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Unitary Owen Points in Cooperative Lot-Sizing Models with Backlogging

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Abstract: This paper analyzes cost sharing in uncapacitated lot-sizing models with backlogging and heterogeneous costs. It is assumed that several firms participate in a consortium aiming at satisfying their demand over the planning horizon with minimal operating cost. Each individual firm has its own ordering channel and holding technology, but cooperation with other firms consists in sharing that information. Therefore, the firms that cooperate can use the best ordering channels and holding technology among members of the consortium. This mode of cooperation is stable. in that allocations of the overall operating cost exist, so that no group of agents benefit from leaving the consortium. Our contribution in the current paper is to present a new family of cost sharing allocations with good properties for enforcing cooperation: the unitary Owen points. Necessary and sufficient conditions are provided for the unitary Owen points to belong to the core of the cooperative game. In addition, we provide empirical evidence, through simulation, showing that, in randomly-generated situations, the above condition is fulfilled in 99% of the cases. Additionally, a relationship between lot-sizing games and a certain family of production-inventory games, through Owen's points of the latter, is described. This interesting relationship enables easily constructing a variety of coalitionally stable allocations for cooperative lot-sizing models.

Keywords: unitary Owen points; cooperation; cost allocation; coalitional stability

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1. Introduction

The economic lot-sizing problem (henceforth ELSP) is a production problem in operations management and inventory theory that has been studied by many researchers for more than 50 years. The ELSP is an extension of the economic order quantity model to the case where there are some goods to be produced over a planning horizon, so that the production lots must be decided, in order to meet certain demand over the given finite horizon. Demand is usually generated from forecasts or by customer orders, or often by a combination of both. This production planning model is a common point for most companies or industries: planning what, when, and how much to produce.

To define the feasible production plans, several parameters of the production system are usually taken into account: the resource capacity (with or without restrictions) and all of the inventory costs involved. The simpler production planning model is known as the single-item uncapacitated lot-sizing model. It corresponds to the production planning of a single item to meet some demand over a discretized planning horizon. Despite its simplicity, it already contains most of the modeling elements that are cited above, with the exception of the capacity constraints.

There are also some other modeling elements that can be found in some situations. Those elements usually complicate the models and make them more difficult to solve. For instance, a competitive model for the allocation of capacity from some shared resources.

In other cases, the products interact through multi-level product structures. Finally, there may be other elements that are needed to refine the model. For instance, the production process may allow demand for finished products to be backlogged.

In this case, it is possible to deliver to a customer later than required, but that delay is pennalized because it has a negative impact on customer satisfaction. The latter is currently happening, for example, with the main producers of vaccines for COVID-19 (BioNTech, Moderna, Oxford), which do not have enough capacity to deliver all of the vaccines requested (Pfizer, Moderna, and Astrazeneca) to all countries on time. The reader is referred to the surveys by [1,2] for a complete and well-organized description of lot-sizing problems.

Over the last 40 years, single-item uncapacitated lot-sizing problems have been studied, trying out different formulations that have enabled more efficient solutions to the problem to be found. Among them, Pochet and Wolsey [3] present a mixed integer programming reformulation of the uncapacitated lot-sizing problem with backlogging, which in an extended space of variables, give strong reformulations using linear programming.

Nevertheless, most of the traditional research on lot-sizing-models focuses on tactical decision-making by single agents using optimization methods. This approach relies on the assumption that the outcome of a particular decision is independent of the decisions of other agents. However, as a result of global interaction, it has also reached supply chain management, an alternative perspective is becoming more common. Specifically, many recent research papers recognize the strategic interaction of multiple agents within supply chains. These agents are often independently owned and motivated companies. The fact that the outcomes from the agents' decisions partly depend on the decisions of other independent agents makes game theory a natural approach to modeling those decisions. In practice, the agents may behave either cooperatively or noncooperatively, and the recent literature contains two comprehensive general reviews, by [4] for both cooperative and noncooperative planning and scheduling games, and by [5,6] for non cooperative lot-sizing games, with and without capacity restrictions, respectively. However, our focus is specifically on cooperative lot-sizing models with backlogging, but without capacity restrictions. Each firm faces demand for a single product in each period and coalitions can pool orders. Firms cooperate by using the best ordering channel and holding technology provided by the participants in the consortium, e.g., they produce, hold inventory, pay backlogged demand, and make orders at the minimum cost among the members of the coalition. Thus, firms aim at satisfying their demand over the planning horizon with minimal operating cost. In principle, sharing private information can be seem as a limitation of this model. However, the reader may notice that this can be easily overcome. To prevent disclosing private information, one can assume that companies communicate through a mediator who helps them to make their optimal decisions without having to share their private information. The mediator implements the cooperation mechanism without disclosing information, reaching a win–win situation for all entities involved and giving rise to acceptable costs allocations.

To illustrate this form of cooperation today, let us consider four automotive companies, Peugeot (P), Citroen (C), Fiat (F), and Iveco (I). They all use the same chassis for their cars and buy it from a Chinese supplier twice a year (two periods). Peugeot is interested in buying a larger quantity of chassis to avoid supply problems from China and a possible increase in transportation costs. In addition, P is able to negotiate with the Chinese supplier and obtain very competitive unit and period purchasing costs for large-scale order sizes. P then proposes to the other companies to place a joint order for chassis. Citroen thinks that this is a good idea, because it has a large warehouse in Vigo where the entire order can always be stored. Iveco proposes to take advantage of its good contacts with transport companies and to fleet a ship to transport the joint orders from China to Vigo. The fixed order and transport costs are included in the set-up costs. Fiat analyzes the proposal and, although it does not have a long delay per period in the delivery of its cars, and its penalty costs are the lowest, it concludes that it turns out profitable. The four companies then reach an agreement and taking the unit purchasing costs of P, the holding costs of C, the set-up costs of I and the backlogging costs of F, they place a joint order with the lowest total cost $c(N)$. We should note that the cooperation of the four companies generates a reduction in the backlogging costs of P, C, and I (to those of F), because they can now reduce delivery times to their customers and leave fewer cars undelivered. We wonder whether there are any kind of unitary prices for the demands, related to inventory costs, which enable coalitional stable allocations of the overall operating costs to be built among the automotive companies, so that no group of firms profits from leaving the consortium.

A possible answer may be to consider the Owen point that applies for PI-problems. The Owen point works very-well as long as there is a strong formulation for the underlying optimization problem, such as PI-problems, because the dual variables (shadow prices) are used to construct the core allocations. However, this does not work for SI-problems, since the corresponding optimization problem has integer variables, and strong duality does not apply in the original space of variables. In this paper, we further extend the idea of dual prices and construct an "ad hoc" price type as the sum of the production, inventory and backlogging costs plus a proportion of the fixed order cost, which depends on the total demand satisfied in that period. They are called unitary prices. These unit prices enable replicating the construction of the Owen point by multiplying such unit prices by the demands and adding in all of the periods. These allocations "a la Owen" are called unitary Owen points. Unfortunately, one cannot always guarantee that unitary Owen points are core allocations. Nevertheless, we provide necessary and sufficient conditions for this situation to hold, i.e., for unitary Owen points to be core allocations and also show, by simulation empirical evidence, that this condition is satisfied in most cases. Furthermore, we consider whether it is possible to relate general SI-situations to simpler situations, where the core is well-known and characterized as in PI-games. In this regard, we prove that the answer to this question is yes: one can use the Owen point of the surplus game, a PI-game that measures the excess in costs that occurs with respect to the minimum unit price.

