T\}$ such that $\phi_{tt'}(c_t, x_{tt''}) = \phi_{t't''}(\phi_{tt'}(c_t, x_{tt'}), x_{t't''})$, if $x_{tt''}$ is the concatenation of $x_{tt'}$ and $x_{t't''}$ (composition or semi-group property), $\phi_{tt}(c_t, x_{tt}) = c_t$, and $(x, y) \in S$ if and only if there exists $c \in C_0$ such that for any $t \in T$, $y(t) = \mu_t(\phi_{0t}(c, x^t), x(t))$. Here x^t denotes $x|_{\{t' | t' < t\}}$ and $x_{tt'}$ denotes $x|_{\{t^* | t \leq t^* < t'\}}$ (where $x \in X$), $X_{tt'}$ denotes

$\{x_{tt'} \mid x \in X\}$, C_t for $t \in T$ are some sets, $c_t \in C_t$, $\phi_{tt'}$ is called a *state-transition function*, and μ_t is called an *output function*. If $C_t = C$ for all $t \in T$, (ϕ, μ) is called a *state-space representation* of S and C is called a *state space*. It is shown that any causal system has a state-space representation [74, Chapter 3, Proposition 2.8]. More general notions of a *pre-dynamical system representation* and a *dynamical system representation* (which also use state-transition functions) are also introduced and studied.

– In the work [121] by B. P. Zeigler two problems associated with the black box view of a system are underscored:

“Firstly, we have the problem of going from structure to behavior: If we know what lies inside the box, we ought to be able to describe, in one or the other way, the behavior of the box as viewed externally. The second area relates to the reverse situation – going from behavior to structure: the problem of trying to infer the internal structure of a black box from external observations.” [121, p. 107].

On one of the levels (I/O system), the interior of a system is modeled using the notion of state. In particular, it is noted:

“The state set is fundamental, as it has to have the property to summarize the past of the system such that the future is uniquely determined by the current state and the future input. This property of the state set is called the semigroup or composition property.” [121, p. 109].

An *I/O system* is defined as a tuple $S = (T, X, \Omega, Y, Q, \Delta, \Lambda)$ where T is a time domain (time base), X, Y are input and output value sets, Ω is a set of *allowable input segments*, i.e. functions defined on a time interval which take values in X, Q is a set of *states*, $\Delta : Q \times \Omega \to Q$ is a *global state transition function*, $\Lambda : Q \times X \to Y$ or $\Lambda : Q \to Y$ is an *output function*. This tuple must satisfy certain constraints: Ω must be closed under concatenation and so-called *left segmentation*, i.e. a left segment (prefix) of an element of Ω is again in Ω, and Δ must satisfy the composition (semigroup) property: $\Delta(q, \omega \bullet \omega') = \Delta(\Delta(q, \omega), \omega')$, where $\bullet$ denotes a concatenation of input segments, assuming ω and ω' are contiguous (the right end of the domain of ω coincides with the left end of the domain of ω'). A relation of this model to a more abstract view of a system (*I/O relation observation*) is considered [121].

Such views are usually consistent with understanding of state in theories which consider it more fundamental and use it in the definition of a system. For example, in the work [54] by R. Kalman it is noted:

"Intuitively speaking, the state is the minimal amount of information about the past history of the system which suffices to predict the effect of the past upon the future."

In the same work a system is defined using the notions of a state space, space of inputs, and transition and output functions which satisfy certain properties (axioms).

All approaches mentioned above insist that if a state of a system is known and fixed at a given time, then for a given future input, a future output of the system is determined uniquely. Thus non-uniqueness of the system's output for a given input can be explained by the freedom of choice of an initial state.

In contrast, in many models considered in computer science (e.g. non-deterministic automata, transition systems, etc.) the notion of state is used in a less restricted sense. A response of a non-deterministic system which starts in a fixed initial state and processes a given input data may not be uniquely determined.

This motivates to look for state-based representations of input/output ("black box") systems which support multiple variants of a state evolution for a given (complete future history of) input and a given initial state. Other desirable features are the ability to represent a sufficiently large class of input/output systems and to take into account partiality of inputs/outputs as functions of time.

A representation that we are looking for is a kind of dynamical system. Formalizations of the notion of a dynamical system of various levels of generality were given in many works, e.g. [12, 78, 36, 11, 15, 76, 94, 37, 100, 57, 55, 74, 71, 111, 27]. Classical approaches to the definition of a dynamical system, such as those proposed by A.A. Markov [78], V.V. Nemytskii and V.V. Stepanov [78] and others (a survey is given in [71]) can be considered as axiomatizations of the properties of systems described by differential equations.

As was noted in the work [37] by O. Hájek, the following properties of ordinary differential equations were of primary concern in various axiomatizations:

1) local existence of solutions;
2) indefinite prolongability (global existence) of solutions;
3) unicity of solutions;
4) autonomness (the right-hand side of the equation does not depend explicitly on time).

However, in a number of works [11, 15, 76, 94, 37, 100, 57, 71], etc., there was a tendency to remove some of these properties from basic assumptions and consider increasingly general classes of dynamical systems. An overview and comparison of many such approaches is given in [71].

In particular, in [37] it was proposed to eliminate all properties 1)-4) from the axiomatization to obtain a far-reaching generalization of dynamical systems. Similar ideas also appeared in some other works [57, 71, 111].

More specifically, in [37, 38] the following notion was introduced: p is called a *process* on P over R, if P is a set, $R \subseteq \mathbb{R}$, and $p \subseteq (P \times R) \times (P \times R)$ satisfies the following properties (infix notation is used for the relation p):

– $(x,\alpha) p (y,\beta)$ implies $\alpha \geq \beta$;
– *Initial-value property*: ${}_\alpha p_\alpha \subseteq I$ for each $\alpha \in R$ (where I is the identity relation on P);
– *Compositivity property*: ${}_\alpha p_\beta \circ {}_\beta p_\gamma = {}_\alpha p_\gamma$, if $\alpha \geq \beta \geq \gamma$ in R, where ${}_\alpha p_\beta$ denotes a binary relation on P such that $x {}_\alpha p_\beta y$ if and only if $(x,\alpha) p (y,\beta)$.

Intuitively, R means a time domain, P is called a phase-space, and a relation $(x,\alpha)p(y,\beta)$ means that there exists a solution/trajectory which takes values x and y at times α and β respectively.

Formally, with a process p there is an associated notion of a *solution*: a partial function $s: R \tilde{\to} P$, the domain of which is a non-empty, but possibly a singleton interval in R, is a *solution* of p, if $s(\alpha)\ {}_{\alpha}p_{\beta}\ s(\beta)$ for all $\alpha,\beta \in dom(s)$, $\alpha \geq \beta$.

The *solution system* S *of a process* p (also denoted as $sol\ p$) is the set of all solutions of p. Solution systems have the following basic properties.

1) Each $s \in S$ is a partial function $s: R \tilde{\to} P$ such that $dom(s)$ is an interval in R.
2) *Partialization property*: $s|_I \in S$ for each $s \in S$ and interval I in R.
3) *Concatenation property*: if $s_1, s_2 \in S$, the domains of s_1, s_2 intersect, and $s_1 \cup s_2$ is a partial function, then $s_1 \cup s_2 \in S$.
4) If $\{s_i\}$ is a monotone family in S, then $\bigcup s_i \in S$.
5) If I is an interval in R, $s: I \to P$, and for each $\alpha,\beta \in I$ there exists $s' \in S$ such that $s(\alpha) = s'(\alpha)$, $s(\beta) = s'(\beta)$, then $s \in S$.

In [37] it is also suggested that a solution system can be defined axiomatically without the notion of a process.

A set S is called a *solution system in* P *over* $R \subseteq \mathbb{R}$ (independently of any process), if S satisfies the properties 1-3 mentioned above. Its members are called solutions. One can associate a process, denoted as $pr\,S$, with such a set by letting $(x,\alpha)\ pr\,S\ (y,\beta)$ if and only if $\alpha \geq \beta$ and there exists $s \in S$ such that $x = s(\alpha)$ and $y = s(\beta)$. But in the general case, neither $S = sol\ pr\,S$, nor $p = pr\ sol\ p$ holds.

If $S = sol\ pr\,S$ holds, S is called *process-complete*. A necessary and sufficient condition for this is the property 5 mentioned above, and a necessary condition is the property 4.

If $p = pr\, sol\, p$ holds, p is called *solution-complete*. A necessary and sufficient condition for this is: $x_\alpha p_\beta y$ if and only if there exists $s \in sol\ p$ such that $x = s(\alpha)$ and $y = s(\beta)$.

We conclude that the notions of process and solution system in the sense of [37] are quite general and take into account the aspects which we are interested in (nondeterminism, partiality, continuous time, no assumptions about the structure of the phase-space P). Among them we prefer the notion of a solution system, because it more explicitly represents a dynamic behavior.

In this chapter our aim is to establish a link between blocks and a notion like solution system. However, we will not use the exact definitions and terminology of [37] for the following main reasons:

– we would like to include the properties 1-4 of a solution system (not only 1-3) in an abstract definition of a dynamical system; we will need a property similar to 4 in this and the next chapter;
– we prefer to use the terms "state space" and "trajectory" instead of "phase space" and "solution" in our context.

We will introduce a notion that is close to a solution system of [37] and call it a *Nondeterministic Complete Markovian System* (NCMS).

Then we will show that strongly nonanticipative blocks have a representation in the form of NCMS.

2.2 Nondeterministic complete Markovian systems (NCMS)

As before, let $T = \mathbb{R}_+$. Denote by $\mathfrak{T}$ the set of all (bounded or unbounded) intervals in T with cardinality greater than one, i.e. $A \in \mathfrak{T}$ if and only if $A \subseteq T$, $[t_1, t_2] \subseteq A$ for all $t_1, t_2 \in A$ such that $t_1 < t_2$, and $\{t_1', t_2'\} \subseteq A$ for some $t_1' \neq t_2'$.

Let Q be a set (a state space) and Tr be some set of functions of the form $s : A \to Q$, where $A \in \mathfrak{T}$. Let us call its elements trajectories.

Definition 2.1. A set of trajectories Tr is closed under proper restrictions (CPR), if $s|_A \in Tr$ for each $s \in Tr$ and $A \in \mathfrak{T}$ such that $A \subseteq dom(s)$.

In order to refer to Definition 2.1 we will use phrases like "Tr is CPR" or "Tr satisfies the CPR property".

Definition 2.2.

1) A trajectory $s_1 \in Tr$ is a subtrajectory of $s_2 \in Tr$ (denoted as $s_1 \sqsubseteq s_2$), if $dom(s_1) \subseteq dom(s_2)$ and $s_1 = s_2|_{dom(s_1)}$.
2) A trajectory $s_1 \in Tr$ is a proper subtrajectory of $s_2 \in Tr$ (denoted as $s_1 \sqsubset s_2$), if $s_1 \sqsubseteq s_2$ and $s_1 \neq s_2$.
3) Trajectories $s_1, s_2 \in Tr$ are incomparable, if neither $s_1 \sqsubseteq s_2$, nor $s_2 \sqsubseteq s_1$.

Lemma 2.1. $(Tr, \sqsubseteq)$ is a (possibly empty) partially ordered set (poset).

Proof. For each $s_1, s_2 \in Tr$, $s_1 \sqsubseteq s_2$ if and only if (the graph of) the function s_1 is a subset of (the graph of) s_2. Then it is obvious that $\sqsubseteq$ is a partial order on Tr. □

Definition 2.3. A CPR set of trajectories Tr is called

1) Markovian (see Fig. 2.1 below), if for each $s_1, s_2 \in Tr$ and $t_0 \in T$ such that $t_0 = \sup dom(s_1) = \inf dom(s_2)$, $s_1(t_0)\downarrow$, $s_2(t_0)\downarrow$, and $s_1(t_0) = s_2(t_0)$, the following function s belongs to Tr:

$$s(t) = \begin{cases} s_1(t), & t \in dom(s_1); \\ s_2(t), & t \in dom(s_2) \end{cases}$$

2) complete, if each non-empty chain in $(Tr, \sqsubseteq)$ has a supremum.

Note that the property 2 differs from chain-completeness [93] in that only non-empty chains must have a supremum.

Because of the CPR property, a supremum of a chain c in the poset $(Tr, \sqsubseteq)$ exists if and only if $s_* \in Tr$, where $s_* : \bigcup_{s \in c} dom(s) \to Q$ is defined as follows: $s_*(t) = s(t)$, if $s \in c$ and $t \in dom(s)$ (this is indeed a function, because c is a chain).

Definition 2.4. A nondeterministic complete Markovian system (NCMS) is a triple (T, Q, Tr), where Q is a set (state space) and Tr (trajectories) is a set of functions $s: T \tilde{\to} Q$ with $dom(s) \in \mathfrak{T}$ such that Tr is CPR, complete, and Markovian (in the sense of Definition 2.3).

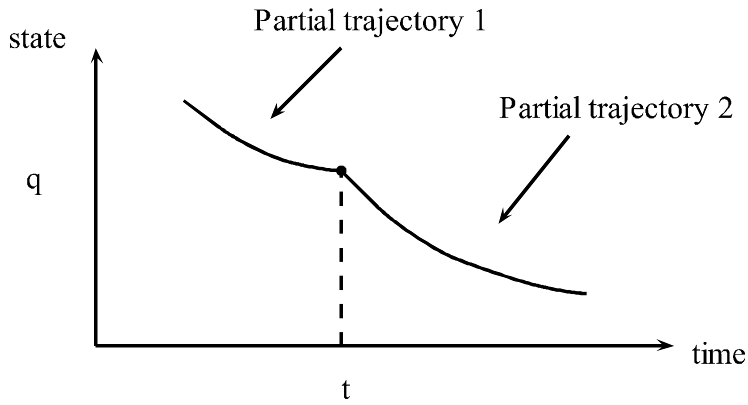

Fig. 2.1. Markovian property of a CPR set of trajectories. If one trajectory ends and another begins in a state q at a time t (both are defined at t), then their concatenation is a trajectory.

The notion of a NCMS is close to the notion of a solution system in the sense of [37] (discussed in Section 2.1), but there are some differences.

- The *time domain* T and the *set of states* Q correspond to the *time domain* R and the *phase-space* P of a solution system (Section 2.1). However, for simplicity we assume that T is fixed to be $\mathbb{R}_+$, while in [37] R can can be any subset of $\mathbb{R}$.
- *Trajectories* correspond to the members of a solution system (*solutions*). However, their domains cannot be singleton sets, while solutions can be defined on singleton sets. This is not a principal difference, but we assume that trajectory domains are not singleton sets for convenience.
- *CPR* property of NCMS corresponds to the *Partialization* property of solution systems (property 2). The difference is that Partialization allows restrictions on singleton sets, while CPR does not allow them.
- *Markovian* property of NCMS basically corresponds to the *Concatenation* property (property 3) of solution systems. By themselves these properties

are not equivalent: the formulation of the Markovian property of NCMS is weaker in the sense that it does not allow one to make a union of two trajectories, if the intersection of their domains is not a singleton set. But using both CPR and Markovian properties, one can make a union of two trajectories even if the intersection of their domains is not a singleton set. The term “Markovian” is meant to indicate that if a system is in a given state, the set of its possible future evolutions does not depend on its past [51] (however, it is not meant to suggest a direct relation to Markov processes in probability theory). The usage of this term in a similar sense can be found in the literature, e.g. [111]. In a more general sense a similar interpretation of a Markov property was considered in [52, 17] in the context of the possibility theory.

– *Completeness* property of NCMS basically corresponds to the (unnamed) property 4 of solution systems (which are associated with processes).

The main reason for considering this notion instead of a solution system is the Completeness property of NCMS (not assumed by default in the process-independent definition of a solution system [37, Definition 2.1]). The results concerning NCMS that we will obtain and use in this and the next chapter significantly depend on it. Moreover, in our opinion, the Markovian property is more convenient than the Concatenation property of solution systems, so we decided to use it in the definition of NCMS.

2.3 Representation of NCMS

In this section we will give a convenient general representation of NCMS. Let us introduce the following terminology.

Definition 2.5. Let $s_1, s_2 : T \tilde{\to} Q$. Then s_1 and s_2 coincide:

1) on a set $A \subseteq T$, if $s_1|_A = s_2|_A$ and $A \subseteq dom(s_1) \cap dom(s_2)$ (this is denoted as $s_1 \doteq_A s_2$);

2) in a left neighborhood of $t \in T$, if $t > 0$ and there exists $t' \in [0,t)$, such that $s_1 \doteq_{(t',t]} s_2$ (this is denoted as $s_1 \doteq_{t-} s_2$);

3) in a right neighborhood of $t \in T$, if there exists $t' > t$, such that $s_1 \doteq_{[t,t')} s_2$ (this is denoted as $s_1 \doteq_{t+} s_2$).

Let Q be a set. Denote by $ST(Q)$ the set of pairs (s,t) where $s : A \to Q$ for some $A \in \mathfrak{T}$ and $t \in A$.

Definition 2.6. A predicate $p : ST(Q) \to Bool$ is called

1) left-local, if $p(s_1,t) \Leftrightarrow p(s_2,t)$ for each $\{(s_1,t),(s_2,t)\} \subseteq ST(Q)$ such that $s_1 \doteq_{t-} s_2$, and, moreover, $p(s,t)$ holds for each (s,t) such that t is the least element of $dom(s)$;

2) right-local, if $p(s_1,t) \Leftrightarrow p(s_2,t)$ for each $\{(s_1,t),(s_2,t)\} \subseteq ST(Q)$ such that $s_1 \doteq_{t+} s_2$, and, moreover, $p(s,t)$ holds for each (s,t) such that t is the greatest element of $dom(s)$.

Let us denote by $LR(Q)$ the set of all pairs (l,r), where $l : ST(Q) \to Bool$ is a left-local predicate and $r : ST(Q) \to Bool$ is a right-local predicate.

Definition 2.7. A pair $(l,r) \in LR(Q)$ is called a LR representation of a NCMS $\Sigma = (T,Q,Tr)$, if $Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\, l(s,t) \wedge r(s,t))\}$.

Theorem 2.1 (About LR representation)

1) Each pair $(l,r) \in LR(Q)$ is a LR representation of a NCMS with the set of states Q.

2) Each NCMS has a LR representation.

Proof.

1) Let $(l,r) \in LR(Q)$. Let $\Sigma = (T,Q,Tr)$, where

$$Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\, l(s,t) \wedge r(s,t))\}.$$

Let us show that Tr is CPR. Let $s \in Tr$, $s : A \to Q$, $A' \in \mathfrak{T}$, and $A' \subseteq A$. Then $dom(s|_{A'}) = A'$ and $l(s,t) \wedge r(s,t)$ for all $t \in A$. If t is a non-maximal element of A', then $s|_{A'} \doteq_{t+} s$ and $r(s,t)$, whence $r(s|_{A'},t)$. Similarly, if t is a non-minimal element of A', then $s|_A \doteq_{t-} s$ and $l(s,t)$, whence $l(s|_{A'},t)$. Moreover, if A' has a minimal element t, then $l(s|_{A'},t)$, because l is left-local. Similarly, if A' has a maximal element t, then $r(s|_{A'},t)$, because r is right-local, Thus $l(s|_{A'},t) \wedge r(s|_{A'},t)$ for all $t \in A' = dom(s|_{A'})$. Then $s|_{A'} \in Tr$.

Let us show that Tr is complete. Let $c \subseteq Tr$ be a non-empty $\sqsubseteq$-chain and $s^* = \bigcup_{s \in c} s$, i.e. the union of (graphs) of functions. Then s^* is a function defined on $\bigcup_{s \in c} dom(s)$. It is sufficient to show that $s^* \in Tr$. We have $l(s,t) \wedge r(s,t)$ for all $s \in c$ and $t \in dom(s)$. Moreover, for each $s \in c$ and t in the interior of $dom(s)$ we have $s \doteq_{t+} s^*$ and $s \doteq_{t-} s^*$. Thus $l(s^*,t) \wedge r(s^*,t)$ for each t in the interior of $dom(s^*)$. Moreover, if $dom(s^*)$ has the least element t_*, then t_* is the least element of $dom(s)$ for some $s \in c$, whence $l(s^*,t_*) \wedge r(s^*,t_*)$, because $s \doteq_{t_*+} s^*$ and $r(s,t_*)$, while $l(s^*,t_*)$ holds automatically. Analogously, we have that if $dom(s^*)$ has the greatest element t^*, then $l(s^*,t^*) \wedge r(s^*,t^*)$. Then $l(s^*,t) \wedge r(s^*,t)$ holds for all $t \in dom(s^*)$. Thus $s^* \in Tr$.

Let us show that Tr is Markovian.

Let $s_1, s_2 \in Tr$, $t_0 = \sup dom(s_1) = \inf dom(s_2)$, $s_1(t_0)\downarrow$, $s_2(t_0)\downarrow$, and $s_1(t_0) = s_2(t_0)$. Let us define $s : dom(s_1) \cup dom(s_2) \to Q$ as $s(t) = s_1(t)$, if $t \in dom(s_1)$ and $s(t) = s_2(t)$, if $t \in dom(s_2)$. Then for $j = 1,2$ we have $l(s_j,t) \wedge r(s_j,t)$ for all $t \in dom(s_j)$. Besides, $s \doteq_{t+} s_1$ for all $t \in dom(s_1) \setminus \{t_0\}$ and $s \doteq_{t-} s_1$ for all non-minimal $t \in dom(s_1)$. Then $l(s,t) \wedge r(s,t)$ for all $t \in dom(s_1) \setminus \{t_0\}$ and $l(s,t_0)$, because l is left-local and r is right-local. Analogously, we have $l(s,t) \wedge r(s,t)$ for all $t \in dom(s_2) \setminus \{t_0\}$ and $r(s,t_0)$. Thus $l(s,t) \wedge r(s,t)$ for all $t \in dom(s)$, whence $s \in Tr$.

Thus Σ is a NCMS and (l,r) is a LR representation of Σ.

2) Let $\Sigma=(T,Q,Tr)$ be a NCMS. Let us define predicates $l:ST(Q)\to Bool$ and $r:ST(Q)\to Bool$ as follows:

- $l(s,t)$ if and only if either t is the least element of $dom(s)$, or there exists $t'<t$ such that $[t',t]\subseteq dom(s)$ and $s|_{[t',t]}\in Tr$;
- $r(s,t)$ if and only if either t is the greatest element of $dom(s)$, or there exists $t'>t$ such that $[t,t']\subseteq dom(s)$ and $s|_{[t,t']}\in Tr$.

Let $Tr'=\{s:A\to Q\mid A\in\mathfrak{T}\wedge(\forall t\in A\; l(s,t)\wedge r(s,t))\}$.

It follows immediately from the CPR property of Tr that l is left-local, r is right-local, and $Tr\subseteq Tr'$.

Let us prove the opposite inclusion $Tr'\subseteq Tr$. Assume that $A\in\mathfrak{T}$, $s:A\to Q$, and $l(s,t)\wedge r(s,t)$ for all $t\in A$.

Consider the following cases:

a) $A=[a,b]$ for some $a<b$. For each $t\in(a,b)$ we have $l(s,t)\wedge r(s,t)$, whence there exists $t'<t$ and $t''>t$ such that $[t',t'']\subseteq dom(s)$ and $s|_{[t',t]}\in Tr$, $s|_{[t,t'']}\in Tr$, whence $s|_{[t',t'']}\in Tr$ by the Markovian property. Denote $O_t=(t',t'')$. Because $a\in dom(s)$, $a\neq\max dom(s)$, and $r(s,a)$, there exists $t''>a$ such that $[a,t'']\subseteq dom(s)$ and $s|_{[a,t'']}\in Tr$. Denote $O_a=[a,t'')$. Similarly, because $b\in dom(s)$, $b\neq\min dom(s)$, and $l(s,b)$, there exists $t'<b$ such that $[t',b]\subseteq dom(s)$ and $s|_{[t',b]}\in Tr$. Denote $O_b=(t',b]$. Thus have we defined O_t for all $t\in A$. Then $(O_t)_{t\in A}$ is an open cover of A in the sense of the topology induced on A from T (O_a and O_b are relatively open). Since A is compact, there exists a finite sub-cover O_{t_i}, $i=1,2,...,k$. Let t_i' denote the left end of O_{t_i} and t_i'' denote the right end of O_{t_i}.

Without loss of generality we can assume that $a=t_1\le t_2\le...\le t_k=b$. By the construction of O_t, $s|_{[t_i',t_i'']}\in Tr$ and $t_i'\le t_i\le t_i''$ for $i=1,2,...,k$. Then it is easy to see that CPR and Markovian properties of Tr imply that $s|_{[a,b]}=s\in Tr$.

b) $A=[a,b)$ for some $a<b$, where $a\in T, b\in T\cup\{+\infty\}$. We have $s|_{[a,t]}\in Tr'$ for all $t\in(a,b)$, because l is left-local and r is right-local. Then $s|_{[a,t]}\in Tr$ for all $t\in(a,b)$ by the previous case a). By the completeness property of Tr we conclude that $s\in Tr$.

c) $A=(a,b]$ for some $a<b$. We have $s|_{[t,b]}\in Tr'$ for all $t\in(a,b)$, because l is left-local and r is right-local. Then $s|_{[t,b]}\in Tr$ for all $t\in(a,b)$ by the case a). Using the completeness property of Tr we conclude that $s\in Tr$.

d) $A=(a,b)$ for some $a<b$ ($a\in T, b\in T\cup\{+\infty\}$). Let us choose an arbitrary $c\in(a,b)$. We have $s|_{(a,c]}\in Tr'$ and $s|_{[c,b)}\in Tr'$, because l is left-local and r is right-local. From the two previous cases b) and c) we obtain that $s|_{(a,c]}\in Tr$ and $s|_{[c,b)}\in Tr$. Then $s\in Tr$ by the Markovian property of Tr.

We conclude that $(l,r)\in LR(Q)$ and $Tr=Tr'$. Thus (l,r) is a LR representation of Σ. □

Informally, this theorem shows that for a NCMS, a global property "is a trajectory" ($s\in Tr$) can be expressed as a conjunction of local properties ($l(s,t)\wedge r(s,t)$) for each time moment.

This theorem has the following corollaries which we will use later.

Lemma 2.2. Let $J\neq\varnothing$ and $((T,Q_j,Tr_j))_{j\in J}$ be an indexed family of NCMS. Then the triple $\Sigma=(T,\bigcap_{j\in J}Q_j,\bigcap_{j\in J}Tr_j)$ is a NCMS.

Proof. Denote $Q=\bigcap_{j\in J}Q_j$, $Tr=\bigcap_{j\in J}Tr_j$. Obviously, each function in Tr takes values in Q. For each $j\in J$ let (l_j,r_j) be a LR representation of (T,Q_j,Tr_j), which exists by Theorem 2.1. Then

$$Tr_j=\{s:A\to Q\mid A\in\mathfrak{T}\wedge(\forall t\in A\, l_j(s,t)\wedge r_j(s,t))\}.$$

Then $ST(Q)\subseteq ST(Q_j)$ for all $j\in J$. Let predicates $l:ST(Q)\to Bool$ and $r:ST(Q)\to Bool$ be defined for each $(s,t)\in ST(Q)$ as $l(s,t)\Leftrightarrow\forall j\in J\, l_j(s,t)$ and $r(s,t)\Leftrightarrow\forall j\in J\, r_j(s,t)$. Because, all l_j, $j\in J$ are left-local, we have that if

$\{(s_1,t),(s_2,t)\} \subseteq ST(Q)$ and $s_1 \doteq_{t-} s_2$, then $l(s_1,t) \Leftrightarrow l(s_2,t)$, and moreover, $l(s,t)$ whenever t is the least element of $dom(s)$. Thus l is left-local. Similarly, because all r_j, $j \in J$ are right-local, we have that if $\{(s_1,t),(s_2,t)\} \subseteq ST(Q)$ and $s_1 \doteq_{t+} s_2$, then $r(s_1,t) \Leftrightarrow r(s_2,t)$, and moreover, $r(s,t)$ whenever t is the greatest element of $dom(s)$. This r is right-local. Then $(l,r) \in LR(Q)$ and by Theorem 2.1, it is an LR representation of a NCMS. Then the triple

$$(T, Q, \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\, l(s,t) \wedge r(s,t))\}$$

is a NCMS. Moreover,

$$\{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\, l(s,t) \wedge r(s,t))\} =$$

$$= \{s : A \to Q \mid A \in \mathfrak{T} \wedge \forall t \in A\, \forall j \in J\, l_j(s,t) \wedge r_j(s,t)\} = \bigcap_{j \in J} Tr_j = Tr.$$

Thus Σ is a NCMS. □

Definition 2.8. A state-restriction of a NCMS $\Sigma = (T, Q, Tr)$ on a set Q', denoted as $\Sigma|_{Q'}$, is a triple $(T, Q \cap Q', \{s \in Tr \mid \forall t \in dom(s)\, s(t) \in Q'\})$.

Lemma 2.3. $\Sigma|_{Q'}$ is a NCMS for each NCMS $\Sigma = (T, Q, Tr)$ and a set Q'.

Proof. Let us define

$$Tr' = \{s \in Tr \mid \forall t \in dom(s)\, s(t) \in Q'\}.$$

Then $\Sigma|_{Q'} = (T, Q \cap Q', Tr')$. Let $l : ST(Q') \to Bool$ and $r : ST(Q') \to Bool$ be predicates which are true for all values of the input argument. Obviously, l is left-local and r is right-local. Let us define

$$Tr'' = \{s : A \to Q' \mid A \in \mathfrak{T} \wedge (\forall t \in A\, l(s,t) \wedge r(s,t))\}.$$

By Theorem 2.1, $\Sigma' = (T, Q', Tr'')$ is a NCMS. Moreover, Tr'' is the set of all functions of the form $s : A \to Q'$ for all $A \in \mathfrak{T}$, whence we have $Tr' = Tr \cap Tr''$. Then from Lemma 2.2 (applied to the case of a two-element indexed family of NCMS) we have that $\Sigma|_{Q'} = (T, Q \cap Q', Tr \cap Tr'')$ is a NCMS. □

2.4 Examples of sets of trajectories and NCMS

Firstly, let us consider some examples of sets of trajectories.

Let $Q = \mathbb{R}$. Consider the following sets of trajectories:

– Tr_{all} is the set of all functions $s : A \to Q$, $A \in \mathfrak{T}$.

– Tr_{cont} is the set of all continuous functions $s \in Tr_{all}$.

– Tr_{diff} is the set of functions $s \in Tr_{all}$ such that s is differentiable on the interior of $dom(s)$.

– Tr_{bnd} is the set of all functions $s \in Tr_{all}$ which are bounded on their domains, i.e. for each $s \in Tr_{bnd}$ there exist $a, b \in \mathbb{R}$, $a < b$ such that $s(t) \in [a,b]$ for all $t \in dom(s)$.

Proposition 2.1. The following holds:

1) $\varnothing$, Tr_{all}, Tr_{cont}, Tr_{diff}, Tr_{bnd}, $Tr_{diff} \cap Tr_{bnd}$ are CPR.

2) $\varnothing$, Tr_{all}, Tr_{cont} are complete and Markovian.

3) Tr_{diff} is complete, but is not Markovian.

4) Tr_{bnd} is Markovian, but is not complete.

5) $Tr_{diff} \cap Tr_{bnd}$ is neither complete, nor Markovian.

Proof.

1) The empty set and Tr_{all} are obviously CPR. The restrictions of continuous, differentiable, bounded, differentiable and bounded functions defined on real intervals onto real sub-intervals are still continuous, differentiable, bounded, differentiable and bounded respectively. Thus Tr_{cont}, Tr_{diff}, Tr_{bnd}, $Tr_{diff} \cap Tr_{bnd}$ are CPR.

2) It follows immediately from Definition 2.3 that $\varnothing$, Tr_{all} are complete and Markovian. To show that Tr_{cont} is complete and Markovian, consider predicates $l, r : ST(Q) \to Bool$ defined as follows:

– $l(s,t)$ if and only if either $\min dom(s)\downarrow = t$, or $t > \inf dom(s)$ and s is left-continuous at t;

– $r(s,t)$ if and only if either $\max dom(s)\downarrow = t$, or $t < \sup dom(s)$ and s is right-continuous at t.

Obviously, $l(s,t)$ is left-local, $r(s,t)$ is right-local. Moreover, $l(s,t)\wedge r(s,t)$ for all $t \in dom(s)$ if and only if s is continuous. Then Theorem 2.1 implies that Tr_{cont} is complete and Markovian.

3) Consider $s_1 : [0,1] \to Q$ and $s_2 : [1,+\infty) \to Q$ such that $s_1(t) = t$ for all $t \in [0,1]$ and $s_2(t) = 1$ for all $t \in [1,+\infty)$. Then $s_1, s_2 \in Tr_{diff} \cap Tr_{bnd}$ and $s_1(1) = s_2(1)$. Let $s(t) = s_1(t)$, if $t \in dom(s_1)$, $s(t) = s_2(t)$, if $t \in dom(s_2)$. Then s is not differentiable at $t = 1$, so $s \notin Tr_{diff}$. Thus Tr_{diff} is not Markovian. Completeness of Tr_{diff} follows from Definition 2.3.

4) Markovian property follows immediately from Definition 2.3. Consider a function $s : T \to Q$, where $s(t) = t$ for all $t \in T$. Then $s|_{[0,t]} \in Tr_{bnd}$ for all $t \in T$, but $s \notin Tr_{bnd}$. Thus Tr_{bnd} is not complete.

5) The same argument as we used in 3) shows that $Tr_{diff} \cap Tr_{bnd}$ is not Markovian. The same argument as we used in 4) shows that $Tr_{diff} \cap Tr_{bnd}$ is not complete. □

Now let us consider some examples of NCMS.

Proposition 2.2. Let $d \in \mathbb{N}$, $Q = \mathbb{R}^d$, and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. Let Tr be the set of all functions $s : A \to Q$, $A \in \mathfrak{T}$ such that on the interior of A the function s is differentiable and satisfies $\frac{d}{dt}s(t) = f(t, s(t))$, and $\partial_- s(t)\downarrow = f(t, s(t))$ holds for $t = \min A$, if $\min A\downarrow$, and $\partial_+ s(t)\downarrow = f(t, s(t))$ holds for $t = \max A$, if $\max A\downarrow$, where $\partial_- s(t)$ and $\partial_+ s(t)$ denote a left and right derivative at t respectively.

Then (T, Q, Tr) is a NCMS.

Proof. Consider the predicates $l, r : ST(Q) \to Bool$ defined as follows:

- $l(s,t) \Leftrightarrow (\min dom(s) \downarrow = t) \vee (t > \inf dom(s) \wedge \partial_{-} s(t) \downarrow = f(t, s(t)))$;
- $r(s,t) \Leftrightarrow (\max dom(s) \downarrow = t) \vee (t < \sup dom(s) \wedge \partial_{+} s(t) \downarrow = f(t, s(t)))$;

Obviously, $l(s,t)$ is left-local and $r(s,t)$ is right-local. Moreover, $l(s,t) \wedge r(s,t)$ for all $t \in dom(s)$ if and only if on the interior of $dom(s)$ the function s is differentiable and satisfies $\frac{d}{dt} s(t) = f(t, s(t))$, and $\partial_{-} s(t) \downarrow = f(t, s(t))$ holds for $t = \min dom(s)$, if $\min dom(s) \downarrow$, and $\partial_{+} s(t) \downarrow = f(t, s(t))$ holds for $t = \max dom(s)$, if $\max dom(s) \downarrow$. Then Theorem 2.1 implies that (T, Q, Tr) is a NCMS. □

Proposition 2.3. Let $(Q, \hat{\to})$ be a state transition system, i.e. Q is a set (states) and $\hat{\to} \subseteq Q \times Q$ is a binary relation (transitions, we will write $q_1 \hat{\to} q_2$, if $(q_1, q_2) \in \hat{\to}$). Suppose that Q is equipped with a discrete topology [77], i.e. open sets are all subsets of Q.

Let Tr be the set of all functions $s : A \to Q$ such that for each non-minimal $t \in A$, $\lim_{\tau \to t-} s(\tau)$ exists and $\lim_{\tau \to t-} s(\tau) = s(t)$, and for each non-maximal $t \in A$, $\lim_{\tau \to t+} s(\tau)$ exists and

$$\begin{cases} \lim_{\tau \to t+} s(\tau) = s(t), & t \notin \mathbb{N}_0, \\ s(t) \hat{\to} \lim_{\tau \to t+} s(\tau), & t \in \mathbb{N}_0. \end{cases}$$

Then (T, Q, Tr) is a NCMS (Fig. 2.2).

Proof. Indeed, consider the predicates $l, r : ST(Q) \to Bool$ such that:

- $l(s,t)$ if and only if either $\min dom(s) \downarrow = t$, or $t > \inf dom(s)$ and $\lim_{\tau \to t-} s(\tau)$ exists and $\lim_{\tau \to t-} s(\tau) = s(t)$;
- $r(s,t)$ if and only if either $\max dom(s) \downarrow = t$, or $t < \sup dom(s)$ and a limit $\lim_{\tau \to t+} s(\tau)$ exists and $\lim_{\tau \to t+} s(\tau) = s(t)$, if $t \notin \mathbb{N}_0$ and $s(t) \hat{\to} \lim_{\tau \to t+} s(\tau)$, if $t \in \mathbb{N}_0$.

Obviously, $l(s,t)$ is left-local and $r(s,t)$ is right-local. Then Theorem 2.1 implies that (T,Q,Tr) is a NCMS. □

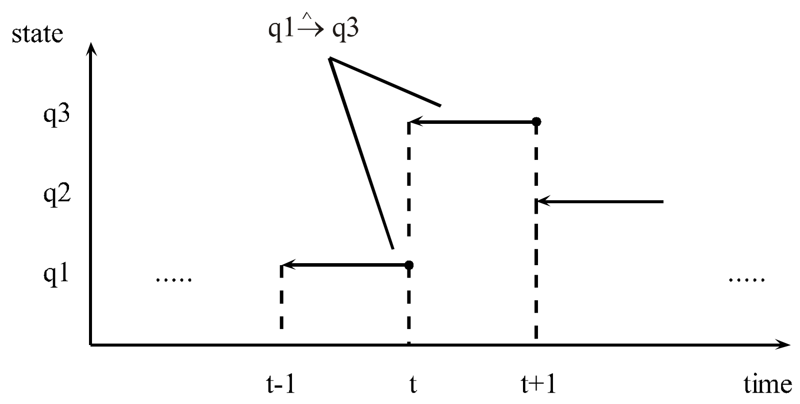

Fig. 2.2. A trajectory which models an execution of a (discrete-time) state transition system $(Q,\hat{\to})$. At non-negative integer time moments the system changes its current state q to a next state q' such that $q \hat{\to} q'$.

2.5 Representation of a strongly nonanticipative block

In this section we will introduce a representation of strongly nonanticipative blocks using NCMS.

As before, let W denote a fixed non-empty set of values.

Definition 2.9. An input-output (I/O) NCMS is an NCMS (T,Q,Tr) such that Q has a form ${}^{I}W \times X \times {}^{O}W$ for some sets I (set of input names), $X \neq \varnothing$ (set of internal states), and O (set of output names). The ${}^{I}W$ is called an input data set and ${}^{O}W$ is called an output data set.

Informally, an I/O NCMS describes possible evolutions (trajectories) of triples (d_{in}, x, d_{out}) of input data ($d_{in} \in {}^{I}W$), internal state ($x \in X$), and output data ($d_{out} \in {}^{O}W$).

Lemma 2.4. Each I/O NCMS (T,Q,Tr) has a unique set of input names, internal states, and output names.

Proof. The proof follows from the fact that if $Q={}^{I_1}W\times X_1\times^{O_1}W={}^{I_2}W\times X_2\times^{O_2}W$ and $X_1,X_2\neq\varnothing$, then $X_1=X_2$, ${}^{I_1}W={}^{I_2}W$, and ${}^{O_1}W={}^{O_2}W$, whence $I_1=I_2$ and $O_1=O_2$, because $W\neq\varnothing$. □

For a I/O NCMS Σ we will denote as $In(\Sigma)$ its unique set of input names, as $Out(\Sigma)$ its set of output names, and as $IState(\Sigma)$ its internal state space.

