THE QUOTIENT ALGEBRA OF COMPACT-BY-APPROXIMABLE OPERATORS ON BANACH SPACES FAILING THE APPROXIMATION PROPERTY

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Abstract

We initiate a study of structural properties of the quotient algebra $\mathcal{K}(X)/\mathcal{A}(X)$ of the compact-by-approximable operators on Banach spaces X failing the approximation property. Our main results and examples include the following: (i) there is a linear isomorphic embedding from c_0 into $\mathcal{K}(Z)/\mathcal{A}(Z)$, where Z belongs to the class of Banach spaces constructed by Willis that have the metric compact approximation property but fail the approximation property, (ii) there is a linear isomorphic embedding from a non-separable space $c_0(\Gamma)$ into $\mathcal{K}(Z_{FJ})/\mathcal{A}(Z_{FJ})$, where Z_{FJ} is a universal compact factorisation space arising from the work of Johnson and Figiel.

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1. Introduction

For Banach spaces X and Y let $\mathcal{K}(X,Y)$ be the Banach space of compact operators $X \to Y$ in the uniform operator norm, and denote by $\mathcal{A}(X,Y) = \overline{\mathcal{F}(X,Y)}$ the closed subspace consisting of the approximable operators $X \to Y$. Here $\mathcal{F}(X,Y)$ is the linear subspace consisting of the bounded finite rank operators $X \to Y$. We put $\mathcal{K}(X) = \mathcal{K}(X,X)$ and $\mathcal{A}(X) = \mathcal{A}(X,X)$ for $Y = X$, so that $\mathcal{A}(X) \subset \mathcal{K}(X)$ are closed two-sided ideals of the Banach algebra $\mathcal{L}(X)$ of the bounded operators on X. Consequently the quotient algebra $\mathfrak{A}_X := \mathcal{K}(X)/\mathcal{A}(X)$ is a non-unital Banach algebra equipped with the quotient norm

$$\|T + \mathcal{A}(X)\| = dist(T, \mathcal{A}(X)) = \inf_{S \in \mathcal{A}(X)} \|T - S\|.$$

It is well known that $\mathcal{A}(X) = \mathcal{K}(X)$ whenever X has the approximation property, so that the quotient $\mathfrak{A}_X$ can only be non-zero within the class of Banach spaces X that fail to have the approximation property. It remains a longstanding problem, see [25, Problem 1.e.9] or [7, Problem 2.7], whether conversely $\mathcal{A}(X) = \mathcal{K}(X)$ will imply that X has the approximation property. Properties of the quotient algebras $\mathfrak{A}_X$ are rather elusive and inaccessible, not least because it is a non-trivial task to construct examples of Banach spaces failing the approximation property, and they have not been much studied. Recently Dales [10] revived the interest in various questions about the size and algebraic structure of $\mathfrak{A}_X$ by collecting results and highlighting a number of problems. It is known, see [9, 2.5.8(iv)], that $\mathfrak{A}_X$ is a radical Banach algebra for any complex Banach space X, that is, any quotient element $S + \mathcal{A}(X)$ is quasi-nilpotent in $\mathfrak{A}_X$. Consequently, from an algebraic perspective the non-trivial quotient algebras $\mathfrak{A}_X$ are natural (non-commutative) radical Banach algebras whose structure is very poorly understood.

It was shown by Bachelis [4] that if E is a Banach space which has the bounded approximation property and E contains a closed linear subspace X which fails the approximation property, then E contains a closed linear subspace Y such that $\mathcal{A}(Y, X) \subsetneq \mathcal{K}(Y, X)$. Moreover, if in addition $E \oplus E$ is isomorphic to E, then E contains the closed linear subspace $Z = Y \oplus X$ for which $\mathcal{A}(Z) \subsetneq \mathcal{K}(Z)$. This extended an earlier result due to Alexander [1] for the sequence spaces $E = \ell^p$ with $2 < p < \infty$.

In this paper we show that in fact the quotient algebra $\mathfrak{A}_X$ is infinite-dimensional for many Banach spaces X. Our results mostly draw on techniques from Banach space theory, which provide less information about the algebraic structure of $\mathfrak{A}_X$. On the other hand, most of our results are independent of the scalar field of X. In section 2 we review for our purposes two general constructions, which yield that the quotient algebra $\mathfrak{A}_X$ is infinite-dimensional for certain classes of Banach spaces. In section 3 we first observe that $\mathfrak{A}_X$ is infinite-dimensional for the class of Banach spaces X, where X has the bounded compact approximation property but fails the approximation property. The first examples of such Banach spaces were constructed by Willis [42], and our main result demonstrates more precisely that $\mathfrak{A}_Z$ always contains a linear isomorphic copy of the sequence space c_0 for the spaces Z from [42]. However, the linear embedding $c_0 \to \mathfrak{A}_Z$ does not preserve much of the algebraic structure. In section 4 we further show that $\mathfrak{A}_X$ is infinite-dimensional for two different universal spaces X. In the first example $X = C_1^*$, where C_1 is the complementably universal separable conjugate space constructed by Johnson [19]. In the second example we show that there is a linear isomorphic embedding of a non-separable sequence space $c_0(\Gamma)$ into $\mathfrak{A}_{Z_{FJ}}$, where Z_{FJ} is a universal compact factorisation space suggested by results of Johnson [18] and Figiel [13].

Recently there has been significant advances in the study of the lattice of closed two-sided ideals $I \subset \mathcal{L}(X)$ for some classical Banach spaces X, see e.g. [37], [14], [21], [20] and their references. Moreover, the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ has been explicitly determined for some special Banach spaces X, see e.g. [2], [40], [28] and [21]. However, the preceding results concern Banach spaces having a Schauder basis, or at least the bounded approximation property, which are disjoint classes of spaces from those relevant for the study of the quotients $\mathfrak{A}_X$.