As compared with the existing literature on lot-sizing games, the contributions of this paper can be summarized, as follows. First of all, when compared with previous papers on the topic, our model extends the results in [7,8] to deal with backlogging and heterogeneous costs, whereas the models in those papers only consider homogeneous costs, and backlogging is not allowed. In addition, we also extend the models in [9,10], since those models do not allow set-up costs, whereas our new model does. Moreover, with respect to the cost sharing literature of production-inventory models, this paper introduces a new family of cost sharing allocations: the unitary Owen points. This family is an extension of the Owen set that enjoys very-good properties in production-inventory problems. We also contribute by providing the necessary and sufficient conditions for unitary Owen points to be core allocations. Furthermore, we empirically show that these conditions are satisfied for almost any SI-situation, which results in an explicit quasi-solution for this class of games. Finally, this paper also proves a new result that establishes a relationship between lot-sizing games and a certain family of PI-games, through the Owen's points of the latter. This relationship enables us to analyze cooperative lot-sizing models using properties of the much simpler and well-known class of PI-games.

The rest of this paper is organized, as follows: the next section reviews the literature of lot-sizing models. Section 3 formulates SI-problems and it shows that SI-games are totally balanced, resorting to a result of [3]. Section 4 describes the unitary Owen points, provides a necessary and sufficient condition for those points to be core allocations, and gives empirical evidence to consider the unitary Owen point as a quasi-solution for SI-games. Section 5 presents a relationship between SI-games and a certain family of PI-games through the Owen's points of the latter. This interesting relationship simplifies the analysis and construction of core allocations for SI-games. Finally, Section 6 presents a research summary and some conclusions.

2. Literature Review

Several papers have tackled problems that are associated with lot-sizing models and the interested reader is referred to [1] for an excellent survey. However, in addition to the references that can be found therein, some interesting recent papers deserve to be cited, since they consider aspects of coordination and cooperation that are directly related with our model. Related with the concept of coordination, Gharaei et al. [11] develop an integrated lot-sizing policy in a multi-level multi-product supply chain under stochastic conditions with limitations. They use generalised Benders decomposition to obtain a satisfactory performance in optimal solution, the number of iterations, dual infeasibility, constraint violation, and complementarity. In addition, Zissis, Ioannou, and Burnetas [12] study the important aspect of coordination in lot-sizing models. They develop a model for coordinating lot-sizing decisions under bilateral information asymmetry while using a mediator. This mediator makes coordination achievable, without enforcing centralized policies. Therefore, individual objectives can be aligned with channel objectives, reducing costs and eliminating inefficiencies.

Concerning the analysis of cooperation in lot-sizing models, several papers have considered the cost sharing aspects of these models. Specifically, Van den Heuvel, Borm, and Hammers [8] focus on the cooperation in economic lot-sizing situations with homogeneous costs, but without backlogging (henceforth, ELS-games). Subsequently, Guadiola, Meca, and Puerto [9,10] present the class of production-inventory games (henceforth, PI-games). PI-games study the cooperation among heterogeneous companies coming from production-inventory situations with discrete demand and backlogging, but without set-up costs. PI-games may be considered to be ELS games without set-up costs, but with backlogging and heterogeneous costs. They prove that the Owen set, i.e., the set of cost allocations that are achievable through shadow prices (the dual solutions to the primal linear optimization problem) (see [13]), is reduced to a unique cost allocation in the class of PI-games. These authors coined the term Owen point to refer to them and prove that the Owen point is always a coalitional stable and consistent cost allocation, in the sense that there is no group of firms that can improve upon or block this point by reducing the aggregated cost for the group (recall that the core of a cooperative cost game consists of all coalitional stable cost allocations, usually called core allocations). More recently, Chen and Zhang [7] consider the ELS-game with general concave ordering costs and they found out that a core allocation can be computed in polynomial time when all retailers have the same cost parameters (again homogeneous costs). Their approach is based on linear programming (LP) duality. Specifically, they prove that there is an optimal dual solution that defines an allocation in the core and point out that it is not necessarily true that every core allocation can be obtained by means of dual solutions.

On the other hand, Dreschel [14] focuses on cooperative lot-sizing games in supply chains. That paper considers several models of cooperation in lot-sizing problems of different complexity that are analyzed regarding theoretical properties, like monotonicity or concavity, and solved with a row generation algorithm to find stable cost allocations. Zeng, Li, and Cai [15] study the ELS-game with perishable inventory. They consider a single supplier and several retailers that collaborate to place joint orders of known and perishable demand in a determinate number of periods. They demonstrate that the associate ELS-game is subadditive and totally balanced. Moreover, they present a core-solution that allocates, in an equal form, the unit cost to each period.

In another paper, Gopaladesikan and Uhan [16] consider cooperative cost-sharing games that arise from ELS-problems with remanufacturing options. While studying the relative strength and integrality gaps of several mathematical optimization formulations of this problem, they show that the core associated to the cost-sharing game is, in general, empty. However, they show that, for two specific cases: large initial quantity of low cost returns and zero setup costs, the cost sharing game has a non-empty core. Finally, they demonstrate that a cost allocation in the core can be computed in polynomial time. Xu and Yang [17] present a competitive cost-sharing method with approximate cost recovering and cross-monotonic for an ELS-game under a weak triangle inequality assumption. They show the effectiveness of the proposed method with numerical results. In addition, Tsao, Chan, and Wu [18] investigate the combined effects of an imperfect production process, learning effect, and lot-sizing integration on a manufacturer-retailer channel for the Nash game and the cooperation game in an imperfect production system. They also solved the problem by a search procedure, and studied the relationship between downstream entities of the supply chain and the upstream to obtain structural and quantitative results. By numerical experiments, they reach the conclusion that the cooperative policy enables further cost reduction under a wide range of parameter settings. We conclude this literature review, recalling that the class of set-up-inventory games (henceforth, SI-games), considered in this paper, is introduced in [19] as a new class of combinatorial optimization games that arise from cooperation in lot-sizing problems with backlogging and heterogeneous costs. That paper proves that the game is balanced, proposes an "ad hoc" parametric family of cost allocations, and provides sufficient conditions for this family to be stable against the coalitional defection of firms.

3. SI-Games: Reformulation and Balancedness

We begin by formulating the set-up-inventory problems with backlogging (SI-problems).

Consider T periods, numbered from 1 to T, where the demand for a single product occurs in each of them. This demand is satisfied by own production, and it can be done during the production periods, in previous periods (inventory), or later periods (backlogging). A fixed cost must be paid in each production period. Therefore, the model includes production, inventory holding, backlogging, and set-up costs. The aim is to find an optimal ordering plan, which is, a feasible ordering plan that minimizes the sum of set-up, production, inventory holding and backlogging costs. Although the model assumes that companies produce their demand, we can interchangeably consider the case where demand is satisfied either by producing or purchasing. One has simply to interpret that the purchasing costs can be ordering costs (set-up costs) and unit purchasing costs (variable costs). The goal is to establish an operational plan in order to satisfy demand at a minimum total cost. The notation of the parameters and decision variables of the model are described in Table 1.

Table 1. Notation table.