For any I/O NCMS $\Sigma=(T,Q,Tr)$ and a state $q\in Q$ we will denote as $in(q)$, $istate(q)$, $out(q)$ the projections of q on the first, second, and third coordinate respectively. Correspondingly, for any $s\in Tr$, $in\circ s$, $istate\circ s$, $out\circ s$, denote a composition of the respective projection map with a trajectory.

For each $i\in Sb(In(\Sigma),W)$ let us denote

- $S(\Sigma,i)=\{s\in Tr\mid dom(s)\in\mathcal{T}_0\wedge in\circ s\preceq i\}$;
- $S_{max}(\Sigma,i)$ is the set of all $\sqsubseteq$-maximal (i.e. non-continuable) trajectories from $S(\Sigma,i)$;
- $S_{init}(\Sigma,i)=\{s(0)\mid s\in S(\Sigma,i)\}$;
- $S_{init}(\Sigma)=\{s(0)\mid s\in Tr\wedge dom(s)\in\mathcal{T}_0\}$.

For each $Q'\subseteq Q$ let us denote:

$$Sel_{1,2}(Q',d,x)=\{q\in Q'\mid\exists d'\ q=(d,x,d')\},$$

i.e. a selection of states from Q' by the value of the first and second component.

For each $Q'\subseteq Q$ and $i\in Sb(In(\Sigma),W)$ let us denote:

$$o_{all}(\Sigma,Q',i)=\begin{cases}\{\bot\}, & Q'=\varnothing \text{ or } i=\bot;\\ \{\{0\}\mapsto out(q)\mid q\in Q'\}, & Q'\neq\varnothing \text{ and } \mathrm{dom(i)}=\{0\};\\ \{out\circ s\mid s\in S_{max}(\Sigma,i)\wedge s(0)\in Q'\}\cup & Q'\neq\varnothing \text{ and}\\ \cup\{\{0\}\mapsto out(q)\mid q\in Q'\setminus S_{init}(\Sigma,i)\}, & \{0\}\subset\mathrm{dom(i)},\end{cases}$$

where $\{0\}\mapsto out(q)$ is a function defined on $\{0\}$ which takes the value $out(q)$.

For each $Q_0 \subseteq Q$ let us denote:

$$O_{all}(\Sigma, Q_0, i) = \begin{cases} \{\perp\}, & dom(i) = \varnothing; \\ \bigcup\limits_{x \in IState(\Sigma)} o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i), & dom(i) \neq \varnothing \end{cases}$$

Definition 2.10. An initial I/O NCMS is a pair (Σ, Q_0) such that $\Sigma = (T, Q, Tr)$ is a I/O NCMS and Q_0 is a set (admissible initial states) such that $S_{init}(\Sigma) \subseteq Q_0 \subseteq Q$.

Definition 2.11. A NCMS representation of a block B is an initial I/O NCMS (Σ, Q_0) such that

1) $In(B) = In(\Sigma)$ and $Out(B) = Out(\Sigma)$;
2) $Op(B)(i) = O_{all}(\Sigma, Q_0, i)$ for all $i \in Sb(In(B), W)$.

Informally, the operation of a block B represented by an initial I/O NCMS (Σ, Q_0) on an input signal bunch i can be described as follows:

1) If $i(0)$ is undefined, then B stops (the output signal bunch is $\perp$).
2) Otherwise, B chooses an arbitrary internal state $x \in IState(\Sigma)$.
3) If there is no admissible initial state $q \in Q_0$ with $in(q) = i(0)$ and $istate(q) = x$ (i.e. $Sel_{1,2}(Q_0, i(0), x) = \varnothing$), then B stops.
4) Otherwise, B chooses an arbitrary $q \in Q_0$ such that $in(q) = i(0)$ and $istate(q) = x$ (i.e. $q \in Sel_{1,2}(Q_0, i(0), x)$).
5) If $dom(i) = \{0\}$ or there is no trajectory s which starts in q and is defined on some interval (of positive length) from $\mathcal{T}_0$, then B outputs $out(q)$ at time 0 and stops.
6) Otherwise, B chooses an arbitrary maximal trajectory s defined on an interval from $\mathcal{T}_0$ such that $s(0) = q$ and $in \circ s \preceq i$ and outputs the signal bunch $out \circ s$.

Theorem 2.2 (About representation of a strongly nonanticipative block). Each strongly nonanticipative block has a NCMS representation.

Theorem 2.3 (Converse theorem about representation of a strongly nonanticipative block). Each initial I/O NCMS is a NCMS representation of a strongly nonanticipative block.

We will prove these two theorems in the next two sections.

2.6 Proof of the theorem about representation of a strongly nonanticipative block

Firstly, let us prove several auxiliary lemmas.

Lemma 2.5. Let (T,Q,Tr) be a NCMS, Q' be a set, $f:Q\to Q'$ be an injective function, and $Tr'=\{f\circ s \mid s\in Tr\}$. Then (T,Q',Tr') is a NCMS.

Proof. Let us show that Tr' is closed under proper restrictions (CPR). Let $s'\in Tr'$, $A\in\mathfrak{T}$, and $A\subseteq dom(s)$. Then $s'=f\circ s$ for some $s\in Tr$, whence $s|_A\in Tr$, because Tr is CPR and $dom(s)=dom(s')$. Thus $s'|_A=f\circ(s|_A)\in Tr$.

Let us show that Tr' is Markovian. Let $s_1',s_2'\in Tr'$, $t^*=\max dom(s_1')=\min dom(s_2')$, and $s_1'(t^*)=s_2'(t^*)$. Then $s_1'=f\circ s_1$, $s_2'=f\circ s_2$ for some $s_1,s_2\in Tr$. Then $f(s_1(t^*))=f(s_2(t^*))$, whence $s_1(t^*)=s_2(t^*)$, because f is injective. Then a function $s:dom(s_1)\cup dom(s_2)\to Q$ such that $s(t)=s_1(t)$ if $t\in dom(s_1)$ and $s(t)=s_2(t)$, if $t\in dom(s_2)$ belongs to Tr. Then $s'=f\circ s\in Tr'$ and $s'(t)=s_1'(t)$, if $t\in dom(s_1')$ and $s'(t)=s_2'(t)$, if $t\in dom(s_2')$.

Let us show that Tr' is complete (in the sense of Definition 2.3). Let $c'\subseteq Tr'$ be a non-empty $\sqsubseteq$-chain. Let $c=\{s\in Tr \mid f\circ s\in c'\}\subseteq Tr$. If $s_1,s_2\in c$, then $f\circ s_1\sqsubseteq f\circ s_2$, or $f\circ s_2\sqsubseteq f\circ s_1$, whence $s_1\sqsubseteq s_2$ or $s_2\sqsubseteq s_1$, because f is injective. Thus c is a $\sqsubseteq$-chain. It is non-empty, because $c'\neq\varnothing$ and for any $s'\in c'\subseteq Tr'$ there exists $s\in Tr$ that $f\circ s=s'$. Then there exists a least upper bound $s^*\in Tr$ of c (when Tr is viewed as poset with respect to $\sqsubseteq$). Let $s'^*=f\circ s^*$. Then $s'^*\in Tr'$, $s'\sqsubseteq s'^*$ for all $s'\in c'$, and $dom(s'^*)=dom(s^*)=\bigcup_{s\in c}dom(s)=\bigcup_{s'\in c'}dom(s')$. Then the graph of s'^* is the union of the graphs of the members of c'. Thus s'^* is the least upper bound of c' (when Tr' is viewed as poset with respect to $\sqsubseteq$). □

Lemma 2.6. Let (T,Q^j,Tr^j), $j \in J$ be an indexed family of NCMS such that $Q^j \cap Q^{j'} = \varnothing$, if $j \neq j'$. Let $Q = \bigcup_{j \in J} Q^j$ and $Tr = \bigcup_{j \in J} Tr^j$. Then (T,Q,Tr) is a NCMS.

Proof. Firstly, let us show that Tr is closed under proper restrictions (CPR). Let $s \in Tr$, $A \in \mathfrak{T}$, and $A \subseteq dom(s)$. Then $s \in Tr^j$ for some $j \in J$, whence $s|_A \in Tr^j \subseteq Tr$.

Secondly, let us show that Tr is Markovian. Let $s_1, s_2 \in Tr$, $t^* = \max dom(s_1) = \min dom(s_2)$, and $s_1(t^*) = s_2(t^*)$. Then $s_1 \in Tr^j$ and $s_2 \in Tr^{j'}$ for some $j, j' \in J$. Then $j = j'$, because otherwise, $Q^j \cap Q^{j'} = \varnothing$ and $s_1(t^*) \neq s_2(t^*)$. Then a function $s : dom(s_1) \cup dom(s_2) \to Q$ such that $s(t) = s_1(t)$, if $t \in dom(s_1)$ and $s(t) = s_2(t)$, if $t \in dom(s_2)$ belongs to $Tr^j \subseteq Tr$.

Finally, let us show that Tr is complete (in the sense of Definition 2.3). This is obvious, if $Tr = \varnothing$, so assume that $Tr \neq \varnothing$. Let $c \subseteq Tr$ be a non-empty $\sqsubseteq$-chain. For each $s \in Tr$ there exists an index $j(s) \in J$ such that $s \in Tr^{j(s)}$. For each $s_1, s_2 \in c$, either $s_1 \sqsubseteq s_2$, or $s_2 \sqsubseteq s_1$, and because $dom(s_1), dom(s_2) \neq \varnothing$ and the sets Q^j are disjoint for different j, we have $j(s_1) = j(s_2)$. Thus all indices $j(s)$, $s \in c$ coincide, so there exists $j \in J$ such that $c \subseteq Tr^j$. Then there exists a least upper bound $s^* \in Tr^j$ of c in the sense of the poset Tr^j (with the ordering $\sqsubseteq$). Then it is easy to see that s^* is a least upper bound of c in the sense of the poset Tr (with the ordering $\sqsubseteq$). □

Lemma 2.7. Let Σ be a I/O NCMS, $i \in Sb(In(\Sigma), W)$, and $s \in S(\Sigma, i)$. Then there exists $s' \in S_{max}(\Sigma, i)$ such that $s \sqsubseteq s'$.

Proof. Consider a set $G = \{s'' \in S(\Sigma, i) \mid s \sqsubseteq s''\}$.

Let $c \subseteq G$ be a non-empty $\sqsubseteq$-chain. Then it has a least upper bound s^* in Tr, because Σ is a NCMS. This implies that $A \subseteq dom(s^*)$ for some $A \in \mathcal{T}_0 \setminus \{\varnothing\}$, whence $0 \in dom(s^*)$. Then $dom(s^*) \in \mathcal{T}_0$, because $s^* \in Tr$. Moreover, $in \circ s^* \preceq i$, because $in \circ s'' \preceq i$ for all $s'' \in c$. Then $s^* \in S(\Sigma, i)$. Obviously, $s \sqsubseteq s^*$, so $s^* \in G$.

We conclude that each non-empty $\sqsubseteq$-chain of elements of G has an upper bound in G. Because $s \in G$, we have $G \neq \varnothing$. Then Zorn's lemma [43] implies that G has some $\sqsubseteq$-maximal element s'. Then $s' \in S_{max}(\Sigma, i)$ and $s \sqsubseteq s'$. □

Lemma 2.8. Let $\Sigma = (T, Q, Tr)$ be a I/O NCMS, $Q' \subseteq Q$, and $i \in Sb(In(\Sigma), W)$. Then

1) $o_{all}(\Sigma, Q', i) \subseteq Sb(Out(\Sigma), W)$;

2) $dom(o) \subseteq dom(i)$ for each $o \in o_{all}(\Sigma, Q', i)$;

3) $o_{all}(\Sigma, Q', i) \neq \varnothing$.

Proof. 1) Let $o \in o_{all}(\Sigma, Q', i)$ be an arbitrary element.

If $Q' = \varnothing$ or $dom(i) \subseteq \{0\}$, then $o = \perp$ or o has a form $\{0\} \mapsto out(q)$ for some $q \in Q$, whence $o \in Sb(Out(\Sigma), W)$, because $out(q) \in^{Out(\Sigma)} W$ for any $q \in Q$.

Consider the case when $\{0\} \subset dom(i)$ and $Q' \neq \varnothing$. Then either o has a form $\{0\} \mapsto out(q)$ for some $q \in Q$, or $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$.

In the former case, $o \in Sb(Out(\Sigma), W)$. In the latter case, $s \in Tr$, $dom(s) \in \mathcal{T}_0$, and $(out \circ s)(t) \in^{Out(\Sigma)} W$ for all $t \in dom(s)$, whence $o = out \circ s \in Sb(Out(\Sigma), W)$. In all cases, $o \in Sb(Out(\Sigma), W)$.

2) Let $o \in o_{all}(\Sigma, Q', i)$ be an arbitrary element.

If $Q' = \varnothing$ or $i = \perp$, then $o = \perp$, whence $dom(o) \subseteq dom(i)$.

If $Q' \neq \varnothing$ and $dom(i) = \{0\}$, then $dom(o) = \{0\} \subseteq dom(i)$.

If $Q' \neq \varnothing$ and $\{0\} \subset dom(i)$, then either $dom(o) = \{0\} \subset dom(i)$, or $dom(o) = dom(out \circ s) = dom(s)$ for some $s \in S_{max}(\Sigma, i)$. In the latter case, $in \circ s \preceq i$, whence $dom(o) = dom(s) \subseteq dom(i)$. In all cases, $dom(o) \subseteq dom(i)$.

3) If $Q' = \varnothing$, or $i = \perp$, or $Q' \neq \varnothing$ and $dom(i) = \{0\}$, then $o_{all}(\Sigma, Q', i) \neq \varnothing$ immediately from the definition of o_{all}.

Consider the case when $Q' \neq \varnothing$ and $\{0\} \subset dom(i)$.
If there exists $q \in Q' \setminus S_{init}(\Sigma, i)$, then $\{0\} \mapsto out(q)$ belongs to $o_{all}(\Sigma, Q', i)$. Otherwise, $Q' \subseteq S_{init}(\Sigma, i)$. Let us choose any $q \in Q'$ (it exists because $Q' \neq \varnothing$). Then $q \in S_{init}(\Sigma, i)$ and there exists $s \in S(\Sigma, i)$ such that $s(0) = q$. Then by Lemma 2.7, there exists $s' \in S_{max}(\Sigma, i)$ such that $s \sqsubseteq s'$, whence $s'(0) = q$. Then $out \circ s' \in o_{all}(\Sigma, Q', i)$. □

Lemma 2.9. Each initial I/O NCMS is a NCMS representation of a unique (up to semantic identity) block.

Proof. Uniqueness up to semantic identity is obvious from Definition 2.11. Let us prove that if (Σ, Q_0) is an initial I/O NCMS, where $\Sigma = (T, Q, Tr)$, then it is a NCMS representation of some block.

Let $i \in Sb(In(\Sigma), W)$. Let us show that $O_{all}(\Sigma, Q_0, i)$ is a non-empty subset of $Sb(Out(\Sigma), W)$ and $dom(o) \subseteq dom(i)$ for all $o \in O_{all}(\Sigma, Q_0, i)$. This is obvious, if $dom(i) = \varnothing$. Consider the case when $dom(i) \neq \varnothing$. Then

$$O_{all}(\Sigma, Q_0, i) = \bigcup_{x \in IState(\Sigma)} o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i).$$

For each $x \in IState(\Sigma)$ we have $Sel_{1,2}(Q_0, i(0), x) \subseteq Q_0 \subseteq Q$. Besides, $IState(\Sigma) \neq \varnothing$. Then Lemma 2.8 implies that $O_{all}(\Sigma, Q_0, i) \in 2^{Sb(Out(\Sigma), W)} \setminus \{\varnothing\}$ and $dom(o) \subseteq dom(i)$ for all $o \in O_{all}(\Sigma, Q_0, i)$. Thus (Σ, Q_0) is a NCMS representation of a block. □

Lemma 2.10. Let B be a deterministic causal block. Then B has a NCMS representation.

Proof. Let us denote $X = \{i \in Sb(In(B), W) \mid \exists t \in T \; dom(i) = [0, t]\}$ and $Q = {}^{In(B)}W \times X \times {}^{Out(B)}W$. Then $X \neq \varnothing$. Let in, $istate$, out denote projection maps from Q on the first, second, and third coordinate respectively.

Let Tr be the set of all functions of the form $s:A\to Q$, where $A\in\mathfrak{T}$, such that the following conditions hold:

a) for each $t\in dom(s)\setminus\{0\}$ we have $dom(istate(s(t)))=[0,t]$ and $istate(s(t))(t)=in(s(t))$, and if $s(0)\downarrow$, then $istate(s(t))(0)=in(s(0))$;

b) for each $t\in dom(s)$ we have $t\in dom(o)$ and $out(s(t))=o(t)$, where o is a unique member of $Op(B)(istate(s(t))$;

c) if $t_1,t_2\in dom(s)\setminus\{0\}$ and $t_1\le t_2$, then $istate(s(t_1))\subseteq istate(s(t_2))$.

Let us show that $\Sigma=(T,Q,Tr)$ is a NCMS.

Firstly, let us check that Tr is closed under proper restrictions (CPR). Let $s\in Tr$, $A\in\mathfrak{T}$, and $A\subseteq dom(s)$. Then s satisfies a)-c). Then $dom(s|_A)=A\in\mathfrak{T}$ and $s|_A$ satisfies a)-c), whence $s|_A\in Tr$.

Secondly, let us check that Tr is Markovian. Assume that $s_1,s_2\in Tr$, $t^*=\max dom(s_1)=\min dom(s_2)$, and $s_1(t^*)=s_2(t^*)$.

Let $s:dom(s_1)\cup dom(s_2)\to Q$ be a function such that $s(t)=s_1(t)$, if $t\in dom(s_1)$ and $s(t)=s_2(t)$, if $t\in dom(s_2)$.

Let us show that s satisfies the condition a). Let $t\in dom(s)\setminus\{0\}$. Then $t\in dom(s_1)\setminus\{0\}$ or $t\in dom(s_2)\setminus\{0\}$ and because s_1,s_2 satisfy the condition a), we have $dom(istate(s(t)))=[0,t]$ and $istate(s(t))(t)=in(s(t))$. Assume that $s(0)\downarrow$. Then $s_1(0)\downarrow$. If $t\in dom(s_1)\setminus\{0\}$, then

$$istate(s(t))(0)=istate(s_1(t))(0)=in(s_1(0))=in(s(0)),$$

because s_1 satisfies a). Otherwise, $t\in dom(s_2)$ and $0<t^*\le t$. Because s_2 satisfies c), we have $istate(s_2(t^*))\subseteq istate(s_2(t))$. Because $t^*\in dom(s_1)\setminus\{0\}$ and $s_1(0)\downarrow$, we have $istate(s_1(t^*))(0)=in(s_1(0))$. Then

$$istate(s(t))(0)=istate(s_2(t))(0)=istate(s_1(t^*))(0)=in(s_1(0))=in(s(0)).$$

Thus s satisfies the condition a).

Moreover, s satisfies b), because $s|_{dom(s_1)} = s_1$, $s|_{dom(s_2)} = s_2$, and s_1, s_2 satisfy the condition b).

Let us show that s satisfies the condition c). Let $t_1, t_2 \in dom(s) \setminus \{0\}$ and $t_1 \le t_2$. If $t_1 \le t^* \le t_2$, then $istate(s(t_1)) \sqsubseteq istate(s(t^*)) \sqsubseteq istate(s(t_2))$, then $istate(s(t_1)) \sqsubseteq istate(s(t_2))$. Otherwise, both t_1, t_2 belong to $dom(s_1)$ or $dom(s_2)$ and $istate(s(t_1)) \sqsubseteq istate(s(t_2))$ also holds. Thus s satisfies the condition c).

We conclude that Tr is Markovian.

Thirdly, let us check that Tr is complete in the sense of Definition 2.3. Let $c \subseteq Tr$ be a non-empty $\sqsubseteq$-chain. Let $s^* : \bigcup_{s \in c} dom(s) \to Q$ be a function such that the graph of s^* is a union of graphs of all elements of c (this is indeed a function, because c is a chain). Then $dom(s^*) \in \mathfrak{T}$ (because $c \ne \varnothing$) and s^* satisfies a)-c) because each $s \in c$ satisfies a)-c). Thus $s^* \in Tr$. It follows that s^* is a least upper bound of c in Tr viewed as a poset with respect to $\sqsubseteq$.

We conclude that Σ is a NCMS.

Let $i \in Sb(In(B), W)$ and $o \in Op(B)(i)$.

Let us show that $out \circ s = o|_{dom(s)}$ for each $s \in S(\Sigma, i)$, and if $s \in S_{max}(\Sigma, i)$, then $out \circ s = o$.

Let $s \in S(\Sigma, i)$. Then $dom(s) \in \mathcal{T}_0$ and $in \circ s \preceq i$ by the definition of $S(\Sigma, i)$, and $istate(s(t))(t) = in(s(t)) = i(t)$ for all $t \in dom(s) \setminus \{0\}$ by the condition a).

If $t', t \in dom(s)$ and $0 < t' \le t$, then $i(t') = istate(s(t'))(t') = istate(s(t))(t')$ by the condition c). Moreover, we have $s(0)\downarrow$, whence $istate(s(t))(0) = in(s(0)) = i(0)$ for each $t \in dom(s) \setminus \{0\}$ by the condition a). Then for each $t \in dom(s) \setminus \{0\}$ we have $istate(s(t)) = i|_{[0,t]}$, because $dom(istate(s(t))) = [0, t]$. Then

$$Op(B)(istate(s(t))) = Op(B)(i|_{[0,t]}) = \{o|_{[0,t]}\},$$

because B is deterministic and causal. Then $out(s(t)) = (o|_{[0,t]})(t)$ and $t \in dom(o)$ for each $t \in dom(s)$ by the condition b). This implies that $dom(s) \subseteq dom(o)$ and for all $t \in dom(s)$, $out(s(t)) = o(t)$. Thus $out \circ s = o|_{dom(s)}$.

We have $\{0\} \subset dom(s) \subseteq dom(o)$, so $dom(o) \in \mathfrak{T}$. Because $in(s(0)) = i(0)$ and $out(s(0)) = o(0)$, it follows that a function $s' : dom(o) \to Q$ such that $s'(0) = s(0)$ and $s'(t) = (i(t), i|_{[0,t]}, o(t))$ for all $t \in dom(o) \setminus \{0\}$ satisfies the conditions a)-c). Moreover, $s' \in Tr$, $dom(s') \in \mathcal{T}_0$, and $in \circ s' = i|_{dom(o)} \preceq i$. Then $s' \in S(\Sigma, i)$. Besides, $s'|_{dom(s)} = s$.

This implies that if $s \in S_{max}(\Sigma, i)$, then $s' = s$ and $out \circ s = o$.

Now let us denote

$$Q_0 = \{(d_{in}, x, d_{out}) \in Q \mid \exists (i,o) \in IO(B)\ \{0\} \in dom(o) \wedge d_{in} = i(0) \wedge d_{out} = o(0)\}.$$

Let us prove that $S_{init}(\Sigma) \subseteq Q_0$. Let $(d_{in}, x, d_{out}) \in S_{init}(\Sigma)$. Then $(d_{in}, x, d_{out}) = s(0)$ for some $s \in Tr$ such that $dom(s) \in \mathcal{T}_0$, then $s \in S(\Sigma, in \circ s)$, whence $out \circ s = o|_{dom(s)}$, where o is the unique member of $Op(B)(in \circ s)$. Then $(in \circ s, o) \in IO(B)$, $\{0\} \in dom(o)$, and $d_{in} = in(s(0)) = (in \circ s)(0)$, and $d_{out} = out(s(0)) = o(0)$. Thus $(d_{in}, x, d_{out}) \in Q_0$.

We conclude that (Σ, Q_0) is an initial I/O NCMS. Obviously, $In(\Sigma) = In(B)$ and $Out(\Sigma) = Out(B)$.

Now let us prove the following property of Q_0:

d) if $(i,o) \in IO(B)$, $q \in Q_0$, $i \neq \perp$, and $in(q) = i(0)$, then $o \neq \perp$ and $out(q) = o(0)$.

Indeed, if $(i,o) \in IO(B)$, $i \neq \perp$, and $q \in Q_0$, then there exists $(i', o') \in IO(B)$ such that $\{0\} \in dom(o') \subseteq dom(i')$, $in(q) = i'(0)$, $out(q) = o'(0)$. Because $i'|_{\{0\}} = i|_{\{0\}}$ and B is deterministic and causal, we have $o'|_{\{0\}} = o|_{\{0\}}$, whence $o \neq \perp$ and $o(0) = o'(0) = out(q)$.

Now let us show that (Σ, Q_0) is a NCMS representation of B.

It is sufficient to show that $Op(B)(i) = O_{all}(\Sigma, Q_0, i)$ for all $i \in Sb(In(B), W)$. This is obvious, if $i = \perp$.

Let $i \in Sb(In(B), W) \setminus \{\perp\}$ and $o \in Op(B)(i)$ be arbitrary elements. Then

$$O_{all}(\Sigma, Q_0, i) = \bigcup_{x \in IState(\Sigma)} o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i).$$

Consider the following cases:

- $Sel_{1,2}(Q_0, i(0), x) = \varnothing$ for some $x \in IState(\Sigma)$. Then there is no pair $(i', o') \in IO(B)$ such that $i'(0) = i(0)$ and $o'(0)\downarrow$. For all $x \in IState(\Sigma)$, $o = \perp$ and $Sel_{1,2}(Q_0, i(0), x) = \varnothing$. Then $O_{all}(\Sigma, Q_0, i) = \{\perp\} = Op(B)(i)$.
- $Sel_{1,2}(Q_0, i(0), x) \neq \varnothing$ for all $x \in IState(\Sigma)$ and $dom(i) = \{0\}$. Then $o(0)\downarrow$ and $out(q) = o(0)$ for each $q \in Sel_{1,2}(Q_0, i(0), x) \subseteq Q_0$ by the property d). Then $o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) = \{\{0\} \mapsto o(0)\} = \{o\}$ for all $x \in IState(\Sigma)$, whence $O_{all}(\Sigma, Q_0, i) = Op(B)(i)$.
- $Sel_{1,2}(Q_0, i(0), x) \neq \varnothing$ for all $x \in IState(\Sigma)$, $\{0\} \subset dom(i)$, and $dom(o) \subseteq \{0\}$. If $in(q) = i(0)$ for some $q \in S_{init}(\Sigma, i)$, then $q = s(0)$ for some $s \in S(\Sigma, i)$, whence $out \circ s = o|_{dom(s)}$ as we have shown above, but this is impossible, because $\{0\} \subset dom(s)$ and $dom(o) \subseteq \{0\}$. Thus $in(q) \neq i(0)$ for each $q \in S_{init}(\Sigma, i)$. Then for each $x \in IState(\Sigma)$ $s(0) \notin Sel_{1,2}(Q_0, i(0), x)$ holds for all $s \in S_{max}(\Sigma, i)$ and $Sel_{1,2}(Q_0, i(0), x) \cap S_{init}(\Sigma, i) = \varnothing$. Then for each $x \in IState(\Sigma)$, $o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) = \{\{0\} \mapsto out(q) \mid q \in Sel_{1,2}(Q_0, i(0), x)\}$, whence $O_{all}(\Sigma, Q_0, i) = \{\{0\} \mapsto out(q) \mid q \in Q_0 \wedge in(q) = i(0)\} \neq \varnothing$. Because for some $q \in Q_0$, $in(q) = i(0)$ by the property d), we have $0 \in dom(o)$ and $O_{all}(\Sigma, Q_0, i) = \{\{0\} \mapsto o(0)\} = \{o\} = Op(B)(i)$.
- $Sel_{1,2}(Q_0, i(0), x) \neq \varnothing$ for all $x \in IState(\Sigma)$ and $\{0\} \subset dom(o)$. We have $dom(o) \in \mathfrak{T}$. Let $x \in IState(\Sigma)$ and $q \in Sel_{1,2}(Q_0, i(0), x)$. Then $in(q) = i(0)$ and have $out(q) = o(0)$ by the property d). It is easy to see

that a function $s':dom(o)\to Q$ such that $s'(0)=q$ and $s'(t)=(i(t),i|_{[0,t]},o(t))$ for all $t\in dom(o)\setminus\{0\}$ satisfies a)-c). Moreover, $s'\in Tr$, $dom(s')\in \mathcal{T}_0$, and $in\circ s'=i|_{dom(o)}\preceq i$. Then $s'\in S(\Sigma,i)$. Then $s'(0)=q\in S_{init}(\Sigma,i)$. Because $q\in Sel_{1,2}(Q_0,i(0),x)$ is arbitrary, we have $Sel_{1,2}(Q_0,i(0),x)\subseteq S_{init}(\Sigma,i)$. Then for each $x\in IState(\Sigma)$, $o_{all}(\Sigma,Sel_{1,2}(Q_0,i(0),x),i)=\{o\}$, because $out\circ s=o$ for any $s\in S_{max}(\Sigma,i)$ as we have shown above and $Sel_{1,2}(Q_0,i(0),x)\neq\varnothing$. Then $O_{all}(\Sigma,Q_0,i)=\{o\}=Op(B)(i)$, because $IState(\Sigma)\neq\varnothing$.

In all possible cases we have $O_{all}(\Sigma,Q_0,i)=Op(B)(i)$. Thus (Σ,Q_0) is a NCMS representation of the block B. □

Let $\Sigma_1=(T,Q_1,Tr_1)$ and $\Sigma_2=(T,Q_2,Tr_2)$ be I/O NCMS such that $In(\Sigma_1)=In(\Sigma_2)$ and $Out(\Sigma_1)=Out(\Sigma_2)$.

Let us introduce the following notions

Definition 2.12.

1) A state embedding from Σ_1 to Σ_2 is a function $f:Q_1\to Q_2$ such that $\{f\circ s\mid s\in Tr_1\}=\{s\in Tr_2\mid \exists t\in dom(s)\,\exists q\in Q_1\; s(t)=f(q)\}$ and there exists an injective function $g:IState(\Sigma_1)\to IState(\Sigma_2)$ such that for all $q\in Q_1$,
$$f(q)=(in(q),g(istate(q)),out(q)).$$
2) A state embedding from an initial I/O NCMS (Σ_1,Q_0^1) to an initial I/O NCMS (Σ_2,Q_0^2) is a state embedding f from Σ_1 to Σ_2 such that for each $q\in Q_1$, $q\in Q_0^1$ if and only if $f(q)\in Q_0^2$.

Note that it follows immediately from this definition that a state embedding from Σ_1 to Σ_2 is an injective function.

Lemma 2.11. Let $\Sigma_1=(T,Q_1,Tr_1)$ and $\Sigma_2=(T,Q_2,Tr_2)$ be I/O NCMS, $In(\Sigma_1)=In(\Sigma_2)$ and $Out(\Sigma_1)=Out(\Sigma_2)$, and f be a state embedding from Σ_1 to Σ_2. Let $i\in Sb(In(\Sigma_1),W)$. Then $S_{max}(\Sigma_2,i)\supseteq\{f\circ s\mid s\in S_{max}(\Sigma_1,i)\}$ and

$$\{q \in S_{init}(\Sigma_2,i) \mid \exists q' \in Q_1\ q = f(q')\} = \{f(q'') \mid q'' \in S_{init}(\Sigma_1,i)\}.$$

Proof. Because $\{f \circ s \mid s \in Tr_1\} \subseteq Tr_2$, we have the following:

$$S(\Sigma_2,i) = \{s \in Tr_2 \mid dom(s) \in \mathcal{T}_0 \wedge in \circ s \preceq i\} \supseteq \{f \circ s \mid s \in Tr_1 \wedge$$
$$\wedge\, dom(f \circ s) \in \mathcal{T}_0 \wedge in \circ (f \circ s) \preceq i\} = \{f \circ s \mid s \in Tr_1 \wedge dom(s) \in \mathcal{T}_0 \wedge in \circ s \preceq i\} =$$
$$= \{f \circ s \mid s \in S(\Sigma_1,i)\}.$$

Let $s \in S_{max}(\Sigma_1,i)$. Then $f \circ s \in S(\Sigma_2,i)$. Suppose that $f \circ s \notin S_{max}(\Sigma_2,i)$. Then $f \circ s \sqsubset s'$ for some $s' \in S(\Sigma_2,i)$. Because $dom(s) \neq \varnothing$, $s'(t) \in \{f(q) \mid q \in Q_1\}$ for some $t \in T$. Then $s' = f \circ s''$ for some $s'' \in Tr_1$, because $\{f \circ s \mid s \in Tr_1\} = \{s \in Tr_2 \mid \exists t \in dom(s)\, \exists q \in Q_1\ s(t) = f(q)\}$. Then $f \circ s \sqsubset f \circ s''$, whence $s \sqsubset s''$, because f is injective. Besides, $dom(s'') = dom(s') \in \mathcal{T}_0$ and $in \circ s'' = in \circ s' \preceq i$. Then $s'' \in S(\Sigma_1,i)$. We get a contradiction with the assumption $s \in S_{max}(\Sigma_1,i)$. We conclude that $f \circ s \in S_{max}(\Sigma_2,i)$. Thus we have $S_{max}(\Sigma_2,i) \supseteq \{f \circ s \mid s \in S_{max}(\Sigma_1,i)\}$.

Now let us show that

$$\{q \in S_{init}(\Sigma_2,i) \mid \exists q' \in Q_1\ q = f(q')\} = \{f(q'') \mid q'' \in S_{init}(\Sigma_1,i)\}.$$

Let $q \in S_{init}(\Sigma_2,i)$ and $q' \in Q_1$ be such that $q = f(q')$. Then $q = s(0)$ for some $s \in S(\Sigma_2,i)$. Then $s \in Tr_2$ and $s(0) = f(q')$, where $q' \in Q_1$. Then there exists $s' \in Tr_1$ such that $s = f \circ s'$. Moreover, $dom(s') = dom(s) \in \mathcal{T}_0$ and $in \circ s' = in \circ s \preceq i$. Then $s' \in S(\Sigma_1,i)$ and $s'(0) \in S_{init}(\Sigma_1,i)$. Thus

$$q = s(0) = f(s'(0)) \in \{f(q'') \mid q'' \in S_{init}(\Sigma_1,i)\}.$$

Conversely, let $q'' \in S_{init}(\Sigma_1,i)$. Then $q'' = s(0)$ for some $s \in S(\Sigma_1,i)$. Then $f \circ s \in S(\Sigma_2,i)$, whence $f(q'') = f(s(0)) = (f \circ s)(0) \in S_{init}(\Sigma_2,i)$. Then because $q'' \in Q_1$, we have $f(q'') \in \{q \in S_{init}(\Sigma_2,i) \mid \exists q' \in Q_1\ q = f(q')\}$. Thus

$$\{q \in S_{init}(\Sigma_2,i) \mid \exists q' \in Q_1\ q = f(q')\} = \{f(q'') \mid q'' \in S_{init}(\Sigma_1,i)\}. \ \square$$

Lemma 2.12. For $j=1,2$ let (Σ_j, Q_0^j) be a NCMS representation of a block B_j. Assume that $In(\Sigma_1)=In(\Sigma_2)$ and $Out(\Sigma_1)=Out(\Sigma_2)$ and there exists a state embedding f from (Σ_1, Q_0^1) to (Σ_2, Q_0^2). Then $B_1 \trianglelefteq B_2$.

Proof. Assume that $\Sigma_1=(T, Q_1, Tr_1)$ and $\Sigma_2=(T, Q_2, Tr_2)$. We have $In(B_1)=In(\Sigma_1)=In(\Sigma_2)=In(B_2)$ and $Out(B_1)=Out(\Sigma_1)=Out(\Sigma_2)=Out(B_2)$. Because f is a state embedding, there exists an injective function $g: IState(\Sigma_1) \to IState(\Sigma_2)$ such that for all $q \in Q$,

$$f(q)=(in(q), g(istate(q)), out(q)).$$

Let $i \in Sb(In(B), W)$. Then for $j=1,2$, $Op(B_j)(i)=O_{all}(\Sigma_j, Q_0^j, i)$. Let us show that $O_{all}(\Sigma_2, Q_0^2, i) \supseteq O_{all}(\Sigma_1, Q_0^1, i)$. This is obvious, if $i=\perp$, so assume $i \neq \perp$.

Let us fix some $x_1 \in IState(\Sigma_1)$. Denote $Q_1'=Sel_{1,2}(Q_0^1, i(0), x_1)$ and $Q_2'=Sel_{1,2}(Q_0^2, i(0), g(x_1))$. Because g is injective and $Q_0^2 \supseteq \{f(q) \mid q \in Q_0^1\}$,

$$\begin{aligned}
Q_2' &= \{q \in Q_0^2 \mid in(q)=i(0) \wedge istate(q)=g(x_1)\} \supseteq \\
&\supseteq \{f(q) \mid q \in Q_0^1 \wedge in(f(q))=i(0) \wedge istate(f(q))=g(x_1)\} = \\
&= \{f(q) \mid q \in Q_0^1 \wedge in(q)=i(0) \wedge g(istate(q))=g(x_1)\} = \\
&= \{f(q) \mid q \in Sel_{1,2}(Q_0^1, i(0), x_1)\} = \{f(q) \mid q \in Q_1'\}.
\end{aligned}$$

Let us show that $Q_2' \neq \varnothing$ if and only if $Q_1' \neq \varnothing$. Indeed, if $Q_1' \neq \varnothing$, then $Q_2' \neq \varnothing$, because $Q_2' \supseteq \{f(q) \mid q \in Q_1'\}$. Conversely, if $Q_2' \neq \varnothing$, then $(i(0), g(x_1), d) \in Q_0^2$ for some $d \in Out(\Sigma_2)$, whence $(i(0), x_1, d) \in Q_1$ and $f((i(0), x_1, d)) \in Q_0^2$, whence, $(i(0), x_1, d) \in Q_0^1$, because f is a state embedding, and finally, $Q_1'=Sel_{1,2}(Q_0^1, i(0), x_1) \neq \varnothing$.

Now let us show that $o_{all}(\Sigma_2, Q_2', i) \supseteq o_{all}(\Sigma_1, Q_1', i)$. This is obvious, if $Q_1'=\varnothing$ or $Q_2'=\varnothing$, because $Q_2' \neq \varnothing$ if and only if $Q_1' \neq \varnothing$ as we have shown above. So let us assume that $Q_1' \neq \varnothing$ and $Q_2' \neq \varnothing$.

Consider the case when $dom(i) = \{0\}$. Because $Q_2' \supseteq \{f(q) \mid q \in Q_1'\}$,

$$o_{all}(\Sigma_2, Q_2', i) = \{\{0\} \mapsto out(q) \mid q \in Q_2'\} \supseteq \{\{0\} \mapsto out(f(q)) \mid q \in Q_1'\} =$$
$$= \{\{0\} \mapsto out(q) \mid q \in Q_1'\} = o_{all}(\Sigma_1, Q_1', i).$$

Now consider the case when $\{0\} \subset dom(i)$. Then

$$o_{all}(\Sigma_2, Q_2', i) = \{out \circ s \mid s \in S_{max}(\Sigma_2, i) \wedge s(0) \in Q_2'\} \cup$$
$$\cup \{\{0\} \mapsto out(q) \mid q \in Q_2' \setminus S_{init}(\Sigma_2, i)\}.$$

By Lemma 2.11 we have $S_{max}(\Sigma_2, i) \supseteq \{f \circ s \mid s \in S_{max}(\Sigma_1, i)\}$ and $\{q \in S_{init}(\Sigma_2, i) \mid \exists q' \in Q_1 \; q = f(q')\} = \{f(q'') \mid q'' \in S_{init}(\Sigma_1, i)\}$.