T	number of periods in the planning horizon
d_t	demand during period t
p_t	unit production costs in period t
h_t	unit inventory carrying costs in period t
b_t	unit backlogging carrying costs in period t
k_t	set-up cost in period t
$M = \sum_{t=1}^{T} d_t$	upper bound on the production
Decision variables	
q_t	production during period t
I_t	inventory at the end of the period t
E_t	backlogged demand at the end of period t
z_t	a binary variable that assumes value 1 if an order is placed at the beginning of period t and 0, otherwise

In the following, we recall the mathematical programming formulation for the set-up-inventory problem (SI-problem). The reader is referred to [3,20] for a detailed discussion of this model. We denote, by $C(d,k,h,b,p)$, the minimum overall operating cost during the planning horizon, then

$$
\begin{aligned}
C(d,k,h,b,p) := \quad & \min \quad \sum_{t=1}^{T} (p_t q_t + h_t I_t + b_t E_t + k_t z_t) \\
& \text{s.t.} \quad I_0 = I_T = E_0 = E_T = 0, \\
& \qquad I_t - E_t = I_{t-1} - E_{t-1} + q_t - d_t, \quad \forall t \in \{1,\dots,T\}, \\
& \qquad q_t \le M \cdot z_t, \quad \forall t \in \{1,\dots,T\}, \\
& \qquad q_t,\ I_t,\ E_t, \text{ non-negative, integer}, \ \forall t \in \{1,\dots,T\}, \\
& \qquad z_t \in \{0,1\},\ \forall t \in \{1,\dots,T\}.
\end{aligned}
$$

The above objective function minimizes the sum of all the considered costs over the planning horizon. The first constraint imposes that the model must start and finish with an empty inventory. The second group of constraints are flow conservation constraints that ensure the correct transition of inventory and backlogged demand among periods. The third group of constraints model that set-up cost is only charged whenever an order is placed. Zangwill [21] proved that there is an optimal solution to $C(d,k,h,b,p)$, such that, for each $t = 1,\dots,T$, $E_t > 0$ and $I_t > 0$, simultaneously; that is, there is always an optimal solution that fulfills the constraints in the formulation of the problem. Therefore, that formulation provides optimal plans to the problem. Actually, this property and formulations were already proposed by [3]. Obviously, it does not mean that other optimal plans dividing the production may not exist, but restriction to these plans is sufficient for achieving an optimal solution.

In order to simplify the notation, we define Z as a matrix in which all costs are included, which is, $Z := (K,H,B,P)$ being K,H,B and P the matrices containing the set-up, inventory carrying, backlogging and unit production costs for all periods $t = 1,\dots,T$. A cooperative cost TU-game is an ordered pair (N,c), where N is a finite set, the set of players, and the characteristic function c is a function from $\mathcal{P}(N)$ to $\mathbb{R}$ with $c(\varnothing) = 0$, where $\mathcal{P}(N)$ is the power set of N (i.e., the set of coalitions in N). We use $x(S)$ to denote $\sum_{i\in S} x_i$ with $x(\varnothing) = 0$ for any cost allocation $x \in \mathbb{R}^n$ and for every coalition $S \subseteq N$.

For each SI-situation represented by its cost matrices (N,D,Z), we associate a cost TU-game (N,c), where, for any nonempty coalition $S \subseteq N, c(S) := C(d^S,k^S,h^S,b^S,p^S)$ with $d^S = \sum_{i\in S} d^i$, where $d^i = \left(d^i_1,\dots,d^i_T\right)$, and the rest of the costs will be the minimum value among all the costs of the players in the coalition S at each one of the periods, serve as an example $p^S = [p^S_1,\dots,p^S_T]'$, where $p^S_t = \min_{i\in S}\{p^i_t\}$ for $t = 1,\dots,T$. Subsequently, for each $S \subseteq N$:

$$
\begin{aligned}
c(S) := \quad & \min \quad \sum_{t=1}^{T} \left(p^S_t q_t + h^S_t I_t + b^S_t E_t + k^S_t z_t\right) \\
& \text{s.t.} \quad I_0 = I_T = E_0 = E_T = 0, \\
& \qquad I_t - E_t = I_{t-1} - E_{t-1} + q_t - d^S_t, \quad \forall t \in \{1,\dots,T\}, \\
& \qquad q_t \le M \cdot z_t, \quad \forall t \in \{1,\dots,T\}, \\
& \qquad q_t,\ I_t,\ E_t, \text{ non-negative, integer}, \ \forall t \in \{1,\dots,T\}, \\
& \qquad z_t \in \{0,1\},\ \forall t \in \{1,\dots,T\}.
\end{aligned}
$$

Every cost TU-game that is defined in this way is what we call a set-up-inventory game (SI-game). The reader may notice that every PI-game (as introduced by [9]) is a SI-game with $k_t = 0$, for all $t \in T$. Moreover, as mentioned above, although the model assumes that companies produce their demand, we can interchangeably consider the case where demand is satisfied by either producing or purchasing. One has simply to interpret that the purchasing costs can be ordering costs (set-up costs) and unit purchasing costs (variable costs). Recall that the core of a game (N,c) consists of those cost allocations that divide the cost of the grand coalition, $c(N)$, in a efficient way, so that no coalition has an incentive to break the consortium because its costs increase. Formally,

$$Core(N,c) = \{x \in \mathbb{R}^n / x(N) = c(N) \text{ and } x(S) \leq c(S) \text{ for all } S \subset N\}.$$

In the following, as is common in cooperative game theory, we call stable allocations the elements of the core. Bondareva [22] and Shapley [23] independently provide a general characterization of games with a nonempty core by means of balancedness. They prove that (N,c) has a nonempty core if and only if it is balanced. In addition, it is a totally balanced game (totally balanced games were introduced by Shapley and Shubik in the study of market games (see [24])) if the core of every subgame is nonempty.

Our goal is to show that SI-games are totally balanced. To do so, we use an easy proof that is based on duality resorting to a result by [3]. Observe that the characteristic function $c(S)$ of these games can be written as the optimal value of the following $LSI(S)$ problem:

$$\begin{aligned} c(S) = \max & \sum_{1\leq t\leq T}\sum_{1\leq\tau\leq T} d^S_\tau p^S_{t\tau}\lambda_{t\tau} + \sum_{1\leq t\leq T} k^S_t z_t \qquad (LSI(S)) \\ \text{s.t. } & d^S_\tau \sum_{1\leq t\leq T} \lambda_{t\tau} = d^S_\tau, \quad \forall \tau \in \{1,\dots,T\}, \\ & \lambda_{t\tau} \leq z_t, \quad \forall t,\tau \in \{1,\dots,T\}, \\ & \lambda_{t\tau}, z_t \in \{0,1\}, \forall t,\tau \in \{1,\dots,T\}. \end{aligned}$$

The variables $\lambda_{t\tau}$ are equal to 1, if and only if demand in period τ is produced in period t and zero otherwise. Likewise, the variables z_t are equal to 1 if and only if there is some production at period t. The cost of covering the demand in period τ if the production is done in period t is given by

$$p^S_{t\tau} = \begin{cases} p^S_t & \text{if } t = \tau, \\ p^S_t + \sum_{i=t}^{\tau-1} h^S_i & \text{if } t < \tau, \\ p^S_t + \sum_{i=\tau}^{t-1} b^S_i & \text{if } t > \tau. \end{cases} \qquad (1)$$

This is the facility location reformulation by [3] of the SI problem. This formulation has a strong dual if the underlying graph of the location problem is a tree ([25]). In this case, the graph is a line and, thus, the mentioned result applies. Let y_τ be the dual variable that is associated with the first constraints and $\beta_{t\tau}$ those that are associated with the second family of constraints, then the dual is:

$$\begin{aligned} c(S) = \max & \sum_{1\leq\tau\leq T} d^S_\tau y_\tau \qquad (DSI(S)) \\ \text{s.t. } & \sum_{1\leq\tau\leq T} \beta_{t\tau} \leq k^S_t, \forall t \in \{1,\dots,T\}, \\ & d^S_\tau y_\tau - \beta_{t\tau} \leq d^S_\tau p^S_{t\tau}, \quad \forall t,\tau \in \{1,\dots,T\}, \\ & \beta_{t\tau} \geq 0, \ \forall t,\tau \in \{1,\dots,T\}, \\ & y_t \text{ free}, \forall t \in \{1,\dots,T\}. \end{aligned}$$

Reference [3] proved that the linear relaxation of a SI-problem, $LSI(S)$, has an integral optimal solution. Hence, the optimal value of its dual problem matches that of the primal one, which is, $v(DSI(S)) = C(d^S, k^S, h^S, b^S, p^S) = c(S)$ for all $S \subseteq N$.

Theorem 1. *Every SI-game is totally balanced.*

Proof. Take a SI-situation (N, D, Z) and the associated SI-game (N, c). Consider (y^*, β^*) to be an optimal solution to dual $DSI(N)$, where $y^* = (y_1^*, ..., y_T^*)$ and $\beta^* = (\beta_{11}^*, ..., \beta_{TT}^*)$. It is known from optimality that

$$\sum_{t=1}^{T} y_t^* d_t^N = v(DSI(N)) = c(N)$$

Note that the solution (y^*, β^*) is also feasible for any dual problem with $S \subseteq N$, since $p^N \leq p^S, h^N \leq h^S, b_t^N \leq b_t^S$ and $k_t^N \leq k_t^S$. Therefore,

$$\sum_{t=1}^{T} y_t^* d_t^S \leq v(DSI(S)) = c(S)$$

Thus, the allocation $(\sum_{t=1}^{T} y_t^* d_t^i)_{i \in N} \in Core(N, c)$.

Note that every subgame of a SI-game is also a SI-game. Hence, we can also conclude that every SI-game is totally balanced. □

4. Unitary Owen Points

In this section, we introduce a new family of cost allocations on the class of SI-games. This family is inspired by the flavour of the Owen point and its relationship with the shadow prices of the dual problems that are associated with SI-problems. To define those cost allocations, it is necessary to describe the set of optimal plans and the unitary prices.

Consider a SI-situation (N, D, Z). A feasible ordering plan for such a situation is defined by $\sigma \in \mathbb{R}^T$, where $\sigma_t \in T \cup \{0\}$ denotes the period where the demand of period t is ordered. We assume the convention that $\sigma_t = 0$ if and only if $d_t = 0$. It means that no order can be placed to satisfy demand at period t, since demand is null there. Moreover, $P^S(\sigma) \in \mathbb{R}^T$ is defined as the operating cost vector that is associated to the ordering plan σ (henceforth: cost-plan vector) for any coalition $S \subseteq N$, where

$$P_t^S(\sigma) = \begin{cases} 0 & \text{if} \quad \sigma_t = 0, \\ p_{\sigma_t t}^S & \text{if} \quad \sigma_t \in \{1, ..., T\}. \end{cases}$$

Given an optimal ordering plan, σ^S, for the SI-problem $C(d^S, k^S, h^S, b^S, p^S)$, the characteristic function is rewritten, as follows: for any non-empty coalition $S \subseteq N$,

$$c(S) = P^S(\sigma^S)' d^S + \delta(\sigma^S)' k^S = \sum_{t=1}^{T} \Big(P_t^S(\sigma^S) d_t^S + \delta_t(\sigma^S) k_t^S \Big),$$

where, $\delta(\sigma^S) = \big(\delta_t(\sigma^S)\big)_{t \in T}$ and

$$\delta_t(\sigma^S) = \begin{cases} 1 & \text{if } \exists r \in T / \sigma_r^S = t\,, \\ 0 & \text{otherwise.} \end{cases}$$

The set of optimal plans is denoted by $\Lambda(N, D, Z) := \Big\{ (\sigma^S)_{S \in \mathcal{P}(N)} \Big\}$ where σ^S is an optimal ordering plan that is associated to $LSI(S)$. Note that the set of optimal plans may be large, since there are often multiple optimal solutions for the program $LSI(S)$. Core allocations that are built from optimal dual variables are known to exhibit some questionable properties, as pointed out, for instance, by [26] or [27]. For this reason, whenever the core is larger than the set of allocations coming from dual variables, it is interesting to provide some alternative core allocations. In the following, we derive alternatives that, under mild conditions, are stable, i.e., core allocations for these situations.

We define the unitary prices as the sum of the production, inventory, and backlogging costs plus a proportion of the fixed order cost which depends on the total demand satisfied in each period.

Definition 1. *Let (N, D, Z) be a SI-situation and $\left(\sigma^S\right)_{S\in\mathcal{P}(N)} \in \Lambda(N, D, Z)$. For each period $t \in T$ and each coalition $S \subseteq N$, the unitary price is defined, as follows:*

$$y_t\left(\sigma^S, d^S, z^S\right) := \begin{cases} 0 & \text{if} \quad \sigma_t^S = 0, \\ P_t^S(\sigma^S) + \frac{k^S_{\sigma_t^S}}{\sum_{m\in Q^S(\sigma_t^S)} d_m^S} & \text{if} \quad \sigma_t^S \neq 0, \end{cases}$$

where $Q^S(t) := \left\{k \in T : \sigma_k^S = t\right\}$ and z^S represent the cost matrix (k^S, h^S, b^S, p^S).

The reader should observe that $Q^S(t)$ is the set of periods that satisfy the demand in period t, for the optimal plan σ^S. Note that, for any coalition $S \subseteq N$, $\sum_{t=1}^T y_t\left(\sigma^S, d^S, z^S\right) \cdot d_t^S = c(S)$.

The next proposition shows that we may construct core allocations from the unitary prices of the grand coalition, as long as they are the cheapest in each period with positive demand. We shall call them unitary Owen points.

Definition 2. *Let (N, D, Z) be a SI-situation and $\left(\sigma^S\right)_{S\in\mathcal{P}(N)} \in \Lambda(N, D, Z)$. The unitary Owen point is given by*

$$\theta\left(\sigma^N, d^N, z^N\right) := \left(\sum_{t=1}^T y_t\left(\sigma^N, d^N, z^N\right) \cdot d_t^i\right)_{i\in N}.$$

Note that every optimal plan generates a unit price for each period of time and, hence, a unitary Owen point.

Observe that, from $y(\sigma^N, d^N, z^N)$, we can build a solution $(y(\sigma^N, d^N, z^N), \beta(\sigma^N))$ with $\beta_{\sigma^N(\tau),\tau} = 0$ if $\sigma^N(\tau) = 0$ and $\beta_{\sigma^N(\tau),\tau} = \frac{k^N_{\sigma^N(\tau)}}{\sum_{m\in Q^N(\sigma^N(\tau))} d_m^N}$ if $\sigma^N(\tau) \neq 0$ satisfying $c(N) = \sum_{\tau=1}^T d_\tau^N y_t(\sigma^N, d^N, z^N)$. However, it may not be a feasible solution of the dual for the grand coalition whenever $P_\tau^N(\sigma^N(\tau)) > p_{t\tau}^N$ for some t. Still, the unitary Owen point that is associated with this dual solution can be a core allocation.

The following example elaborates on a SI-situation with three players and two periods. The unitary Owen point for the corresponding SI-game is a core allocation, but this allocation does not come from optimal dual prices.