Because f is injective and $Q_2' \supseteq \{f(q) \mid q \in Q_1'\}$,

$$\{out \circ s \mid s \in S_{max}(\Sigma_2, i) \wedge s(0) \in Q_{2'}\} \supseteq \{out \circ (f \circ s) \mid s \in S_{max}(\Sigma_1, i) \wedge$$
$$\wedge f(s(0)) \in Q_2'\} \supseteq \{out \circ s \mid s \in S_{max}(\Sigma_1, i) \wedge s(0) \in Q_1'\}.$$

Moreover, because $Q_1' \subseteq Q_1$ and f is injective, we have

$$Q_2' \setminus S_{init}(\Sigma_2, i) \supseteq \{f(q) \mid q \in Q_1'\} \setminus S_{init}(\Sigma_2, i) =$$
$$= \{f(q) \mid q \in Q_1'\} \setminus \{q \in S_{init}(\Sigma_2, i) \mid \exists q' \in Q_1 \; q = f(q')\} =$$
$$= \{f(q) \mid q \in Q_1'\} \setminus \{f(q'') \mid q'' \in S_{init}(\Sigma_1, i)\} =$$
$$= \{f(q) \mid q \in Q_1' \setminus S_{init}(\Sigma_1, i)\}.$$

Then

$$\{\{0\} \mapsto out(q) \mid q \in Q_2' \setminus S_{init}(\Sigma_2, i)\} \supseteq \{\{0\} \mapsto out(f(q)) \mid$$
$$q \in Q_1' \setminus S_{init}(\Sigma_1, i)\} = \{\{0\} \mapsto out(q) \mid q \in Q_1' \setminus S_{init}(\Sigma_1, i)\}.$$

Finally, we have $o_{all}(\Sigma_2, Q_2', i) \supseteq o_{all}(\Sigma_1, Q_1', i)$.

We conclude that for each $x_1 \in IState(\Sigma_1)$,

$$o_{all}(\Sigma_2, Sel_{1,2}(Q_0^2, i(0), g(x_1)), i) \supseteq o_{all}(\Sigma_1, Sel_{1,2}(Q_0^1, i(0), x_1), i).$$

Then because $i \neq \perp$ by our assumption, we have

$$Op(B_2)(i) = O_{all}(\Sigma_2, Q_0^2, i) = \bigcup_{x_2 \in IState(\Sigma_2)} o_{all}(\Sigma_2, Sel_{1,2}(Q_0^2, i(0), x_2), i) \supseteq$$
$$\supseteq \bigcup_{x_1 \in IState(\Sigma_1)} o_{all}(\Sigma_2, Sel_{1,2}(Q_0^2, i(0), g(x_1)), i) \supseteq$$

$$\supseteq \bigcup_{x_1 \in IState(\Sigma_1)} o_{all}(\Sigma_1, Sel_{1,2}(Q_0^1, i(0), x_1), i) = O_{all}(\Sigma_1, Q_0^1, i) = Op(B_1)(i).$$

We conclude that $B_1 \trianglelefteq B_2$. □

Definition 2.13. A disjoint union of an indexed family of initial I/O NCMS $((\Sigma_j, Q_0^j))_{j \in J}$, where $J \neq \varnothing$ and $\Sigma_j = (T, Q_j, Tr_j)$ for each $j \in J$, is a pair (Σ, Q_0), where $\Sigma = (T, Q, Tr)$ and

1) $Q = {}^{IN} W \times (\bigcup_{j \in J} \{j\} \times IState(\Sigma_j)) \times^{OUT} W$, where $IN = \bigcup_{j \in J} In(\Sigma_j)$, and $OUT = \bigcup_{j \in J} Out(\Sigma_j)$;
2) $Tr = \{f_j \circ s \mid j \in J \wedge s \in Tr_j\}$;
3) $Q_0 = \{f_j(q) \mid j \in J \wedge q \in Q_0^j\}$;

where for each $j \in J$, $f_j : Q_j \to Q$ is a function such that

$$f_j(q) = (in(q), (j, istate(q)), out(q)), q \in Q_j.$$

Lemma 2.13. Let (Σ, Q_0) be a disjoint union of an indexed family of initial I/O NCMS $((\Sigma_j, Q_0^j))_{j \in J}$, where $J \neq \varnothing$. Then (Σ, Q_0) is an initial I/O NCMS.

Proof. Assume $\Sigma = (T, Q, Tr)$ and $\Sigma_j = (T, Q_j, Tr_j)$ for each $j \in J$. For each $j \in J$, let $f_j : Q_j \to Q$ be defined as in Definition 2.13.

Let us show that Σ is a NCMS. Let $IN = \bigcup_{j \in J} In(\Sigma_j)$, $OUT = \bigcup_{j \in J} Out(\Sigma_j)$, and for each $j \in J$ let $Q'_j = {}^{IN} W \times (\{j\} \times IState(\Sigma_j)) \times^{OUT} W$.

Then f_j is an injective function from Q_j to Q'_j. Then because Σ_j is a NCMS, the triple (T, Q'_j, Tr'_j), where $Tr'_j = \{f_j \circ s \mid s \in Tr_j\}$, is a NCMS by Lemma 2.5. For each $j, j' \in J$ such that $j \neq j'$ we have $Q'_j \cap Q'_{j'} = \varnothing$. Moreover, $Q = \bigcup_{j \in J} Q'_j$ and $Tr = \bigcup_{j \in J} Tr'_j$ by Definition 2.13. Then $\Sigma = (T, Q, Tr)$ is a NCMS by Lemma 2.6. Because $\bigcup_{j \in J} \{j\} \times IState(\Sigma_j) \neq \varnothing$, Σ is an I/O NCMS.

For each $j \in J$, (Σ_j, Q_0^j) is an initial I/O NCMS, so $S_{init}(\Sigma_j) \subseteq Q_0^j \subseteq Q_j$. Then

$$S_{init}(\Sigma) = \{s(0) \mid s \in Tr \wedge dom(s) \in \mathcal{T}_0\} =$$
$$= \{(f_j \circ s)(0) \mid j \in J \wedge s \in Tr_j \wedge dom(f_j \circ s) \in \mathcal{T}_0\} =$$
$$= \{f_j(q) \mid j \in J \wedge q \in S_{init}(\Sigma_j)\} \subseteq \{f_j(q) \mid j \in J \wedge q \in Q_0^j\} = Q_0.$$

Obviously, $Q_0 \subseteq Q$, so $S_{init}(\Sigma) \subseteq Q_0 \subseteq Q$. We conclude that (Σ, Q_0) is an initial I/O NCMS. □

Definition 2.14.

1) A complete set of sub-blocks of a block B is a set $\mathcal{B}$ of sub-blocks of B such that $IO(B) = \bigcup_{B' \in \mathcal{B}} IO(B')$.
2) A complete indexed family of sub-blocks of a block B is an indexed family $(B_j)_{j \in J}$ such that $\{B_j \mid j \in J\}$ is a complete set of sub-blocks of B.

Lemma 2.14. Let $(B_j)_{j \in J}$ be a complete indexed family of sub-blocks of a block B, where $J \neq \varnothing$. Assume that for each $j \in J$, B_j has a NCMS representation (Σ_j, Q_0^j). Let (Σ, Q_0) be a disjoint union of $((\Sigma_j, Q_0^j))_{j \in J}$. Then (Σ, Q_0) is a NCMS representation of B.

Proof. Assume that $\Sigma_j = (T, Q_j, Tr_j)$ for each $j \in J$ and $\Sigma = (T, Q, Tr)$.

By Lemma 2.13, (Σ, Q_0) is an initial I/O NCMS, whence by Lemma 2.9, there exists a block B' (unique up to semantic identity) such that (Σ, Q_0) is a NCMS representation of B'. Because $B_j \trianglelefteq B$, for each $j \in J$ we have $In(\Sigma_j) = In(B_j) = In(B)$ and $Out(\Sigma_j) = Out(B_j) = Out(B)$.

Because $J \neq \varnothing$, we have $In(B') = In(\Sigma) = \bigcup_{j \in J} In(\Sigma_j) = In(B)$ and $Out(B') = Out(\Sigma) = \bigcup_{j \in J} Out(\Sigma_j) = Out(B)$.

For each $j \in J$, let $g_j : IState(\Sigma_j) \to IState(\Sigma)$ and $f_j : Q_j \to Q$ be functions such that $g_j(x) = (j, x)$ for all $x \in IState(\Sigma_j)$, and for each $q \in Q_j$,

$$f_j(q) = (in(q), g_j(istate(q)), out(q)).$$

Let us show that $B \trianglelefteq B'$. Let us fix $j \in J$. Obviously, $In(\Sigma_j) = In(\Sigma)$, $Out(\Sigma_j) = Out(\Sigma)$, and the function g_j is injective. Because $\{f_{j'}(q) \mid q \in Q_{j'}\} \cap \{f_{j''}(q) \mid q \in Q_{j''}\} = \varnothing$, if $j', j'' \in J$ and $j' \neq j''$, from the item 2 of Definition 2.13 it follows that

$$\{f_j \circ s \mid s \in Tr_j\} = \{s \in Tr \mid \exists t \in dom(s)\, \exists q \in Q_j\; s(t) = f_j(q)\}.$$

Then f_j is a state embedding from Σ_j to Σ. Moreover, from the item 3 of Definition 2.13 it follows that for each $q \in Q_j$, $q \in Q_0^j$ if and only if $f_j(q) \in Q_0$, because f_j is injective and $\{f_{j'}(q) \mid q \in Q_{j'}\} \cap \{f_{j''}(q) \mid q \in Q_{j''}\} = \varnothing$ for all $j', j'' \in J$ such that $j' \neq j''$. Thus f_j is a state embedding from (Σ_j, Q_0^j) to (Σ, Q_0). Then $B_j \trianglelefteq B'$ by Lemma 2.12. Because $j \in J$ is arbitrary, we conclude that $B \trianglelefteq B'$.

Now let us show that $B' \trianglelefteq B$. Let $(i,o) \in IO(B')$. Then $o \in O_{all}(\Sigma, Q_0, i)$.

If $i = \perp$, then $o = \perp$ and $(i,o) \in IO(B)$ (because B is a block).

Consider the case when $i \neq \perp$. Then there exists an element $x^* \in IState(\Sigma) = \bigcup_{j \in J} \{j\} \times IState(\Sigma_j)$ such that $o \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x^*), i)$. Then there exists $j \in J$ and $x_j^* \in IState(\Sigma_j)$ such that $x^* = (j, x_j^*)$.

Let $Q_j' = {}^{In(\Sigma)}W \times (\{j\} \times IState(\Sigma_j)) \times {}^{Out(\Sigma)}W$ and $\Sigma_j' = \Sigma|_{Q_j'}$. Then Σ_j' is a NCMS by Lemma 2.3. Let us denote by Tr_j' the set of trajectories of Σ_j'. Obviously, Q_j' is the set of states of Σ_j' and $In(\Sigma_j') = In(\Sigma)$, $Out(\Sigma_j') = Out(\Sigma)$. Besides, Σ_j' is an I/O NCMS and

$$S_{init}(\Sigma_j') = S_{init}(\Sigma|_{Q_j'}) \subseteq S_{init}(\Sigma) \cap Q_j' \subseteq Q_0 \cap Q_j' \subseteq Q_j',$$

because (Σ, Q_0) is an initial I/O NCMS. Denote $Q_{0,j}' = Q_0 \cap Q_j'$. Then $(\Sigma_j', Q_{0,j}')$ is an initial I/O NCMS. Moreover, $x^* \in IState(\Sigma_j')$.

Let us prove that $o \in O_{all}(\Sigma_j', Q_{0,j}', i)$. Denote $Q' = Sel_{1,2}(Q_0, i(0), x^*)$.

Firstly, let us show that $Sel_{1,2}(Q'_{0,j}, i(0), x^*) = Q'$. Indeed, if $q \in Q'_{0,j} = Q_0 \cap Q'_j$ and $in(q) = i(0)$, $istate(q) = x^*$, then $q \in Q'$. Conversely, if $q \in Q'$, then $q \in Q_0$, $in(q) = i(0), istate(q) = (j, x_j^*)$, whence $q \in Q'_j$ and $q \in Sel_{1,2}(Q'_{0,j}, i(0), x^*)$.

Secondly, let us show that $o \in o_{all}(\Sigma'_j, Q', i)$. Note that $o \in o_{all}(\Sigma, Q', i)$. If $Q' = \varnothing$ or $dom(i) \subseteq \{0\}$, from the definition of o_{all} we have $o \in o_{all}(\Sigma, Q', i) = o_{all}(\Sigma'_j, Q', i)$. Consider the case when $Q' \neq \varnothing$ and $\{0\} \subset dom(i)$. Because $o \in o_{all}(\Sigma, Q', i)$, the following two sub-cases are possible:

a) $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$ such that $s(0) \in Q'$. Then $s \in Tr$, $dom(s) \in \mathcal{T}_0$, and $in \circ s \preceq i$. Because $s(0) \in Q'$, we have $istate(s(0)) = x^* = (j, x_j^*)$. From the item 2 of Definition 2.13 for Σ it follows that the first component of the value $istate(s(t))$ is j for all $t \in dom(s)$. Then $s(t) \in Q'_j$ for all $t \in dom(s)$, whence $s \in Tr'_j$. Then $s \in S(\Sigma'_j, i)$. Moreover, $s \in S_{max}(\Sigma'_j, i)$, because otherwise, $s \sqsubset s'$ for some $s' \in S(\Sigma'_j, i)$, whence $s' \in S(\Sigma, i)$, and we get a contradiction with $s \in S_{max}(\Sigma, i)$. Thus $o \in o_{all}(\Sigma'_j, Q', i)$ by definition of o_{all}.

b) $o = \{0\} \mapsto out(q)$ for some $q \in Q' \setminus S_{init}(\Sigma, i)$. Then $q \neq s(0)$ for all $s \in Tr$ such that $dom(s) \in \mathcal{T}_0$ and $in \circ s \preceq i$. Because $\Sigma'_j = \Sigma|_{Q'_j}$, we have, in particular, $q \neq s(0)$ for all $s \in Tr'_j$ such that $dom(s) \in \mathcal{T}_0$ and $in \circ s \preceq i$. Then $q \in Q' \setminus S_{init}(\Sigma'_j, i)$, whence $o \in o_{all}(\Sigma'_j, Q', i)$.

Thus we conclude that $o \in o_{all}(\Sigma'_j, Q', i)$, $Q' = Sel_{1,2}(Q'_{0,j}, i(0), x^*)$, and $x^* \in IState(\Sigma'_j)$, whence $o \in O_{all}(\Sigma'_j, Q'_{0,j}, i)$, because $i \neq \perp$ by assumption.

By Lemma 2.9, there exists a block B'_j such that $(\Sigma'_j, Q'_{0,j})$ is a NCMS representation of B'_j. Let $g: IState(\Sigma'_j) \to IState(\Sigma_j)$ and $f: Q'_j \to Q_j$ be functions such that $g((j,x)) = x$ for all $x \in IState(\Sigma'_j)$ and $f(q) = (in(q), g(istate(q)), out(q))$ for all $q \in Q'_j$. Obviously, $In(\Sigma'_j) = In(\Sigma_j)$, $Out(\Sigma'_j) = Out(\Sigma_j)$, and g is injective. Moreover, f is an inverse of f_j, whence

$$\{f \circ s \mid s \in Tr'_j\} = \{f \circ s \mid s \in Tr \wedge \forall t \in dom(s)\ s(t) \in Q'_j\} =$$
$$= \{f \circ (f_{j'} \circ s) \mid j' \in J \wedge s \in Tr_{j'} \wedge \forall t \in dom(s)\ (f_{j'} \circ s)(t) \in Q'_j\} = Tr_j.$$

Because $dom(s) \neq \emptyset$ for each $s \in Tr_j$, and $s(t) = f(f_j(s(t)))$ and $f_j(s(t)) \in Q'_j$ for each $t \in dom(s)$, we have

$$\{f \circ s \mid s \in Tr'_j\} = \{s \in Tr_j \mid \exists t \in dom(s)\ \exists q \in Q'_j\ s(t) = f(q)\}.$$

Then f is a state embedding from Σ'_j to Σ_j. Moreover, for each $q \in Q'_j$, $q \in Q'_{0,j} = Q_0 \cap Q'_j$ if and only if $q = f_j(q')$ for some $q' \in Q_0^j$ if and only if $f(q) \in Q_0^j$. Then f is a state embedding from $(\Sigma'_j, Q'_{0,j})$ to (Σ_j, Q_0^j). Then $B'_j \trianglelefteq B_j$ by Lemma 2.12. As we have shown above, $o \in O_{all}(\Sigma'_j, Q'_{0,j}, i) = Op(B'_j)(i)$, so $o \in Op(B_j)(i)$, whence $(i,o) \in IO(B)$. We conclude that $B' \trianglelefteq B$.

We have shown that $B \trianglelefteq B'$ and $B' \trianglelefteq B$. Then B and B' are semantically identical. Then (Σ, Q_0) is a NCMS representation of B. □

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. Let B be a strongly nonanticipative block. Let us show that B has a NCMS representation.

Let $\mathcal{R}$ be the set of all relations $R \subseteq IO(B)$ such that R is an I/O relation of a deterministic causal block. For each $R \in \mathcal{R}$ let us define a block B_R such that $IO(B_R) = R$, $In(B_R) = In(B)$, $Out(B_R) = Out(B)$. Then B_R is a deterministic causal block for each $R \in \mathcal{R}$ and $IO(B) = \bigcup_{R \in \mathcal{R}} IO(B_R)$, because B is strongly nonanticipative. Then $(B_R)_{R \in \mathcal{R}}$ is a complete indexed family of sub-blocks of B and $\mathcal{R} \neq \emptyset$. By Lemma 2.10, for each $R \in \mathcal{R}$ there exists an initial I/O NCMS

(Σ_R, Q_0^R) which is a NCMS representation of B_R. Let (Σ, Q_0) be a disjoint union of $((\Sigma_R, Q_0^R))_{R\in\mathcal{R}}$. Then by Lemma 2.14, (Σ, Q_0) is a NCMS representation of B. □

2.7 Proof of the converse theorem about representation of a strongly nonanticipative block

In this section we give a proof of Theorem 2.3.

Lemma 2.15. Assume that a block B has a NCMS representation, $(i,o)\in IO(B)$, and $(i',o')\preceq^2(i,o)$. Then $(i',o')\in IO(B)$.

Proof. Let (Σ, Q_0) be a NCMS representation of B, where $\Sigma=(T,Q,Tr)$.

Because $(i',o')\preceq^2(i,o)$, we have $i'=i|_A$ an $o'=o|_A$ for some $A\in\mathcal{T}_0$. If $i=\perp$ or $A=\varnothing$, then $i'=o'=\perp$, so $(i',o')\in IO(B)$.

Let us assume that $i\neq\perp$ and $A\neq\varnothing$. Then $o\in Op(B)(i)=O_{all}(\Sigma,Q_0,i)$ and there exists $x\in IState(\Sigma)$ such that $o\in o_{all}(\Sigma,Q',i)$, where $Q'=Sel_{1,2}(Q_0,i(0),x)$. Moreover, $i'(0)\downarrow=i(0)$ and $Q'=Sel_{1,2}(Q_0,i'(0),x)$, whence

$$o_{all}(\Sigma,Q',i')\subseteq O_{all}(\Sigma,Q_0,i')=Op(B)(i').$$

Because $o\in o_{all}(\Sigma,Q',i)$, the following cases are possible.

1) $Q'=\varnothing$ and $o=\perp$. Then $o'=\perp\in o_{all}(\Sigma,Q',i')\subseteq Op(B)(i')$.

2) $dom(i)=\{0\}$ and $o=\{0\}\mapsto out(q)$ for some $q\in Q'$. Then $i'=i$ and $o'=o$, because $A\neq\varnothing$. Then $o'=o\in Op(B)(i)=Op(B)(i')$.

3) $\{0\}\subset dom(i)$ and $o=\{0\}\mapsto out(q)$ for some $q\in Q'\setminus S_{init}(\Sigma,i)$. Then $o'=o$, because $A\neq\varnothing$. If $A=\{0\}$, then $dom(i')=\{0\}$ and $q\in Q'$, so

$$o'=o=\{0\}\mapsto out(q)\in o_{all}(\Sigma,Q',i')\subseteq Op(B)(i').$$

Consider the case when $A\neq\{0\}$. Then $\{0\}\subset A$ and $\{0\}\subset dom(i')$, because $\{dom(i),A\}\subset\mathcal{T}_0$. Moreover, because $i'\preceq i$ and $q\notin S_{init}(\Sigma,i)$, we have $q\notin S_{init}(\Sigma,i')$, whence $q\in Q'\setminus S_{init}(\Sigma,i')$. Then $o'=o=\{0\}\mapsto out(q)\in o_{all}(\Sigma,Q',i')\subseteq Op(B)(i')$.

4) $\{0\} \subset dom(i)$ and $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$ such that $s(0) \in Q'$.

If $A = \{0\}$, then $dom(i') = \{0\}$ and $s(0) \in Q'$, so

$$o' = o|_{\{0\}} = \{0\} \mapsto out(s(0)) \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i').$$

Consider the case when $A \neq \{0\}$. Then $\{0\} \subset A$. Then $\{0\} \subset dom(i')$, because $\{dom(i), A\} \subset \mathcal{T}_0$ and $\{0\} \subset dom(i)$. By the CPR property of Σ, we have $s|_A \in Tr$. Besides, $in \circ (s|_A) = (in \circ s)|_A \preceq i|_A = i'$, because $in \circ s \preceq i$, so $s|_A \in S(\Sigma, i')$. Denote $s' = s|_A$. Because $s' \in S(\Sigma, i')$, by Lemma 2.7 there exists $\hat{s} \in S_{max}(\Sigma, i')$ such that $s' \sqsubseteq \hat{s}$. Because $in \circ \hat{s} \preceq i' = i|_A$, we have $dom(\hat{s}) \subseteq A$. Then

$$dom(s) \cap A = dom(s|_A) \subseteq dom(\hat{s}) \subseteq A.$$

Because $\{dom(s), A\} \subset \mathcal{T}_0$, we have either $A = dom(s|_A) = dom(\hat{s})$, or $dom(s) = dom(s|_A) \subseteq dom(\hat{s})$. In the former case, $s' = s|_A = \hat{s}$, because $s' \sqsubseteq \hat{s}$. In the latter case, $dom(s) \subseteq A$, whence $s = s|_A = s' \sqsubseteq \hat{s}$. Moreover, $\hat{s} \in S_{max}(\Sigma, i') \subseteq S(\Sigma, i)$, because $i' \preceq i$, and $s \in S_{max}(\Sigma, i)$, so $s = s' = \hat{s}$. So in both cases, $s' = \hat{s} \in S_{max}(\Sigma, i')$. Moreover, $o' = o|_A = out \circ s'$ and $s'(0) = s(0) \in Q'$. Then

$$o' = out \circ s' \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i').$$

In all possible cases $o' \in Op(B)(i')$. We conclude that $(i', o') \in IO(B)$. □

Lemma 2.16. Assume that a block B has a NCMS representation, $o \in Op(B)(i)$, and $i \preceq i'$. Then there exists $o' \in Op(B)(i')$ such that $(i, o) \preceq^2 (i', o')$.

Proof. Let (Σ, Q_0) be a NCMS representation of B, where $\Sigma = (T, Q, Tr)$.

Assume that $i = \perp$. Then $o = \perp$. We have $Op(B)(i') \neq \varnothing$. Let us choose an arbitrary $o' \in Op(B)(i')$. Then $(i, o) \preceq^2 (i', o')$.

Now let us assume that $i \neq \perp$. Then $o \in Op(B)(i) = O_{all}(\Sigma, Q_0, i)$ and there exists $x \in IState(\Sigma)$ such that $o \in o_{all}(\Sigma, Q', i)$, where $Q' = Sel_{1,2}(Q_0, i(0), x)$. Moreover, $i'(0) \downarrow = i(0)$ and $Q' = Sel_{1,2}(Q_0, i'(0), x)$, whence

$$o_{all}(\Sigma, Q', i') \subseteq O_{all}(\Sigma, Q_0, i') = Op(B)(i').$$

Because $o \in o_{all}(\Sigma, Q', i)$, the following cases are possible.

1) $Q' = \varnothing$ and $o = \perp$. Let $o' = \perp$. Then $o' = \perp \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i')$. Moreover, $i = i'|_{dom(i)}$ and $o = o'|_{dom(i)}$, so $(i,o) \preceq^2 (i',o')$.

2) $dom(i) = \{0\}$ and $o = \{0\} \mapsto out(q)$ for some $q \in Q'$.

If $dom(i') = \{0\}$, then $i = i'$, so for $o' = o$ we have $o' \in Op(B)(i')$ and $(i,o) \preceq^2 (i',o')$.

Consider the case when $\{0\} \subset dom(i')$.

If $q \notin S_{init}(\Sigma, i')$, then $o = \{0\} \mapsto out(q) \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i')$, so for $o' = o$ we have $i = i'|_{dom(i)}$ and $o = o'|_{dom(i)}$ (because $dom(o) \subseteq dom(i)$), whence $(i,o) \preceq^2 (i',o')$ and $o' \in Op(B)(i')$.

If $q \in S_{init}(\Sigma, i')$, then by Lemma 2.7 there exists $s \in S_{max}(\Sigma, i')$ such that $s(0) = q \in Q'$. Then $out \circ s \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i')$. Let $o' = out \circ s$. Then $i = i'|_{\{0\}}$ and $o = o'|_{\{0\}}$, because $dom(i) = \{0\}$ and $o'(0) = out(s(0)) = out(q) = o(0)$, so $(i,o) \preceq^2 (i',o')$. Besides, $o' \in Op(B)(i')$.

3) $\{0\} \subset dom(i)$ and $o = \{0\} \mapsto out(q)$ for some $q \in Q' \setminus S_{init}(\Sigma, i)$.

Let us show that $q \notin S_{init}(\Sigma, i')$. Suppose that $q \in S_{init}(\Sigma, i')$. Then there exists $s \in S(\Sigma, i')$ such that $s(0) = q$. Then $in \circ (s|_{dom(i)}) \preceq i'|_{dom(i)} = i$ and $s|_{dom(i)} \in Tr$ by the CPR property, so $s|_{dom(i)} \in S(\Sigma, i)$ and $q = (s|_{dom(i)})(0) \in S_{init}(\Sigma, i)$. This contradicts the assumption $q \in Q' \setminus S_{init}(\Sigma, i)$. Thus $q \notin S_{init}(\Sigma, i')$.

Then $\{0\} \subset dom(i')$ and $q \in Q' \setminus S_{init}(\Sigma, i')$. Let $o' = o$. Then $o' = \{0\} \mapsto out(q) \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i')$. Moreover, $i = i'|_{dom(i)}$ and $o = o'|_{dom(i)}$, so $(i,o) \preceq^2 (i',o')$.

4) $\{0\} \subset dom(i)$ and $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$ such that $s(0) \in Q'$.

We have $in \circ s \preceq i \preceq i'$, so $s \in S(\Sigma, i')$. By Lemma 2.7 there exists $s' \in S_{max}(\Sigma, i')$ such that $s \sqsubseteq s'$. Let $o' = out \circ s'$. Then $s'(0) = s(0) \in Q'$ and $\{0\} \subset dom(i')$, whence $o' = out \circ s' \in o_{all}(\Sigma, Q', i') \subseteq Op(B)(i')$. We have

$in \circ s'|_{dom(i)} \preceq i'|_{dom(i)} = i$ and $s'|_{dom(i)} \in Tr$ by the CPR property, so $s'|_{dom(i)} \in S(\Sigma, i)$. Also, $s \sqsubseteq s'|_{dom(i)}$, because $dom(s) \subseteq dom(i)$. Then $s = s'|_{dom(i)}$, because $s \in S_{max}(\Sigma, i)$. Then $o = o'|_{dom(i)}$. Moreover, $i = i'|_{dom(i)}$, so $(i,o) \preceq^2 (i',o')$.

In all cases there exists $o' \in Op(B)(i')$ such that $(i,o) \preceq^2 (i',o')$. □

Lemma 2.17. If a block B has a NCMS representation, then it is weakly nonanticipative.

Proof. Follows from Lemma 2.15, Lemma 2.16, and Theorem 1.1. □

Lemma 2.18. Assume that a block B is weakly nonanticipative and $dom(o) \subseteq \{0\}$ for each $(i,o) \subseteq IO(B)$. Then B is strongly nonanticipative.

Proof. Let us fix $(i_*, o_*) \in IO(B)$. Then $(i_*|_{\{0\}}, o_*|_{\{0\}}) \preceq^2 (i_*, o_*)$, so $(i_*|_{\{0\}}, o_*|_{\{0\}}) \in IO(B)$ by Theorem 1.1, because B is weakly nonanticipative.

Let $I = \{i \in Sb(In(B), W) \mid dom(i) \subseteq \{0\}\}$. We have $i_*|_{\{0\}} \in I$ and $o_* = o_*|_{\{0\}} \in Op(B)(i_*|_{\{0\}})$, because $dom(o_*) \subseteq \{0\}$. For each $i \in I$ we have $Op(B)(i) \neq \varnothing$, so there exists a (selector) function $f : I \to Sb(Out(B), W)$ such that $f(i) \in Op(B)(i)$ for all $i \in I$ and $f(i_*|_{\{0\}}) = o_*$.

For each $i \in Sb(In(B), W)$ let $O(i) = \{f(i|_{\{0\}})\}$. Then $O(i) \neq \varnothing$ and if $o \in O(i)$, then $o = f(i|_{\{0\}}) \in Op(B)(i|_{\{0\}})$, so $dom(o) \subseteq dom(i|_{\{0\}}) \subseteq dom(i)$. Then there exists a block B' such that $In(B') = In(B)$, $Out(B') = Out(B)$, and $Op(B')(i) = O(i)$ for all $i \in Sb(In(B), W)$.

The block B' is deterministic, because $O(i)$ is a singleton for each i. Moreover, if $i_1, i_2 \in Sb(In(B), W)$, $t \in T$, $i_1|_{[0,t]} = i_2|_{[0,t]}$, $o_1 \in Op(B')(i_1)$, and $o_2 \in Op(B')(i_2)$, then $o_1 = f(i_1|_{\{0\}}) = f(i_2|_{\{0\}}) = o_2$, whence $o_1|_{[0,t]} = o_2|_{[0,t]}$. Thus the block B' is causal.

Let us show that $B' \trianglelefteq B$. Let $(i,o) \in IO(B')$. If $i = \perp$, then $o = \perp$ and $(i,o) \in IO(B)$.

Assume that $i \neq \bot$. Then $o = f(i|_{\{0\}}) \in Op(B)(i|_{\{0\}})$. We have $i|_{\{0\}} \preceq i$, so by Theorem 1.1 there exists $o' \in Op(B)(i)$ such that $(i|_{\{0\}}, o) \preceq^2 (i, o')$ (because B is weakly nonanticipative). Then $i|_{\{0\}} = i|_A$ and $o = o'|_A$ for some $A \in \mathcal{T}_0$. Then $0 \in A$, because $i \neq \bot$. Then $dom(o') \subseteq \{0\} \subseteq A$, whence $o' = o'|_A = o$. Then $(i, o) \in IO(B)$.

Because $(i, o) \in IO(B')$ is arbitrary, $B' \trianglelefteq B$.

Moreover, we have $Op(B')(i_*) = O(i_*) = \{f(i_*|_{\{0\}})\} = \{o_*\}$, so $(i_*, o_*) \in IO(B')$.

We conclude that for each $(i_*, o_*) \in IO(B)$ there exists a deterministic causal sub-block $B' \trianglelefteq B$ such that $(i_*, o_*) \in IO(B')$. Thus B is strongly nonanticipative. □

Lemma 2.19. Assume that a block B has a NCMS representation, $(i_*, o_*) \in IO(B)$, $\{0\} \subset dom(i_*)$, and $dom(o_*) = \{0\}$. Then there exists a sub-block $B' \trianglelefteq B$ such that B' has a NCMS representation and $Op(B')(i_*) = \{o_*\}$.

Proof. Let (Σ, Q_0) be a NCMS representation of B, where $\Sigma = (T, Q, Tr)$. Then $o_* \in O_{all}(\Sigma, Q_0, i_*)$, $i_* \neq \bot$, and there exists $x_* \in IState(\Sigma)$ such that $o_* \in o_{all}(\Sigma, Q'_*, i_*)$, where $Q'_* = Sel_{1,2}(Q_0, i_*(0), x_*)$. Because $dom(o_*) = \{0\}$, there exists $q_* \in Q'_* \setminus S_{init}(\Sigma, i_*)$ such that $o_* = \{0\} \mapsto out(q_*)$.

Let $Q'_0 = \{q \in Q_0 \mid in(q) \neq i_*(0)\} \cup (\{i_*(0)\} \times IState(\Sigma) \times \{o_*(0)\})$.

On the set of all function of the form $s : A \to Q$, where $A \in \mathfrak{T}$ let us define a predicate P such that

$$P(s) \Leftrightarrow (range(s|_{\{0\}}) \subseteq Q'_0) \wedge (\forall t \in T \setminus \{0\} (in \circ (s|_{[0,t)}) \neq i_*|_{[0,t)}).$$

For each $s : A \to Q$, where $A \in \mathfrak{T}$, let us define a function $F(s) : A \to Q$:

- $F(s)(0) = q_*$, if $0 \in A$ and $in(s(0)) = i_*(0)$;
- $F(s)(0) = s(0)$, if $0 \in A$ and $in(s(0)) \neq i_*(0)$;
- $F(s)(t) = s(t)$ for all $t \in A \setminus \{0\}$.

Let $(l, r) \in LR(Q)$ be a LR representation of Σ (which exists by Theorem 2.1). Let $r' : ST(Q) \to Bool$ be a predicate such that

- $r'(s, 0) \Leftrightarrow r(F(s), 0) \wedge P(s)$, if $(s, 0) \in ST(Q)$;

– $r'(s,t) \Leftrightarrow r(s,t)$, if $(s,t) \in ST(Q)$ and $t > 0$.

Let us show that r' is right-local. Let $(s_1,t_0),(s_2,t_0) \in ST(Q)$ and $s_1 \doteq_{t_0+} s_2$. If $t_0 > 0$, then $r'(s_1,t_0) \Leftrightarrow r(s_1,t_0) \Leftrightarrow r(s_2,t_0) \Leftrightarrow r'(s_2,t_0)$, because r is right-local.

Consider the case when $t_0 = 0$. Then there exists $t' > 0$ such that $s_1 \doteq_{[0,t')} s_2$. Let us show that $\neg r'(s_1,0) \Rightarrow \neg r'(s_2,0)$. Assume that $\neg r'(s_1,0)$. Then either $\neg r(F(s_1),0)$, or $\neg P(s_1)$. In the former case, $F(s_1) \doteq_{[0,t')} F(s_2)$ by the definition of F, because $s_1 \doteq_{[0,t')} s_2$, whence $\neg r(F(s_2),0)$, because r is right-local, so $\neg r'(s_2,0)$. In the latter case, i.e. $\neg P(s_1)$, we have either $s_1(0) \notin Q'_0$, or there exists $t > 0$ such that $in \circ (s_1 |_{[0,t)}) = i_* |_{[0,t)}$. If $s_1(0) \notin Q'_0$, then $s_2(0) = s_1(0) \notin Q'_0$, so $\neg P(s_2)$ and $\neg r'(s_2,0)$. Otherwise, there exists $t > 0$ such that $in \circ (s_1 |_{[0,t)}) = i_* |_{[0,t)}$, so $i_* |_{[0,\min\{t,t'\})} = in \circ (s_1 |_{[0,\min\{t,t'\})}) = in \circ (s_2 |_{[0,\min\{t,t'\})})$, so $\neg P(s_2)$ and $\neg r'(s_2,0)$.

Thus $\neg r'(s_1,0) \Rightarrow \neg r'(s_2,0)$ in all cases, so $r'(s_2,0) \Rightarrow r'(s_1,0)$.

Thus we have shown that $r'(s_2,0) \Rightarrow r'(s_1,0)$ whenever $s_1 \doteq_{t_0+} s_2$. Then we have $r'(s_1,0) \Leftrightarrow r'(s_2,0)$ whenever $s_1 \doteq_{t_0+} s_2$. Moreover, if $(s,t) \in ST(Q)$ and $\max dom(s) \downarrow = t$, then $t > 0$ and $r(s,t)$, so $r'(s,t)$ holds.

We conclude that r' is right-local.

Then $(l,r') \in LR(Q)$ is a LR representation of some NCMS $\Sigma' = (T,Q,Tr')$ by Theorem 2.1. Then $Tr' = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l(s,t) \wedge r'(s,t))\}$.

Let us show that $s \in Tr' \Leftrightarrow (F(s) \in Tr \wedge P(s))$ holds for each function of the form $s : A \to Q$, where $A \in \mathfrak{T}$. Consider the following cases.

a) $s(0)\uparrow$. Then $F(s) = s$. Also, $P(s)$, because $range(s |_{\{0\}}) = \varnothing$ and $i_*(0)\downarrow$, so $r'(s,t) \Leftrightarrow r(s,t)$ for all $t \in dom(s)$, whence $s \in Tr' \Leftrightarrow (F(s) \in Tr \wedge P(s))$.

b) $s(0)\downarrow$. Then we have $r'(s,0) \Leftrightarrow r(F(s),0) \wedge P(s)$. Moreover, $F(s) |_{T\setminus\{0\}} = s |_{T\setminus\{0\}}$, whence $l(F(s),t) \Leftrightarrow l(s,t)$ and $r(F(s),t) \Leftrightarrow r(s,t) \Leftrightarrow r'(s,t)$ for all $t \in dom(s) \setminus \{0\}$. Also, $l(F(s),0)$ and $l(s,0)$, so $l(F(s),0) \Leftrightarrow l(s,0)$. Then

$$(\forall t \in dom(s)\, l(s,t) \wedge r'(s,t)) \Leftrightarrow (\forall t \in dom(s)\, l(F(s),t) \wedge r(F(s),t)) \wedge P(s),$$

whence $s \in Tr' \Leftrightarrow (F(s) \in Tr \wedge P(s))$.

We conclude that $s \in Tr' \Leftrightarrow (F(s) \in Tr \wedge P(s))$ for each $s : A \to Q$, $A \in \mathfrak{T}$.

Obviously, $Q_0' \subseteq Q$. Let us show that $S_{init}(\Sigma') \subseteq Q_0'$. Let $q \in S_{init}(\Sigma')$. Then $s(0) \downarrow = q$ for some $s \in Tr'$. Then $F(s) \in Tr \wedge P(s)$ holds. Then $range(s|_{\{0\}}) \subseteq Q_0'$, because $P(s)$. Then $q = s(0) \in Q_0'$.

We conclude that $S_{init}(\Sigma') \subseteq Q_0' \subseteq Q$, so (Σ', Q_0') is an initial I/O NCMS. By Lemma 2.9, it is a NCMS representation of some block B'. Then $In(B) = In(B')$ and $Out(B) = Out(B')$.

Let us show that $Op(B')(i) \subseteq Op(B)(i)$ for all $i \in Sb(In(B), W)$.

Assume that $i \in Sb(In(B), W)$ and $o \in Op(B')(i)$. Let us show that $o \in Op(B)(i)$. This is obvious, if $i = \perp$, so assume that $i \neq \perp$. Then there exists $x \in IState(\Sigma') = IState(\Sigma)$ such that $o \in o_{all}(\Sigma', Q', i)$, where $Q' = Sel_{1,2}(Q_0', i(0), x)$.

Then the following cases are possible.

1) $Q' = \varnothing$ and $o = \perp$. Then $i(0) \neq i_*(0)$, so $Sel_{1,2}(Q_0, i(0), x) = \varnothing$ and

$$o = \perp \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) \subseteq O_{all}(\Sigma, Q_0, i) = Op(B)(i).$$

2) $dom(i) = \{0\}$ and $o = \{0\} \mapsto out(q)$ for some $q \in Q'$.

If $i(0) \neq i_*(0)$, then $q \in Sel_{1,2}(Q_0, i(0), x)$, so

$$o = \{0\} \mapsto out(q) \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) \subseteq Op(B)(i).$$

Consider the case when $i(0) = i_*(0)$. Because $q \in Q_0'$ and $in(q) = i(0) = i_*(0)$, we have $out(q) = o_*(0)$, so $o = o_*$ and $i = i_*|_{\{0\}}$. Because $(i_*|_{\{0\}}, o_*) \preceq^2 (i_*, o_*)$, $(i_*, o_*) \in IO(B)$, by Lemma 2.15, we have $(i_*|_{\{0\}}, o_*) \in IO(B)$, so $o \in Op(B)(i)$.

3) $\{0\} \subset dom(i)$ and $o = \{0\} \mapsto out(q)$ for some $q \in Q' \setminus S_{init}(\Sigma', i)$.