Example 1. *Consider the following SI-situation with two periods and three players and the associated SI-game, as shown in Table 2:*

Table 2. SI-situation with two periods and three players and the associated SI-game.

	d_1^S	d_2^S	p_1^S	p_2^S	h_1^S	b_1^S	k_1^S	k_2^S	c
$\{1\}$	2	1	9	9	6	4	6	8	39
$\{2\}$	8	2	9	6	9	7	7	9	100
$\{3\}$	6	1	5	6	3	5	6	10	44
$\{1,2\}$	10	3	9	6	6	4	6	8	122
$\{1,3\}$	8	2	5	6	3	4	6	8	62
$\{2,3\}$	14	3	5	6	3	5	6	9	100
$\{1,2,3\}$	16	4	5	6	3	4	6	8	118

The optimal plan for coalition N is $\sigma^N = (1, 1)$ with $p^N(\sigma^N) = (5, 8)$ and $y\left(\sigma^N, d^N, z^N\right) = \left(5 + \frac{3}{10}, 8 + \frac{3}{10}\right)$. The unitary Owen point $\theta\left(\sigma^N, d^N, z^N\right) = \left(18 + \frac{9}{10}, 59, 40 + \frac{1}{10}\right) \in Core(N, c)$. Note that $(y(\sigma^N, d^N, z^N), \beta(\sigma^N))$ with $\beta_{21}(\sigma^N) = \frac{3}{10}, \beta_{22}(\sigma^N) = \frac{3}{10}$ and $\beta_{t\tau}(\sigma^N) = 0$ for all the remaining t and τ, is not feasible for the dual problem $DSI(N)$. Indeed, it violates the constraint $d_2^N y_2(\sigma^N, d^N, z^N) - \beta_{22}(\sigma^N) \leq d_2^N p_{22}^N$, since this is equivalent to $32 + \frac{9}{10} \leq 24$.

Therefore, it is clear that the unitary Owen point can provide core allocation, which does not come from optimal dual prices, although it is not clear under which conditions this unitary price fulfills this property. The following result provides an easy sufficient condition for this to happen.

Proposition 1. *Let (N, D, Z) be a SI-situation, $\left(\sigma^S\right)_{S\in\mathcal{P}(N)} \in \Lambda(N, D, Z)$, and (N, c) the associated SI-game. If $y_t\left(\sigma^N, d^N, z^N\right) \leq y_t\left(\sigma^S, d^S, z^S\right)$ for all $t \in T$ and for all $S \subset N$ with $d_t^S \neq 0$, then*

$$\theta\left(\sigma^N, d^N, z^N\right) \in Core(N, c).$$

Proof. It is straightforward from the definition of the unitary Owen point. □

It would be reasonable that, the larger a coalition, the lower its unit prices, since its members operate with the best technology available in the group. Unfortunately, this condition is not always satisfied as Example 3 shows. Therefore, we are interested in finding stronger conditions than the one that is given in Proposition 1. This question is addressed below.

In order to simplify the notation, for each $t \in T$, we define:

- Cost difference per demand unit between coalition S and R in a period t:

$$a_t^{SR} := P_t^S(\sigma^S) - P_t^R(\sigma^R).$$

  Note that $a_t^{SR} + a_t^{RS} = 0$.
- Aggregate demand of coalition $S \subseteq N$ in all of those periods that satisfy its demand in period t:

$$\alpha_t(S) := \sum_{m\in Q^N(t)} d_m^S.$$

- Aggregate order cost of coalition $S \subseteq N$:

$$k(S) := \sum_{t\in T^S} k_t^S,$$

  where $T^S := \left\{t \in T\ \delta_t(\sigma^S) = 1\right\}$ is the set of ordering periods.

The next theorem provides the necessary and sufficient conditions for the unitary Owen point to be a core allocation. These conditions state an upper bound for the average cost savings per unit demand in the grand coalition, for those periods where an order is placed. Such an upper bound is related to the savings in fixed order costs.

Theorem 2. *Let (N, D, Z) be a SI-situation, $\left(\sigma^S\right)_{S\in\mathcal{P}(N)} \in \Lambda(N, D, Z)$, and (N, c) the associated SI-game. $\theta\left(\sigma^N, d^N, z^N\right) \in Core(N, c)$ if and only if there are real weights β_t^S, for any $S \subsetneq N$ and any $t \in T^N$ with $\alpha_t(S) > 0$, satisfying that*

$$\sum_{j\in Q^N(t)} \frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \leq \beta_t^S \cdot \frac{k(S)}{\alpha_t(S)} - \frac{k_t^N}{\alpha_t(N)}$$

with $\sum_{t\in T^N:\alpha_t(S)>0} \beta_t^S \leq 1$.

Proof. (if) Take $\left(\sigma^S\right)_{S\in\mathcal{P}(N)} \in \Lambda(N, D, Z)$ and consider a coalition $S \subsetneq N$. We must prove that $\theta\left(\sigma^N, d^N, z^N\right) \in Core(N, c)$, e.g., $\sum_{i\in S} \theta_i\left(\sigma^N, d^N, z^N\right) - c(S) \leq 0$. Indeed,

$$\sum_{i\in S} \theta_i\left(\sigma^N, d^N, z^N\right) - c(S)$$

$$= \sum_{t=1}^{T} y_t\left(\sigma^N, d^N, z^N\right) \cdot d_t^S - \sum_{t=1}^{T} y_t\left(\sigma^S, d^S, z^S\right) \cdot d_t^S$$
$$= \sum_{t=1}^{T} \left(a_t^{NS} \cdot d_t^S + \frac{k_{\sigma_t^N}^N \cdot d_j^S}{\sum_{m \in Q^N(\sigma_t^N)} d_m^N} - \frac{k_{\sigma_t^S}^S \cdot d_j^S}{\sum_{m \in Q^S(\sigma_t^S)} d_m^S} \right)$$
$$= \sum_{t \in T^N} \left(\sum_{j \in Q^N(t)} \left(a_j^{NS} \cdot d_j^S + \frac{k_t^N \cdot d_j^S}{\alpha_t(N)} \right) \right) - k(S)$$
$$= \sum_{t \in T^N : \alpha_t(S) > 0} \left(\alpha_t(S) \cdot \sum_{j \in Q^N(t)} \left(\frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \right) + \frac{k_t^N \cdot \alpha_t(S)}{\alpha_t(N)} \right) - k(S)$$
$$\leq \sum_{t \in T^N : \alpha_t(S) > 0} \left(\beta_t^S \cdot \frac{\alpha_t(S) \cdot k(S)}{\alpha_t(S)} - \frac{\alpha_t(S) \cdot k_t^N}{\alpha_t(N)} + \frac{k_t^N \cdot \alpha_t(S)}{\alpha_t(N)} \right) - k(S)$$
$$= k(S) \cdot \sum_{t \in T^N : \alpha_t(S) > 0} \beta_t^S - k(S) \leq 0$$

(only if) Consider now that $\theta(\sigma^N, d^N, z^N) \in Core(N, c)$. Then, for all $S \subset N$, $\sum_{i \in S} \theta_i(\sigma^N, d^N, z^N) - c(S) \leq 0$ which is equivalent to

$$\sum_{t \in T^N} \left(\sum_{j \in Q^N(t)} \left(a_j^{NS} \cdot d_j^S + \frac{k_t^N \cdot d_j^S}{\alpha_t(N)} \right) \right) \leq k(S).$$

For all $t \in T^N$ and every coalition $S \subsetneq N$ with $\alpha_t(S) > 0$, there are always real weights β_t^S with $\sum_{t \in T^N} \beta_t^S \leq 1$, satisfying

$$\sum_{j \in Q^N(t)} \left(a_j^{NS} \cdot d_j^S + \frac{k_t^N \cdot d_j^S}{\alpha_t(N)} \right) \leq \beta_t^S \cdot k(S),$$
$$\sum_{j \in Q^N(t)} \left(\frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \right) + \frac{k_t^N \cdot \alpha_t(S)}{\alpha_t(N) \cdot \alpha_t(S)} \leq \frac{\beta_t^S \cdot k(S)}{\alpha_t(S)},$$
$$\sum_{j \in Q^N(t)} \frac{a_j^{NS} \cdot d_j^S}{\alpha_t(S)} \leq \beta_t^S \cdot \frac{k(S)}{\alpha_t(S)} - \frac{k_t^N}{\alpha_t(N)}.$$

□

The next example illustrates Proposition 1 and Theorem 2. It shows how unitary Owen points are calculated by using unitary prices.

Example 2. *Consider the following SI-situation with three periods and three players with the associated SI-game, as shown in Table 3:*

Table 3. SI-situation with three periods and three players with the associated SI-game.

	d_1^s	d_2^s	d_3^s	p_1^s	p_2^s	p_3^s	h_1^s	h_2^s	b_1^s	b_2^s	k_1^s	k_2^s	k_3^s	c
{1}	1	3	1	1	1	1	1	1	1	1	3	1	5	8
{2}	2	1	1	2	3	4	1	1	1	1	1	4	8	12
{3}	2	1	3	2	3	5	1	1	1	1	1	1	7	20
{1,2}	3	4	2	1	1	1	1	1	1	1	1	1	5	13
{1,3}	3	4	4	1	1	1	1	1	1	1	1	1	5	17
{2,3}	4	2	4	2	3	4	1	1	1	1	1	1	7	31
{1,2,3}	5	5	5	1	1	1	1	1	1	1	1	1	5	22

An optimal plan is given by Table 4:

Table 4. An optimal plan for the SI-situation.

	σ_1^S	σ_2^S	σ_3^S	$P_1^S(\sigma^S)$	$P_2^S(\sigma^S)$	$P_3^S(\sigma^S)$	$k(S)$
$\{1\}$	2	2	2	2	1	2	1
$\{2\}$	1	1	1	2	3	4	1
$\{3\}$	1	1	1	2	3	4	1
$\{1,2\}$	1	2	2	1	1	2	2
$\{1,3\}$	1	2	2	1	1	2	2
$\{2,3\}$	1	1	1	2	3	4	1
$\{1,2,3\}$	1	2	2	1	1	2	2

Thus, the unitary prices for the optimal plan above are described in Table 5:

Table 5. The unitary prices associated to the optimal plan above.

	$y_1(\sigma^S, d^S, z^S)$	$y_2(\sigma^S, d^S, z^S)$	$y_3(\sigma^S, d^S, z^S)$
$\{1\}$	$2+\frac{1}{5}$	$1+\frac{1}{5}$	$2+\frac{1}{5}$
$\{2\}$	$2+\frac{1}{4}$	$3+\frac{1}{4}$	$4+\frac{1}{4}$
$\{3\}$	$2+\frac{1}{6}$	$3+\frac{1}{6}$	$4+\frac{1}{6}$
$\{1,2\}$	$1+\frac{1}{3}$	$1+\frac{1}{6}$	$2+\frac{1}{6}$
$\{1,3\}$	$1+\frac{1}{3}$	$1+\frac{1}{8}$	$2+\frac{1}{8}$
$\{2,3\}$	$2+\frac{1}{10}$	$3+\frac{1}{10}$	$4+\frac{1}{10}$
$\{1,2,3\}$	$1+\frac{1}{5}$	$1+\frac{1}{10}$	$2+\frac{1}{10}$

One can observe that $y_t(\sigma^N, d^N, z^N) \leq y_t(\sigma^S, d^S, z^S)$ for all $t \in T$ and, so, by Proposition 1, $\theta(\sigma^N, d^N, z^N) = (6.6, 5.6, 9.8) \in Core(N, c)$.

On the other hand, the ordering plan for the grand coalition $\tilde{\sigma}^N = (1,2,3)$ belongs to an optimal plan and the associated unit prices are $y_1(\tilde{\sigma}^N, d^N, z^N) = 1 + \frac{1}{5}$, $y_2(\tilde{\sigma}^N, d^N, z^N) = 1 + \frac{1}{5}$ and $y_3(\tilde{\sigma}^N, d^N, z^N) = 1 + \frac{5}{5}$. Note that for this plan $T^N = \{1,2,3\}$. Theorem 2 is applied here for the weights that are given in Table 6.

Table 6. Weights associated to the optimal plan.

	$\beta_1^S \geq$	$\beta_2^S \geq$	$\beta_3^S \geq$	$\sum_{t \in T^N} \beta_t^S$
$\{1\}$	$\frac{-4}{5}$	$\frac{3}{5}$	0	$\frac{-1}{5}$
$\{2\}$	$\frac{-8}{5}$	$\frac{-9}{5}$	-2	$\frac{-27}{5}$
$\{3\}$	$\frac{-8}{5}$	$\frac{-9}{5}$	-6	$\frac{-47}{5}$
$\{1,2\}$	$\frac{3}{10}$	$\frac{4}{10}$	0	$\frac{7}{10}$
$\{1,3\}$	$\frac{3}{10}$	$\frac{4}{10}$	0	$\frac{7}{10}$
$\{2,3\}$	$\frac{-16}{5}$	$\frac{-18}{5}$	-8	$\frac{-27}{5}$

Hence, it follows that $\theta(\tilde{\sigma}^N, d^N, z^N) = (6'8, 5'6, 9'6) \in Core(N, c)$.

This section is completed with a third example showing that, if any of the conditions either of the Proposition 1 or Theorem 2 fail, the unitary Owen points are no longer core allocations.

Example 3. *Now, consider the following SI-situation with three periods, two players, and the associated 2-player SI-game, given by Table 7:*

Table 7. SI-situation with three periods and two players with the associated SI-game.

	d_1^S	d_2^S	d_3^S	p_1^S	p_2^S	p_3^S	h_1^S	h_2^S	b_1^S	b_2^S	k_1^S	k_2^S	k_3^S	c
$\{1\}$	0	10	10	1	1	1	1	1	1	1	1	50	15	46
$\{2\}$	0	35	0	1	1	1	1	1	1	1	1	50	15	71
$\{1,2\}$	0	45	10	1	1	1	1	1	1	1	1	50	15	115

There is a single optimal ordering plan that is shown in Table 8:

Table 8. An optimal plan for SI-situation.

	σ_1^S	σ_2^S	σ_3^S	$P_1^S(\sigma^S)$	$P_2^S(\sigma^S)$	$P_3^S(\sigma^S)$	$k(S)$
$\{1\}$	0	1	3	0	2	1	16
$\{2\}$	0	1	0	0	2	0	1
$\{1,2\}$	0	2	2	0	1	2	50

The unitary prices for the optimal plan above are described in Table 9:

Table 9. The unitary prices associated to the optimal plan above.

	$y_1(\sigma^S, d^S)$	$y_2(\sigma^S, d^S)$	$y_3(\sigma^S, d^S)$
$\{1\}$	0	$2+\frac{1}{10}$	$1+\frac{15}{10}$
$\{2\}$	0	$2+\frac{1}{35}$	0
$\{1,2\}$	0	$1+\frac{50}{55}$	$2+\frac{50}{55}$

Note that $\theta(\sigma^N, d^N, z^N) = \left(30+\frac{200}{11}, 35+\frac{350}{11}\right) = (48'\widehat{18}, 66'\widehat{81})$ *is not a core allocation. Theorem 2 fails here, because* $T^N = \{2\}$ *and* $\beta_2^{\{1\}} \geq \frac{5}{4}$.

Numerical Experiments

At first glance, the reader might think that the conditions of Theorem 2 are too restrictive, i.e., they are only satisfied by a small family of SI-situations. However, an empirical analysis simulating SI-situations shows that most of the instances satisfy those conditions. Indeed, we start by randomly generating (using the uniform probability distribution) a first set of 100,000 instances of SI-situations, so that, for every player and for each period, the data range in $d_t^i \in [0,30]$, $p_t^i, h_t^i, b_t^i \in [0,10]$, and $k_t^i \in [0,50]$. Table 10 shows the percentage of SI-situations for which the Unitary Owen point belongs to the core of the corresponding SI-game.

Table 10. Percentage of instances fulfilling the condition of Theorem 2 for the first set of instances.