Consider the following sub-cases.

3.1) $in(q) \neq i_*(0)$. Then because $q \in Q_0'$, we have $q \in Q_0$, so $q \in Sel_{1,2}(Q_0, i(0), x)$. Let us show that $q \notin S_{init}(\Sigma, i)$. Suppose $q \in S_{init}(\Sigma, i)$. Then

$q = s(0)$ for some $s \in S(\Sigma, i)$. Then $in(s(0)) \neq i_*(0)$, so $F(s) = s$. Moreover, $range(s|_{\{0\}}) \subseteq Q'_0$, because $q \in Q'_0$, and $(\forall t \in T \setminus \{0\} (in \circ (s|_{[0,t)}) \neq i_*|_{[0,t)})$, because $in(s(0)) \neq i_*(0)$. Then $P(s)$ holds and $F(s) = s \in Tr$, so $s \in Tr'$. Besides, $in \circ s \preceq i$, so $s \in S(\Sigma', i)$ and $q \in S_{init}(\Sigma', i)$. This contradicts the assumption $q \in Q' \setminus S_{init}(\Sigma', i)$. Thus $q \notin S_{init}(\Sigma, i)$. Then because $q \in Sel_{1,2}(Q_0, i(0), x)$, we have

$$o = \{0\} \mapsto out(q) \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) \subseteq Op(B)(i).$$

3.2) $in(q) = i_*(0)$. Then because $q \in Q'_0$, we have $out(q) = o_*(0)$, so $o = o_*$.

Consider the case when $i|_{[0,\bar{t})} = i_*|_{[0,\bar{t})}$ for some $\bar{t} > 0$. Then because $(i|_{[0,\bar{t})}, o) = (i_*|_{[0,\bar{t})}, o_*|_{[0,\bar{t})}) \preceq^2 (i_*, o_*)$ and $(i_*, o_*) \in IO(B)$, by Lemma 2.15, we have $(i|_{[0,\bar{t})}, o) \in IO(B)$. Then there exists $o' \in Op(B)(i)$ such that $(i|_{[0,\bar{t})}, o) \preceq^2 (i, o')$ by Lemma 2.16. Then $o = o'$ by Lemma 1.3, because $(i|_{[0,\bar{t})}, o)$ is an abnormal I/O pair. Then $o \in Op(B)(i)$.

Now consider the case when $i|_{[0,t)} \neq i_*|_{[0,t)}$ for all $t > 0$.

Let us show that $q_* \notin S_{init}(\Sigma, i)$. Suppose that $q_* \in S_{init}(\Sigma, i)$. Then $q_* = s(0)$ for some $s \in S(\Sigma, i)$. Let $s' : dom(s) \to Q$ be a function such that $s'(0) = q$, $s'(t) = s(t)$ for $t \in dom(s) \setminus \{0\}$. Then $in(s'(0)) = in(q) = i_*(0)$, so $F(s')(0) = q_* = s(0)$, $F(s')|_{T \setminus \{0\}} = s'|_{T \setminus \{0\}} = s|_{T \setminus \{0\}}$. Then $F(s') = s \in Tr$. Also, $range(s'|_{\{0\}}) = \{q\} \subseteq Q'_0$. We have $in \circ s' \preceq i$, because $in \circ s \preceq i$ and $in(s'(0)) = in(q) = i_*(0) = i(0)$. Because $i|_{[0,t)} \neq i_*|_{[0,t)}$ for all $t > 0$, this implies that $in \circ (s'|_{[0,t)}) \neq i_*|_{[0,t)}$ for all $t > 0$. Then $P(s')$ holds. So we have $F(s') \in Tr \wedge P(s')$. Thus $s' \in Tr'$. Then because $s'(0)\downarrow$ and $in \circ s' \preceq i$, we have $s' \in S(\Sigma', i)$. Then $q = s'(0) \in S_{init}(\Sigma', i)$, which contradicts the assumption $q \in Q' \setminus S_{init}(\Sigma', i)$.

Thus $q_* \notin S_{init}(\Sigma, i)$. Because $q_* \in Sel_{1,2}(Q_0, i_*(0), x_*)$, $i_*(0) = in(q) = i(0)$, we have $\{0\} \mapsto out(q_*) \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x_*), i) \subseteq Op(B)(i)$. Because $q \in Q'_0$, $in(q) = i_*(0)$, we have $out(q) = o_*(0) = out(q_*)$. Then $o = \{0\} \mapsto out(q) \in Op(B)(i)$.

4) $\{0\} \subset dom(i)$ and $o = out \circ s$ for some $s \in S_{max}(\Sigma', i)$ such that $s(0) \in Q'$.

Then $s \in Tr'$, so $F(s) \in Tr \wedge P(s)$ holds. Also, we have $in(q_*) = i_*(0)$, which implies that $in \circ F(s) = in \circ s$. Then $in \circ F(s) \preceq i$, so $F(s) \in S(\Sigma, i)$.

Let us show that $F(s) \in S_{max}(\Sigma, i)$. Let $s' \in S(\Sigma, i)$ be any element such that $F(s) \sqsubseteq s'$. Let $s'' : dom(s') \to Q$ be a function such that $s''(0) = s(0)$ and $s''(t) = s'(t)$ for all $t \in dom(s') \setminus \{0\}$. Then $F(s'')(0) = F(s)(0) = s'(0)$ and $F(s'')(t) = s''(t) = s'(t)$ for all $t \in dom(s') \setminus \{0\}$, so $F(s'') = s' \in Tr$. Moreover, $in \circ s'' = in \circ s' \preceq i$ holds, because $in(s''(0)) = in(s(0)) = i(0)$. Because $P(s)$ holds, we have $(in \circ s)|_{[0,t)} \neq i_*|_{[0,t)}$ for all $t > 0$. Then because $in \circ s \preceq i$ and $in \circ s'' \preceq i$, we have $(in \circ s'')|_{[0,t)} \neq i_*|_{[0,t)}$ for all $t > 0$. From this and $range(s''|_{\{0\}}) = \{s(0)\} \subseteq Q' \subseteq Q'_0$, we have $P(s'')$. Thus $F(s'') \in Tr \wedge P(s'')$. Then $s'' \in Tr'$, because $dom(s'') = dom(s') \in \mathfrak{T}$. Then $s'' \in S(\Sigma', i)$, because $in \circ s'' \preceq i$. Also, $s''(t) = s'(t) = F(s)(t) = s(t)$ for $t \in dom(s) \setminus \{0\}$ and $s''(0) = s(0)$, so $s \sqsubseteq s''$. Then $s = s''$, because $s \in S_{max}(\Sigma', i)$. As we have shown above, $F(s'') = s'$, so $s' = F(s)$.

We have shown that for any $s' \in S(\Sigma, i)$, if $F(s) \sqsubseteq s'$, then $s' = F(s)$. Then because $F(s) \in S(\Sigma, i)$, we have $F(s) \in S_{max}(\Sigma, i)$. Consider the following cases.

4.1) $in(s(0)) \neq i_*(0)$. Then $s = F(s) \in S_{max}(\Sigma, i)$, and because $s(0) \in Q' \subseteq Q'_0$, we have $s(0) \in Q_0$ and $s(0) \in Sel_{1,2}(Q_0, i(0), x)$. Then

$$o = out \circ s \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) \subseteq Op(B)(i).$$

4.2) $in(s(0)) = i_*(0)$. Because $s(0) \in Q'$, we have $i(0) = in(s(0)) = i_*(0)$. Then $F(s)(0) = q_* \in Sel_{1,2}(Q_0, i_*(0), x_*) = Sel_{1,2}(Q_0, i(0), x_*)$. Then

$$out \circ F(s) \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x_*), i) \subseteq Op(B)(i).$$

Because $s(0) \in Q'_0$ and $in(s(0)) = i_*(0)$, we have $out(s(0)) = o_*(0)$. Then $out(F(s)(0)) = out(q_*) = out(o_*(0)) = out(s(0))$. Also, $out(F(s)(t)) = out(s(t))$ for all $t \in dom(s) \setminus \{0\}$. Thus $o = out \circ s = out \circ F(s) \in Op(B)(i)$.

We conclude that $Op(B')(i) \subseteq Op(B)(i)$ for all i. Thus $B' \trianglelefteq B$.

Let us show that $Op(B')(i_*) \subseteq \{o_*\}$. Assume that $i \in Sb(In(B),W)$ and $o \in Op(B')(i_*)$. Because $\{0\} \subset dom(i_*)$, there exists $x \in IState(\Sigma')$ such that $o \in o_{all}(\Sigma',Q',i_*)$, where $Q' = Sel_{1,2}(Q_0',i_*(0),x)$. Then $Q' \neq \emptyset$.

Then the following cases are possible.

a) $o = out \circ s$ for some $s \in S_{max}(\Sigma',i_*)$ such that $s(0) \in Q'$. Then $s \in Tr'$, $s(0)\downarrow$, and $in \circ s \preceq i_*$. Then $F(s) \in Tr \wedge P(s)$, so $\forall t \in T \setminus \{0\} (in \circ (s|_{[0,t)}) \neq i_*|_{[0,t)})$, but this contradicts the relation $in \circ s \preceq i_*$, because $\{0\} \subset dom(s)$ and $dom(s) \in \mathcal{T}_0$.

b) $o = \{0\} \mapsto out(q)$ for some $q \in Q' \setminus S_{init}(\Sigma',i_*)$. Then $q \in Q_0'$ and $in(q) = i_*(0)$, so $out(q) = o_*(0)$. Then $o = o_*$.

We conclude that $Op(B')(i_*) \subseteq \{o_*\}$. Then $Op(B')(i_*) = \{o_*\}$. Thus B' satisfies the statement of the lemma. □

Lemma 2.20. Assume that a block B has a NCMS representation, $(i_*,o_*) \in IO(B)$, and $\{0\} \subset dom(o_*)$. Then there exists a deterministic block B' such that B' has a NCMS representation, $In(B') = In(B)$, $Out(B') = Out(B)$, $Op(B')(i) \subseteq Op(B)(i)$ for each $i \in Sb(In(B),W)$ such that $i(0)\downarrow = i_*(0)$, and $(i_*,o_*) \in IO(B')$.

Proof. Let (Σ,Q_0) be a NCMS representation of B, where $\Sigma = (T,Q,Tr)$. Then $o_* \in O_{all}(\Sigma,Q_0,i_*)$. Then $i_* \neq \perp$, because $\{0\} \subset dom(o_*) \subseteq dom(i_*)$. Then there exists $x_* \in IState(\Sigma)$ such that $o_* \in o_{all}(\Sigma, Sel_{1,2}(Q_0,i_*(0),x_*),i_*)$. Then because $\{0\} \subset dom(o_*)$, there exists $s_* \in S_{max}(\Sigma,i_*)$ such that $s_*(0) \in Sel_{1,2}(Q_0,i_*(0),x_*)$ and $o_* = out \circ s_*$. Then $s_* \in Tr$.

Let $\mathcal{X}$ be the set of all sets $X \subseteq Tr$ such that

a) $s_* \in X$;

b) $0 \in dom(s)$ and $s(0) = s_*(0)$ for each $s \in X$;

c) for each $s \in X$ and $t \in T \setminus \{0\}$, $s|_{[0,t)} \in X$ and $s|_{[0,t]} \in X$;

d) for each $s_1, s_2 \in X$, if $in \circ s_1 = in \circ s_2$, then $s_1 = s_2$.

It follows immediately that

$$\{s_* |_{[0,t)} \mid t \in T \setminus \{0\}\} \cup \{s_* |_{[0,t]} \mid t \in T \setminus \{0\}\} \cup \{s_*\} \in \mathcal{X},$$

and $\bigcup c \in \mathcal{X}$ for each non-empty $\subseteq$-chain $c \subseteq \mathcal{X}$. Then Zorn's lemma implies that $\mathcal{X}$ has some $\subseteq$-maximal element X^*.

Let us show that each non-empty $\sqsubseteq$-chain in X^* has a supremum in X^*. Let $C \subseteq X^*$ be a non-empty $\sqsubseteq$-chain. Let s_0 be a function (the graph of) which is a union of (graphs of) elements of C (s_0 is indeed a function, because C is a $\sqsubseteq$-chain). Then $s_0 \in Tr$ by the completeness and CPR properties of the NCMS Σ. Besides, $0 \in dom(s_0)$.

Suppose that $s_0 \notin X^*$. Let $X' = X^* \cup \{s_0\}$. Then $X^* \subset X'$, whence $X' \notin \mathcal{X}$, because X^* is $\subseteq$-maximal in $\mathcal{X}$. To get a contradiction, let us show that $X' \in \mathcal{X}$. We have $X' \subseteq Tr$, because $s_0 \in Tr$. The conditions a) and b) are obviously satisfied for X'. If $t > 0$ and $s_0 |_{[0,t)} \neq s_0$, then $dom(s) \cap (dom(s_0) \setminus dom(s_0 |_{[0,t)})) \neq \varnothing$ for some $s \in C$ (because $dom(s_0) = \bigcup_{s \in C} dom(s)$), which implies that $s_0 |_{[0,t)} \sqsubseteq s$, (because $s(0) \downarrow$ and $dom(s) \in \mathfrak{T}$), whence $s_0 |_{[0,t)} \in X^*$, because $s \in X^*$ and X^* satisfies the condition c). Similarly, if $t > 0$ and $s_0 |_{[0,t]} \neq s_0$, then $s_0 |_{[0,t]} \sqsubseteq s$ for some $s \in C$ and $s_0 |_{[0,t]} \in X^*$. Thus X' satisfies the condition c). Then because $X' \notin \mathcal{X}$, this implies that X' does not satisfy the condition d), i.e. there exist $s_1, s_2 \in X' = X^* \cup \{s_0\}$ such that $s_1 \neq s_2$ and $in \circ s_1 = in \circ s_2$. Because X^* satisfies the condition d), we have that one of the elements of $\{s_1, s_2\}$ belongs to X^* and another one coincides with s_0. Without loss of generality we can assume that $s_1 \in X^*$ and $s_2 = s_0$. Then $in \circ s_1 = in \circ s_0$ and $s_1 \neq s_0$. Then $dom(s_1) = dom(s_0)$ and there exists $t \in dom(s_0)$ such that $s_1 |_{[0,t]} \neq s_0 |_{[0,t]}$. Then $t > 0$, because $s_0(0) = s_*(0) = s_1(0)$. If $s_0 |_{[0,t]} \neq s_0$, then $s_0 |_{[0,t]} \in C \subseteq X^*$ and $s_1 |_{[0,t]} \in X^*$, and

$in \circ (s_1 |_{[0,t]}) = in \circ (s_0 |_{[0,t]})$, which contradicts the condition d) for the set X^*. Thus $s_0 |_{[0,t]} = s_0$, and because $t \in dom(s_0)$, t is the greatest element of $dom(s_0)$. Then $[0,t] = dom(s_0) = \bigcup_{s \in C} dom(s)$, whence there exists $s \in C$ such that $t \in dom(s)$. Then $dom(s) = [0,t]$ and s is the $\sqsubseteq$-greatest element of C, because C is a $\sqsubseteq$-chain. Thus $s_0 = s \in C$. Then $s_0 \in X^*$, $s_1 \in X^*$, $s_1 \neq s_0$, and $in \circ s_1 = in \circ s_0$. But this contradicts the condition d) for the set X^*.

We conclude that $s_0 \in X^*$. It follows immediately from the definition of s_0 that s_0 is a $\sqsubseteq$-supremum of C. Because C is arbitrary, it follows that each non-empty $\sqsubseteq$-chain in X^* has a supremum in X^*.

Let $Y = IState(\Sigma) \times X^*$ and $Q' = {}^{In(B)}W \times Y \times {}^{Out(B)}W$. Then $Y \neq \varnothing$, because $X^* \neq \varnothing$. For each $s \in X^*$ and $y \in Y$ let $f_s^y : dom(s) \to Q'$ be a function such that

$$f_s^y(0) = \big(in(s(0)), y, out(s(0))\big);$$

$$f_s^y(t) = \big(in(s(t)), (istate(s(t)), s |_{[0,t]}), out(s(t))\big), \text{ if } t \in dom(s) \setminus \{0\}.$$

Note that because $X^* \in \mathcal{X}$, for each $s \in X^*$ we have $0 \in dom(s)$, $s \in Tr$, and $s |_{[0,t]} \in X^*$ for all $t > 0$. This implies that f_s^y indeed takes values in Q'.

Let us define the following set:

$$Tr' = \{\tilde{s} \mid \exists s \in X^* \; \exists y \in Y \; \exists A \in \mathfrak{T} \; A \subseteq dom(s) \wedge \tilde{s} = f_s^y |_A\}.$$

Because $dom(f_s^y) = dom(s)$ and $range(f_s^y) \subseteq Q'$ for all $s \in X^*$ and $y \in Y$, we have that Tr' is the set of all functions of the form $\tilde{s} : A \to Q'$, where $A \in \mathfrak{T}$, such that there exist $s \in X^*$ and $y \in Y$ such that $\tilde{s} \sqsubseteq f_s^y$.

Let $\Sigma' = (T, Q', Tr')$. Let us show that Σ' is a NCMS.

Let us show that Tr' satisfies the CPR property. Let $\tilde{s} \in Tr'$, $A \in \mathfrak{T}$, and $A \subseteq dom(\tilde{s})$. Then there exist $s \in X^*$ and $y \in Y$ such that $\tilde{s} \sqsubseteq f_s^y$. Then $dom(\tilde{s} |_A) = A \in \mathfrak{T}$ and $\tilde{s} |_A \sqsubseteq f_s^y$. Thus $\tilde{s} |_A \in Tr'$.

Let us show that Tr' satisfies the Markovian property. Assume that $\tilde{s}_1, \tilde{s}_2 \in Tr'$ and $t_0 = \max dom(\tilde{s}_1) = \min dom(\tilde{s}_2)$, and $\tilde{s}_1(t_0) = \tilde{s}_2(t_0)$. Let $\tilde{s} : dom(\tilde{s}_1) \cup dom(\tilde{s}_2) \to Q'$ be a function such that $\tilde{s}(t) = s_1(t_0)$, if $t \le t_0$ and $\tilde{s}(t) = s_2(t_0)$, if $t > t_0$. Because $\tilde{s}_1, \tilde{s}_2 \in Tr'$, there exist $s_1, s_2 \in X^*$ and $y_1, y_2 \in Y$ such that $\tilde{s}_j \sqsubseteq f_{s_j}^{y_j}$ for $j = 1,2$. Then $f_{s_j}^{y_j}(t_0) \downarrow = \tilde{s}_j(t_0)$ for $j = 1,2$. Then $f_{s_1}^{y_1}(t_0) = f_{s_2}^{y_2}(t_0)$. We have $t_0 > 0$, because $t_0 = \max dom(\tilde{s}_1)$ and $dom(\tilde{s}_1) \in \mathfrak{T}$, so from the definition of f_s^y we have $s_1 |_{[0,t_0]} = s_2 |_{[0,t_0]}$. Also, $t_0 \in dom(s_1) \cap dom(s_2)$, because $dom(f_{s_j}^{y_j}) = dom(s_j)$ for $j = 1,2$. Then $[0, t_0] \subseteq dom(s_1) \cap dom(s_2)$, because $s_1, s_2 \in X^*$. Then $s_1(t) = s_2(t)$ for all $t \in [0, t_0]$.

Let us show that $\tilde{s}_1 \sqsubseteq f_{s_2}^{y_1}$. Because $dom(\tilde{s}_1) \subseteq [0, t_0] \subseteq dom(s_2)$, for all $t \in dom(\tilde{s}_1) \setminus \{0\}$ we have

$$\tilde{s}_1(t) = f_{s_1}^{y_1}(t) = \left(in(s_1(t)), (istate(s_1(t)), s_1 |_{[0,t]}), out(s_1(t))\right) =$$
$$= \left(in(s_2(t)), (istate(s_2(t)), s_2 |_{[0,t]}), out(s_2(t))\right).$$

Moreover, $\tilde{s}_1(t) = f_{s_1}^{y_1}(0) = \left(in(s(0)), y, out(s(0))\right)$. Then $f_{s_2}^{y_1}(t) \downarrow = \tilde{s}_1(t)$ for all $t \in dom(\tilde{s}_1)$. Thus $\tilde{s}_1 \sqsubseteq f_{s_2}^{y_1}$.

Besides, $\tilde{s}_2 \sqsubseteq f_{s_2}^{y_1}$, because $\tilde{s}_2 \sqsubseteq f_{s_2}^{y_2}$ and $f_{s_2}^{y_2} |_{T \setminus \{0\}} = f_{s_2}^{y_1} |_{T \setminus \{0\}}$, and $0 \notin dom(\tilde{s}_2)$. Thus $\tilde{s}_1 \sqsubseteq f_{s_2}^{y_1}$ and $\tilde{s}_2 \sqsubseteq f_{s_2}^{y_1}$. Then $\tilde{s} \sqsubseteq f_{s_2}^{y_1}$ by the definition of $\tilde{s}$. Moreover, $dom(\tilde{s}) \in \mathfrak{T}$, so $\tilde{s} \in Tr'$. Thus Tr' satisfies the Markovian property.

Let us show that Tr' is complete in the sense of Definition 2.3.

Let $c' \subseteq Tr'$ be a non-empty $\sqsubseteq$-chain. Let $\tilde{s}^*$ a function (the graph of) which is the union of (graphs of) elements of c'. Then $dom(\tilde{s}^*) = \bigcup_{\tilde{s} \in c'} dom(c') \in \mathfrak{T}$.

Let us show that $\tilde{s}^* \in Tr'$. Because $c' \subseteq Tr'$, there exist functions $\varphi : c' \to X^*$ and $\gamma : c' \to Y$ such that $\tilde{s} \sqsubseteq f_{\varphi(\tilde{s})}^{\gamma(\tilde{s})}$ for each $\tilde{s} \in c'$.

For each $A \in \mathfrak{T}$ denote $\Phi(A) = \bigcup_{\tau \in A \setminus \{0\}} [0, \tau]$. Let $c = \{\varphi(\tilde{s})|_{\Phi(dom(\tilde{s}))} \mid \tilde{s} \in c'\}$. Then for any $A \in \mathfrak{T}$, the set $\Phi(A)$ has a form $[0, t)$ or $[0, t]$ for some $t > 0$ (because $A \neq \varnothing$ and $A \neq \{0\}$). Because $range(\varphi) \subseteq X^* \in \mathcal{X}$, this implies that $c \subseteq X^*$. Moreover, $c \neq \varnothing$, because $c' \neq \varnothing$.

Let us show that c is a $\sqsubseteq$-chain. Because c' is a $\sqsubseteq$-chain, it is sufficient to show that if $\tilde{s}_1, \tilde{s}_2 \in c'$ and $\tilde{s}_1 \sqsubseteq \tilde{s}_2$, then $\varphi(\tilde{s}_1)|_{\Phi(dom(\tilde{s}_1))} \sqsubseteq \varphi(\tilde{s}_2)|_{\Phi(dom(\tilde{s}_2))}$. Assume that $\tilde{s}_1, \tilde{s}_2 \in c'$ and $\tilde{s}_1 \sqsubseteq \tilde{s}_2$. Denote $s_j = \varphi(\tilde{s}_j)$ and $y_j = \gamma(\tilde{s}_j)$ for $j = 1,2$. Let $t \in dom(s_1) \cap \Phi(dom(\tilde{s}_1))$. Then $t \in [0, \tau]$ for some $\tau \in dom(\tilde{s}_1) \setminus \{0\}$ by the definition of Φ. Because $\tilde{s}_1 \sqsubseteq \tilde{s}_2$, we have $\tilde{s}_2(\tau) \downarrow = \tilde{s}_1(\tau)$. Moreover, $\tilde{s}_j \sqsubseteq f_{\varphi(\tilde{s}_j)}^{\gamma(\tilde{s}_j)} = f_{s_j}^{y_j}$ for $j = 1,2$. Then $f_{s_j}^{y_j}(\tau) \downarrow = \tilde{s}_j(\tau)$ for $j = 1,2$. Then $f_{s_1}^{y_1}(\tau) = f_{s_2}^{y_2}(\tau)$. Then $[0, \tau] \subseteq dom(s_1) \cap dom(s_2)$, and because $\tau > 0$, we have $s_1|_{[0,\tau]} = s_2|_{[0,\tau]}$. Then $s_2(t) \downarrow = s_1(t)$. Because $t \in dom(s_1) \cap \Phi(dom(\tilde{s}_1))$ is arbitrary, we have $s_1|_{\Phi(dom(\tilde{s}_1))} \sqsubseteq s_2$. Moreover, $\Phi(dom(\tilde{s}_1)) \subseteq \Phi(dom(\tilde{s}_2))$, because $\tilde{s}_1 \sqsubseteq \tilde{s}_2$. Then $s_1|_{\Phi(dom(\tilde{s}_1))} \sqsubseteq s_2|_{\Phi(dom(\tilde{s}_2))}$. Thus $\varphi(\tilde{s}_1)|_{\Phi(dom(\tilde{s}_1))} \sqsubseteq \varphi(\tilde{s}_2)|_{\Phi(dom(\tilde{s}_2))}$. We conclude that c is a $\sqsubseteq$-chain.

Thus c is a non-empty $\sqsubseteq$-chain in X^*. As we have shown above, this implies that c has a supremum in X^*. Denote this supremum as s^*. Then $s \sqsubseteq s^*$ for all $s \in c$ and $s^* \in X^*$. If $\tilde{s}^*(0) \downarrow$, let us denote $y^* = istate(\tilde{s}^*(0))$, otherwise, let y^* be an arbitrary element of Y (which exists, because $Y \neq \varnothing$).

Let us show that $\tilde{s}^* \sqsubseteq f_{s^*}^{y^*}$. It is sufficient to show that $\tilde{s} \sqsubseteq f_{s^*}^{y^*}$ for each $\tilde{s} \in c'$. Indeed, let $\tilde{s} \in c'$, $s = \varphi(\tilde{s}) \in X^*$, and $y = \psi(\tilde{s})$. Let $t \in dom(\tilde{s}) \setminus \{0\}$. Then

$$f_s^y(t) \downarrow = \big(in(s(t)), (istate(s(t)), s|_{[0,t]}), out(s(t))\big) = \tilde{s}(t).$$

Then $t \in \Phi(dom(\tilde{s})) \cap dom(s)$, whence $s|_{\Phi(dom(\tilde{s}))}(t) \downarrow$. Because $s \in X^*$ this implies that $[0, t] \subseteq dom(s|_{\Phi(dom(\tilde{s}))})$. Moreover, $\varphi(\tilde{s})|_{\Phi(dom(\tilde{s}))} = s|_{\Phi(dom(\tilde{s}))} \in c$, and

s^* is a supremum of c, so have $s^*(\tau)\downarrow= s(\tau)$ for all $\tau\in[0,t]$. Then $s^*(t)=s(t)$ and $s^*|_{[0,t]}= s|_{[0,t]}$. Then $f_{s^*}^{y^*}(t)\downarrow= f_s^y(t)=\tilde{s}(t)$. Because $t\in dom(\tilde{s})\setminus\{0\}$ is arbitrary, we have $\tilde{s}|_{T\setminus\{0\}}\sqsubseteq f_{s^*}^{y^*}$. If $\tilde{s}(0)\uparrow$, then $\tilde{s}\sqsubseteq f_{s^*}^{y^*}$. Let us show that $\tilde{s}\sqsubseteq f_{s^*}^{y^*}$, if $\tilde{s}(0)\downarrow$.

Suppose that $\tilde{s}(0)\downarrow$ Then $f_s^y(0)\downarrow=\tilde{s}(0)$, whence $s(0)\downarrow$ and $\tilde{s}(0)=(in(s(0)),y,out(s(0)))$. Then $\varphi(\tilde{s})|_{\Phi(dom(\tilde{s}))}(0)\downarrow= s(0)$, so $s^*(0)\downarrow= s(0)$, because $\varphi(\tilde{s})|_{\Phi(dom(\tilde{s}))}\in c$. Because $\tilde{s}^*$ is the union of elements of c' and $\tilde{s}\in c'$, we have $\tilde{s}^*(0)\downarrow=\tilde{s}(0)$. Then $y^*=istate(\tilde{s}^*(0))=istate(\tilde{s}(0))=y$, whence

$$f_{s^*}^{y^*}(0)\downarrow=(in(s^*(0)),y^*,out(s^*(0)))=(in(s(0)),y,out(s(0)))=\tilde{s}(0).$$

Then $\tilde{s}|_{\{0\}}\sqsubseteq f_{s^*}^{y^*}$, and because $\tilde{s}|_{T\setminus\{0\}}\sqsubseteq f_{s^*}^{y^*}$, we have $\tilde{s}\sqsubseteq f_{s^*}^{y^*}$.

We conclude that $\tilde{s}\sqsubseteq f_{s^*}^{y^*}$ for each $\tilde{s}\in c'$. Then $\tilde{s}^*\sqsubseteq f_{s^*}^{y^*}$. Because $dom(\tilde{s}^*)\in\mathfrak{T}$, this implies that $\tilde{s}^*\in Tr'$. Then it follows that c' has a least upper bound in Tr' ($\tilde{s}^*$ is its least upper bound).

Because c' is an arbitrary non-empty $\sqsubseteq$-chain, we conclude that Tr' is complete in the sense of Definition 2.3.

Thus Σ' is a NCMS. The definition of Q' implies that Σ' is an I/O NCMS.

For each $d_{in}\in Sb(In(B),W)$ denote

$$O_0(d_{in})=\{d_{out}\mid\exists x\in IState(\Sigma)\ (d_{in},x,d_{out})\in Q_0)\};$$

$$D_0=\{d_{in}\in Sb(In(B),W)\mid O_0(d_{in})\neq\varnothing\}.$$

Note that because $s_*\in Tr$, $s_*(0)\downarrow$, and (Σ,Q_0) is an initial I/O NCMS, we have

$$(in(s_*(0)),istate(s_*(0)),out(s_*(0)))=s_*(0)\in S_{init}(\Sigma)\subseteq Q_0,$$

whence $out(s_*(0))\in O_0(in(s_*(0)))$ and $in(s_*(0))\in D_0$.

Then there exists a function $\eta:D_0\to Sb(Out(B),W)$ (selector) such that $\eta(in(s_*(0)))=out(s_*(0))$ and $\eta(d_{in})\in O_0(d_{in})$ for each $d_{in}\in D_0$.

Let us define

$$Q'_0 = \{(d_{in}, y, d_{out}) \mid d_{in} \in D_0 \wedge y \in Y \wedge d_{out} = \eta(d_{in})\}.$$

Obviously, $Q'_0 \subseteq Q'$.

Let us show that $in(q) = in(s_*(0))$ and $out(q) = out(s_*(0))$ for each $q \in S_{init}(\Sigma')$. Let $q \in S_{init}(\Sigma')$. Then there exists $\tilde{s} \in Tr'$ such that $\tilde{s}(0) \downarrow= q$. Then $\tilde{s} \sqsubseteq f_s^y$ for some $s \in X^*$ and $y \in Y$. Then $f_s^y(0) \downarrow= (in(s(0)), y, out(s(0))) = q$. Because $s \in X^*$ and $X^* \in \mathcal{X}$, we have $s(0) = s_*(0)$. Then $in(q) = in(s_*(0))$ and $out(q) = out(s_*(0))$.

Then for each $q \in S_{init}(\Sigma')$ we have

$$\eta(in(q)) \downarrow= \eta(in(s_*(0))) = out(s_*(0)) = out(q),$$

whence $S_{init}(\Sigma') \subseteq Q'_0$.

Thus $S_{init}(\Sigma') \subseteq Q'_0 \subseteq Q'$. Then (Σ', Q'_0) is an initial I/O NCMS. Then by Lemma 2.9, (Σ', Q'_0) is a NCMS representation of some block B'. Then $In(B') = In(\Sigma') = In(B)$, $Out(B') = Out(\Sigma') = Out(B)$, and $Op(B)(i) = O_{all}(\Sigma', Q'_0, i)$ for all $i \in Sb(In(B), W)$.

Let us show that the block B' is deterministic. Suppose that there exist $i \in Sb(In(B'), W)$ and $\{o_1, o_2\} \subseteq Op(B')(i)$ such that $o_1 \neq o_2$. Then $i \neq \perp$, because otherwise, $o_1 = o_2 = \perp$. Then $\{o_1, o_2\} \subseteq O_{all}(\Sigma', Q'_0, i)$ and there exist $y_1, y_2 \in Y$ such that $o_j \in o_{all}(\Sigma', Sel_{1,2}(Q'_0, i(0), y_j), i)$ for $j = 1,2$. Denote $Q'_j = Sel_{1,2}(Q'_0, i(0), y_j)$, $j = 1,2$. If $i(0) \in D_0$, then $Q'_j = (i(0), y_j, \eta(i(0)))$ for $j = 1,2$, and otherwise, $Q'_1 = Q'_2 = \varnothing$. Thus $Q'_1 \neq \varnothing$ if and only if $Q'_2 \neq \varnothing$. Besides, because $o_1 \neq o_2$ and $o_j \in o_{all}(\Sigma', Q'_j, i)$ for $j = 1,2$, at least one of Q'_1 and Q'_2 is non-empty (otherwise, $o_1 = o_2 = \perp$). Thus both Q'_1 and Q'_2 are non-empty and $i(0) \in D_0$.

Let us show that $dom(i) \neq \{0\}$. Suppose that $dom(i) = \{0\}$. Because $Q'_1 \neq \varnothing$, $Q'_2 \neq \varnothing$, and $o_j \in o_{all}(\Sigma', Q'_j, i)$ for $j = 1,2$, for each $j = 1,2$ there exists $q_j \in Q'_j$ such that $o_j = \{0\} \mapsto out(q_j)$. For $j = 1,2$ we have $q_j \in Q'_j$, whence

$in(q_j)=i(0)\in D_0$. Then $out(q_j)=\eta(i(0))$ for $j=1,2$, because $q_j\in Q_0'$. Then $out(q_1)=out(q_2)$, but this contradicts the assumption $o_1\neq o_2$. Thus $dom(i)\neq\{0\}$.

Because $i\neq\perp$ and $dom(i)\neq\{0\}$, we have $\{0\}\subset dom(i)$. Then for each $j=1,2$, because $Q_j'\neq\varnothing$ and $o_j\in o_{all}(\Sigma',Q_j',i)$, we have either $o_j=\{0\}\mapsto out(q_j)$ for some $q_j\in Q_j'\setminus S_{init}(\Sigma',i)$, or $o_j=out\circ\tilde{s}_j$ for some $\tilde{s}_j\in S_{max}(\Sigma',i)$ such that $\tilde{s}_j(0)\in Q_j'$. Consider the following cases.

1) For both $j=1,2$ there exist $q_j\in Q_j'\setminus S_{init}(\Sigma',i)$ such that $o_j=\{0\}\mapsto out(q_j)$. Then for $j=1,2$ we have $in(q_j)=i(0)\in D_0$, whence $out(q_j)=\eta(i(0))$, because $q_j\in Q_0'$. Then $out(q_1)=out(q_2)$ and $o_1=o_2$, but this contradicts the assumption $o_1\neq o_2$.

2) There exists a (single) index $j\in\{1,2\}$ such that $o_j=\{0\}\mapsto out(q_j)$ for some $q_j\in Q_j'\setminus S_{init}(\Sigma',i)$ and $o_{3-j}=out\circ\tilde{s}_{3-j}$ for some $\tilde{s}_{3-j}\in S_{max}(\Sigma',i)$ such that $\tilde{s}_{3-j}(0)\in Q_{3-j}'$. Denote $q=\tilde{s}_{3-j}(0)$. Then $q\in S_{init}(\Sigma')$, so, as we have shown above, $in(q)=in(s_*(0))$ and $out(q)=out(s_*(0))$. Besides, $q\in Q_{3-j}'=Sel_{1,2}(Q_0',i(0),y_{3-j})$, so $i(0)=in(q)=in(s_*(0))$. Because $q_j\in Q_j'$, we have $in(q_j)=i(0)=in(s_*(0))$. Because $q_j\in Q_0'$ and $in(s_*(0))\in D_0$, we have $out(q_j)=\eta(in(q_j))=out(s_*(0))$. Then let $\tilde{s}:dom(\tilde{s}_{3-j})\to Q'$ be a function such that $\tilde{s}(0)=q_j$ and $\tilde{s}(t)=\tilde{s}_{3-j}(t)$, if $t\in dom(\tilde{s}_{3-j})\setminus\{0\}$. Because $\tilde{s}_{3-j}\in Tr'$, there exists $s\in X^*$ and $y\in Y$ such that $\tilde{s}_{3-j}\sqsubseteq f_s^y$. Then $s(0)\downarrow$ and $f_s^y(0)=(in(s(0)),y,out(s(0)))=\tilde{s}_{3-j}(0)=q$. Let $z=istate(q_j)\in Y$. Then $f_s^z(t)=f_s^y(t)$ for all $t\in dom(s)\setminus\{0\}$ and

$$f_s^z(0)=(in(s(0)),z,out(s(0)))=(in(q),z,out(q))=$$
$$=(in(s_*(0)),z,out(s_*(0)))=(in(q_j),istate(q_j),out(q_j))=q_j.$$

Then $\tilde{s}\sqsubseteq f_s^z$, because $\tilde{s}_{3-j}\sqsubseteq f_s^y$. Then because $dom(\tilde{s})=dom(\tilde{s}_{3-j})\in\mathfrak{T}$, we have $\tilde{s}\in Tr'$. Moreover, $in(\tilde{s}(0))=in(q_j)=i(0)$, and $in(\tilde{s}(t))=in(\tilde{s}_{3-j}(t))=i(t)$ for all

$t \in dom(\tilde{s}) \setminus \{0\}$, because $\tilde{s}_{3-j} \in S_{max}(\Sigma', i)$. Then $in \circ \tilde{s} \preceq i$. Besides, $dom(\tilde{s}) \in \mathcal{T}_0$, so $\tilde{s} \in S(\Sigma', i)$. Moreover, $\tilde{s}(0) = q_j$, so $q_j \in S_{init}(\Sigma', i)$. We have a contradiction with the assumption $q_j \in Q'_j \setminus S_{init}(\Sigma', i)$.

3) For both $j = 1,2$ there exist $\tilde{s}_j \in S_{max}(\Sigma', i)$ such that $\tilde{s}_j(0) \in Q'_j$ and $o_j = out \circ \tilde{s}_j$. Then there exist $z_1, z_2 \in Y$, $s_1, s_2 \in X^*$ such that $\tilde{s}_j \sqsubseteq f_{s_j}^{z_j}$ for $j = 1,2$. Then $dom(\tilde{s}_j) \subseteq dom(s_j)$ for $j = 1,2$ and $in(\tilde{s}_j(t)) = in(f_{s_j}^{z_j}(t)) = in(s_j(t))$ for each $j = 1,2$, $t \in dom(\tilde{s}_j)$. Also, $in \circ \tilde{s}_j \preceq i$ for $j = 1,2$. Denote $A = dom(\tilde{s}_1) \cap dom(\tilde{s}_2)$. Then $in(s_1(t)) = in(\tilde{s}_1(t)) = i(t) = in(\tilde{s}_2(t)) = in(s_2(t))$ for all $t \in A$. Moreover, either $A = T$, or A has a form $[0, t)$ or $[0, t)$ for some $t \in T \setminus \{0\}$. Then because $s_1, s_2 \in X^*$ and $X^* \in \mathcal{X}$, we have $s_1|_A \in X^*$ and $s_2|_A \in X^*$. Besides, $in \circ (s_1|_A) = in \circ (s_2|_A)$, whence $s_1|_A = s_2|_A$ by the property d) of the set $X^* \in \mathcal{X}$. Then for each $t \in A \setminus \{0\}$,

$$\tilde{s}_1(t) = f_{s_1}^{z_1}(t) = (in(s_1(t)), (istate(s_1(t)), s_1|_{[0,t]}), out(s_1(t))) =$$

$$= (in(s_2(t)), (istate(s_2(t)), s_2|_{[0,t]}), out(s_2(t))) = f_{s_2}^{z_2}(t) = \tilde{s}_2(t).$$

Because $dom(\tilde{s}_1), dom(\tilde{s}_2) \in \mathcal{T}_0$, we have $A \in \{dom(\tilde{s}_1), dom(\tilde{s}_2)\}$. Then $A = dom(\tilde{s}_k)$ for some $k \in \{1,2\}$. Then $\tilde{s}_k|_{T \setminus \{0\}} \sqsubseteq \tilde{s}_{3-k}|_{T \setminus \{0\}}$.