Players	$T=2$	$T=3$	$T=4$	$T=5$
2	99.934%	99.979%	99.993%	100%
3	99.942%	99.983%	99.989%	99.995%
4	99.950%	99.991%	99.996%	99.999%
5	99.974%	99.982%	99.992%	99.998%
6	99.974%	99.993%	99.998%	99.999%
7	99.985%	99.996%	99.999%	100%

It can be seen that the the larger the number of players and periods, the higher the percentage that some unitary Owen point belongs to the core. In case that we impose that the demand and the costs are greater than zero: $d_t^i \in [1,30]$, $p_t^i, h_t^i, b_t^i \in [1,10]$, and $k_t^i \in [1,50]$, the results even improve significantly, as Table 11 shows.

In the previous simulation, the range of variation for the costs have been chosen, so that those costs are actually relevant in determining the optimal plans for each coalition. In addition, if the the set-up costs are large when compared to the other costs, as, for instance, for $d_t^i \in [0,10]$, $p_t^i, h_t^i, b_t^i \in [0,10]$ and $k_t^i \in [50,500]$ the percentage of instances where the unitary Owen point is a core allocation is close to 99.995%, even for the case of two players and two periods. Moreover, if the demand is larger, as happens in the following situation $d_t^i \in [10,50]$, $p_t^i, h_t^i, b_t^i \in [0,10]$ and $k_t^i \in [0,50]$, percentages of "success" also increase close to 1 (99.999%).

Table 11. Percentage of instances fulfilling the condition of Theorem 2 for instances with positive costs.

Players	$T = 2$	$T = 3$	$T = 4$	$T = 5$
2	99.984%	99.996%	99.999%	100%
3	99.997%	99.995%	99.999%	99.999%
4	99.998%	99.996%	99.999%	99.998%
5	100%	99.999%	100%	100%
6	100%	100%	100%	100%

5. SI-Games and PI-Games

To complete the paper, we provide a relationship between a generic SI-game and a specific family of PI-games through Owen's points of the latter. We use Owen points from an *ad hoc* family of PI-situations constructed from core allocations of the so-called surplus game, which measures the excess in costs that occurs with respect to the minimum unit price. This interesting relationship simplifies the analysis and construction of core allocations for SI-games.

First, we introduce the minimum unitary prices for every optimal plan. Denote, by $\Delta := \left(\sigma^S\right)_{S \in \mathcal{P}(N)}$, an optimal plan in $\Lambda(N, D, Z)$.

Definition 3. *Let (N, D, Z) be a SI-situation. The minimum unitary price for Δ, in each period $t \in T$, is*

$$y_t^*(\Delta) = \min_{\substack{S \subseteq N \\ d_t^S \neq 0}} \{y_t\left(\sigma^S, d^S, z^S\right)\}.$$

Second, for each coalition, we measure the excess in costs that occurs with respect to the minimum unit prices. The resulting cost game is what we have called surplus game.

Definition 4. *Let (N, D, Z) be a SI-situation and (N, c) the associated SI-game. For any $\Delta \in \Lambda(N, D, Z)$, the surplus game (N, c^Δ) is defined for all $S \subseteq N$, as*

$$c^\Delta(S) := c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S.$$

Note that the surplus game is a non-negative cost game that measures the increase in costs by the influence of set-up costs. The first result of this section shows that surplus games are always balanced.

Proposition 2. *Every surplus game (N, c^Δ) is balanced.*

Proof. It follows from Theorem 1 that every SI-game (N, c) is balanced. Take a core allocation $x \in \mathbb{R}^N$ for it. For each $S \subset N$, it holds that

$$\begin{aligned} x(S) \quad \leq \quad & c(S) \Longleftrightarrow x(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \leq c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \\ \Longleftrightarrow \quad & x(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \leq c^{\Delta}(S) \\ \Longleftrightarrow \quad & \sum_{i \in S} \left(x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right) \leq c^{\Delta}(S). \end{aligned}$$

Moreover $x(N) = c(N)$, which easily implies that $\sum_{i \in N} \left(x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right) = c^{\Delta}(N)$. Hence, we conclude that $\left(x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c^{\Delta})$. □

In the following, we use this game to construct core allocations for SI-games by means of the Owen points of the surplus game, which is an easy PI-game. The next result provides a necessary and sufficient condition for this purpose: the set-up costs cannot contribute to any increase in costs for the grand coalition. In other words, there are no costs exceeding the unit prices of the grand coalition.

Proposition 3. *Let (N, c) be a SI-game. For any $\Delta \in \Lambda(N, D, Z)$,*

$$c^{\Delta}(N) = 0 \textit{ if and only if } \left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c).$$

Proof. (If) If $c^{\Delta}(N) = 0$ then $\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^N = c(N)$. For each $S \subset N$, $\sum_{i \in S} \left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)$ $= \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S \leq \sum_{t=1}^{T} y_t(\sigma^S, d^S) \cdot d_t^S = c(S)$. Thus, $\left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c)$. (Only if) If $\left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right)_{i \in N} \in Core(N, c)$, it is satisfied that $\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^N = c(N)$, and, so, $c^{\Delta}(N) = 0$. □

The main theorem in this section shows that the core of any SI-game consists of the Owen points of certain PI-games that were obtained from core allocations of surplus games. To state this theorem, it is necessary to describe a procedure for constructing a PI-situation from core allocations of surplus games.

Consider a SI-situation (N, D, Z), the associated SI-game (N, c), and the surplus game (N, c^{Δ}), for $\Delta \in \Lambda(N, D, Z)$. For any $\alpha \in Core(N, c^{\Delta})$, $(N, \overline{D}(\alpha), \overline{Z})$ is a PI-situation with $\overline{Z} = (\overline{K}, \overline{H}, \overline{B}, \overline{P})$ and $\overline{D}(\alpha) = [\overline{d}^1, \ldots, \overline{d}^n]'$, $\overline{K} = 0$, $\overline{H} = [M, \ldots, M]'$, $\overline{B} = [M, \ldots, M]'$, $\overline{P} = [\overline{p}, \ldots, \overline{p}]'$, with $\overline{p} = (y_1^*(\Delta), ..., y_T^*(\Delta), 1)$, $\overline{d}^i = (d_1^i, ..., d_T^i, \alpha_i)$ for all $i \in N$ and $M \in \mathbb{R}^N$ large enough. This procedure shows that any SI-situation can be transformed into multiple PI-situations just by using the core of the surplus games.

Theorem 3. *Let (N, c) be a SI-game and (N, c^{Δ}) the associated surplus game for $\Delta \in \Lambda(N, D, Z)$. Thus,*

$$Core(N, c) = \left\{ Owen\left(N, \overline{D}(\alpha), \overline{Z}\right) : \alpha \in Core(N, c^{\Delta}) \right\}.$$

Proof. As (N, c^{Δ}) is balanced, there is at least one $\alpha \in \mathbb{R}^N$, such that $\alpha(S) \leq c^{\Delta}(S) = c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S$ for all $S \subset N$ and $\alpha(N) = c(N) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^N$. Consider a PI-situation $(N, \overline{D}(\alpha), \overline{Z})$ with $T + 1$ periods, where

$$\overline{D}(\alpha) = [\overline{d}^1, \ldots, \overline{d}^n]', \overline{K} = 0, \overline{H} = [M, \ldots, M]', \overline{B} = [M, \ldots, M]', \overline{P} = [\overline{p}, \ldots, \overline{p}]'$$

with $\overline{p} = (y_1^*(\Delta), \ldots, y_T^*(\Delta), 1)$, $\overline{d}^i = (d_1^i, \ldots, d_T^i, \alpha_i)$ for all $i \in N$ and $M \in \mathbb{R}^N$ large enough. For each $i \in N$, $Owen_i(N, \overline{D}(\alpha), \overline{Z}) = \sum_{t=1}^{T+1} y_t^*(N) d_t^i = \left(\sum_{t=1}^{T} y_t^*(\Delta) d_t^i\right) + \alpha_i$. Subsequently, for all $S \subset N$:

$$\begin{aligned}\sum_{i \in S} Owen_i(N, \overline{D}(\alpha), \overline{Z}) &= \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S + \alpha(S) \\ &\leq \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S + c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S = c(S).\end{aligned}$$

Moreover, $\sum_{i \in N} Owen_i(N, \overline{D}(\alpha), \overline{Z}) = \sum_{t=1}^{T} y_t^*(\Delta) d_t^N + \alpha(N) = \sum_{t=1}^{T} y_t^*(\Delta) d_t^N + c(N) - \sum_{t=1}^{T} y_t^*(\Delta) d_t^N = c(N)$. Thus $Owen(N, \overline{D}(\alpha), \overline{Z}) \in Core(N, c)$.

On the other hand, if $x \in Core(N, c)$, for each $S \subset N$, it holds that

$$\begin{aligned}x(S) &\leq c(S); \\ x(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S &\leq c(S) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S; \\ \sum_{i \in S} x_i - \sum_{i \in S} \left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^{\{i\}} \right) &\leq c^\Delta(S); \\ \sum_{i \in S} \left(x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^{\{i\}} \right) &\leq c^\Delta(S);\end{aligned}$$

Moreover, $x(N) = c(N) \Leftrightarrow \sum_{i \in N} \left(x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i \right) = c^\Delta(N)$. Thus, for each $x \in Core(N, c)$, we can take $\alpha_i := x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i$ for all $i \in N$, such that $\alpha \in Core(N, c^\Delta)$. From there, it easily follows that $Owen(N, \overline{D}(\alpha), \overline{Z}) = \left(\left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i\right) + \alpha_i\right)_{i \in N} = x$. □

We illustrate the procedure above with the Example 3 that is shown above.

Example 4. *Consider the two-player SI-game that is given in Example 3. We have shown that the unitary Owen point is not a core allocation for this example. It can be easily checked that the minimal unit prices are those shown in Table 12.*

Table 12. Minimal unit prices.

$y_1^*(\Delta)$	$y_2^*(\Delta)$	$y_3^*(\Delta)$
0	$1 + \frac{50}{55}$	$1 + \frac{15}{10}$

Thus, the surplus game is given by in Table 13

Table 13. The surplus game.

	c	c^Δ
$\{1\}$	46	$1 + \frac{10}{11}$
$\{2\}$	71	$4 + \frac{2}{11}$
$\{1,2\}$	115	$4 + \frac{1}{11}$

Consider a core allocation from the surplus game, for instance, the nucleolus $\eta(N, c^\Delta) = (\frac{10}{11}, 3\frac{2}{11})$. *We obtain a core allocation for the SI-game just by calculating the Owen point of the* *associated PI-situation* $\left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right)$. *Thus,* $Owen\left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right) = (45, 70)$. *It can be concluded that*

$$\eta(N, c) = Owen\Big(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\Big).$$

In the above example, the nucleolus of the surplus game leads to the nucleolus of the SI-game through the Owen point. The last result in the paper shows that this close relationship between both nucleoli always holds, e.g., the nucleolus of any SI-game matches the Owen point for the PI-situation that is obtained from the nucleolus of the surplus game.

Proposition 4. *Let* (N, D, Z) *be a SI-situation,* (N, c) *the associated SI-game, and* (N, c^{Δ}) *the surplus game for* $\Delta \in \Lambda(N, D, Z)$. *Thus,*

$$Owen\Big(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\Big) = \eta(N, c).$$

Proof. It is known that $x \in Core(N, c)$ if and only if $x^{\Delta} := \left(x_i - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i\right)_{i \in N} \in Core(N, c^{\Delta})$. Thus, the excess vectors, $e(S, x)$, and $e_{\Delta}(S, x^{\Delta})$ coincide.

For each coalition $S \subseteq N$, it holds that:

$$\begin{aligned} e(S, \eta(N, c)) &= c(S) - \sum_{i \in S} \eta_i(N, c) \\ &= c^{\Delta}(S) + \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S - \sum_{i \in S} \eta_i(N, c) \\ &= c^{\Delta}(S) - \sum_{i \in S} \left(\eta_i(N, c) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i\right). \end{aligned}$$

Therefore, $\eta_i(N, c^{\Delta}) = \eta_i(N, c) - \sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^i$ for all $i \in N$, because otherwise $\eta(N, c)$ would not be the nucleolus. Moreover,

$$\begin{aligned} c^{\Delta}(S) - \eta_i(N, c^{\Delta}) &= c(S) - \left(\sum_{t=1}^{T} y_t^*(\Delta) \cdot d_t^S + \sum_{i \in S} \eta_i(N, c^{\Delta})\right) \\ &= c(S) - \sum_{i \in S} Owen_i\Big(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\Big) \\ &= e(S, Owen\Big(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\Big)). \end{aligned}$$

This implies that $Owen\left(N, \overline{D}(\eta(N, c^{\Delta})), \overline{Z}\right) = \eta(N, c)$. □

6. Discussion

The study of cooperation in lot-sizing problems with backlogging and heterogeneous costs has been previously considered by [19]. The authors prove that there are always stable allocations of the overall operating cost among the firms, so that no group of agents benefit from leaving the consortium. They propose a parametric family of cost allocations and provide sufficient conditions for this to be a stable family against coalitional defections of firms and focus on those periods of the time horizon that are consolidated, analyzing their effect on the stability of cost allocations.

This paper completes the study of those cooperative lot-sizing models by presenting unitary Owen points. As mentioned, the Owen point works very well for constructing core-allocations in the class of PI-games. Unfortunately, no longer works for SI-problems. In spite of that, here we have managed to construct a particular kind of prices, which we call unitary prices, based on the production, inventory, and backlogging costs, and a proportion of the fixed order cost, which depends on the total demand satisfied in each period. These unit prices resemble the Owen point and allow to replicate its construction, so that these allocations "a la Owen" are called unitary Owen points. These quantities can be understood as approximate dual prices that allow pricing each firm resources, in order to distribute, in a stable manner, the overall operating costs. Necessary and sufficient conditions are provided for the unitary Owen points to belong to the core of the cooperative game. In addition, we provide empirical evidence, through simulation, showing that, in randomly generated situations, the above condition is fulfilled in 99% of cases. Finally, a relationship between lot-sizing games and a certain family of production-inventory games, through Owen's points of the latter, is described. This interesting relationship enables us to easily derive and interpret a variety of coalitionally stable allocations for cooperative lot-sizing models. The growing literature that is devoted to the study of cooperation in lot-sizing models shows that there are always ways to allocate the minimum cost that results from cooperation that are coalitionally stable. In addition, a few algorithms have been proposed to determine some of these allocations. The main contribution of this paper is that it presents an explicit cost allocation for cooperative lot-sizing models with backlogging and heterogeneous costs that is coalitional stable and consistent in 99% of cases.

Moreover, the analysis of cooperation in general lot-sizing models offers a vast field for future research. We propose the following directions for future research. The first direction is to find a set of properties the determine the unitary Owen point by means of an axiomatic characterization. The second direction is to consider lot-sizing models with capacity constraints, which is, to study capacitated lot-sizing models with backlogging. We believe that adding restrictions on the companies' production capacity could create incentives for them to compete with each other, in a first stage, and to cooperate later, once the production capacity has been decided. Finally, it would also be interesting to study the cooperation in lot-sizing models with limited information sharing. Full information sharing us a typical assumption in cooperative models. However, in supply chains settings, companies may not release all of their relevant information. Furthermore, it should be possible to design mechanisms to encourage the firms to release full and true information.

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