Let us show that $\tilde{s}_k|_{T \setminus \{0\}} \neq \tilde{s}_{3-k}|_{T \setminus \{0\}}$. Suppose that $\tilde{s}_k|_{T \setminus \{0\}} = \tilde{s}_{3-k}|_{T \setminus \{0\}}$. Then $o_k|_{T \setminus \{0\}} = o_{3-k}|_{T \setminus \{0\}}$. For each $j = 1,2$ we have $\tilde{s}_j(0) \in S_{init}(\Sigma')$, whence $o_j(0) = out(\tilde{s}_j(0)) = out(s_*(0))$. Then $o_1(0) = o_2(0)$ and $o_1 = o_2$, so we have a contradiction with the assumption $o_1 \neq o_2$. Thus $\tilde{s}_k|_{T \setminus \{0\}} \neq \tilde{s}_{3-k}|_{T \setminus \{0\}}$.

So we have $\tilde{s}_k|_{T \setminus \{0\}} \sqsubset \tilde{s}_{3-k}|_{T \setminus \{0\}}$. Let us define a function $\tilde{s} : dom(\tilde{s}_{3-k}) \to Q'$ as follows: $\tilde{s}(0) = \tilde{s}_k(0)$ and $\tilde{s}(t) = \tilde{s}_{3-k}(t)$, if $t \in dom(\tilde{s}_{3-k}) \setminus \{0\}$. For $j = 1,2$, $\tilde{s}_j \sqsubseteq f_{s_j}^{z_j}$ and $\tilde{s}_j(0) \in S_{init}(\Sigma')$, whence $in(s_j(0)) = in(\tilde{s}_j(0)) = in(s_*(0))$ and $out(s_j(0)) = out(\tilde{s}_j(0)) = out(s_*(0))$. Then $\tilde{s}(0) = \tilde{s}_k(0) = f_{s_k}^{z_k}(0) = f_{s_{3-k}}^{z_k}(0)$ and

$\widetilde{s}(t)=\widetilde{s}_{3-k}(t)=f_{s_{3-k}}^{z_{3-k}}(t)=f_{s_{3-k}}^{z_k}(t)$ for $t\in dom(\widetilde{s})\setminus\{0\}$. Thus $\widetilde{s}\sqsubseteq f_{s_{3-k}}^{z_k}$. Because $dom(\widetilde{s})=dom(\widetilde{s}_{3-j})\in\mathfrak{T}$, we have $\widetilde{s}\in Tr'$. Moreover, $dom(\widetilde{s})\in\mathcal{T}_0$. Because $in\circ\widetilde{s}_k\preceq i$ and $in\circ\widetilde{s}_{3-k}\preceq i$, we have $in\circ\widetilde{s}\preceq i$. Thus $\widetilde{s}\in S(\Sigma',i)$. Besides, $\widetilde{s}_k\sqsubset\widetilde{s}$, because $\widetilde{s}_k|_{T\setminus\{0\}}\sqsubset\widetilde{s}_{3-k}|_{T\setminus\{0\}}=\widetilde{s}|_{T\setminus\{0\}}$ and $\widetilde{s}(0)=\widetilde{s}_k(0)$. But the relation $\widetilde{s}_k\sqsubset\widetilde{s}$ contradicts the assumption $\widetilde{s}_k\in S_{max}(\Sigma',i)$.

In all cases 1)-3) we have a contradiction. Thus for each $i\in Sb(In(B'),W)$ and $\{o_1,o_2\}\subseteq Op(B')(i)$ we have $o_1=o_2$.

We conclude that the block B' is deterministic.

Let us show that $Op(B')(i)\subseteq Op(B)(i)$ for each $i\in Sb(In(B),W)$ such that $i(0)\downarrow=i_*(0)$.

Assume that $i\in Sb(In(B),W)$, $i(0)\downarrow=i_*(0)$, and $o\in Op(B')(i)$. Let us show that $o\in Op(B)(i)$. We have $o\in O_{all}(\Sigma',Q_0',i)$. Because $i\neq\bot$, there exists $y_*\in Y$ such that $o\in o_{all}(\Sigma',Sel_{1,2}(Q_0',i(0),y_*),i)$.

Let $Q_1'=Sel_{1,2}(Q_0',i(0),y_*)$. Because $s_*\in S_{max}(\Sigma,i_*)$, we have $in(s_*(0))=i_*(0)=i(0)$. Then $Q_1'=\{(in(s_*(0)),y_*,out(s_*(0)))\}\neq\varnothing$, because $\eta(in(s_*(0)))=out(s_*(0))$.

Besides, $s_*(0)\in S_{init}(\Sigma)\subseteq Q_0$, because (Σ,Q_0) is an initial I/O NCMS.

Let $x_*=istate(s_*(0))\in IState(\Sigma)$ and $Q_1=Sel_{1,2}(Q_0,i(0),x_*)$. Then

$$s_*(0)\in Sel_{1,2}(Q_0,in(s_*(0)),x_*)=Q_1.$$

Let us show that if $s_*(0)\in S_{init}(\Sigma,i)$, then $q\in S_{init}(\Sigma',i)$. Assume that $s_*(0)\in S_{init}(\Sigma,i)$. Then $s_*(0)=\bar{s}(0)$ for some $\bar{s}\in S(\Sigma,i)$. Let

$$X=X^*\cup\{\bar{s}|_{[0,t)}\mid t\in T\setminus\{0\}\}\cup\{\bar{s}|_{[0,t]}\mid t\in T\setminus\{0\}\}\cup\{\bar{s}\}.$$

Then $X\subseteq Tr$. Because $X^*\in\mathcal{X}$, we have $s_*\in X$, $s(0)\downarrow=s_*(0)$ for all $s\in X$, and for each $s\in X$ and $t\in T\setminus\{0\}$, $s|_{[0,t]}\in X$ and $s|_{[0,t)}\in X$, so X satisfies the properties a)-c) of the elements of $\mathcal{X}$. Because X^* is $\subseteq$-maximal in $\mathcal{X}$, either X does not satisfy the property $d)$ of the elements of $\mathcal{X}$, or $X = X^*$.

In the former case, there exists $s_1, s_2 \in X$ such that $in \circ s_1 = in \circ s_2$ and $s_1 \neq s_2$. Because $X^* \in \mathcal{X}$ and different elements of $X \setminus X^*$ have different domains, we have that for some $k \in \{1,2\}$, $s_k \in X^*$ and $s_{3-k} \in X \setminus X^*$. Then $in \circ s_k = in \circ s_{3-k} \preceq in \circ \bar{s} \preceq i$, $f_{s_k}^{y_*} \in Tr'$, $dom(f_{s_k}^{y_*}) = dom(s_k)$, and $s_k(0) \downarrow = s_*(0)$. Then for $\tilde{s} = f_{s_k}^{y_*}$ we have $\tilde{s} \in Tr'$, $\tilde{s}(0) \downarrow = f_{s_k}^{y_*}(0) = (in(s_*(0)), y_*, out(s_*(0))) = q$, because $q \in Q_1'$, and $in \circ \tilde{s} = in \circ f_{s_k}^{y_*} = in \circ s_k \preceq i$. Then $\tilde{s} \in S(\Sigma', i)$ and $\tilde{s}(0) = q$, whence $q \in S_{init}(\Sigma', i)$.

In the latter case, $X = X^*$, so $\bar{s} \in X^*$, $in \circ \bar{s} \preceq i$, and $\bar{s}(0) \downarrow = s_*(0)$. Then for $\tilde{s} = f_{\bar{s}}^{y_*}$ we have $\tilde{s} \in Tr'$, $\tilde{s}(0) \downarrow = f_{\bar{s}}^{y_*}(0) = (in(s_*(0)), y_*, out(s_*(0))) = q$, because $q \in Q_1'$, and $in \circ \tilde{s} = in \circ f_{\bar{s}}^{y_*} = in \circ \bar{s} \preceq i$. Then $\tilde{s} \in S(\Sigma', i)$, $\tilde{s}(0) = q$, so $q \in S_{init}(\Sigma', i)$.

In both cases, $q \in S_{init}(\Sigma', i)$.

We conclude that if $s_*(0) \in S_{init}(\Sigma, i)$, then $q \in S_{init}(\Sigma', i)$.

Because $Q_1' \neq \varnothing$ and $o \in o_{all}(\Sigma', Q_1', i)$, the following cases are possible.

1) $dom(i) = \{0\}$ and $o = \{0\} \mapsto out(q)$ for some $q \in Q_1'$. Then $q = (in(s_*(0)), y_*, out(s_*(0)))$, so $out(q) = out(s_*(0))$. Because $s_*(0) \in Q_1$, we have

$$o = \{0\} \mapsto out(s_*(0)) \in o_{all}(\Sigma, Q_1, i) \subseteq O_{all}(\Sigma, Q_0, i) = Op(B)(i).$$

2) $\{0\} \subset dom(i)$ and $o = \{0\} \mapsto out(q)$ for some $q \in Q_1' \setminus S_{init}(\Sigma', i)$. As we have shown above, if $s_*(0) \in S_{init}(\Sigma, i)$, then $q \in S_{init}(\Sigma', i)$. Then because $q \notin S_{init}(\Sigma', i)$, we have $s_*(0) \notin S_{init}(\Sigma, i)$. Besides, $s_*(0) \in Q_1 = Sel_{1,2}(Q_0, i(0), x_*)$, so $s_*(0) \in Q_1 \setminus S_{init}(\Sigma, i)$. Also, $out(q) = out(s_*(0))$, because $q \in Q_1'$. Then because $\{0\} \subset dom(i)$, we have

$$o = \{0\} \mapsto out(q) = \{0\} \mapsto out(s_*(0)) \in o_{all}(\Sigma, Q_1, i) \subseteq O_{all}(\Sigma, Q_0, i) = Op(B)(i).$$

3) $o = out \circ \tilde{s}$ for some $\tilde{s} \in S_{max}(\Sigma', i)$ such that $\tilde{s}(0) \in Q_1'$. Then $\tilde{s}(0) = (in(s_*(0)), y_*, out(s_*(0)))$, $\tilde{s} \in Tr'$, and there exists $\bar{s} \in X^*$ such that $\tilde{s} \sqsubseteq f_{\bar{s}}^{y_*}$.

Let $s=\bar{s}|_{dom(\tilde{s})}$. Because $dom(\tilde{s})$ either coincides with T, or has a form $[0,t]$ or $[0,t)$ for some $t\in T\setminus\{0\}$, we have $s\in X^*$. Also, $dom(\tilde{s})=dom(s)=dom(f_s^{y_*})$ and $f_s^{y_*}(t)=f_{\bar{s}}^{y_*}(t)$ for all $t\in dom(s)$, so $\tilde{s}=f_{\bar{s}}^{y_*}|_{dom(\tilde{s})}=f_s^{y_*}$. Because $\tilde{s}\in S_{max}(\Sigma',i)$, we have $s(0)\downarrow$ and $in\circ s=in\circ f_s^{y_*}=in\circ\tilde{s}\preceq i$. Also, $s\in X^*\subseteq Tr$, so $s\in S(\Sigma,i)$.

By Lemma 2.7, there exists $\hat{s}\in S_{max}(\Sigma,i)$ such that $s\sqsubseteq\hat{s}$.

Let us show that $\hat{s}\in X^*$. Suppose that $\hat{s}\notin X^*$. Let

$$X_1=\{\hat{s}|_{[0,t)}\mid t\in T\setminus\{0\}\}\cup\{\hat{s}|_{[0,t]}\mid t\in T\setminus\{0\}\}\cup\{\hat{s}\}$$

and $X=X^*\cup X_1$. We have $X\subseteq Tr$ and $\hat{s}(0)=s(0)=\bar{s}(0)=s_*(0)$, because $\bar{s}\in X^*$. Then X satisfies the properties a)-c) of the elements of $\mathcal{X}$. Because $\hat{s}\notin X^*$ and X^* is $\subseteq$-maximal in $\mathcal{X}$, the set X does not satisfy the property d) of the elements of $\mathcal{X}$. Then there exist $s_1,s_2\in X$ such that $in\circ s_1=in\circ s_2$ and $s_1\neq s_2$. Because $X^*\in\mathcal{X}$ and different elements of $X\setminus X^*$ have different domains, for some $k\in\{1,2\}$, $s_k\in X^*$ and $s_{3-k}\in X\setminus X^*$. If $s_{3-k}\sqsubseteq s$, then s_{3-k} either coincides with $s\in X^*$, or has a form $s|_{[0,t]}$ or $s|_{[0,t)}$ for some $t>0$, whence $s_{3-k}\in X^*$. In both cases we have a contradiction with $s_{3-k}\in X\setminus X^*$, so $s\sqsubset s_{3-k}$, because X_1 is a $\sqsubseteq$-chain and $s,s_{3-k}\in X_1$. Because $dom(s)\subseteq dom(s_{3-k})$, we have $in\circ(s_k|_{dom(s)})=in\circ(s_{3-k}|_{dom(s)})=in\circ(\hat{s}|_{dom(s)})=in\circ s$. Then because $s\in X^*$ and $s_k\in X^*$, we have $s_k|_{dom(s)}\in X^*$ and $s_k|_{dom(s)}=s$. Then $s\sqsubset s_k$, because $dom(s)\subset dom(s_{3-k})=dom(s_k)$. This implies that $\tilde{s}=f_s^{y_*}\sqsubset f_{s_k}^{y_*}$, because $s_k\in X^*$. Also, we have $f_{s_k}^{y_*}\in Tr'$ and $in\circ f_{s_k}^{y_*}=in\circ s_k=in\circ s_{3-k}\preceq in\circ\hat{s}\preceq i$, so $f_{s_k}^{y_*}\in S(\Sigma',i)$. This contradicts the inclusion $\tilde{s}\in S_{max}(\Sigma',i)$, because $\tilde{s}\sqsubset f_{s_k}^{y_*}$.

We conclude that $\hat{s}\in X^*$.

Then because $s \sqsubseteq \hat{s}$, we have $\tilde{s} = f_s^{y_*} \sqsubseteq f_{\hat{s}}^{y_*}$. Also, $f_{\hat{s}}^{y_*} \in Tr'$ and $in \circ f_{\hat{s}}^{y_*} = in \circ \hat{s} \preceq i$, so $f_{\hat{s}}^{y_*} \in S(\Sigma', i)$. Then because $\tilde{s} \in S_{max}(\Sigma', i)$, we have $\tilde{s} = f_s^{y*} = f_{\hat{s}}^{y_*}$. This implies that $dom(s) = dom(\hat{s})$. Then $s = \hat{s}$, because $s \sqsubseteq \hat{s}$.

We conclude that $s = \hat{s} \in S_{max}(\Sigma, i)$. Moreover, $s(0) \in S_{init}(\Sigma) \subseteq Q_0$ and $in(s(0)) = in(f_s^{y_*}(0)) = in(\tilde{s}(0)) = i(0)$, whence $s(0) \in Sel_{1,2}(Q_0, i(0), x)$, where $x = istate(s(0))$. Besides, $out \circ s = out \circ f_s^{y_*} = out \circ \tilde{s} = o$. Then

$$o = out \circ s \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) \subseteq O_{all}(\Sigma, Q_0, i) = Op(B)(i).$$

In all cases 1)-3) above, $o \in Op(B)(i)$.

We conclude that $Op(B')(i) \subseteq Op(B)(i)$ for each $i \in Sb(In(B), W)$ such that $i(0) \downarrow = i_*(0)$.

Let us show that $(i_*, o_*) \in IO(B')$. We have $s_* \in X^*$, $s_* \in S_{max}(\Sigma, i_*)$, and $o_* = out \circ s_*$. Let $y \in Y$ be an arbitrary element. Because $s_* \in X^*$, we have $f_{s_*}^y \in Tr'$. Denote $\tilde{s} = f_{s_*}^y$. Then $dom(\tilde{s}) = dom(s_*)$ and $in \circ \tilde{s} = in \circ f_{s_*}^y = in \circ s_* \preceq i_*$, so $\tilde{s} \in S(\Sigma', i_*)$. By Lemma 2.7, there exists $\tilde{s}' \in S_{max}(\Sigma', i_*)$ such that $\tilde{s} \sqsubseteq \tilde{s}'$. Because $\tilde{s}' \in Tr'$, there exists $s \in X^*$ and $y' \in Y$ such that $\tilde{s}' \sqsubseteq f_s^{y'}$. Then $f_{s_*}^y \sqsubseteq f_s^{y'}$. This implies that $dom(s_*) \subseteq dom(s)$, $y = y'$, and $s_*(t) = s(t)$ for all $t \in dom(s_*) \setminus \{0\}$. Moreover, $s(0) = s_*(0)$, because $s \in X^*$. Then $s_* \sqsubseteq s$. Denote $A = dom(\tilde{s}')$. Then $dom(s_*) = dom(\tilde{s}) \subseteq dom(\tilde{s}') = A$, so $s_* \sqsubseteq s|_A$. Besides, $s|_A \in X^*$, because $s \in X^*$. Then we have $f_{s|_A}^y \sqsubseteq f_s^y$. Moreover, $dom(f_{s|_A}^y) = dom(s|_A) = A$, because $A = dom(\tilde{s}') \subseteq dom(f_s^{y'}) = dom(s)$. Then because $\tilde{s}' \sqsubseteq f_s^{y'} = f_s^y$ and $A = dom(\tilde{s}')$, we have $f_{s|_A}^y = \tilde{s}'$. Then $in \circ (s|_A) = in \circ (f_{s|_A}^y) = in \circ \tilde{s}' \preceq i_*$. Besides, $s|_A \in X^* \subseteq Tr$, whence $s|_A \in S(\Sigma, i_*)$. Because $s_* \sqsubseteq s|_A$ and $s_* \in S_{max}(\Sigma, i_*)$, we have $s_* = s|_A$. Then $\tilde{s} = f_{s_*}^y = f_{s|_A}^y = \tilde{s}' \in S_{max}(\Sigma', i_*)$. We have $\tilde{s}(0) \in S_{init}(\Sigma') \subseteq Q_0'$, because (Σ', Q_0') is an initial I/O NCMS. Moreover, $in(\tilde{s}(0)) = in(s_*(0)) = i_*(0)$ and $istate(\tilde{s}(0)) = istate(f_{s_*}^{y}(0)) = y \in IState(\Sigma')$, whence $\tilde{s}(0) \in Sel_{1,2}(Q_0', i_*(0), y)$. Moreover, $out \circ \tilde{s} = out \circ f_{s_*}^{y} = out \circ s_* = o_*$. Then

$$o_* = out \circ \tilde{s} \in o_{all}(\Sigma', Sel_{1,2}(Q_0', i_*(0), y), i_*) \subseteq O_{all}(\Sigma', Q_0', i_*) = Op(B')(i_*).$$

Thus $(i_*, o_*) \in IO(B')$. □

Lemma 2.21. Assume that a block B has a NCMS representation, $(i_*, o_*) \in IO(B)$, and $i_*(0)\downarrow$. Then there exists a deterministic causal block B' such that $In(B') = In(B)$, $Out(B') = Out(B)$, $Op(B')(i) \subseteq Op(B)(i)$ for each $i \in Sb(In(B), W)$ such that $i(0)\downarrow = i_*(0)$, and $(i_*, o_*) \in IO(B')$.

Proof. Consider the following cases.

1) Either $dom(i_*) = \{0\}$, or $o_* = \perp$, and also the inclusion $dom(o) \subseteq \{0\}$ holds for each $(i, o) \in IO(B)$ such that $(i_*, o_*) \preceq^2 (i, o)$.

Let us define a function $O : Sb(In(B), W) \to 2^{Sb(Out(B), W)}$ as follows: $O(\perp) = \{\perp\}$ and $O(i) = \{o_*\}$, if $i \neq \perp$. Then $O(i)$ is a singleton set for each i. Moreover, we have $dom(o_*) \subseteq \{0\}$, so $dom(o) \subseteq dom(i)$ holds for all i, o such that $o \in O(i)$. Then there exists a deterministic block B' such that $In(B) = In(B')$, $Out(B) = Out(B')$, and $Op(B') = O$. If $o_1 \in Op(B')(i_1)$ and $o_2 \in Op(B')(i_2)$ for some i_1, i_2 such that $i_1|_{[0,t]} = i_2|_{[0,t]}$ for some $t \in T$, then $i_1 = \perp$ if and only if $i_2 = \perp$, so $o_1 = o_2$, whence $o_1|_{[0,t]} = o_2|_{[0,t]}$. Thus B' is causal.

Moreover, $o_* \in O(i_*) = Op(B')(i_*)$, because $i_* \neq \perp$. Then $(i_*, o_*) \in IO(B')$.

Let $i \in Sb(In(B), W)$ and $i(0)\downarrow = i_*(0)$.

Consider the case when $o_* = \perp$. Then because $(i_*, o_*) \in IO(B)$ and $(i_*|_{\{0\}}, \perp) \preceq^2 (i_*, o_*)$, we have $(i_*|_{\{0\}}, \perp) \in IO(B)$ by Lemma 2.15. Then because $i_*|_{\{0\}} \preceq i$, by Lemma 2.16 there exists $o' \in Op(B)(i)$ such that $(i_*|_{\{0\}}, \perp) \preceq^2 (i, o')$. Then because $i_*(0)\downarrow$, we have $o' = \perp$. Then $\{\perp\} = \{o_*\} = Op(B')(i) \subseteq Op(B)(i)$.

Consider the case when $o_* \neq \perp$. Then $dom(i_*) = \{0\}$ and because $(i_*, o_*) \in IO(B)$ and $i_* = i_* |_{\{0\}} \preceq i$, by Lemma 2.16 there exists $o' \in Op(B)(i)$ such that $(i_*, o_*) \preceq^2 (i, o')$. Then $dom(o') \subseteq \{0\}$ and $dom(o_*) = \{0\}$, so $o' = o_*$ and $\{o_*\} = Op(B')(i) \subseteq Op(B)(i)$.

Thus B' satisfies the statement of the lemma.

2) $\{0\} \subset dom(i_*)$, $o_* \neq \perp$, and the inclusion $dom(o) \subseteq \{0\}$ holds for each $(i, o) \in IO(B)$ such that $(i_*, o_*) \preceq^2 (i, o)$.

Then $\{0\} \subset dom(i_*)$ and $dom(o_*) = \{0\}$, so by Lemma 2.19 there exists a sub-block $B' \trianglelefteq B$ such that B' has a NCMS representation and $Op(B')(i_*) = \{o_*\}$.

By Lemma 2.17, B' is weakly nonanticipative. Consider the following cases.

2.1) There exists $(i_0, o_0) \in IO(B')$ such that $i_0(0) \downarrow = i_*(0)$ and $\{0\} \subset dom(o_0)$. Then by Lemma 2.20 (applied to B'), there exists a deterministic block B'' which has a NCMS representation, such that $In(B'') = In(B') = In(B)$, $Out(B'') = Out(B') = Out(B)$, $Op(B'')(i) \subseteq Op(B')(i) \subseteq Op(B)(i)$ for each $i \in Sb(In(B'), W)$ such that $i(0) \downarrow = i_0(0) = i_*(0)$, and $(i_0, o_0) \in IO(B'')$. Then $Op(B'')(i_*) \subseteq Op(B')(i_*) = \{o_*\}$, so $(i_*, o_*) \in IO(B'')$. Besides, B'' is causal by Lemma 2.17 and Lemma 1.5. Then B'' satisfies the statement of the lemma.

2.2) For each $(i, o) \subseteq IO(B')$, if $i(0) \downarrow = i_*(0)$, then $\{0\} \subset dom(o)$ is not satisfied (which is implies the inclusion $dom(o) \subseteq \{0\}$).

Let B'_0 be a block such that $In(B'_0) = In(B)$, $Out(B'_0) = Out(B)$, and $Op(B'_0)(i) = Op(B')(i)$, if $i(0) \downarrow = i_*(0)$, and $Op(B'_0)(i) = \{\perp\}$, otherwise. Obviously, B'_0 is indeed correctly defined as a block.

Let us show that B'_0 is weakly nonanticipative. Let $A \in \mathcal{T}_0$, $i_1, i_2 \in Sb(In(B'_0), W)$, and $i_1 |_A = i_2 |_A$. If $A = \varnothing$ or $i_1 = \perp$ or $i_2 = \perp$, then

$$\{o |_A \mid o \in Op(B'_0)(i_1)\} = \{\perp\} = \{o |_A \mid o \in Op(B'_0)(i_2)\}.$$

Assume that $0 \in A \cap dom(i_1) \cap dom(i_2)$. If $i_1(0) = i_*(0)$, then $i_2(0) = i_*(0)$, whence $Op(B'_0)(i_j) = Op(B')(i_j)$ for $j = 1,2$, and so

$$\{o|_A \mid o \in Op(B'_0)(i_1)\} = \{o|_A \mid o \in Op(B'_0)(i_2)\}$$

because B' is weakly nonanticipative. Otherwise, $i_2(0) = i_1(0) \neq i_*(0)$, whence $Op(B'_0)(i_1) = Op(B'_0)(i_2) = \{\perp\}$. Then

$$\{o|_A \mid o \in Op(B'_0)(i_1)\} = \{\perp\} = \{o|_A \mid o \in Op(B'_0)(i_2)\}.$$

We conclude that B'_0 is weakly nonanticipative. Moreover, $dom(o) \subseteq \{0\}$ for each $(i,o) \in IO(B'_0)$. Then B'_0 is strongly nonanticipative by Lemma 2.18, so it has some deterministic causal sub-block $B'' \trianglelefteq B'_0$ (because $IO(B'_0) \neq \emptyset$). Then $Op(B'')(i_*) \subseteq Op(B'_0)(i_*) = Op(B')(i_*) = \{o_*\}$, whence $(i_*, o_*) \in IO(B'')$. Besides, $In(B'') = In(B)$, $Out(B'') = Out(B)$, and for each $i \in Sb(In(B), W)$ such that $i(0) \downarrow= i_*(0)$ we have $Op(B'')(i) \subseteq Op(B'_0)(i) = Op(B')(i) \subseteq Op(B)(i)$. Then B'' satisfies the statement of the lemma.

3) There exists $(i_0, o_0) \in IO(B)$ such that $(i_*, o_*) \preceq^2 (i_0, o_0)$ and $dom(o_0) \subseteq \{0\}$ does not hold. Then $\{0\} \subset dom(o_0)$, $i_0(0) \downarrow= i_*(0)$, and by Lemma 2.20 there exists a deterministic block B' which has a NCMS representation, such that $In(B') = In(B)$, $Out(B') = Out(B)$, $Op(B')(i) \subseteq Op(B)(i)$ for each $i \in Sb(In(B), W)$ such that $i(0) \downarrow= i_0(0) = i_*(0)$, and $(i_0, o_0) \in IO(B')$. Then B' is weakly nonanticipative by Lemma 2.17, so it is causal by Lemma 1.5. Because $(i_0, o_0) \in IO(B')$ and $(i_*, o_*) \preceq^2 (i_0, o_0)$, we have $(i_*, o_*) \in IO(B')$ by Theorem 1.1. Then B' satisfies the statement of the lemma. □

Lemma 2.22. Assume that a block B has a NCMS representation. Then B is strongly nonanticipative.

Proof. Let us fix an arbitrary $(i_0, o_0) \in IO(B)$.

If $i_0 = \perp$, then let $i_* = \{0\} \mapsto []$ and o_* be an arbitrary member of $Op(B)(i_*)$. Otherwise, i.e. if $i_0 \neq \perp$, then let $i_* = i_0$ and $o_* = o_0$. In both cases we have defined a pair (i_*, o_*) such that $(i_*, o_*) \in IO(B)$ and $i_* \neq \perp$.

Denote $D = {}^{In(B)} W$. For each each $d \in D$ let $i_d = \{0\} \mapsto d$, if $d \neq i_*(0)$ and $i_d = i_*$, if $d = i_*(0)$. Then $i_d(0) \downarrow = d$ and $Op(B)(i_d) \neq \varnothing$ for each $d \in D$ and $o_* \in Op(B)(i_{i_*(0)})$. Then there exists a (selector) function $f : D \to Sb(Out(B), W)$ such that $f(d) \in Op(B)(i_d)$ for each $d \in D$ and $f(i_*(0)) = o_*$.

Then by Lemma 2.21, for each $d \in D$ let us choose a deterministic causal block B_d such that $In(B_d) = In(B)$, $Out(B_d) = Out(B)$, $Op(B_d)(i) \subseteq Op(B)(i)$ for each $i \in Sb(In(B), W)$ such that $i(0) \downarrow = i_d(0)$, and $(i_d, f(d)) \in IO(B_d)$.

Let $O : Sb(In(B), W) \to 2^{Sb(Out(B), W)}$ be a function such that $O(i) = Op(B_{i(0)})(i)$, if $i \neq \perp$ and $O(\perp) = \{\perp\}$.

Then $O(i) \neq \varnothing$ for all i and $dom(o) \subseteq dom(i)$ whenever $o \in O(i)$. Then there exists a block B' such that $In(B') = In(B)$, $Out(B') = Out(B)$, $Op(B') = O$.;

Because for each $d \in D$ the block B_d is deterministic, B' is deterministic.

Let us show that $B' \trianglelefteq B$. Let $(i, o) \in IO(B')$. If $i = \perp$, then $(i, o) = (\perp, \perp) \in IO(B)$. Otherwise, $o \in O(i) = Op(B_{i(0)})(i) \subseteq Op(B)(i)$, because $i(0) = i_{i(0)}(0)$, whence $(i, o) \in IO(B)$. Thus $B' \trianglelefteq B$.

Let us show that B' is causal. Let $i, i' \in Sb(In(B'), W)$, $t \in T$, $i|_{[0,t]} = i'|_{[0,t]}$, $o \in Op(B')(i)$, and $o' \in Op(B')(i')$. If $i = \perp$ or $i' = \perp$, then $i = i' = o = o' = \perp$, so $o|_{[0,t]} = o'|_{[0,t]}$. Consider the case when $i \neq \perp$ and $i' \neq \perp$. Then $i(0)\downarrow$, $i'(0)\downarrow$, and $i(0) = i'(0)$. Denote $d = i(0)$. Then $o \in Op(B')(i) = Op(B_d)(i)$ and $o' \in Op(B')(i') = Op(B_d)(i')$,whence $o|_{[0,t]} = o'|_{[0,t]}$, because B_d is causal.

We conclude that B' is causal. Moreover,

$$Op(B')(i_*) = Op(B_{i_*(0)})(i_*) = Op(B_{i_*(0)})(i_{i_*(0)}) = \{f(i_*(0))\} = \{o_*\}.$$

Then $(i_*, o_*) \in IO(B')$. If $i_0 \neq \perp$, this implies that $(i_0, o_0) = (i_*, o_*) \in IO(B')$. Otherwise, i.e. if $i_0 = \perp$, then $(i_0, o_0) = (\perp, \perp) \in IO(B')$.

We conclude that for each $(i_0, o_0) \in IO(B)$ there exists a deterministic causal sub-block $B' \trianglelefteq B$ such that $(i_0, o_0) \in IO(B')$. Thus B is strongly nonanticipative. □

Now we can prove Theorem 2.3.

Proof of Theorem 2.3. Let (Σ, Q_0) be an initial I/O NCMS. By Lemma 2.9, it is a NCMS representation of some block B. Then by Lemma 2.22, B is strongly nonanticipative. □

2.8 Strongly nonanticipative blocks, NCMS, and predicate pairs

As we have shown above (Theorem 2.2), each strongly nonanticipative block has a representation in the form of an initial I/O NCMS (NCMS representation). Conversely, each initial I/O NCMS is a NCMS representation of a strongly nonanticipative block (Theorem 2.3). This is illustrated in Fig. 2.3.

An initial I/O NCMS consists of an I/O NCMS and a set of admissible initial states. An I/O NCMS is a NCMS, in which the set of states has a special form: ${}^I W \times X \times {}^O W$, where I, O are sets of input and output names, X is a non-empty set of internal states, and W is a set of signal values.

An I/O NCMS is a kind of NCMS, so by Theorem 2.1 it can be represented by a left-local/right-local predicate pair (LR representation). An LR representation of a NCMS is illustrated in Fig. 2.4.

We will use the results described above in the next chapter to derive criteria of the existence of total I/O pairs of strongly nonanticipative blocks.

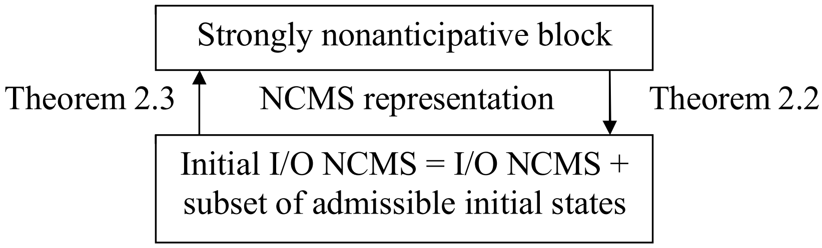

Fig. 2.3. An illustration of a NCMS representation of a block.

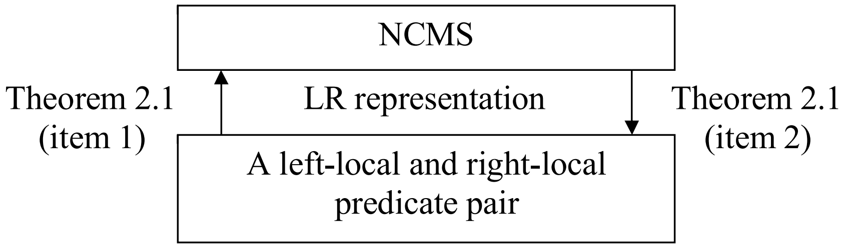

Fig. 2.4. An illustration of a LR representation of a NCMS.

2.9 Conclusions from the chapter

We have introduced a class of abstract dynamical systems that we called Non-deterministic Complete Markovian Systems (NCMS) on the basis of the notion of a solution system in the sense of [**37**] and investigated their basic properties (the existence of a LR representation).

We have defined a special kind of NCMS, namely input-output (I/O) NCMS, and also introduced a notion of an initial I/O NCMS as a pair of an I/O NCMS and a set of initial states.

We have defined a notion of a NCMS representation of a (strongly nonanticipative) block as an initial I/O NCMS. We have shown that each strongly nonanticipative block has a NCMS representation and that each initial I/O NCMS is a representation of a strongly nonanticipative block.

CHAPTER 3
EXISTENCE OF TOTAL I/O PAIRS OF A STRONGLY NONANTICIPATIVE BLOCK

3.1 Overview

In Chapter 1 we defined the notion of a block which allows partially defined inputs and outputs. The operation of a block can be described by a set of input-output pairs (i,o) (which we denoted as $IO(B)$) which are partial functions of time with possibly different domains (but such that $dom(o) \subseteq dom(i)$). However, as we have mentioned in Chapter 1, several approaches to mathematical systems theory consider the case of total input-output pairs ($dom(i) = dom(o) = T$) particularly important. This motivates to investigate the properties of the set of total input-output pairs as a subset of the set of all input-output pairs of a block.

One of the most basic questions that can be asked about total input-output pairs of a block is their existence.

In this chapter we consider the following question:

(a) *How can one prove that a given strongly nonanticipative block B has a total I/O pair (if B indeed has a total I/O pair) ?*

Using the same techniques which we will use to answer this question, in this chapter we will also give an answer to the following question:

(b) *How can one prove that for a given input signal bunch $i \in Sb(In(B),W)$, where $dom(i) = T$, there exists $o \in Op(B)(i)$ with $dom(o) = T$?*

That is, to prove that a block admits a total output for a given total input. Due to the fact that we interpret the case $dom(o) \subset dom(i)$ as an abnormal termination of a block on the input i, this can be interpreted as proving that it is possible for a block to process the input i normally.

3.2 Using the NCMS representation

The following two theorems show that the questions (a) and (b) formulated in the previous section can be reduced to the problem of proving the existence of global trajectories of NCMS.

Definition 3.1. A trajectory s of a NCMS Σ is called global, if $dom(s)=T$.

Theorem 3.1. Let B be a strongly nonanticipative block and (Σ,Q_0) be its NCMS representation, where $\Sigma=(T,Q,Tr)$. Then B has a total I/O pair if and only if Σ has a global trajectory.

Proof.

Let us prove the "if" part. Assume that $s\in Tr$ and $dom(s)=T$. Let $q_0=s(0)$, $x=istate(q_0)$, $i=in\circ s$, $o=out\circ s$, and $Q'=Sel_{1,2}(Q_0,i(0),x)$. Then $q_0\in S_{init}(\Sigma)\subseteq Q_0$, whence $q_0\in Q'$, so $Q'\neq\varnothing$. Besides, $s\in S_{max}(\Sigma,i)$, because $dom(s)=T$ and $in\circ s=i\preceq i$. Then because $s(0)\in Q'$, we have $o=out\circ s\in o_{all}(\Sigma,Q',i)$ by the definition of o_{all}. Then $o\in O_{all}(\Sigma,Q_0,i)=Op(B)(i)$, because $i\neq\perp$ and (Σ,Q_0) is a NCMS representation of B. Then $(i,o)\in IO(B)$ and $dom(i)=dom(o)=T$. Thus B has a total I/O pair.

Let us prove the "only if" part. Assume that B has a total I/O pair $(i,o)\in IO(B)$. Because (Σ,Q_0) is a NCMS representation of B and $i\neq\perp$, we have $o\in O_{all}(\Sigma,Q_0,i)$. Then there is $x\in IState(\Sigma)$ such that $o\in o_{all}(\Sigma,Q',i)$, where $Q'=Sel_{1,2}(Q_0,i(0),x)$. Then $o=out\circ s$ for some $s\in S_{max}(\Sigma,i)$ such that $s(0)\in Q'$, because $dom(o)=T$. Then $s\in Tr$ and $dom(s)=T$, so s is a global trajectory. □

Theorem 3.2. Let B be a strongly nonanticipative block and (Σ,Q_0) be its NCMS representation, where $\Sigma=(T,Q,Tr)$. Let $i\in Sb(In(B),W)$ and $dom(i)=T$. Let (l,r) be a LR representation of Σ and $l':ST(Q)\to Bool$ and $r':ST(Q)\to Bool$ be predicates such that

$$l'(s,t)\Leftrightarrow l(s,t)\wedge(\min dom(s)\downarrow=t\vee in(s(t))=i(t)),$$

$$r'(s,t) \Leftrightarrow r(s,t) \wedge (\max dom(s) \downarrow= t \vee in(s(t)) = i(t)).$$

Then

1) $(l',r') \in LR(Q)$;

2) If (l',r') is a LR representation of a NCMS $\Sigma' = (T,Q,Tr')$, then $\{o \in Op(B)(i) \mid dom(o) = T\} \neq \varnothing$ if and only if Σ' has a global trajectory.

Proof.

1) Let us show that l' is left-local. Assume that $(s_1,t),(s_2,t) \in ST(Q)$ and $s_1 \doteq_{t-} s_2$. Then t is not the least element of either $dom(s_1)$, or $dom(s_2)$, whence $l(s_1,t) \Leftrightarrow l(s_2,t)$ and $in(s_1(t)) = i(t)$ if and only if $in(s_2(t)) = i(t)$, because $s_1(t) = s_2(t)$. Then $l'(s_1,t) \Leftrightarrow l'(s_2,t)$. Moreover, if $(s,t) \in ST(Q)$ and t is the least element of $dom(s)$, then $l(s,t)$, whence $l'(s,t)$. Thus l' is right-local.

Let us show that r' is right-local. Assume that $(s_1,t),(s_2,t) \in ST(Q)$ and $s_1 \doteq_{t+} s_2$. Then t is not the greatest element of either $dom(s_1)$, or $dom(s_2)$, whence $r(s_1,t) \Leftrightarrow r(s_2,t)$ and $in(s_1(t)) = i(t)$ if and only if $in(s_2(t)) = i(t)$, because $s_1(t) = s_2(t)$. Then $r'(s_1,t) \Leftrightarrow r'(s_2,t)$. Moreover, if $(s,t) \in ST(Q)$ and t is the greatest element of $dom(s)$, then $r(s,t)$, whence $r'(s,t)$. Thus r' is right-local.

2) Assume that (l',r') is a LR representation of a NCMS $\Sigma' = (T,Q,Tr')$. Then $Tr' = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\, l'(s,t) \wedge r'(s,t))\}$.

Firstly, let us show that $\{s \in Tr' \mid dom(s) \in \mathcal{T}_0\} = S(\Sigma,i)$.

Let $s \in Tr'$ and $dom(s) \in \mathcal{T}_0$. Then $l'(s,t) \wedge r'(s,t)$ for all $t \in dom(s)$. Then $l(s,t) \wedge r(s,t)$ for all $t \in dom(s)$, whence $s \in Tr$. Moreover, $in(s(t)) = i(t)$ for all non-minimal $t \in dom(s)$ and $in(s(t)) = i(t)$ for all non-maximal $t \in dom(s)$, so $in(s(t)) = i(t)$ for all $t \in dom(s)$ (because $dom(s)$ is not a singleton). Then $in \circ s \preceq i$, whence $s \in S(\Sigma,i)$.

Conversely, let $s \in S(\Sigma,i)$. Then $s \in Tr$ and $dom(s) \in \mathcal{T}_0$, whence $l(s,t) \wedge r(s,t)$ for all $t \in dom(s)$. Moreover, $in(s(t)) = i(t)$ for all $t \in dom(s)$. Then $l'(s,t) \wedge r'(s,t)$ for all $t \in dom(s)$, whence $s \in Tr'$.

We conclude that $\{s \in Tr' \mid dom(s) \in \mathcal{T}_0\} = S(\Sigma, i)$.

Now let us show that $\{o \in Op(B)(i) \mid dom(o) = T\} \neq \varnothing$ if and only if there exists $s \in Tr'$ such that $dom(s) = T$.

Let us prove the "if" part. Assume that $s \in Tr'$ and $dom(s) = T$. Then $s \in S(\Sigma, i)$. Let $q_0 = s(0)$, $x = istate(q_0)$, $o = out \circ s$, and $Q' = Sel_{1,2}(Q_0, i(0), x)$. Then $q_0 \in S_{init}(\Sigma') \subseteq S_{init}(\Sigma) \subseteq Q_0$, whence $q_0 \in Q'$, so $Q' \neq \varnothing$. Besides, $s \in S_{max}(\Sigma, i)$, because $dom(s) = T$. Then $o = out \circ s \in o_{all}(\Sigma, Q', i)$ by the definition of o_{all}, because $s(0) \in Q'$. Then $o \in O_{all}(\Sigma, Q_0, i) = Op(B)(i)$, because $i \neq \perp$ and (Σ, Q_0) is a NCMS representation of B. Moreover, $dom(o) = T$. Thus $\{o \in Op(B)(i) \mid dom(o) = T\} \neq \varnothing$.

Let us prove the "only if" part. Assume that $o \in Op(B)(i)$ and $dom(o) = T$. Because (Σ, Q_0) is a NCMS representation of B and $i \neq \perp$, we have $o \in O_{all}(\Sigma, Q_0, i)$. Then there is $x \in IState(\Sigma)$ such that $o \in o_{all}(\Sigma, Q', i)$, where $Q' = Sel_{1,2}(Q_0, i(0), x)$. Then $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$ such that $s(0) \in Q'$, because $dom(o) = T$. Then $s \in S(\Sigma, i)$, whence $s \in Tr'$ and $dom(s) = T$, so s is a global trajectory of Σ'. □

Now we will focus on the problem of existence of global trajectories of a NCMS.

3.3 Existence of globally defined trajectories of NCMS

An obvious method of proving the existence of a global trajectory of a NCMS with a given LR representation (l, r) is to choose (guess) some global trajectory candidate function $s : T \to Q$ and prove that $\forall t \in T \; l(s, t) \wedge r(s, t)$.

As an alternative to guessing an entire global trajectory one can try to find/guess for each t a partial trajectory s_t defined in a neighborhood of t which satisfies $l(s_t, t) \wedge r(s_t, t)$ in such a way that all s_t, $t \in T$ can be glued together into a total function. An important aspect here is that the admissible choices of s_t, $s_{t'}$ for distant time moments $t, t' \in T$ (i.e. such that s_t, $s_{t'}$ appear as subtrajectories of some global trajectory) can be dependent.

However, this method can be generalized: instead of guessing an exact global trajectory or its exact locally defined subtrajectories, one can guess some "region" (subset of trajectories) which presumably contains a global trajectory and has some convenient representation. It is desirable that for this region the proof of the existence of a global trajectory can be accomplished by finding/guessing locally defined trajectories in a neighborhood of each time moment independently, or at least so that when choosing a local trajectory in a neighborhood of a time moment t one does not need to care about a choice of a local trajectory in a neighborhood of a distant time moment.

We formalize the described generalized method of proving the existence of global trajectories of a NCMS as follows.

Let $\Sigma = (T, Q, Tr)$ be a fixed NCMS.

Definition 3.2. Σ satisfies

1) the local forward extensibility (LFE) property, if for each $s \in Tr$ of the form $s:[a,b] \to Q$ ($a < b$) there exists a trajectory $s':[a,b'] \to Q$ such that $s' \in Tr$, $s \sqsubseteq s'$, and $b' > b$ (i.e. s' is a continuation of s).
2) the global forward extensibility (GFE) property, if for each trajectory s of the form $s:[a,b] \to Q$ there exists a trajectory $s':[a,+\infty) \to Q$ such that $s \sqsubseteq s'$.

Theorem 3.3. Let (l, r) be a LR representation of Σ. Then Σ has a global trajectory if and only if there exists a pair $(l', r') \in LR(Q)$ such that

1) $l'(s,t) \Rightarrow l(s,t)$ and $r'(s,t) \Rightarrow r(s,t)$ for all $(s,t) \in ST(Q)$;
2) $\forall t \in [0,\varepsilon]\, l'(s,t) \wedge r'(s,t)$ for some $\varepsilon > 0$ and a function $s:[0,\varepsilon] \to Q$;
3) if (l', r') is a LR representation of a NCMS Σ', then Σ' satisfies GFE.

Proof.

Let us prove the "if" part. Assume that 1)-3) hold. By 2) there exists $\varepsilon > 0$ and $s:[0,\varepsilon] \to Q$ such that $l'(s,t) \wedge r'(s,t)$ for all $t \in [0,\varepsilon]$. Let $\Sigma' = (T, Q, Tr')$ be a NCMS such that (l', r') is a LR representation of Σ' (which exists, because $(l',r') \in LR(Q)$). Then by 3), Σ' satisfies GFE. Besides, $s \in Tr'$. Then there exists $s':[0,+\infty) \to Q$ such that $s' \in Tr'$ and $s \sqsubseteq s'$. Then $l'(s,t) \wedge r'(s,t)$ for all $t \in T$, whence $s' \in Tr$, because of 1), so Σ has a global trajectory.

Let us prove the "only if" part. Assume that Σ has a global trajectory $s^* \in Tr$. Let $l': ST(Q) \to Bool$ and $r': ST(Q) \to Bool$ be predicates such that

$$l'(s,t) \Leftrightarrow l(s,t) \wedge (\min dom(s) \downarrow= t \vee s(t) = s^*(t)),$$

$$r'(s,t) \Leftrightarrow r(s,t) \wedge (\max dom(s) \downarrow= t \vee s(t) = s^*(t)).$$

In the same way as in the proof of the item 1) of Theorem 3.2, it is straightforward to show that l' is left-local and r' is right-local. Then $(l',r') \in LR(Q)$. Obviously, $l'(s,t) \Rightarrow l(s,t)$ and $r'(s,t) \Rightarrow r(s,t)$ for all $(s,t) \in ST(Q)$, so 1) holds. Besides, we have $l(s^*,t) \wedge r(s^*,t)$ for all $t \in T$, because $s^* \in Tr$, whence $l'(s^*,t) \wedge r'(s^*,t)$ for all $t \in T$. Then 2) also holds. Assume that (l',r') is a LR representation of a NCMS Σ'. Let us show that Σ' satisfies GFE. Let $s:[a,b] \to Q$ $(a < b)$ be a trajectory of Σ'. Then $l'(s,t) \wedge r'(s,t)$ for all $t \in dom(s)$. Then $s(t) = s^*(t)$ for all $t \in (a,b]$ and $s(t) = s^*(t)$ for all $t \in [a,b)$, so $s(t) = s^*(t)$ for all $t \in [a,b]$. Then $s \sqsubseteq s^*$. Besides, s^* is a trajectory of Σ'. Let $s' = s^*|_{[a,+\infty)}$. Then s' is a trajectory of Σ' by the CPR property and $s \sqsubseteq s'$. Thus Σ' satisfies the GFE property. □

Theorem 3.3 means that the existence of a global trajectory of a NCMS Σ with a LR representation (l,r) can be proved using the following approach:

1) Choose/guess a pair $(l',r') \in LR(Q)$ such that $l'(s,t) \Rightarrow l(s,t)$ and $r'(s,t) \Rightarrow r(s,t)$ for all $(s,t) \in ST(Q)$. This pair is a LR representation of a NCMS $\Sigma' = (T, Q, Tr')$, where

$$Tr' = \{s: A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l'(s,t) \wedge r'(s,t))\}.$$

The set $Tr' \subseteq Tr$ plays the role of a region which presumably contains a global trajectory.

2) If it is possible to find a function s on a small segment $[0,\varepsilon]$ which satisfies $l'(s,t) \wedge r'(s,t)$ for $t \in [0,\varepsilon]$ (i.e. s is a trajectory of Σ') and prove that Σ' satisfies GFE, then Σ has a global trajectory.

To complete this method of proving the existence of a global trajectory, in the next section we will show that the GFE property of a NCMS can be proven by proving the existence of certain locally defined trajectories independently in a neighborhood of each time moment.

3.4 Reduction of the GFE property to the LFE property

As above, let $\Sigma = (T, Q, Tr)$ be a fixed NCMS.

Definition 3.3. A right dead-end path (in Σ) is a trajectory $s:[a,b) \to Q$, where $a,b \in T$, $a < b$, such that there is no $s':[a,b] \to Q$, $s' \in Tr$ such that $s \sqsubset s'$ (i.e. s cannot be extended to a trajectory on $[a,b]$).

Definition 3.4. An escape from a right dead-end path $s:[a,b) \to Q$ (in Σ) is a trajectory $s':[c,d) \to Q$ (where $d \in T \cup \{+\infty\}$) or $s':[c,d] \to Q$ (where $d \in T$) such that $c \in (a,b)$, $d > b$, and $s(c) = s'(c)$. An escape s' is called infinite, if $d = +\infty$.

Definition 3.5. A right dead-end path $s:[a,b) \to Q$ in Σ is called strongly escapable, if there exists an infinite escape from s.

Lemma 3.1. If $s:[a,b) \to Q$ is a right dead-end path and $c \in (a,b)$, then $s|_{[c,b)}$ is a right dead-end path.

The proof follows immediately from the CPR and Markovian properties of Σ.

Lemma 3.2. Σ satisfies GFE if and only if Σ satisfies LFE and each right dead-end path is strongly escapable.

Proof.

Let us prove the "if" part. Assume that Σ satisfies LFE and each right dead-end path in Σ is strongly escapable.

Let us prove that Σ satisfies GFE. Let $s:[a,b]\to Q$ be a trajectory. Let us denote $S=\{s'\in Tr \mid s\sqsubseteq s' \wedge \min dom(s)\downarrow \wedge \min dom(s)=a\}$. Then each nonempty $\sqsubseteq$-chain of elements of S has an upper bound in S, because of the completeness property of Σ. Because $S\neq\varnothing$, Zorn's lemma implies that S has some maximal element s^* (with respect to $\sqsubseteq$). Because of the LFE property, $dom(s^*)$ cannot be a closed bounded segment. Then either $dom(s^*)=[a,+\infty)$, or $dom(s^*)=[a,y)$ for some $y\in T$. Consider the latter case, i.e. $dom(s^*)=[a,y)$, $y\in T$, $a<y$. Because s^* is maximal in S, s^* cannot be extended to a trajectory on $[a,y]$. Hence s^* is a right dead-end path. Moreover, $y>b$, because $s\sqsubseteq s^*$. Then $b\in(a,y)$ and by Lemma 3.1, $s^*|_{[b,y)}$ is a right dead-end path. Then there exists some infinite escape $s_1:[c,+\infty)\to Q$ from $s^*|_{[b,y)}$ (where $c\in(b,y)$, $s_1(c)=s^*(c)$).

Let us define $s_2:[a,+\infty)\to Q$ as follows:

$$s_2(t)=\begin{cases} s^*(t), & t\in[a,c] \\ s_1(t), & t>c \end{cases}$$

Then $s_2\in Tr$ by the CPR and Markovian properties of Σ. Moreover, $s\sqsubseteq s_2$, because $c>b$ and $s\sqsubseteq s^*$.

We conclude that in any case, either $dom(s^*)=[a,+\infty)$ and $s\sqsubseteq s^*$, or there exists a trajectory $s_2:[a,+\infty)\to Q$ such that $s\sqsubseteq s_2$. Because s is arbitrary, Σ satisfies the GFE property.

Let us prove the "only if" part. Assume that Σ satisfies the GFE property. Then Σ satisfies the LFE property because of the CPR property of Σ.

Let us prove that each right dead-end path is strongly escapable. Let $s:[a,b)\to Q$ $(a<b)$ be a right dead-end path. Let $c\in(a,b)$. Then $s|_{[a,c]}\in Tr$ by the CPR property of Σ. Then there exists a trajectory $s':[a,+\infty)\to Q$ such that

$s|_{[a,c]} \sqsubseteq s'$ by the GFE property. Let $s'' = s'|_{[c,+\infty)}$. Then $s'' \in Tr$ by the CPR property of Σ and $s''(c) = s(c)$. Then s'' is an infinite escape from s. Thus each dead-end path is strongly escapable. □

Now we will consider conditions under which each right dead-end path is strongly escapable.

Definition 3.6. A function $\alpha : [0,+\infty) \to [0,+\infty)$ is of class K_∞, if it is continuous, strictly increasing, $\lim_{x \to +\infty} \alpha(x) = +\infty$, and $\alpha(0) = 0$.

Definition 3.7.

1) A right extensibility measure is a function $f^+ : \mathbb{R} \times \mathbb{R} \tilde{\to} \mathbb{R}$ such that $A = \{(x, y) \in T \times T \mid x \le y\} \subseteq dom(f^+)$, $f(x, y) \ge 0$ for all $(x, y) \in A$, $f^+|_A$ is strictly decreasing in the first argument and strictly increasing in the second argument, and for each $x \ge 0$, $f^+(x, x) = x$ and $\lim_{y \to +\infty} f^+(x, y) = +\infty$.
2) A right extensibility measure f^+ is called normal, if f^+ is continuous on $\{(x, y) \in T \times T \mid x \le y\}$ and there exists a function $\alpha \in K_\infty$ such that $\alpha(y) < y$ for all $y > 0$ and the function $y \mapsto f^+(\alpha(y), y)$ is of class K_∞.

Let us fix a right extensibility measure f^+. Note that $f^+(x, y) > f^+(y, y) = y$ for all $x, y \ge 0$ such that $x < y$.

Definition 3.8. A right dead-end path $s : [a,b) \to Q$ is called f^+-escapable (Fig. 3.1), if there exists an escape $s' : [c,d] \to Q$ from s such that $d \ge f^+(c,b)$.

Informally, this definition means that the value of a right extensibility measure gives a lower estimate on how long an escape from a right dead-end path can be. The first argument of the right extensibility measure is the time at which the escape starts (i.e. the left end of its domain) and the second argument is the time at which the right dead-end path becomes undefined (the right end of its domain).

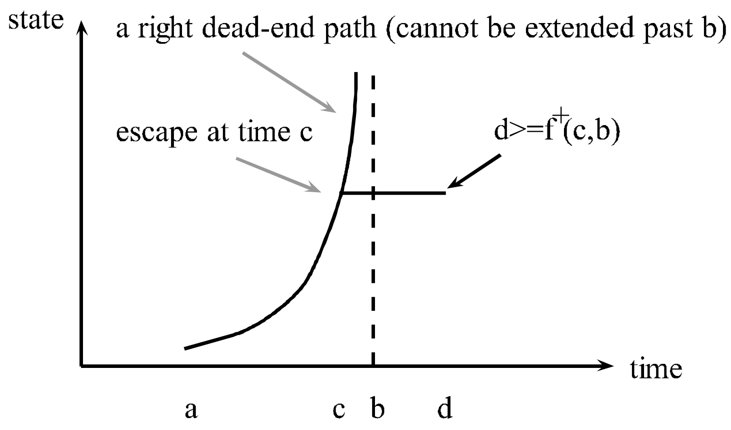

Fig. 3.1. An f^+-escapable right dead-end path $s:[a,b)\to Q$ (displayed here as a curve) and a corresponding escape $s':[c,d]\to Q$ (displayed here as a horizontal segment) such that $d \geq f^+(c,b)$.

Theorem 3.4 (About right dead-end path). Assume that Σ satisfies the LFE property and f^+ is a normal right extensibility measure. Then each right dead-end path is strongly escapable if and only if each right dead-end path is f^+-escapable.

We will give a proof of this theorem in a separate section (Section 3.5), because it is longer than other proofs in this section.

The following lemma gives an example of a right extensibility measure.

Lemma 3.3. For each $n \in \mathbb{N}$ the function $f_n^+(x,y) = y + (y-x)^n$ $(x, y \in \mathbb{R})$ is a normal right extensibility measure.

Proof. Obviously, f_n^+ is defined and non-negative on the set $A = \{(x,y) \in T \times T \mid x \leq y\}$, $f_n^+|_A$ is strictly decreasing in the first argument and is strictly increasing in the second argument, $f_n^+(x,x) = x$, and $\lim_{y\to+\infty} f_n^+(x,y) = +\infty$, so f_n^+ is a right extensibility measure. Besides, f_n^+ is continuous on $\mathbb{R}\times\mathbb{R}$. Let $\alpha:[0,+\infty)\to[0,+\infty)$ be a function $\alpha(y) = y/2$. Then $\alpha \in K_\infty$, $\alpha(y) < y$ for all $y > 0$, and $f_n^+(\alpha(y), y) = y + (y/2)^n$ is a continuous, strictly increasing, unbounded function which takes zero value at zero, so $y \mapsto f^{+}(\alpha(y), y)$ is of class K_{∞}. So f_n^{+} is a normal right extensibility measure. □

Note that for $f_1^{+}(x, y) = 2y - x$, a right dead-end path $s:[a,b) \to Q$ is f_1^{+}-escapable, if there exists an escape $s':[c,d] \to Q$ with $d - b \geq b - c$.

Now let us give a criterion for the GFE property.

Theorem 3.5. Let (l, r) be an LR representation of Σ and f^{+} be a normal right extensibility measure. Then Σ satisfies the GFE property if and only if for each $t > 0$ there exists $\varepsilon \in (0, t]$ such that for each $t_0 \in [t - \varepsilon, t)$ and $s:[t_0, t] \to Q$ the following holds:

1) $\left(\forall \tau \in [t_0, t]\, l(s,\tau) \wedge r(s,\tau)\right) \Rightarrow \exists t_1 > t$

$$\exists s':[t, t_1] \to Q\; s'(t) = s(t) \wedge (\forall \tau \in dom(s')\, l(s',\tau) \wedge r(s',\tau));$$

2) $\left(\forall \tau \in [t_0, t)\, l(s,\tau) \wedge r(s,\tau)\right) \wedge \neg l(s,t) \Rightarrow \exists t_1 \in (t_0, t)$

$$\exists s':[t_1, f^{+}(t_1, t)] \to Q\; s'(t_1) = s(t_1) \wedge (\forall \tau \in dom(s')\, l(s',\tau) \wedge r(s',\tau)).$$

Proof. Let us prove the "if" part.

Assume that for each $t > 0$ there exists $\varepsilon \in (0, t]$ such that 1) and 2) hold for each $t_0 \in [t - \varepsilon, t)$ and $s:[t_0, t] \to Q$.

Let us show that Σ satisfies GFE.

Firstly, let us show that Σ satisfies LFE. Let $\bar{s}:[a,b] \to Q$ be a trajectory of Σ (where $a, b \in T$, $a < b$). Then $b > 0$. Then for $t = b$ there exists $\varepsilon \in (0, t]$ such that the property 1) holds for each $t_0 \in [t - \varepsilon, t)$ and $s:[t_0, t] \to Q$. Let $t_0 = \max\{a, t - \varepsilon\}$ and $s = \bar{s}|_{[t_0, t]}$. Then $s \in Tr$ by the CPR property and $l(s,\tau) \wedge r(s,\tau)$ for all $\tau \in [t_0, t]$, and by the property 1) there exists $t_1 > t = b$ and $s':[t, t_1] \to Q$ such that $s'(t) = s(t) = \bar{s}(t)$ and $l(s',\tau) \wedge r(s',\tau)$ for all $\tau \in dom(s')$. Then $s' \in Tr$. Let us define $s'':[a, t_1] \to Q$ as follows: $s''(\tau) = \bar{s}(\tau)$, if $\tau \in [a,b]$ and $s''(\tau) = s'(\tau)$, if $\tau \in [b, t_1]$. Then $s'' \in Tr$ by the Markovian property. Moreover, $\bar{s} \sqsubseteq s''$ and $t_1 > b$. So Σ satisfies LFE.

Secondly, let us show that each right dead-end path in Σ is f^+-escapable. Let $\bar{s}:[a,b)\to Q$ be a right dead-end path in Σ (where $a,b\in T$, $a<b$). Then $b>0$. Then for $t=b$ there exists $\varepsilon\in(0,t]$ such that the property 2) holds for each $t_0\in[t-\varepsilon,t)$ and $s:[t_0,t]\to Q$. Let $t_0=\max\{a,t-\varepsilon\}$ and s be some continuation of $\bar{s}|_{[t_0,t)}$ on $[t_0,t]$. Then $s|_{[t_0,t)}\in Tr$ by the CPR property and $l(s,\tau)\wedge r(s,\tau)$ for all $\tau\in[t_0,t)$. Besides, $\neg l(s,t)$, because $\bar{s}$ is a dead-end path and $r(s,t)$ holds. Then by the property 2) there exists $t_1\in(t_0,t)$ and a function $s':[t_1,f^+(t_1,t)]\to Q$ such that $s'(t_1)=s(t_1)$ and $l(s',\tau)\wedge r(s',\tau)$ for all $\tau\in dom(s')$. Then $s'\in Tr$. Moreover, $t_1\in(a,b)$, $s'(t_1)=s(t_1)=\bar{s}(t_1)$, and $\max dom(s')\geq f^+(t_1,b)$. Thus s' is an escape from $\bar{s}$. Then $\bar{s}$ is f^+-escapable.

Thus by Theorem 3.4, each right dead-end path in Σ is strongly escapable. Then by Lemma 3.2, Σ satisfies GFE.

Now let us prove the "Only if" part. Assume that Σ satisfies GFE. Let $t>0$. Let us choose an arbitrary $\varepsilon\in(0,t]$. Assume that $t_0\in[t-\varepsilon,t)$ and $s:[t_0,t]\to Q$.

Let us prove the property 1). Assume that $l(s,\tau)\wedge r(s,\tau)$ for all $\tau\in[t_0,t]$. Then $s\in Tr$ and by GFE there exists $s_1:[t_0,+\infty]\to Q$ such that $s_1\in Tr$ and $s\sqsubseteq s_1$. Let $t_1=t+1$ and $s'=s|_{[t,t_1]}$. Then $s'\in Tr$ by the CPR property and $s'(t)=s(t)$ and $l(s',\tau)\wedge r(s',\tau)$ for all $\tau\in dom(s')$.

Let us prove the property 2). Assume that $l(s,\tau)\wedge r(s,\tau)$ for all $\tau\in[t_0,t)$. Then $s|_{[t_0,t)}\in Tr$. Consider the case when $s|_{[t_0,t)}$ is a right dead-end path in Σ. Then by Lemma 3.2 it is strongly escapable, so there exists $t_1\in(t_0,t)$ and $s_1:[t_1,+\infty)\to Q$ such that $s_1(t_1)=s(t)$ and $s_1\in Tr$. Let $s'=s_1|_{[t_1,f^+(t_1,t)]}$. Then $s'\in Tr$ by the CPR property and $s'(t_1)=s(t_1)$ and $l(s',\tau)\wedge r(s',\tau)$ for all $\tau\in dom(s')$.

Now consider the case when $s|_{[t_0,t)}$ is not a right dead-end path. Then there exists $s_0:[t_0,t]\to Q$ such that $s_0\in Tr$ and $s|_{[t_0,t)}\sqsubseteq s_0$. Then by GFE there exists $s_1:[t_0,+\infty]\to Q$ such that $s_1\in Tr$ and $s_0\sqsubseteq s_1$. Let us choose an arbitrary $t_1\in(t_0,t)$ and define $s'=s_1|_{[t_1,f^+(t_1,t)]}$. Then $s'\in Tr$ by the CPR property and $s'(t_1)=s_1(t_1)=s_0(t_1)=s(t_1)$ and $l(s',\tau)\wedge r(s',\tau)$ for all $\tau\in dom(s')$.

Thus in both cases the property 2) holds. □

Note that in this theorem the first condition basically means the LFE property and the second condition expresses the existence of an escape of a length given by f^+. This theorem means that to prove the GFE property, it is sufficient to prove the existence of certain locally defined trajectories independently in a neighborhood of each time moment.

3.5 Proof of the theorem about a right dead-end path

In this section we will give a proof of Theorem 3.4.

Let $\Sigma=(T,Q,Tr)$ be a fixed NCMS and f^+ be a fixed normal right extensibility measure. Let us introduce several auxiliary definitions and lemmas.

Definition 3.9. A right t_0-bunch (in Σ) is a non-empty set $A\subseteq Tr$ such that $\min(dom(s))\downarrow= t_0$ for each $s\in A$ and $s_1 \doteq_{t_0+} s_2$ for all $s_1,s_2\in A$.

For each non-empty set $A\subseteq Tr$ denote

$$|A|^+=\sup_{s\in A}(\sup dom(s)).$$

We assume that $|A|^+=+\infty$, if $\sup(dom(s))=+\infty$ for some $s\in A$.

Definition 3.10. A (right) t_0-bunch A is called bounded, if $|A|^+<+\infty$. Otherwise it is called unbounded.

Lemma 3.4. There exists a function $g^{+}: T \times T \tilde{\to} T$ defined on $\{(x, y) \in T \times T \mid x \leq y\}$ such that

1) g^{+} is strictly increasing in both arguments;

2) $g^{+}(x,x) = x$ and $x < g^{+}(x, y) < y$ for all $x, y \in T$ such that $x < y$;

3) $g^{+}(x, f^{+}(x, y)) = y$ for all $x, y \in T$ such that $x \leq y$.

Proof. For each fixed $x \geq 0$ let $h_x : [x,+\infty) \to \mathbb{R}$ be a function such that $h_x(y) = f^{+}(x, y)$ for all $y \in [x,+\infty)$. Then because f^{+} is a normal right extensibility measure, we have that h_x is strictly increasing, continuous, maps the set $[x,+\infty)$ to itself, is unbounded from above, and $h_x(x) = x$. Therefore, it has a strictly increasing inverse h_x^{-1} which is defined on $[x,+\infty)$. Let us define $g^{+}: T \times T \tilde{\to} T$ as follows: $g^{+}(x, y) = h_x^{-1}(y)$ for all $(x, y) \in T \times T$ such that $x \leq y$. Then g^{+} is strictly increasing in the second argument and $g^{+}(x, f^{+}(x, y)) = y$ for all $x, y \in T$ such that $x \leq y$. If $x_1, x_2, y \in T$ and $x_1 < x_2 \leq y$, then $f^{+}(x_1, g^{+}(x_1, y)) = y = f^{+}(x_2, g^{+}(x_2, y)) < f^{+}(x_1, g^{+}(x_2, y))$, which implies that $g^{+}(x_1, y) < g^{+}(x_2, y)$, so g^{+} is strictly increasing in the first argument. Thus the condition 1 and 3 of the lemma are satisfied.

Let us prove the condition 2. Indeed, $g^{+}(x, x) = h_x^{-1}(x) = x$ for all $x \geq 0$. Besides, if $x, y \in T$ and $x < y$, then $x = g^{+}(x, x) < g^{+}(x, y) < g^{+}(y, y) = y$, because g^{+} is strictly increasing in both arguments as we have shown above. □

Let us fix a function g^{+} which is described in Lemma 3.4.

Definition 3.11. A bounded right t_0-bunch A is called g^{+}-convergent, if for each $t' \in (t_0, |A|^{+})$ and $s_1, s_2 \in A$ the following holds:

if $\min\{\sup(dom(s_1)), \sup(dom(s_2))\} \geq g^{+}(t', |A|^{+})$, then $s_1 \doteq_{[t_0, t')} s_2$.

We will need the following notion [58]: if X is some set and $g: X \to X$, a function $f: X \to X$ which satisfies the equation $\underbrace{f \circ f \circ ... \circ f}_{N \text{ times}} = g$ ($N \in \mathbb{N}$) is called an N-th order iterative root of g. The existence of iterative roots can be established in some cases using the following theorem which was proved in [58].

Theorem 3.6 [58, Theorem 11.2.2]. Let $X \subset \mathbb{R}$ be an interval and let f be a strictly increasing and continuous self-mapping of X. Then f possesses strictly increasing and continuous iterative roots of all orders.

Note that here the interval X can be unbounded.

Let $\alpha \in K_\infty$ be a function such that $\alpha(y) < y$ for all $y > 0$ and the function $y \mapsto f^+(\alpha(y), y)$ is of class K_∞ (such a function exists by Definition 3.7).

Then by Theorem 3.6 there exists a continuous and strictly increasing function ξ on $[0,+\infty)$ such that for all $x \geq 0$,

$$\xi(\xi(x)) = f^+(\alpha(x), x) \tag{3.1}$$

Lemma 3.5. ξ is of class K_∞ and $\xi(x) > x$ for all $x > 0$.

Proof. For all $x > 0$ we have $\xi(\xi(x)) = f^+(\alpha(x), x) > f^+(x, x) = x$, because $\alpha(x) < x$. Suppose that $\xi(x_0) \leq x_0$ for some $x_0 > 0$. Then $x_0 < \xi(\xi(x_0)) \leq \xi(x_0)$ by monotonicity of ξ. This contradics the assumption $\xi(x_0) \leq x_0$. Thus $\xi(x) > x$ for all $x > 0$. Moreover, because, $\xi(\xi(0)) = f^+(\alpha(0), 0) = f^+(0,0) = 0$, we have $\xi(0) = 0$. Besides, $\lim_{x \to +\infty} \xi(x) = +\infty$ and ξ is continuous and strictly increasing. We conclude that ξ is of class K_∞ and $\xi(x) > x$ for all $x > 0$. □

Let ϕ be a strictly increasing and continuous function such that

$$\phi(\phi(x)) = \xi(x)$$

for all $x > 0$ (it exists by Theorem 3.6).

Then $\phi(x) > x$ for all $x > 0$ (because otherwise, there exists $x_0 > 0$ with $\phi(x_0) \leq x_0$ and $\xi(x_0) = \phi(\phi(x_0)) \leq \phi(x_0) \leq x_0$ – a contradiction with Lemma 3.5).

Let ψ be a strictly increasing and continuous function such that

$$\psi(\psi(x)) = \phi(x)$$

for all $x > 0$ (which exists by Theorem 3.6).

Then $\psi(x) > x$ for all $x > 0$, because $\phi(x) > x$ for all $x > 0$, whence

$$x < \psi(x) < \psi(\psi(x)) = \phi(x) < \phi(\phi(x)) = \xi(x). \quad (3.2)$$

For any set of sets S and a binary relation $\prec \subseteq S \times S$ denote:

– $Ch(S,\prec)$ is the set of all $\subseteq$-chains $c \subseteq S$ (i.e. $A \subseteq B$ or $B \subseteq A$ for each $A, B \in c$) such that

  1) the union of elements of each non-empty bounded subset (in the sense of $\subseteq$) of c belongs to c, i.e. for each $c' \in 2^c \setminus \{\varnothing\}$, if there exists $X \in c$ such that $\bigcup c' \subseteq X$, then $\bigcup c' \in c$. Note this implies that c is a Dedekind-complete poset with resepct to $\subseteq$ [97, p. 87] (i.e. every nonempty bounded subset has a supremum);

  2) for each non-maximal $A \in c$ (i.e. $A \subset A'$ for some $A' \in c$) there exists $A' \in c$ such that $A \subseteq A'$ and $A \prec A'$.

– $\leq$ is a binary relation on $Ch(S,\prec)$ such that $c_1 \leq c_2$ if and only if $c_1 \subseteq c_2$, and $A \subseteq B$ for all $A \in c_1$ and $B \in c_2 \setminus c_1$.

For each $t_0 \geq 0$ let us define:

– $S^+_{t_0}$ is the set of all bounded g^+-convergent right t_0-bunches (in Σ);

– $\prec^+$ is a binary relation on $S^+_{t_0}$ such that $A \prec^+ B$ if and only if $|A|^+ < |B|^+ < \psi(|A|^+)$.

Let us prove some general properties of $Ch(S,\prec)$.

Lemma 3.6.

1) $\leq$ is a partial order on $Ch(S,\prec)$.

2) Each chain in the poset $(Ch(S,\prec),\leq)$ has an upper bound.

Proof. The statement 1. follows immediately from definition of $\leq$.

Let us show the statement 2. Let $C \subseteq Ch(S,\prec)$ be a $\leq$-chain. Let us show that $c = \bigcup C \in Ch(S,\prec)$. It is straightforward to check that c is a $\subseteq$-chain.

Let us check that for each non-empty $c' \subseteq c$, if there exists $X \in c$ such that $\bigcup c' \subseteq X$, then $\bigcup c' \in c$. Assume $c' \subseteq c$, $c' \neq \varnothing$, $X \in c$ and $\bigcup c' \subseteq X$. Then there exists $c_X \in C$ such that $X \in c_X$, because $X \in c = \bigcup C$. Firstly, let us show that $c' \subseteq c_X$. Let $A \in c'$. Because $A \in c' \subseteq c = \bigcup C$, there exists $c_A \in C$ such that $A \in c_A$. Now assume that $A \notin c_X$. Then $c_A \not\leq c_X$ (otherwise $A \in c_A \subseteq c_X$). Then $c_X \leq c_A$, because C is a $\leq$-chain and $c_A, c_X \in C$. Then $X \subseteq A$, because $X \in c_X$, $A \in c_A \setminus c_X$. Then $A \subseteq \bigcup c' \subseteq X \subseteq A$. Then we have a contradiction $A = X \in c_X$. Thus $A \in c_X$ and we conclude that $c' \subseteq c_X$. Now we have that $c' \subseteq c_X$, $c' \neq \varnothing$ and there exists $X \in c_X$ such that $\bigcup c' \subseteq X$. Then $\bigcup c' \in c_X$ by the definition of $Ch(S,\prec)$, because $c_X \in C \subseteq Ch(S,\prec)$. Thus $\bigcup c' \in \bigcup C = c$.

Let us check that for each non-maximal $A \in c$ (with respect to $\subseteq$) there exists $A' \in c$ such that $A \subseteq A'$ and $A \prec A'$. Assume that $A \in c$ is non-maximal. Then there exists $B \in c$ such that $A \subset B$. Then there exist $c_A, c_B \in C$ such that $A \in c_A$, $B \in c_B$, because $c = \bigcup C$. Moreover, either $c_A \leq c_B$ or $c_B \leq c_A$. If $A \in c_B$, then A is a non-maximal element of c_B. Then there exists $A' \in c_B$ such that $A \subseteq A'$ and $A \prec A'$, because $c_B \in Ch(S,\prec)$. Then $A' \in \bigcup C = c$. On the other hand, if $A \notin c_B$, then $c_B \leq c_A$ (because otherwise $A \in c_B$) and $B \subseteq A$, because $B \in c_B$ and $A \in c_A \setminus c_B$. This contradicts the inclusion $A \subset B$ given above. We conclude that there exists $A' \in c$ such that $A \subseteq A'$ and $A \prec A'$.

Thus $c \in Ch(S,\prec)$ by the definition of $Ch(S,\prec)$. □

Lemma 3.7. Let c_m be a $\leq$-maximal element of $Ch(S,\prec)$ and $X = \bigcup c_m \in S$. Then the following holds:

1) $X \in c_m$.

2) There is no set $Y \in S$ such that $X \subset Y$ and $X \prec Y$.

Proof. Let us prove 1). Let $c'_m = c_m \cup \{X\}$.

Let us show that $c'_m \in Ch(S,\prec)$. We have $c'_m \subseteq S$ and $A \subseteq X$ for all $A \in c_m$, because $X = \bigcup c_m$. Moreover, c_m is a $\subseteq$-chain. Thus $c'_m = c_m \cup \{X\}$ is a $\subseteq$-chain.

Let us check that for each non-empty $c' \subseteq c'_m$, if there exists $X' \in c'_m$ such that $\bigcup c' \subseteq X'$, then $\bigcup c' \in c'_m$. Assume that $c' \subseteq c'_m$, $c' \neq \varnothing$. If $X \in c'$, then $X \subseteq \bigcup c' \subseteq \bigcup c'_m = (\bigcup c_m) \cup X = X$ and $\bigcup c' \in c'_m$. Consider the case when $X \notin c'$. Then $c' \subseteq c_m$ and because c_m is a $\subseteq$-chain, for each $A \in c_m$, either $A \subseteq B$ holds for some $B \in c'$, or $B \subseteq A$ holds for all $B \in c'$. Then for each $A \in c_m$, either $A \subseteq \bigcup c'$ or $\bigcup c' \subseteq A$. If $\bigcup c' \subseteq A$ for some $A \in c_m$, then by taking into account that $c' \subseteq c_m$, $c' \neq \varnothing$ and $c_m \in Ch(S,\prec)$, we have $\bigcup c' \in c_m \subseteq c'_m$. If $A \subseteq \bigcup c'$ for all $A \in c_m$, then $X = \bigcup c_m \subseteq \bigcup c' \subseteq \bigcup c_m = X$, because $c' \subseteq c_m$. Then $\bigcup c' \in c'_m$. Thus in all cases $\bigcup c' \in c'_m$.

Let us check that for each non-maximal $A \in c'_m$ (with respect to $\subseteq$) there exists $A' \in c'_m$ such that $A \subseteq A'$ and $A \prec A'$. Assume that $A \in c'_m$ is non-maximal. Then $A \neq X$. Then $A \in c_m$. Moreover, A is a non-maximal element of c_m, because otherwise $X = \bigcup c_m = A$. Then because $c_m \in Ch(S,\prec)$, there exists $A' \in c_m \subseteq c'_m$ such that $A \subseteq A'$ and $A \prec A'$.

Thus $c'_m \in Ch(S,\prec)$ by definition of $Ch(S,\prec)$.

We have $c_m \subseteq c'_m$ and $A \subseteq B$ for all $A \in c_m$ and $B \in c'_m \setminus c_m$, because $c'_m \setminus c_m \subseteq \{X\}$ and $A \subseteq \bigcup c_m = X$. Then $c_m \leq c'_m$. Then $c_m = c'_m$, because $c'_m \in Ch(S,\prec)$ and c_m is a $\leq$-maximal element of $Ch(S,\prec)$. Thus $X \in c_m$.

Now let us prove 2) by contradiction. Assume that there exists $Y \in S$ such that $X \subset Y$ and $X \prec Y$. Let $c'_m = c_m \cup \{Y\}$. Let us show that $c'_m \in Ch(S,\prec)$.

We have $c'_m \subseteq S$. Also, $A \subseteq Y$ for all $A \in c_m$, because $\bigcup c_m = X \subseteq Y$. Moreover, c_m is a $\subseteq$-chain. Thus $c'_m = c_m \cup \{Y\}$ is a $\subseteq$-chain.

Let us check that for each non-empty $c' \subseteq c'_m$, if there exists $X' \in c'_m$ such that $\bigcup c' \subseteq X'$, then $\bigcup c' \in c'_m$. Let $c' \subseteq c_m$, $c' \neq \varnothing$. If $Y \in c'$, then $Y \subseteq \bigcup c' \subseteq \bigcup c'_m = (\bigcup c_m) \cup Y = X \cup Y = Y$ and $\bigcup c' \in c'_m$. Consider the case when $Y \notin c'$. Then $c' \subseteq c_m$ and $\bigcup c' \subseteq \bigcup c_m = X$. Moreover, $X \in c_m$ by the statement 1) of this lemma. From this and from $c' \subseteq c_m$, $c' \neq \varnothing$ and $c_m \in Ch(S, \prec)$, we have $\bigcup c' \in c_m \subseteq c'_m$. Thus in both cases we have $\bigcup c' \in c'_m$.

Let us check that for each non-maximal $A \in c'_m$ (with respect to $\subseteq$) there exists $A' \in c'_m$ such that $A \subseteq A'$ and $A \prec A'$. Assume $A \in c'_m$ is non-maximal element. Then $A \neq Y$, because Y is a maximal element of c'_m. Then $A \in c_m$. If A is a non-maximal element of c_m, then there exists $A' \in c_m \subseteq c'_m$ such that $A \subseteq A'$ and $A \prec A'$, because $c_m \in Ch(S, \prec)$. If A is a maximal in c_m, then $X = \bigcup c_m = A$, $Y \in c'_m$, $A \subseteq Y$ and $A \prec Y$. Thus $c'_m \in Ch(S, \prec)$.

We have $c_m \subseteq c'_m$ and $A \subseteq B$ for all $A \in c_m$ and $B \in c'_m \setminus c_m$, because $c'_m \setminus c_m \subseteq \{Y\}$ and $A \subseteq \bigcup c_m = X \subseteq Y$. Then $c_m \leq c'_m$. Also, we have $X = \bigcup c_m \neq Y = \bigcup c'_m$, because $X \subset Y$. Then $c_m \neq c'_m$. Then c_m is not a $\leq$-maximal element of $Ch(S, \prec)$, because $c_m \leq c'_m$ and $c'_m \in Ch(S, \prec) \setminus \{c_m\}$. We have a contradiction with assumptions of the lemma. Thus there is no set $Y \in S$ such that $X \subset Y$ and $X \prec Y$. □

Let us consider some properties of the set $Ch(S^+_{t_0}, \prec^+) \setminus \{\varnothing\}$ for a fixed $t_0 \in T$. Note for each element c of this set, $\bigcup c \neq \varnothing$, because $\varnothing \notin S^+_{t_0}$.

Lemma 3.8. If $c \in Ch(S^+_{t_0}, \prec^+) \setminus \{\varnothing\}$ and $\left|\bigcup c\right|^+ < +\infty$, then $\bigcup c \in S^+_{t_0}$.

Proof. Let $c \in Ch(S^+_{t_0}, \prec^+) \setminus \{\varnothing\}$, $X = \bigcup c$, and $|X|^+ < +\infty$. Then $X \neq \varnothing$.

Let us show that X is a bounded right t_0-bunch. For each $s \in X$ there exists $A \in c$ such that $s \in A$. Then $\min(dom(s)) \downarrow= t_0$, because A is a right t_0-bunch. Let

$s_1, s_2 \in X$. Then there exist $A_1, A_2 \in c$ such that $s_1 \in A_1$ and $s_2 \in A_2$. Then $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, because c is a $\subseteq$-chain. Moreover, $A_1, A_2 \in S_{t_0}^+$. If $A_1 \subseteq A_2$, then $s_1, s_2 \in A_2$ for $i = 1,2$. Then $s_1 \doteq_{t_0+} s_2$, because A_2 is a right t_0-bunch. Similarly, if $A_2 \subseteq A_1$, then $s_1 \doteq_{t_0+} s_2$, because A_1 is a right t_0-bunch. In both cases $s_1 \doteq_{t_0+} s_2$. Thus X is a bounded right t_0-bunch, because $|X|^+ < +\infty$.

Let us show that X is g^+-convergent. Let $t' \in (t_0, |X|^+)$, $s_1, s_2 \in X$. Then there exist $A_1, A_2 \in c$ such that $s_1 \in A_1$, $s_2 \in A_2$. Then $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, because c is a chain. Also, A_1, A_2 are bounded t_0-bunches, because $A_1, A_2 \in S_{t_0}^+$.

Let $t_i = \sup(dom(s_i))$, $i = 1,2$. Assume that $\min\{t_1, t_2\} \ge g^+(t', |X|^+)$.

Let us show that $s_1 \doteq_{[t_0, t')} s_2$.

Consider the case $A_1 \subseteq A_2$. Then $s_1, s_2 \in A_2$ and $t_1, t_2 \le |A_2|^+$. Then

$$|X|^+ \ge |A_2|^+ \ge \min\{t_1, t_2\} \ge g^+(t', |X|^+) > t',$$

because $A_2 \subseteq X$ and $t' < |X|^+$. If $t' \in (t_0, |A_2|^+)$, then $s_1 \doteq_{[t_0, t')} s_2$, because A_2 is g^+-convergent. Otherwise, $t' = |A_2|^+$. Then

$$\min\{t_1, t_2\} \ge g^+(t', |X|^+) \ge g^+(t', |A_2|^+) = |A_2|^+,$$

by monotonicity of g^+, whence $t_1 = t_2 = |A_2|^+$, because $t_1, t_2 \le |A_2|^+$.

For each $t'' \in (t_0, |A_2|^+)$ we have

$$\min\{t_1, t_2\} = |A_2|^+ > g^+(t'', |A_2|^+).$$

Then $s_1 \doteq_{[t_0, t'')} s_2$, because A_2 is g^+-convergent. Then $s_1 \doteq_{[t_0, t')} s_2$, because $t'' \in (t_0, |A_2|^+) = (t_0, t')$ is arbitrary.

In the case $A_2 \subseteq A_1$ we can show that $s_1 \doteq_{[t_0, t')} s_2$ using analogous arguments. Thus X is g^+-convergent. Then $X \in S_{t_0}^+$ by the definition of $S_{t_0}^+$. □

Let us define a prefix relation $\hat{\sqsubseteq}$ on Tr: $s_1 \hat{\sqsubseteq} s_2$ if and only if $s_1 \sqsubseteq s_2$ and for each $t_1 \in dom(s_1)$ and $t_2 \in dom(s_2) \setminus dom(s_1)$ we have $t_1 < t_2$.

Let us define a prefix closure operation pcl on 2^{Tr}:

$$pcl(A) = \{s \in Tr \mid \exists s' \in A \; s \hat{\sqsubseteq} s'\}, \text{ if } A \subseteq Tr.$$

Lemma 3.9.

1) $\hat{\sqsubseteq}$ is a partial order on Tr.
2) pcl is a closure operator, i.e. $A \subseteq pcl(A)$ (extensivity), $pcl(A) \subseteq pcl(A')$, if $A \subseteq A'$ (monotonicity), and $pcl(pcl(A)) = pcl(A)$ (idempotence).

Proof.

1) It is obvious that $\hat{\sqsubseteq}$ is reflexive and anitsymmetric. Let us check that $\hat{\sqsubseteq}$ is transitive. Assume $s_1 \hat{\sqsubseteq} s_2$ and $s_2 \hat{\sqsubseteq} s_3$. Then $s_1 \sqsubseteq s_2$, $s_2 \sqsubseteq s_3$, whence $s_1 \sqsubseteq s_3$. Besides, if $t_1 \in dom(s_1)$ and $t_3 \in dom(s_3) \setminus dom(s_1)$, then either $t_3 \in dom(s_2)$, whence $t_1 < t_3$, because $s_1 \hat{\sqsubseteq} s_2$, or $t_3 \in dom(s_3) \setminus dom(s_2)$, whence $t_1 < t_3$, because $s_2 \hat{\sqsubseteq} s_3$ and $t_1 \in dom(s_1) \subseteq dom(s_2)$. Thus $\hat{\sqsubseteq}$ is transitive.

2) Monotonicity of pcl follows from its definition. Moreover, pcl is extensive and idempotent, because $\hat{\sqsubseteq}$ is reflexive and transitive. □

Lemma 3.10. $pcl(A) \in S_{t_0}^{+}$ for each $A \in S_{t_0}^{+}$.

Proof. Let $A \in S_{t_0}^{+}$. Then A is a bounded g^{+}-convergent right t_0-bunch.

Let $A' = pcl(A)$. Then $A \subseteq A' \subseteq Tr$, so $A' \neq \varnothing$. Let $s \in A'$. Then there exists $s' \in A$ such that $s \hat{\sqsubseteq} s'$. Then $\min dom(s') \downarrow = t_0$, and because $s \hat{\sqsubseteq} s'$ and $dom(s_1)$ is a nonempty subset of $dom(s_2)$, we conclude that $\min dom(s) \downarrow = t_0$. Then $[t_0, t_1] \subseteq dom(s)$ for some $t_1 > t_0$ and $s \sqsubseteq s'$, whence $s =_{t_0 +} s'$. Because $s \in A'$ is arbitrary and $s' =_{t_0 +} s''$ for all $s', s'' \in A$, we have that $s' =_{t_0 +} s''$ for all $s', s'' \in A'$. Thus A' is a right t_0-bunch.

Because $A' \neq \varnothing$ and for each $s \in A'$ there exists $s' \in A$ such that $s \sqsubseteq s'$, we have $|A'|^+ \leq |A|^+$. On the other hand, $|A|^+ \leq |A'|^+$, because $A \subseteq A'$. Thus $|A|^+ = |A'|^+$ and A' is a bounded right t_0-bunch.

Let us show that A' is g^+-convergent. Let $t' \in (t_0, |A'|^+) = (t_0, |A|^+)$ and $s_1, s_2 \in A'$. Assume that

$$\min\{\sup(dom(s_1)), \sup(dom(s_2))\} \geq g^+(t', |A'|^+) = g^+(t', |A|^+).$$

Then there exist $s_1', s_2' \in A$ such that $s_1 \hat{\sqsubseteq} s_1'$ and $s_2 \hat{\sqsubseteq} s_2'$. Then

$$\min\{\sup(dom(s_1')), \sup(dom(s_2'))\} \geq g^+(t', |A|^+),$$

so $s_1' \doteq_{[t_0,t')} s_2'$, because A is g^+-convergent. Then $s_1 \doteq_{[t_0,t')} s_2$, because $[t_0, t') \subseteq [t_0, g^+(t', |A|^+)) \subseteq dom(s_1) \cap dom(s_2)$ (by the item 2 of Lemma 3.4), $s_1 \hat{\sqsubseteq} s_1'$, and $s_2 \hat{\sqsubseteq} s_2'$. Thus A' is g^+-convergent. We conclude that $A' = pcl(A) \in S_{t_0}^+$. □

Lemma 3.11. If $c \in Ch(S_{t_0}^+, \prec^+)$, then $\{pcl(A) \mid A \in c\} \in Ch(S_{t_0}^+, \prec^+)$.

Proof. Let $c \in Ch(S_{t_0}^+, \prec^+)$ and $\hat{c} = \{pcl(A) \mid A \in c\}$.

By Lemma 3.10 we have $\hat{c} \subseteq S_{t_0}^+$. Besides, $\hat{c}$ is a non-empty $\subseteq$-chain, because c is a non-empty $\subseteq$-chain and pcl is monotone.

Let us show that the union of elements of each non-empty bounded subset of $\hat{c}$ belongs to $\hat{c}$. Assume that $c' \in 2^{\hat{c}} \setminus \{\varnothing\}$ and there exists $X \in \hat{c}$ such that $\bigcup c' \subseteq X$. Then there exists $Y \in c$ such that $X = pcl(Y)$ and there exists a non-empty set $c'' \subseteq c$ such that $c' = \{pcl(A) \mid A \in c''\}$. Then

$$\bigcup\{pcl(A) \mid A \in c''\} = \bigcup c' \subseteq X = pcl(Y).$$

From the definition of pcl we have

$$\bigcup\{pcl(A) \mid A \in c''\} = pcl(\bigcup\{A \mid A \in c''\}) = pcl(\bigcup c'').$$

If there exists $Z \in c$ such that $\bigcup c'' \subseteq Z$ (i.e. c'' is a bounded subset of c), then $\bigcup c'' \in c$, because $c \in Ch(S_{t_0}^+, \prec^+)$. From this we have $\bigcup c' = pcl(\bigcup c'') \in \hat{c}$.

Otherwise, $\bigcup c'' = \bigcup c$, because c is a chain. Then

$$pcl(\bigcup c) = pcl(\bigcup c'') = \bigcup c' \subseteq X = pcl(Y) \subseteq pcl(\bigcup c)$$

by monotonicity of pcl, whence $\bigcup c' = X \in \hat{c}$.

In both cases, $\bigcup c' \in \hat{c}$. Thus the union of elements of a non-empty bounded subset of $\hat{c}$ belongs to $\hat{c}$.

Let A be a non-maximal element of $\hat{c}$. Then there exists $B \in c$ such that $A = pcl(B)$. If B is a maximal element of c, then $B = \bigcup c$, because c is a chain, and $A = pcl(B) = pcl(\bigcup c) \supseteq \bigcup \{pcl(A') \mid A' \in c\} = \bigcup \hat{c}$, which contradicts the assumption that A is non-maximal. Thus B is a non-maximal element of c.

Then there exists $B' \in c$ such that $B \subseteq B'$ and $B \prec^+ B'$. Then $pcl(B') \in \hat{c}$ and $A \subseteq pcl(B')$. Moreover, $|B|^+ < |B'|^+ < \psi(|B|^+)$, whence $|A|^+ < |pcl(B')|^+ < \psi(|A|^+)$, because $|A|^+ = |pcl(B)|^+ = |B|^+$ and $|B'|^+ = |pcl(B')|^+$. Thus $A \prec^+ pcl(B')$.

We conclude that $\hat{c} = \{pcl(A) \mid A \in c\} \in Ch(S_{t_0}^+, \prec^+)$. □

Lemma 3.12. If $c \in Ch(S_{t_0}^+, \prec^+) \setminus \{\varnothing\}$ and $|\bigcup c|^+ = +\infty$, then there exists a trajectory $s_* : [t_0, +\infty) \to Q$ and $A \in c$ such that $s_* \doteq_{t_0+} s'$ for all $s' \in A$.

Proof. Let $c \in Ch(S_{t_0}^+, \prec^+) \setminus \{\varnothing\}$ and $|\bigcup c|^+ = +\infty$. Let $\hat{c} = \{pcl(A) \mid A \in c\}$. Because $c \neq \varnothing$, we have $\hat{c} \in Ch(S_{t_0}^+, \prec^+) \setminus \{\varnothing\}$ by Lemma 3.11. Moreover, $|\bigcup \hat{c}|^+ = +\infty$, because $c \subseteq \hat{c}$.

Let us construct a $\subseteq$-monotone sequence $A_n \in \hat{c}$, $n \in \mathbb{N}$ and a sequence $s_n \in A_n$, $n \in \mathbb{N}$ as follows. Lemma 3.5 implies that the function ξ has an inverse function ξ^{-1} which is defined and strictly increasing on $[0, +\infty)$. Moreover, $\xi^{-1}(x) < x$ for all $x > 0$. Let us choose $A_1 \in \hat{c}$ arbitrarily and choose $s_1 \in A_1$ in such a way that $\sup dom(s_1) = \xi^{-1}(|A_1|^+)$ (this is possible, because $A_1 \neq \varnothing$, $0 < \xi^{-1}(|A_1|^+) < |A_1|^+$, and A_1 is prefix-closed, i.e. $pcl(A_1) = A_1$).

Suppose that elements $A_1,\dots,A_n$ and $s_1,\dots,s_n$ are already constructed. Let us construct A_{n+1}, s_{n+1} in the following way.

Let $C=\{A'\in\hat{c}\mid A_n\subseteq A'\wedge A_n\prec^+ A'\}$. Then $C\neq\varnothing$, because A_n is not a $\subseteq$-maximal element of $\hat{c}$ ($\hat{c}$ has no maximal elements, because $|\bigcup\hat{c}|^+=+\infty$). Let $A^*=\bigcup C$. Then $|A^*|^+\le\psi(|A_n|^+)<+\infty$, because $|A'|^+<\psi(|A_n|^+)$ for all $A'\in C$. Let us choose $X\in\hat{c}$ such that $|X|^+>|A^*|^+$. Because $\hat{c}$ is a $\subseteq$-chain, $A'\subseteq X$ for all $A'\in C\subseteq\hat{c}$. Then $A^*\subseteq X$. Then $A^*\in\hat{c}$, because $\hat{c}$ is Dedekind-complete. Then there exists $B\in\hat{c}$ such that $A^*\subseteq B$ and $A^*\prec^+ B$, because A^* is a non-maximal element of $\hat{c}$ and $\hat{c}\in Ch(S^+_{t_0},\prec^+)$. Let us define $A_{n+1}=B$. Then $A_n\subseteq A_{n+1}$ and $A_{n+1}\in\hat{c}$. Also, let us choose $s_{n+1}\in A_{n+1}$ such that $\sup dom(s_{n+1})=\xi^{-1}(|A_{n+1}|^+)$ (this is possible, because $A_{n+1}\neq\varnothing$, $\xi^{-1}(|A_{n+1}|^+)<|A_{n+1}|^+$, and A_{n+1} is prefix-closed).

We have defined sequences A_n and s_n, $n\ge 1$. The sequence A_n is obviously $\subseteq$-monotone. Let us show that for each $n\ge 1$,

$$|A_n|^+<\psi(|A_n|^+)\le|A_{n+1}|^+<\xi(|A_n|^+). \tag{3.3}$$

Let $n\ge 1$. Like above, let $C=\{A'\in\hat{c}\mid A_n\subseteq A'\wedge A_n\prec^+ A'\}$ and $A^*=\bigcup C$. Then $|A_{n+1}|^+\le\psi(\psi(|A_n|^+))$, because $|A^*|^+\le\psi(|A_n|^+)$ and $A^*\prec^+ A_{n+1}$. Moreover, $A_{n+1}\notin C$ and $|A_*|^+<|A_{n+1}|^+\le\psi(|A_*|^+)$, because $A^*\subseteq A_{n+1}$ and $A^*\prec^+ A_{n+1}$. Then $A_n\not\prec^+ A_{n+1}$ by the definition of C, because $A_{n+1}\in\hat{c}$, and $A_n\subseteq A_{n+1}$. Then $\psi(|A_n|^+)\le|A_{n+1}|^+$ or $|A_{n+1}|^+\le|A_n|^+$ by the definition of $\prec^+$. However, $|A_{n+1}|^+\ge|A^*|^+>|A_n|^+$, because $C\neq\varnothing$. Thus $\psi(|A_n|^+)\le|A_{n+1}|^+\le\psi(\psi(|A_n|^+))$. From this and (3.2) we finally have (3.3).

The sequence $|A_n|^+$ is monotone. If it is bounded from above, then its limit is a fixed point of ψ, because ψ is continuous. But $\psi(x)>x$ for all $x>0$, whence

$$\lim_{n\to\infty}|A_n|^+=+\infty.$$

By the construction of s_n, $\sup dom(s_n) = \xi^{-1}(|A_n|^+)$ for all $n \geq 1$, thus

$$\lim_{n\to\infty} \sup dom(s_n) = +\infty. \qquad (3.4)$$

From (3.1) we have that for all $x \geq 0$,

$$g^+(\alpha(x), \xi(\xi(x))) = g^+(\alpha(x), f^+(\alpha(x), x)) = x > \alpha(x).$$

Let us prove that for each $n \geq 1$,

$$s_n(t) = s_{n+1}(t) \text{ for all } t < \alpha(\sup dom(s_n)). \qquad (3.5)$$

Let $n \geq 1$ and $x = \sup dom(s_n)$, $a = |A_n|^+$, $b = |A_{n+1}|^+$. Then $x = \xi^{-1}(a)$ and $a < b < \xi(a)$ by (3.3). Then

$$x \geq g^+(\alpha(x), \xi(\xi(x))) = g^+(\alpha(x), \xi(a)) \geq g^+(\alpha(x), b).$$

by monotonicity of g^+. Then

$$\min\{\sup dom(s_n), \sup dom(s_{n+1})\} = \min\{\xi^{-1}(|A_n|^+), \xi^{-1}(|A_{n+1}|^+)\} =$$

$$= \xi^{-1}(|A_n|^+) = x \geq g^+(\alpha(x), |A_{n+1}|^+).$$

Because $x \in (0, |A_{n+1}|^+)$ and $\alpha(x) \in (0, |A_{n+1}|^+)$, $s_n, s_{n+1} \in A_{n+1}$, and A_{n+1} is g^+-convergent, we have $s_n(t) = s_{n+1}(t)$ for all $t < \alpha(x) = \alpha(\sup dom(s_n))$.

Let us define a function s_* on $[t_0, +\infty)$ such that for each $t \geq t_0$, $s_*(t) = s_{m(t)}(t)$ where $m(t) = \min\{n \in \mathbb{N} \mid t \in [t_0, \alpha(\sup dom(s_n)))\}$. Because α is unbounded and $\alpha(y) < y$ for all $y > 0$, from (3.4) it follows that $s_*(t)$ is defined for all $t \geq t_0$.

The sequence $\sup dom(s_n)$, $n \geq 1$ is monotone (by construction of s_n) and α is monotone, therefore (3.5) implies that $s_m(t) = s_n(t)$ for all m, $n \geq m$ and $t < \alpha(\sup dom(s_m))$. Then $s_*(t) = s_n(t)$ for each t such that $t < \alpha(\sup dom(s_{m(t)}))$ and $n \geq m(t)$. But $t < \alpha(\sup dom(s_{m(t)}))$ for all $t \geq t_0$. Thus

$$s_*(t) = s_n(t) \text{ for all } t \geq t_0 \text{ and } n \geq m(t).$$

It is easy to see that the function $m(t)$ is monotonically non-decreasing, so for each $t \geq t_0$ and $\tau \in [t_0, t+1]$, $s_*(\tau) = s_{m(t+1)}(\tau)$, whence $s_* \doteq_{t+} s_{m(t+1)}$ and if $t > t_0$, then $s_* \doteq_{t-} s_{m(t+1)}$. Let $(l,r) \in LR(Q)$ be a LR representation of Σ. Because for all n, $s_n \in Tr$ and $\min(dom(s_n)) = t_0$ (each s_n belongs to some right t_0-bunch), and for each $t \geq t_0$, $\sup(dom(s_{m(t+1)})) \geq t+1$, we have $l(s_{m(t+1)},t) \wedge r(s_{m(t+1)},t)$ for all $t \geq t_0$. Then because l is left-local and r is right-local, $l(s_*,t) \wedge r(s_*,t)$ for all $t \geq t_0$. Hence $s_* \in Tr$. Moreover, $s_* \doteq_{t_0+} s_{m(t_0+1)}$. Let $A \in c$ be a set such that $A_{m(t_0+1)} = pcl(A)$. Because A is a right t_0-bunch, we have $s_* \doteq_{t_0+} s'$ for all $s' \in A \subseteq A_{m(t_0+1)}$. □

Lemma 3.13. Assume that Σ satisfies the LFE property and each right dead-end path is f^+-escapable. Then for each $X \in S_{t_0}^+$ there exists $\bar{s} \in Tr$ such that $X \cup \{\bar{s}\} \in S_{t_0}^+$ and $X \prec^+ X \cup \{\bar{s}\}$.

Proof. Assume $X \in S_{t_0}^+$. Then $X \neq \varnothing$ and $|X|^+ < +\infty$. Denote $t_m = |X|^+$ and

$$H = \{(t,q) \in T \times Q \mid t \geq t_0 \wedge$$
$$\exists t' \in (t,t_m)\ \exists s \in X\ (s(t) \downarrow = q \wedge \sup(dom(s)) \geq g^+(t',t_m))\}.$$

Denote $H_1 = dom(H)$. Let us show that $H_1 = [t_0,t_m)$. The inclusion $H_1 \subseteq [t_0,t_m)$ follows from the definition of H. Let $t \in [t_0,t_m)$. Let us choose any $t' \in (t,t_m)$. Then $g^+(t',t_m) < t_m$. Because $t_m = |X|^+$, there exists $s \in X$ such that $\sup(dom(s)) \geq g^+(t',t_m) \geq t' > t$. Because $t \geq t_0$, we have $t \in dom(s)$ and $(t,s(t)) \in H$. Then $[t_0,t_m) \subseteq H_1$, because $t \in [t_0,t_m)$ is arbitrary.

Let us show that H is a functional binary relation. Assume that $(t,q_1) \in H$ and $(t,q_2) \in H$. Then there exist t_1', t_2' and $s_1, s_2 \in X$ such that $t_i' \in (t,t_m)$, $\sup(dom(s_i)) \geq g^+(t_i',t_m)$, and $q_i = s_i(t)$ for $i = 1,2$. Let $t' = \min\{t_1', t_2'\}$. Then $t' \in (t_0, |X|^+)$, because $t \geq t_0$. Moreover,

$$\min\{\sup(dom(s_1)), \sup(dom(s_2))\} \geq \min\{g^+(t_1',t_m), g^+(t_2',t_m)\} \geq g^+(t',t_m),$$

by the monotonicity of g^+. Then $s_1(t)=s_2(t)$ for $t\in[t_0,t')$, because X is g^+-convergent. Then $s_1(t)=q_1=q_2=s_2(t)$, because $t\in[t_0,t_1')$ and $t\in[t_0,t_2')$.

We conclude that H is a graph of some function $s_*:[t_0,t_m)\to Q$.

Let $(l,r)\in LR(Q)$ be a LR representation of Σ. Let us show that $s_*\in Tr$. Let $t\in(t_0,t_m)$. Then $(t,s_*(t))\in H$ and there exists $t'\in(t,t_m)$ and $s\in X$ such that $s_*(t)=s(t)$ and $\sup(dom(s))\geq g^+(t',t_m)\geq t'>t$. For each $\tau\in[t_0,t')$,

$$\tau\geq t_0\wedge t'\in(\tau,t_m)\wedge\sup(dom(s))\geq g^+(t',t_m)\wedge s\in X.$$

Then $(\tau,s(\tau))\in H$. Hence $s(\tau)=s_*(\tau)$ for all $\tau\in[t_0,t')$. Then $s\doteq_{t-}s_*$ and $s\doteq_{t+}s_*$, because $t\in(t_0,t')$. Because $t_0<t<\sup(dom(s))$ and $s\in Tr$, we have $l(s,t)\wedge r(s,t)$. Then $l(s_*,t)\wedge r(s_*,t)$, because l is left-local and r is right-local. Thus $l(s_*,t)\wedge r(s_*,t)$ for each $t\in(t_0,t_m)$.

Moreover, because $t_0\in H_1$, there exists $t'\in(t_0,t_m)$ and $s\in X$ such that $\sup(dom(s))\geq g^+(t',t_m)$ and $s_*(t_0)=s(t_0)$. Then for each $t\in(t_0,t')$ we have $t\geq t_0$ and $t'\in(t,t_m)$. Hence $(t,s(t))\in H$ for each $t\in(t_0,t')$. Then $s(t)=s_*(t)$ for $t\in[t_0,t')$. Then $r(s_*,t_0)$, because $r(s,t_0)$. We conclude that $s_*\in Tr$. Moreover, $s_*\doteq_{t_0+}s$ for all $s\in X$, because X is a right t_0-bunch.

Consider the case when s_* is not a dead-end path, i.e. there exists a continuation s_*' of s_* to $[t_0,t_m]$. Then by the LFE and CPR properties there exists $t_m'\in(t_m,\psi(t_m))$ and a trajectory $\bar{s}:[t_0,t_m']\to Q$ such that $\bar{s}\sqsubseteq s_*'$. Then using monotonicity of g^+ it is straightforward to show that $X\cup\{\bar{s}\}$ is a bounded g^+-convergent right t_0-bunch (i.e. $X\cup\{\bar{s}\}\in S_{t_0}^+$), and $X\prec^+ X\cup\{\bar{s}\}$.

Consider the case when s_* is a right dead-end path. Then s_* is f^+-escapable. Let us choose $\tau\in(t_0,t_m)$ such that $f^+(\tau,t_m)<\psi(t_m)$ (this is possible, because $f^+(t_m,t_m)=t_m$, the function $\tau\mapsto f^+(\tau,t_m)$ is continuous on $(t_0,t_m]$, and $\psi(t_m)>t_m$).

The CPR property and Lemma 3.1 imply that there exists an escape s_e from s_* of the form $s_e:[t_e,t'_e)\to Q$, where $t_e\in(\tau,t_m)$ and $t'_e=f^+(t_e,t_m)$.

Because $t_e\ge\tau$, we have

$$t_m<t'_e=f^+(t_e,t_m)\le f^+(\tau,t_m)<\psi(t_m).$$

Let us define a function $\bar{s}:[t_0,t'_e)\to Q$ as follows:

$$\bar{s}(t)=\begin{cases}s_*(t), & t\in[t_0,t_e)\\ s_e(t), & t\in[t_e,t'_e)\end{cases}.$$

The Markovian property implies that $\bar{s}\in Tr$. Moreover, $X\cup\{\bar{s}\}$ is a bounded right t_0-bunch, because $\bar{s}\doteq_{t_0+}s_*$ and $dom(\bar{s})$ is bounded. Also, $X\prec^+X\cup\{\bar{s}\}$, because $|X|^+=t_m<t'_e=|X\cup\{\bar{s}\}|^+<\psi(t_m)=\psi(|X|^+)$.

Let us prove $X\cup\{\bar{s}\}$ is g^+-convergent. Assume that $t'\in(t_0,t'_e)$, $s_1,s_2\in X\cup\{\bar{s}\}$, $t_i=\sup(dom(s_i))$, $i=1,2$, and $\min\{t_1,t_2\}\ge g^+(t',t'_e)$. Let us show that $s_1\doteq_{[t_0,t')}s_2$. Consider the following cases.

- Suppose that $s_1,s_2\in X$, $t'<t_m$. Then $\min\{t_1,t_2\}\ge g^+(t',t'_e)\ge g^+(t',t_m)$. Then $s_1(t)=s_2(t)$ for all $t\in[t_0,t')$, because X is g^+-convergent.
- Suppose that $s_1,s_2\in X$ and $t'\ge t_m$. Then $\min\{t_1,t_2\}\ge g^+(t',t'_e)\ge t'\ge t_m$. Then $t'=t_1=t_2=t_m$, because $s_1,s_2\in X$. The definition of H implies that $(t,s_1(t))\in H$ and $(t,s_2(t))\in H$ for all $t\in[t_0,t_m)$, because $t_m\ge g^+(t'',t_m)$ for all $t''<t_m$. Thus $s_1\doteq_{[t_0,t')}s_2$.
- Suppose that $\{s_1,s_2\}\not\subseteq X$. The case $s_1=s_2=\bar{s}$ is trivial, so assume either $s_1\in X$ and $s_2=\bar{s}$, or $s_2\in X$ and $s_1=\bar{s}$. We consider only the former case, because the latter case is analogous. Let $s_1\in X$, $s_2=\bar{s}$. Then $t_m\ge t_1=\min\{t_1,t_2\}\ge g^+(t',t'_e)\ge t'$. Also, $t_1\ge g^+(t',t_m)$ because $t'\le t_m<t'_e$. We have $(t,s_1(t))\in H$ for all $t\in[t_0,t')$ by definition of H,

because $t_m \geq g^+(t'',t_m)$ for all $t'' < t_m$. Hence $s_1 \doteq_{[t_0,t')} s_*$. Assume that $t' > t_e$. Then $t_1 \geq g^+(t',t'_e) \geq g^+(t_e,t'_e) = t_m$, because $t'_e = f^+(t_e,t_m)$. Then $g^+(t',t'_e) = g^+(t_e,t'_e) = t_m$. Then $t_e \geq t'$ by monotonicity of g^+. This contradicts assumption $t' > t_e$. Thus $t' \leq t_e$. Moreover, $s_*(t) = \bar{s}(t)$ for all $t \in [t_0,t_e]$. Thus $s_1(t) = \bar{s}(t) = s_2(t)$ for all $t \in [t_0,t')$. □

Now we have lemmas that are necessary to prove Theorem 3.4.

Proof of Theorem 3.4. The "Only if" part of theorem follows from the CPR property, so let us prove the "If" part.

Assume that Σ satisfies the LFE property and each right dead-end path is f^+-escapable. Let $s:[t_0,t^*) \to Q$ be a right dead-end path in Σ, $A_0 = \{s\}$, and $c_0 = \{A_0\}$. It follows immediately that $A_0 \in S^+_{t_0}$ and $c_0 \in Ch(S^+_{t_0},\prec^+)$. From Lemma 3.6 and Zorn's lemma it follows that there exists a maximal element $c_m \in Ch(S^+_{t_0},\prec^+)$ (with respect to $\leq$) such that $c_0 \leq c_m$.

Let $X = \bigcup c_m$. Then $X \neq \varnothing$, $c_m \neq \varnothing$, and $A_0 \subseteq X$, because $c_0 \subseteq c_m$.

Let us show that $|X|^+ = +\infty$. Suppose that $|X|^+ < +\infty$. Then $X \in S^+_{t_0}$ by Lemma 3.8. Then by Lemma 3.13 there exists $\bar{s}$ such that $X \cup \{\bar{s}\} \in S^+_{t_0}$ and $X \prec^+ X \cup \{\bar{s}\}$. Then $X \subset X \cup \{\bar{s}\}$, but this contradicts Lemma 3.7, so $|X|^+ = +\infty$.

Then by Lemma 3.12 there exists a trajectory $s_*:[t_0,+\infty) \to Q$ and $A \in c_m$ such that $s_* \doteq_{t_0+} s'$ for all $s' \in A$.

Because c_m is a $\subseteq$-chain, $A \in c_m$, and $A_0 = \{s\} \in c_0 \leq c_m$, we have either $\{s\} \subseteq A$, or $A \subseteq \{s\}$. Because $A \in c_m \subseteq S^+_{t_0}$ and $\varnothing \notin S^+_{t_0}$, we have $A \neq \varnothing$, so in both cases $s \in A$. Then $s_* \doteq_{t_0+} s$. Then s_* is an infinite escape from s.

Because s is an arbitrary right dead-end path, we conclude that each right dead-end path is strongly escapable. □

3.6 Example of an application of the theorem about right dead-end path

Although Theorem 3.3 and Theorem 3.5 together give an explicit criterion for the existence of a global trajectory of a NCMS with a given LR representation, proofs of the existence of global trajectories of NCMS which are not represented using LR representation can be accomplished using Theorem 3.4. In this section we give an example with illustrates this application of Theorem 3.4.

3.6.1 Informal considerations

Informally, consider the following situation: a system S travels in accordance with a known law of motion and a (hazardous) object H moves along a fixed trajectory independently of S. If H becomes sufficiently close to S, the system S tries to perform a maneuver to avoid collision with H. We are interested in conditions under which S can travel during an unbounded time interval while avoiding collisions with H.

3.6.2 Semiformal considerations

Consider the following semiformal clarification of the described situation. Suppose that the behavior of S is modeled by a control system of the form

$$\frac{d}{dt}y(t) = g(t, y(t), u(t)) \tag{3.6}$$

where $y(t) \in \mathbb{R}^n$ is a position of S at time t and $u(t)$ is an input control of S which influences the trajectory of S and can be used to perform a maneuver, and the position of H at each time is described by a function $z : T \to \mathbb{R}^n$.

We are going to find conditions under which there exists a function u and a corresponding solution y of (3.6) such that u and y are defined on T, $y(t) \neq z(t)$ for all $t \in T$, and u is constant over each time interval where $y(t) - z(t) \notin D$, where D is a given (fixed) set (D defines a region near $z(t)$, where the input control of S can be varied in order to perform a maneuver).

To simplify the problem, let us introduce a new variable $x(t) = y(t) - z(t)$ and assume that z is differentiable. Then (3.6) can be rewritten in the form

$$\frac{d}{dt}x(t) = g(t, x(t) + z(t), u(t)) - \frac{d}{dt}z(t) \quad (3.7)$$

After introducing a new function $f(t,x,u) = g(t, x + z(t), u) - \frac{d}{dt}z(t)$ we can rewrite the equation (3.7) as

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)) \quad (3.8)$$

The problem is to propose sufficient conditions under which there exist functions u and x defined on T which satisfy (3.8), $x(t) \neq \mathbf{0}$ for all $t \in T$, and u is constant over each time interval where $x(t) \notin D$, where $\mathbf{0}$ is the null vector in $\mathbb{R}^n$.

Similar and related problems were considered e.g. in [29] and were studied using control-theoretic methods. However, in this example we will demonstrate a direct application of Theorem 3.4 in this situation.

3.6.3 Formal considerations

Let us formulate the described problem formally in terms of NCMS.

Let $n, m \in \mathbb{N}$, $n \geq 2$, $x^* \in \mathbb{R}^n$, and $D \subseteq \mathbb{R}^n$. Let $U \subset \mathbb{R}^m$ be a non-empty compact set, $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$ or $\mathbb{R}^m$, and $f : T \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be a function such that

– f is continuous and bounded on $T \times \mathbb{R}^n \times U$;

– there exists a number $L > 0$ such that $\|f(t, x_1, u) - f(t, x_2, u)\| \leq L\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^n$, $t \in T$, and $u \in U$ (i.e. f is Lipschitz-continuous in x).

Under these conditions Caratheodory existence theorem [26] implies that for each $t_0 \in T$ and $x_0 \in \mathbb{R}^n$, and a Lebesgue-measurable [95] function $u : [t_0, +\infty) \to U$ the problem

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)) \quad (3.9)$$

$$x(t_0) = x_0 \quad (3.10)$$

has a Caratheodory solution defined for all $t \geq t_0$, i.e. a function $t \mapsto x(t;t_0;x_0;u)$ which is absolutely continuous [95] on every segment $[a,b] \subset [t_0,+\infty)$, satisfies the equation (3.9) a.e. (almost everywhere [95] in the sense of Lebesgue measure), and satisfies (3.10). Moreover, this solution is unique in the sense that for any function $x:[t_0,t_1) \to \mathbb{R}^n$, which is absolutely continuous on every segment $[a,b] \subset [t_0,t_1)$, satisfies (3.9) a.e. on $[t_0,t_1)$ and satisfies (3.10), $x(t) = x(t;t_0;x_0;u)$ holds for $t \in [t_0,t_1)$.

Let $Q = \mathbb{R}^n \times U$. Denote by $proj_1 : Q \to \mathbb{R}^n$, $proj_2 : Q \to U$ the projections on the first and second component, i.e. $proj_1((x_0,u_0)) = x_0$ and $proj_2((x_0,u_0)) = u_0$.

Let Tr be the set of all functions $s : A \to Q$, where $A \in \mathfrak{T}$, such that the following conditions are satisfied, where $x = proj_1 \circ s$ and $u = proj_2 \circ s$:

1) u is Lebesgue-measurable;

2) x is absolutely continuous on each segment $[a,b] \subseteq A$ $(a < b)$ and satisfies the equation $\frac{d}{dt}x(t) = f(t,x(t),u(t))$ a.e. on A;

3) $x(t) \neq \mathbf{0}$ for all $t \in A$;

4) for each non-maximal $t \in A$ such that $x(t) \notin D$ there exists $t' \in (t,+\infty) \cap A$ such that $u(t'') = u(t)$ for all $t'' \in (t,t')$.

5) for each non-minimal $t \in A$ such that $x(t) \notin D$ there exists $t' \in (0,t) \cap A$ such that $u(t'') = u(t)$ for all $t'' \in (t',t)$.

It follows straightforwardly from this definition that $\Sigma = (T,Q,Tr)$ is a NCMS (i.e. Tr is a CPR, Markovian, and complete set of trajectories).

The problem is to give a sufficient condition which ensures that Σ has a global trajectory.

Proposition 3.1.

1) Σ satisfies the LFE property.

2) There exists $s \in Tr$ and $\varepsilon > 0$ such that $dom(s) = [0,\varepsilon]$.

Proof. 1) Let $s:[a,b]\to Q$ be a trajectory, $x = proj_1 \circ s$, and $u = proj_2 \circ s$. Let $u':[a,+\infty)\to U$ be a function such that $u'(t)=u(t)$, if $t\in[a,b]$ and $u'(t)=u(b)$, if $t>b$. Then $u = u'|_{[a,b]}$, u' is measurable, and $x(t)=x(t;a;x(a);u')$ for all $t\in[a,b]$. Let $b'=b+1$ and $x':[a,b']\to\mathbb{R}^n$ be a function such that $x'(t)=x(t;a;x(a);u')$ for $t\in[a,b']$. Then $x = x'|_{[a,b]}$. Because $x'(t)\neq\mathbf{0}$ for all $t\in[a,b]$ and x' is continuous, there exists $b''\in(b,b']$ such that $x'(t)\neq\mathbf{0}$ for all $t\in[a,b'']$. Let $s':[a,b'']\to Q$ be a function such that $s'(t)=(x'(t),u'(t))$ for all $t\in[a,b'']$. Then it follows immediately that s' satisfies the conditions 1-4, so $s'\in Tr$. Besides, $s \sqsubseteq s'$. Thus Σ satisfies LFE.

2) Let us choose any $x_0\in\mathbb{R}^n\setminus\{\mathbf{0}\}$ and $u_0\in U$ and define $x:T\to\mathbb{R}^n$ as $x(t)=x(t;0;x_0;u_0)$ for all $t\in T$. Then x is continuous and $x(0)=x_0\neq\mathbf{0}$, so there exists $\varepsilon>0$ such that $x(t)\neq\mathbf{0}$ for all $t\in[0,\varepsilon]$. Let $s:[0,\varepsilon]\to Q$ be a function $s(t)=(x(t),u_0)$, $t\in[0,\varepsilon]$. Then $s\in Tr$. □

Proposition 3.2. Assume that:

1) for each $t\in T$ there exist $u_1,u_2\in U$ such that $f(t,\mathbf{0},u_1)$, $f(t,\mathbf{0},u_2)$ are (nonzero) noncollinear vectors, i.e. $k_1 f(t,\mathbf{0},u_1)+k_2 f(t,\mathbf{0},u_2)\neq\mathbf{0}$ whenever $k_1,k_2\in\mathbb{R}$ are not both zero;
2) for each $s\in Tr$ defined on a set of the form $[t_1,t_2)$, if $\lim_{t\to t_2-}(proj_1\circ s)(t)=\mathbf{0}$, then $proj_1(s(t))\in D$ for some $t\in[t_1,t_2)$.

Then each right dead-end path in Σ is f_1^+-escapable, where $f_1^+(x,y)=2y-x$ is a right extensibility measure defined in Lemma 3.3.

Proof.

Let $M' = 1+\sup\{\|f(t',x',u')\| \mid (t',x',u')\in T\times\mathbb{R}^n\times U\}$. Then $0<M'<+\infty$, because f is bounded.

Let $s:[a,b)\to Q$ be a right dead-end path and $x = proj_1\circ s$, $u = proj_2\circ s$. Let $u':[a,+\infty)\to U$ be a function such that $u'(t)=u(t)$, if $t\in[a,b)$ and

$u'(t)=u(a)$, if $t \ge b$. Then $u=u'|_{[a,b)}$, u' is measurable, and $x(t)=x(t;a;x(a);u')$ for all $t \in [a,b)$. Then there exists a limit $x_l = \lim_{t\to b-} x(t) = x(b;a;x(a);u') \in \mathbb{R}^n$.

Firstly, consider the case when $x_l \neq \mathbf{0}$. Then $\|x_l\|>0$. Let us choose an arbitrary $t_0 \in (a,b)$ such that $b-t_0 < \|x_l\|/(4M')$ and $\|x(t_0)-x_l\| < \|x_l\|/2$ (this is possible, because $x_l = \lim_{t\to b-} x(t)$). Let $u'':[t_0,+\infty)\to U$ and $x'':[t_0,+\infty)\to \mathbb{R}^n$ be functions such that $u''(t)=u(t_0)$ for all $t \ge t_0$ and $x''(t)=x(t;t_0;x(t_0);u'')$ for all $t \ge t_0$. Then $\|x''(t_0)\| = \|x(t_0)-x_l+x_l\| \ge \|x_l\| - \|x(t_0)-x_l\| > \|x_l\|/2 > 2M'(b-t_0)$. Then for all $t \ge t_0$ we have

$$\|x''(t)\| = \left\| x''(t_0) + \int_{t_0}^{t} f(t,x''(t),u''(t))dt \right\| \ge \|x''(t_0)\| - \int_{t_0}^{t} \|f(t,x''(t),u''(t))\| dt >$$

$$> 2M'(b-t_0) - M'(t-t_0) = M'(2b-t_0-t).$$

Let $d = 2b - t_0$. Then $d > t_0$ because $t_0 < b$. Then $x''(t) \neq \mathbf{0}$ for all $t \in [t_0,d]$. Let $s_* : [t_0,d] \to Q$ be a function such that $s_*(t) = (x''(t),u''(t))$ for all $t \in [t_0,d]$. It follows immediately that $s_* \in Tr$. Also, $s_*(t_0)=s(t_0)$ and $d = 2b-t_0 = f_1^+(t_0,b)$. Then s_* is an escape from s and s is f_1^+-escapable.

Now consider the case when $x_l = \mathbf{0}$.

Let us choose $u_1,u_2 \in U$ such that $v_1 = f(b,\mathbf{0},u_1)$ and $v_2 = f(b,\mathbf{0},u_2)$ are noncollinear (this is possible by the assumption 1 of the lemma). Then the function $h(k_1,k_2) = \|k_1 v_1 + k_2 v_2\|$ attains some minimal value $M>0$ on $\{(k_1,k_2)\in \mathbb{R}\times\mathbb{R} \mid |k_1|+|k_2|=1\}$. Then for all k_1,k_2 such that $k_1 \neq 0$ or $k_2 \neq 0$,

$$h(k_1,k_2) = (|k_1|+|k_2|)h(k_1(|k_1|+|k_2|)^{-1}, k_2(|k_1|+|k_2|)^{-1}) \ge M(|k_1|+|k_2|).$$

Let $\varepsilon = M/2 > 0$. Because f is continuous, there exists $\delta > 0$ such that for each $j=1,2$, $t \in T$, and $x_0 \in \mathbb{R}^n$ such that $|b-t| + \|x_0\| < \delta$ we have $\|f(t,x_0,u_j) - v_j\| = \|f(t,x_0,u_j) - f(b,\mathbf{0},u_j)\| < \varepsilon$. Let $R = \delta/4$, $t_1 = \max\{b-R, a\}$, and $t_2 = b + R$. Then $R > 0$, $a \le t_1 < b < t_2$ and for all $j = 1,2$, $t \in [t_1, t_2]$ and x_0 such that $\|x_0\| \le R$, $\|f(t, x_0, u_j) - v_j\| < \varepsilon$.

Let us choose an arbitrary $c \in (t_1, b)$ such that $b - c < \min\{R/(2M'), R/2\}$. Then $s|_{[c,b)} \in Tr$ by the CPR property and $\lim_{t \to t_2 -}(proj_1 \circ s|_{[c,b)})(t) = x_l = \mathbf{0}$, so by the assumption 2 there exists $t_0 \in [c, b)$ such that $proj_1(s(t_0)) = x(t_0) \in D$.

Let $x_1 : [t_0, t_2] \to \mathbb{R}^n$ and $x_2 : [t_0, t_2] \to \mathbb{R}^n$ be functions such that $x_1(t) = x(t; t_0; x(t_0); u_1)$ and $x_2(t) = x(t; t_0; x(t_0); u_2)$ for all $t \in [t_0, t_2]$. Denote $d_j(t) = f(t, x_j(t), u_j) - v_j$ for each $j = 1,2$ and $t \in [t_0, t_2]$.

Then the following two cases are possible.

a) There exists $j \in \{1,2\}$ such that $\mathbf{0} \notin range(x_j)$. Let us choose any $d \in (\max\{2b - t_0, t_0\}, t_2)$ (this is possible, because $t_0 < b < t_2$ and $2b - t_0 \le 2b - c < b + R/2 < b + R = t_2$). Then let $s_* : [t_0, d] \to Q$ be a function such that $s_*(t_0) = s(t_0) = (x(t_0), u(t_0))$ and $s_*(t) = (x_j(t), u_j)$ for all $t \in (t_0, d]$. Because $x_j(t_0) = x(t_0) \in D$ and $x_j(t) \ne \mathbf{0}$ for all $t \in [t_0, t_2] \supset [t_0, d]$, we have that $s_* \in Tr$. Besides, $s_*(t_0) = s(t_0)$ and $d > 2b - t_0 = f_1^+(t_0, b)$, so s_* is an escape from s and s is f_1^+-escapable.

b) $\mathbf{0} \in range(x_1) \cap range(x_2)$. Then because x_1, x_2 are continuous, there exist $t'_j = \min\{t \in [t_0, t_2] \mid x_j(t) = \mathbf{0}\}$ for $j = 1,2$. Moreover, $t'_j \in (t_0, t_2]$ for $j = 1,2$, because $x_1(t_0) = x_2(t_0) = x(t_0) \ne \mathbf{0}$.

If we suppose that $\|x_j(t)\| < R$ for each $j = 1,2$ and $t \in [t_0, t'_j]$, then $\|d_j(t)\| = \|f(t, x_j(t), u_j) - v_j\| < \varepsilon$ for each $j = 1,2$ and $t \in [t_0, t'_j]$, whence

$$\|\mathbf{0} - \mathbf{0}\| = \|x_1(t'_1) - x_2(t'_2)\| = \left\| x(t_0) + \int_{t_0}^{t'_1} f(t, x_1(t), u_1)dt - x(t_0) - \int_{t_0}^{t'_2} f(t, x_2(t), u_2)dt \right\| =$$

$$= \left\| \int_{t_0}^{t'_1} v_1 + d_1(t)dt - \int_{t_0}^{t'_2} v_2 + d_2(t)dt \right\| = \left\| v_1(t'_1 - t_0) - v_2(t'_2 - t_0) + \int_{t_0}^{t'_1} d_1(t)dt - \int_{t_0}^{t'_2} d_2(t)dt \right\| \ge$$

$$\geq \|v_1(t_1'-t_0)-v_2(t_2'-t_0)\| - \int_{t_0}^{t_1'} \|d_1(t)\| dt - \int_{t_0}^{t_2'} \|d_2(t)\| dt \geq$$

$$\geq M(|t_1'-t_0|+|t_2'-t_0|) - \varepsilon(t_1'-t_0) - \varepsilon(t_2'-t_0) = \frac{M}{2}(t_1'-t_0+t_2'-t_0) > 0\,.$$

We have a contradiction, so there exists $j \in \{1,2\}$ and $t'' \in [t_0, t_j']$ such that $\|x_j(t'')\| \geq R$. This implies that

$$R \leq \|x_j(t'')\| = \|x_j(t_j') - x_j(t'')\| = \left\| \int_{t''}^{t_j'} f(t, x_j(t), u_j) dt \right\| \leq M'(t_j' - t'')\,.$$

Then $t_j' - t_0 \geq t_j' - t'' \geq R / M' > 2(b-c) \geq 2(b-t_0)$, so $t_j' > 2b - t_0$. Let us choose any $d \in (\max\{2b-t_0, t_0\}, t_j')$. Let $s_* : [t_0, d] \to Q$ be a function such that $s_*(t_0) = s(t_0) = (x(t_0), u(t_0))$ and $s_*(t) = (x_j(t), u_j)$ for all $t \in (t_0, d]$. Because $x_j(t_0) = x(t_0) \in D$ and $x_j(t) \neq \mathbf{0}$ for all $t \in [t_0, t_j') \supset [t_0, d]$, we have $s_* \in Tr$. Besides, $s_*(t_0) = s(t_0)$ and $d > 2b - t_0 = f_1^+(t_0, b)$, so s_* is an escape from s and s is f_1^+-escapable. □

Proposition 3.3. Assume that:

1) for each $t \in T$ there exist $u_1, u_2 \in U$ such that $f(t,\mathbf{0},u_1)$ and $f(t,\mathbf{0},u_2)$ are noncollinear;
2) $\{\mathbf{0}\}$ is a path-component [77] of $\{\mathbf{0}\} \cup (\mathbb{R}^n \setminus D)$.

Then Σ has a global trajectory.

Proof.

Let us show that the assumption 2 of Proposition 3.2 holds. Let $s \in Tr$, $dom(s) = [t_1, t_2)$ $(t_1 < t_2)$, $\lim_{t \to t_2 -} (proj_1 \circ s)(t) = \mathbf{0}$. Denote $x = proj_1 \circ s$. Suppose that $x(t) \notin D$ for all $t \in [t_1, t_2)$. Let $\gamma : [0,1] \to \{\mathbf{0}\} \cup (\mathbb{R}^n \setminus D)$ be a function such that $\gamma(\varepsilon) = x(t_1 + \varepsilon(t_2 - t_1))$, if $\varepsilon \in [0,1)$ and $\gamma(1) = \mathbf{0}$. Then γ is continuous, so there is a path from $\gamma(0) = x(t_1) \neq \mathbf{0}$ to $\mathbf{0}$ in $\{\mathbf{0}\} \cup (\mathbb{R}^n \setminus D)$ (considered as a topological subspace of $\mathbb{R}^n$). This contradicts the assumption that $\{\mathbf{0}\}$ is a path-component of $\{\mathbf{0}\} \cup (\mathbb{R}^n \setminus D)$. Thus $x(t) \in D$ for some $t \in [t_1, t_2)$.

The assumption 1 of Proposition 3.2 also holds, so by Proposition 3.1, Proposition 3.2, Lemma 3.3, Theorem 3.4, and Lemma 3.2, Σ satisfies GFE. Besides, by Proposition 3.1, there exists $s \in Tr$ with $dom(s) = [0, \varepsilon]$ for some $\varepsilon > 0$, so by the GFE property, Σ has a global trajectory. □

3.7 Related work

In such domains as the theory of differential equations, control theory, viability theory [8], the problems of global existence of solutions of initial value problems for various classes of differential equations [19, 26, 32, 5] and inclusions [7, 8, 30, 70, 109, 98], existence of global non-Zeno executions of hybrid systems [85, 33, 42, 18] were studied for many specific classes of systems. Although such results are practically relevant, the classes of systems considered are usually of a lower level of abstraction than the class of NCMS and thus cannot be applied to the problem of existence of total I/O pairs of strongly nonanticipative blocks in the general case. On the other hand, the results presented in this chapter hold for arbitrary strongly nonanticipative blocks and NCMS.

3.8 Conclusions from the chapter

We have considered the questions of how one can prove that a given strongly nonanticipative block B has a total I/O pair (if B indeed has a total I/O pair) and how one can prove that for a given input signal bunch $i \in Sb(In(B), W)$, where $dom(i) = T$, there exists $o \in Op(B)(i)$ with $dom(o) = T$. We have reduced these questions to the problem of proving the existence of global trajectories of a NCMS (Theorem 3.1, Theorem 3.2).

We have proposed a method of proving the existence of a global trajectory of a NCMS (Theorem 3.3) which is based on finding a subset of trajectories which satisfy the global forward extensibility (GFE) property.

We have proposed a criterion (Theorem 3.5) which can be used to prove the GFE property of NCMS by proving the existence of certain locally defined trajectories independently in a neighborhood of each time moment.

CONCLUSIONS

In the thesis we have given a solution to the problem of investigation of abstract systems which admit inputs and outputs as partial functions of time. The systems of this kind can be used for giving formal semantics and determining properties of specification and development languages for cyber-physical systems, real-time information processing systems, and other similar systems.

In the work we have proposed to apply the principles of the composition-nominative approach to abstract systems which admit inputs and outputs as partial functions of time. We have obtained the results listed below.

1) A new class of abstract time systems with partially defined inputs and outputs called blocks was introduced. Basic properties of the systems of this class were studied.

2) On the basis of the notions of causality (nonanticipation) which were considered in the works by T. Windeknecht, M. Mesarovic, Y. Takahara the notions of a strongly and weakly nonanticipative block were introduced.

3) On the basis of the notion of a solution system by O. Hájek a class of abstract dynamical systems called initial Nondeterministic Complete Markovian Systems (NCMS) was introduced.

4) Theorems about representation of strongly nonanticipative blocks using NCMS were proved. It was shown that each strongly nonanticipative block has a NCMS representation and that each initial I/O NCMS is a representation of a strongly nonanticipative block.

5) General criteria for the existence of total input-output pairs of a strongly nonanticipative block and the existence of a total output for a given total input of a strongly nonanticipative block were obtained.

6) A general criterion for the existence of global trajectories of NCMS was obtained. This criterion expresses the existence of global trajectories in terms of conditions of the existence of locally defined trajectories of NCMS.

REFERENCES

[1] Abadi M. An old-fashioned recipe for real time / Martin Abadi, Leslie Lamport // Real-Time: Theory in Practice / Springer, 1992. — P. 1–27.

[2] Alur R. The algorithmic analysis of hybrid systems / Rajeev Alur, Costas Courcoubetis, Nicolas Halbwachs [et al.] // Theoretical computer science. — 1995. — Vol. 138, no. 1. — P. 3–34.

[3] Alur R. Modularity for timed and hybrid systems / Rajeev Alur, Thomas A. Henzinger // CONCUR'97: Concurrency Theory. — Springer, 1997. — P. 74–88.

[4] Heymann M. Analysis of Zeno behaviors in a class of hybrid systems / Michael Heymann, Feng Lin, George Meyer, Stefan Resmerita // Automatic Control, IEEE Transactions on. — 2005. — Vol. 50, no. 3. — P. 376–383.

[5] Angeli D. Forward completeness, unboundedness observability, and their lyapunov characterizations / David Angeli, Eduardo D. Sontag // Systems & Control Letters. — 1999. — Vol. 38, no. 4. — P. 209–217.

[6] Arbib M.A. A common framework for automata theory and control theory / Michael A Arbib // Journal of the Society for Industrial & Applied Mathematics, Series A: Control. — 1965. — Vol. 3, no. 2. — P. 206–222.

[7] Aubin J.P. Viability Theory (Modern Birkhauser Classics) / Jean-Pierre Aubin. — Birkhauser Boston, 2009. — 484 p.

[8] Aubin J.P. Differential inclusions: set-valued maps and viability theory / Jean Pierre Aubin, Arrigo Cellina. — Springer-Verlag GmbH, 1984. — 342 p.

[9] Baheti R. Cyber-physical systems / Radhakisan Baheti, Helen Gill // The Impact of Control Technology. — 2011. — P. 161–166.

[10] Ball J. Finite time blow-up in nonlinear problems / J.M. Ball // Nonlinear Evolution Equations. — 1978. — P. 189–205.

[11] Barbashin E. Introduction to stability theory / E.A. Barbashin. — Moscow: Nauka, 1967. — 223 p (in Russian).

[12] Birkhoff G.D. Dynamical systems / George David Birkhoff. — American Mathematical Society, 1927. — 312 p.

[13] Blow-Up in Quasilinear Parabolic Equations (De Gruyter Expositions in Mathematics) / Alexander A. Samarskii, Victor A. Galaktionov, Sergei P. Kurdyumov, Alexander P. Mikhailov. — Walter De Gruyter Inc., 1995. — 533 p.

[14] Branicky M. S. Studies in Hybrid Systems: Modeling, Analysis, and Control : Ph.D. thesis / Michael S. Branicky. – Massachusetts Institute of Technology, 1995. – 198 p.

[15] Bushaw D. Dynamical polysystems and optimisation / D.D. Bushaw // J. Differential Equations. — 1963. — Vol. 2. — P. 351–356.

[16] Buslenko N. Lectures on theory of complex systems / N.P. Buslenko, V.V. Kalashnikov, Kovalenko I.N. — Moscow: Soviet Radio, 1973. — 441 p (in Russian).

[17] Bychkov A. Theoretically possibilistic approach to modeling of systems with uncertain structure / A.S. Bychkov, Ie.V. Ivanov, V.S. Kasyanyuk // Mathematical machines and systems. — 2012. — No. 4. — P. 115–124 (in Russian).

[18] Cai C. Smooth Lyapunov functions for hybrid systems—part i: Existence is equivalent to robustness / C. Cai, A.R. Teel, R. Goebel // IEEE Transactions on Automatic Control. — 2007. — Vol. 52, no. 7. — P. 1264–1277.

[19] Coddington E.A. Theory of Ordinary Differential Equations / Earl A. Coddington, Norman Levinson. — Krieger Pub Co, 1984. — 429 p.

[20] Cyber-physical systems – a concept map [Electronic Resource]. – Mode of access : URL : http://cyberphysicalsystems.org/. – Title from the screen.

[21] Deng K. The role of critical exponents in blow-up theorems: the sequel / Keng Deng, Howard A. Levine // Journal of Mathematical Analysis and Applications. — 2000. — Vol. 243, no. 1. — P. 85–126.

[22] Drack M. Ludwig von Bertalanffy's early system approach / Manfred Drack // Systems Research and Behavioral Science. — 2009. — Vol. 26, no. 5. — P. 563–572.

[23] Emelyanov S. Variable Structure Control Systems / S.V. Emelyanov. — Moscow: Nauka, 1967. — 336 p (in Russian).

[24] The architecture analysis & design language (AADL): An introduction : Rep. : CMU/SEI-2006-TN-011 / Carnegie-Mellon University ; Executor: Peter H. Feiler, David P. Gluch, John J. Hudak : 2006. — 145 p.

[25] Feiler P.H. The SAE architecture analysis & design language (AADL). A standard for engineering performance critical systems / Peter H. Feiler, Bruce A. Lewis, Steve Vestal // IEEE International Symposium on Computer-Aided Control Systems Design. — Munich, 2006. — P. 1206–1211.

[26] Filippov A.F. Differential Equations with Discontinuous Righthand Sides: Control Systems (Mathematics and its Applications) / A.F. Filippov. — Springer, 1988. — 304 p.

[27] Filippov V.V. The topological structure of solution spaces of ordinary differential equations / V.V. Filippov // Russian Mathematical Surveys. — 1993. — Vol. 48. — P. 101–154.

[28] Foo N. Y. Realization for causal nondeterministic input-output systems / Norman Y. Foo, Pavlos Peppas // Studia Logica. — 2001. — Vol. 67. — P. 419–437.

[29] Frankowska H. Optimal control under state constraints / H. Frankowska // Proceedings of the International Congress of Mathematicians, Hyderabad, India, 19-27 August 2010. — 2010. — P. 2915–2942.

[30] Frankowska H.A. measurable upper semicontinuous viability theorem for tubes / Hélène Frankowska, Sawomir Plaskacz // Nonlinear analysis. — 1996. — Vol. 26. — P. 565–582.

[31] Galaktionov V. The problem of blow-up in nonlinear parabolic equations / Victor Galaktionov, Juan-Luis Vazquez // Discrete and Continuous Dynamical Systems. — 2002. —Vol. 8. — P. 399–433.

[32] Gliklikh Y. E. Necessary and sufficient conditions for global-in-time existence of solutions of ordinary, stochastic, and parabolic differential equations / Yuri E Gliklikh // Abstract and Applied Analysis. — 2006. —Vol. 2006. — P. 1–17.

[33] Goebel R. Hybrid dynamical systems / R. Goebel, R. G. Sanfelice, A. Teel // Control Systems, IEEE. — 2009. — Vol. 29. — P. 28–93.

[34] Goode H.H. System engineering: an introduction to the design of large-scale systems / Harry H. Goode, Robert Engel Machol. — McGraw-Hill, 1957. – 551 p.

[35] Goriely A. Integrability and nonintegrability of dynamical systems / Alain Goriely. — World Scientific Publishing Company, 2001. — Vol. 19. – 415 p.

[36] Gottschalk W.H. Topological dynamics / Walter Helbig Gottschalk, Gustav Arnold Hedlund. — American Mathematical Society Colloquium Publications, 1955. — 163 p.

[37] Hájek O. Theory of processes, I / Otomar Hájek // Czechoslovak Mathematical Journal. — 1967. — Vol. 17. — P. 159–199.

[38] Hájek O. Theory of processes, II / Otomar Hájek // Czechoslovak Mathematical Journal. — 1967. — Vol. 17. — P. 372–398.

[39] Hall A.D. A Methodology for Systems Engineering / A. D. Hall. — Van Nostrand, 1962. — 478 p.

[40] Hall A.D. Definition of system / Arthur D Hall, Robert E Fagen // General Systems. — 1956. — Vol. 1. — P. 18–28.

[41] Hause M. The SysML modelling language [Electronic resource] / M. Hause. – Mode of access : URL : http://www.omgsysml.org/The_SysML_Modelling_Language.pdf. – Title from the screen.

[42] Henzinger T. A. The theory of hybrid automata / T. A. Henzinger // Proc. Eleventh Annual IEEE Symp Logic in Computer Science LICS ’96. — IEEE, 1996. — P. 278–292.

[43] Herrlich H. Axiom of Choice / Horst Herrlich. — Springer, 2006. — 212 p.

[44] Hinrichsen D. Mathematical Systems Theory I: Modelling, State Space Analysis, Stability and Robustness (Part 1) / Diederich Hinrichsen, Anthony J. Pritchard. — Springer, 2011. — 804 p.

[45] Hugues J. AADL for cyber-physical systems: Semantics and beyond, validate what’s next [Electronic Resource]. – Mode of access : URL: http://chess.eecs.berkeley.edu/pubs/836.html. – Title from the screen.

[46] Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems / Rajeev Alur, Costas Courcoubetis, Thomas A. Henzinger, Pei Hsin Ho. — Springer-Verlag, 1993. — P. 209–229.

[47] Islam S. Toward integration of two system theories by Mesarovic and Wymore / Sirajul Islam // International Journal of General Systems. — 1974. — Vol. 1, no. 1. — P. 35–40.

[48] Ivanov E.V. Algebras of complex-named data / E.V. Ivanov // Proceedings of the International Conference “Theoretical and Applied Aspects of Program Systems Development” (TAAPSD’2009), December 8-10, 2009, Kyiv, Ukraine. — 2009. — P. 191–196 (in Ukrainian).

[49] Ivanov Ie. Properties of algebras of complex-named data / Ievgen Ivanov // Science Notes of the National University of Kyiv-Mohyla Academy. — 2010. — Vol. 112. — P. 95–102 (in Ukrainian).

[50] Ivanov Ie. Operational semantics of programs over complex-named data / Ievgen Ivanov // Visnyk (Bulletin) of Taras Shevchenko National University of Kyiv, Phys.-Math. Series. — 2011. — No. 1. — P. 133–136.

[51] Ivanov Ie. A criterion for existence of global-in-time trajectories of non-deterministic Markovian systems / Ievgen Ivanov // Communications in Computer and Information Science. — 2013. — Vol. 347. — P. 111–130.

[52] Ivanov Ie. Hybrid possibilistic systems with nondeterministic continuous evolutions and switchings / Ievgen Ivanov, Mykola Nikitchenko, Louis Feraud // Acta Electrotechnica et Informatica. — 2012. — Vol. 12. — P. 49–59.

[53] John F. Nonlinear Wave Equations, Formation of Singularities (University Lecture Series) / Fritz John. — American Mathematical Society, 1990. — 64 p.

[54] Kalman R.E. Mathematical description of linear dynamical systems / Rudolf Emil Kalman // Journal of the Society for Industrial & Applied Mathematics, Series A: Control. — 1963. — Vol. 1. — P. 152–192.

[55] Kalman R.E. Topics in Mathematical System Theory (Pure & Applied Mathematics S.) / Rudolf E. Kalman, Peter L. Falb, Michael A. Arbib. — McGraw-Hill Education, 1969. — 358 p.

[56] Klir G.J. Facets of Systems Science (IFSR International Series on Systems Science and Engineering) / George J. Klir. — Springer, 2001. — 740 p.

[57] Kloeden P.E. General control systems / P. E. Kloeden. — Springer-Verlag, 1978. — P. 119–137.

[58] Kuczma M. Iterative Functional Equations (Issue 32 of Encyclopedia of Mathematics and its Applications) / Marek Kuczma, Bogdan Choczewski, Roman Ger. — Cambridge University Press, 1990. — 552 p.

[59] Kuhtenko A. On abstract levels of description of complex systems / A.I. Kuhtenko // Methodological problems of systems engineering. Matherials of the 1st USSR symposium. — Leningrad: Sudostroenie, 1970. — P. 34—36 (in Russian).

[60] Lakshmikantham V. Theory of Impulsive Differential Equations / Vangipuram Lakshmikantham, Drumi D. Banov, Pavel S. Simeonov. — World Scientific Publishing Company, 1989. — 288 p.

[61] Lee E.A. Computing needs time / Edward A. Lee // Communications of the ACM. — 2009. — Vol. 52. — P. 70–79.

[62] Lee E. A. Introduction to embedded systems: A cyber-physical systems approach / Edward Ashford Lee, Sanjit Arunkumarr Seshia. — Lulu.com, 2013. — 516 p.

[63] Levine H.A. The role of critical exponents in blowup theorems / Howard A. Levine // Siam Review. — 1990. — Vol. 32, no. 2. — P. 262–288.

[64] Levine W.S. The control handbook / William S. Levine. — CRC Press, 1996. — 1540 p.

[65] Liberzon D. Switching in Systems and Control (Systems & Control: Foundations & Applications) / Daniel Liberzon. — Birkhauser Boston Inc., 2003. — 246 p.

[66] Lin Y. General systems theory: A mathematical approach / Yi Lin. — Springer, 1999. — 395 p.

[67] Liu X. Modeling timed concurrent systems / Xiaojun Liu, Eleftherios Matsikoudis, Edward A. Lee // Proeedings of CONCUR 2006–Concurrency Theory. — Springer, 2006. — P. 1–15.

[68] Lynch N. Hybrid I/O automata / Nancy Lynch, Roberto Segala, Frits Vaandrager // Information and Computation. — 2003. — Vol. 185. — P. 105–157.

[69] Maler O. From timed to hybrid systems / Oded Maler, Zohar Manna, Amir Pnueli // Real-time: theory in practice, Lecture Notes in Computer Science. — 1992. — Vol. 600. — P. 447–484.

[70] Marques M.D. Viability results for nonautonomous differential inclusions / Manuel D.P. Monteiro Marques // Journal of Convex Analysis. — 2000. — Vol. 7, no. 2. — P. 437–443.

[71] Matrosov V.M. The method of comparison in mathematical systems theory / V. M. Matrosov, L.Yu. Anapolskiy, S.N. Vasilyev. — Novosibirsk: Nauka, 1980. — 480 p (in Russian).

[72] On fixed points of strictly causal functions : Rep. : UCB/EECS-2013-27 / EECS Department, University of California, Berkeley ; Executor: Eleftherios Matsikoudis, Edward A. Lee : 2013.

[73] Mesarovic M.D. General Systems Theory: Mathematical Foundations (Mathematics in Science and Engineering) / Mihajlo D. Mesarovic, Y. Takahara. — Elsevier Science, 1975. — 322 p.

[74] Mesarovic M.D. Abstract Systems Theory (Lecture Notes in Control and Information Sciences) / Mihailo D. Mesarovic, Yasuhiko Takahara. — Springer, 1989. — 439 p.

[75] Ivanov Ie. Modeling programs over complex-named data by term rewriting systems / Ievgen Ivanov, M. Nikitchenko, L. Feraud, M. Strecker // Proceedings of the International Conference "Theoretical and Applied Aspects of Program Systems Development" (TAAPSD'2010), October 4-8, 2010, Kyiv, Ukraine. — 2010. — P. 40–49.

[76] Movchan A.A. Stability of processes in two metrics / A.A Movchan // Prikladnaya Matematika i Mekhanika. — 1960. — No. 6. — P. 988–1001 (in Russian).

[77] Munkres J.R. Topology / James R. Munkres. — Prentice Hall, 2000. — 537 p.

[78] Nemytskii V.V. Qualitative theory of differential equations / V.V. Nemytskii, V.V. Stepanov. — Moscow: GITTL, 1949. — 550 p (in Russian).

[79] Nerode A. Models for hybrid systems: Automata, topologies, controllability, observability / Anil Nerode, Wolf Kohn. — Springer-Verlag, 1993. — Jun. — P. 317–356.

[80] Nikitchenko M. Properties of composition-nominative languages of programs with associative denaming / M.S. Nikitchenko, E.V. Ivanov //

Proceedings of XVI All-Ukrainian scientific conference Modern problems of applied mathematics and informatics, October 8-9, 2009, Lviv, Ukraine. — 2009. — P. 157–158 (in Ukrainian).

[81] Nikitchenko M. Composition-nominative languages of programs with associative denaming / M.S. Nikitchenko, Ievgen Ivanov // Visnyk (Bulletin) of the Lviv University. Ser. Appl. Math. Inform. — 2010. — Vol. 16. — P. 124–139 (in Ukrainian).

[82] Nikitchenko M. Programming with nominative data / M. Nikitchenko, Ievgen Ivanov // Proceedings of CSE'2010 International Scientific Conference on Computer Science and Engineering, September 20-22, 2010, Kosice, Slovakia. — 2010. — P. 30–39.

[83] Nikitchenko M. Stability and monotonicity of programs with respect to structure transformations of data / M.S. Nikitchenko, Ievge Ivanov // Problems in programming. — 2010. — Vol. No. 2-3. — P. 58–67 (in Ukrainian).

[84] A composition nominative approach to program semantics : Rep. / IT-TR 1998-020, Technical University of Denmark ; Executor: Nikolaj S Nikitchenko : 1998.

[85] Heemels W. On the existence and uniqueness of solution trajectories to hybrid dynamical systems / W. Heemels, M. K. Camlibel, A. J. Van der Schaft, J. M. Schumacher // Nonlinear and Hybrid Control in Automotive Applications. — 2002. — P. 391–422.

[86] Oppenheim A.V. Signals and systems / Alan V. Oppenheim, Alan S. Willsky, Syed Hamid Nawab. — Prentice-Hall, 1983. — 796 p.

[87] Orava P. Causality and state concepts in dynamical systems theory / P.J. Orava // International Journal of Systems Science. — 1973. — Vol. 4, no. 4. — P. 679–690.

[88] Orava P.J. Theoretical Systems Ecology: Advances and Case Studies / P. J. Orava. — Academic Press Inc., 1979. — P. 214–232.

[89] Padulo L. System theory: a unified state-space approach to continuous and discrete systems / L. Padulo, M.A. Arbib. — W.B. Saunders Company, 1974. — 779 p.

[90] Polderman J.W. Introduction to mathematical systems theory: a behavioral approach / Jan W. Polderman, Jan C. Willems. — Springer, Berlin, 1997. — 424 p.

[91] Red’ko V. Composition aspects of programmology. II / V.N. Red’ko, N.S. Nikitchenko // Cybernetics and Systems Analysis. — 1988. — Vol. 24. — P. 33–41.

[92] Red’ko V.N. Composition aspects of programmology. I / V.N. Red’ko, N.S. Nikitchenko // Cybernetics and Systems Analysis. — 1987. — Vol. 23. — P. 627–637.

[93] Roman S. Lattices and ordered sets / Steven Roman. — Springer, 2008. — 305 p.

[94] Roxin E. Local definition of generalized control systems. / Emilio Roxin // The Michigan Mathematical Journal. — 1966. — Vol. 13. — P. 91–96.

[95] Royden H. Real analysis / H.L. Royden. — Macmillan, 1968. — 349 p.

[96] Samoilenko A. Impulsive differential equations / A.M. Samoilenko, N.A. Perestyuk. — World Scientific (Singapore and River Edge, NJ), 1995. — 472 p.

[97] Schechter E. Handbook of Analysis and Its Foundations / Eric Schechter. — Academic Press, 1996. — 883 p.

[98] Seah S.W. Existence of solutions and asymptotic equilibrium of multivalued differential systems / Sek Wui Seah // Journal of Mathematical Analysis and Applications. — 1982. — Vol. 89. — P. 648–663.

[99] Seising R. Cybernetics, system(s) theory, information theory and fuzzy sets and systems in the 1950s and 1960s / Rudolf Seising // Information Sciences. — 2010. — Vol. 180. — P. 4459–4476.

[100] Sell G.R. Differential equations without uniqueness and classical topological dynamics / George R. Sell // Journal of Differential Equations. — 1973. — Vol. 14. — P. 42–56.

[101] Shannon C. A mathematical theory of communication / Claude Shannon // The Bell System Technical Journal. — 1948. — Vol. 27. — P. 379–423.

[102] Simulink – simulation and model-based design [Electronic Resource]. – Mode of access : URL : http://www.mathworks.com/products/simulink. – Title from the screen.

[103] Caml M.K. Solution concepts for hybrid dynamical systems / M.K. Caml, W.P.M.H. Heemels, A.J. Schaft van der, J.M. Schumacher // Proceedings of the 15th IFAC World Congress / Ed. by E.F. Camacho, L. Basañez, J.A. Puente de la. — Vol. 15. — IFAC, 2002. — P. 1217–1222.

[104] Shi J. A survey of cyber-physical systems / Jianhua Shi, Jiafu Wan, Hehua Yan, Hui Suo // Wireless Communications and Signal Processing (WCSP), 2011 International Conference. — IEEE, 2011. — P. 1–6.

[105] Tavernini L. Differential automata and their discrete simulators. / Lucio Tavernini // Nonlinear Anal. Theory Methods Applic. — 1987. — Vol. 11, no. 6. — P. 665–683.

[106] Utkin V.I. Sliding Modes in Control and Optimization (Communications and Control Engineering) / Vadim I. Utkin. — Springer, 1992. — 286 p.

[107] Von Bertalanffy L. General system theory: Foundations, development, applications / Ludwig Von Bertalanffy. — George Braziller New York, 1968. — 296 p.

[108] Von Bertalanffy L. The history and status of general systems theory. / Ludwig Von Bertalanffy // Academy of Management Journal. — 1972. — Vol. 15, no. 4. — P. 407–426.

[109] Vrabie I.I. A Nagumo type viability theorem / I.I. Vrabie // Scientific Annals of the “Alexandru Ioan Cuza” University of Iasi, Mathematics Series. — 2005. — Vol. 51. — P. 293–308.

[110] Wiener N. Cybernetics, Second Edition: or the Control and Communication in the Animal and the Machine / Norbert Wiener. — MIT Press, 1965. — 212 p.

[111] Willems J.C. Paradigms and puzzles in the theory of dynamical systems / J. C. Willems // IEEE Transactions on Automatic Control. — 1991. — Vol. 36, no. 3. — P. 259–294.

[112] Windeknecht T. Mathematical systems theory: Causality / T.G. Windeknecht // Mathematical systems theory. — 1967. — Vol. 1, no. 4. — P. 279–288.

[113] Witsenhausen, H. A class of hybrid-state continuous-time dynamic systems / Hans Witsenhausen // IEEE Transactions on Automatic Control. — 1966. — Vol. 11. — P. 161–167.

[114] Wolff K.E. Towards a conceptual system theory / Karl Erich Wolff // Proceedings of the World Multiconference on Systemics, Cybernetics and Informatics, 2000. — P. 124–132.

[115] Wymore A.W. A mathematical theory of systems engineering: the elements / A. Wayne Wymore. — Wiley, 1967. — 353 p.

[116] Wymore A.W. Model-Based Systems Engineering / A. Wayne Wymore. — CRC Press, 1993. — 710 p.

[117] Zadeh L.A. From circuit theory to system theory / Lotfi A. Zadeh // Proceedings of the IRE. — 1962. — Vol. 50. — P. 856–865.

[118] Zadeh L.A. The concepts of system, aggregate, and state in system theory // System Theory, Zadeh, L. A. and E. Polak, eds. / Lotfi A. Zadeh. – McGraw-Hill, New York, 1969. – P. 3-49.

[119] Zadeh L.A. Linear System Theory: The State Space Approach / Lotfi Asker Zadeh, Charles A. Desoer. — McGraw-Hill, 1963. — 628 p.

[120] Zeigler B.P. The hierarchy of system specifications and the problem of structural inference / Bernard P. Zeigler // PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association. Volume One: Contributed Papers. — The University of Chicago Press on behalf of the Philosophy of Science Association, 1976. — P. 227–239.

[121] Zeigler B.P. Theory of modeling and simulation: integrating discrete event and continuous complex dynamic systems / Bernard P. Zeigler, Herbert Praehofer, Tag Gon Kim. — Academic press, 2000. — 510 p.

[122] Zeno hybrid systems / Jun Zhang, Karl Henrik Johansson, John Lygeros, Shankar Sastry // International Journal of Robust and Nonlinear Control. — 2001. — Vol. 11. — P. 435–451.