2. General constructions

The standing assumption of this paper is that the scalar field of the underlying infinite-dimensional Banach space X is $\mathbb{R}$ or $\mathbb{C}$, unless otherwise specified, since $\mathfrak{A}_X = \mathcal{K}(X)/\mathcal{A}(X)$ is a Banach algebra in both cases. Note that for real scalars the quotient elements $S+\mathcal{A}(X)$ are still quasi-nilpotent in $\mathfrak{A}_X^{\#}$ for any $S \in \mathcal{K}(X)$, where $\mathfrak{A}_X^{\#}$ denotes $\mathfrak{A}_X$ with an adjoined identity element. In fact, classical Fredholm theory, see e.g. [5, 1.4.7 and 1.4.9] or [29, chapter 1.4], applies also to real Banach spaces X and the argument in [9, 2.5.8(iv)] remains valid.

In this section we review for our subsequent use two direct sum constructions (perhaps known in some form) that produce Banach spaces Z for which the quotient algebra $\mathfrak{A}_Z$ is infinite-dimensional, by starting from any Banach space X that fails the approximation property. The drawback is that in general X only embeds as a complemented subspace in Z, where Z is typically quite different from X. We also discuss the algebraic relevance of these constructions.

We first recall the various approximation properties of Banach spaces that will be required. The Banach space X has the approximation property (A.P. in short) if for any compact subset $K \subset X$ and any $\varepsilon > 0$ there is a finite rank operator $T \in \mathcal{F}(X)$ such that

$$\sup_{x \in K} \|x - Tx\| \leq \varepsilon. \tag{1}$$

If one instead allows approximation by $T \in \mathcal{K}(X)$ in condition (1), then X is said to have the compact approximation property (C.A.P.). Moreover, X has the bounded approximation property (B.A.P.), respectively the bounded compact approximation property (B.C.A.P.), if there is a uniform constant $M < \infty$ so that the approximating operator T from (1) can be chosen to satisfy $\|T\| \leq M$. Finally, X has the metric C.A.P. if above $M = 1$.

We refer e.g. to [25, 1.e] and the survey [7] for useful general information about these approximation properties. Recall that any Banach space with a Schauder basis has the B.A.P., and that the first example of a Banach space without the A.P. was constructed by Enflo [11]. The references [25, 2.d], [26, 1.g] and [34, 10.4] contain the detailed constructions of some Banach spaces failing the A.P. Presently the space $\mathcal{L}(\ell^2)$ of the bounded operators on the Hilbert space ℓ^2 is the most explicit space known to fail the A.P. by a result of Szankowski [39]. Moreover, the Calkin algebra $\mathcal{L}(\ell^2)/\mathcal{K}(\ell^2)$ also fails the A.P. [15].

To motivate our first construction recall a classical fact due to Grothendieck. If the Banach space X fails to have the A.P., then there is a linear subspace $Y \subset X$ and a complete norm $|\cdot|$ on Y, such that the inclusion map $J : (Y, |\cdot|) \to X$ is a compact non-approximable operator. For this fact see e.g. the argument for the implication (v) $\Rightarrow$ (i) in [25, Thm. 1.e.4]. It easily follows that $\mathcal{A}(X \oplus Y) \subsetneq \mathcal{K}(X \oplus Y)$, where $Y = (Y, |\cdot|)$.

This fact can be generalised in many ways, and the following construction implies that $\mathfrak{A}_Z$ is infinite-dimensional, where e.g. Z is the countable direct ℓ^p-sum $X \oplus \big(\oplus_{n\in\mathbb{N}} Y\big)_{\ell^p}$, and X and Y are as above. Here $1 \le p \le \infty$, where for notational unity the case $p = \infty$ denotes a direct c_0-sum equipped with the supremum norm.

Proposition 2.1. *Suppose that X and Y_n ($n \in \mathbb{N}$) are Banach spaces such that one of the following conditions holds:*
(i) $\mathcal{A}(Y_n, X) \subsetneq \mathcal{K}(Y_n, X)$, $\quad n \in \mathbb{N}$,
(ii) $\mathcal{A}(X, Y_n) \subsetneq \mathcal{K}(X, Y_n)$, $\quad n \in \mathbb{N}$.
Let $Z = X \oplus \big(\oplus_{n\in\mathbb{N}} Y_n\big)_{\ell^p}$ for $1 \le p \le \infty$. Then the quotient $\mathfrak{A}_Z$ is infinite-dimensional.

Proof. Let $P_0 : Z \to X$ and $P_n : Z \to Y_n$ be the natural projections, respectively $J_0 : X \to Z$ and $J_n : Y_n \to Z$ the corresponding inclusion maps for $n \in \mathbb{N}$.

Suppose that (i) holds. We may after a normalisation find compact operators $S_n \in \mathcal{K}(Y_n, X)$ so that

$$dist(S_n, \mathcal{A}(Y_n, X)) = 1, \quad n \in \mathbb{N}.$$

Consider the compact operators $T_n = J_0 S_n P_n \in \mathcal{K}(Z)$ for $n \in \mathbb{N}$. We claim that

$$dist(T_n - T_m, \mathcal{A}(Z)) \ge 1 \tag{2}$$

for any $n, m \in \mathbb{N}$ with $n \neq m$. In particular, (2) says that $(T_n + \mathcal{A}(Z)) \subset \mathfrak{A}_Z$ is a bounded sequence which does not have any convergent subsequences, whence $\mathfrak{A}_Z$ is infinite-dimensional.

Towards the desired estimate suppose that $V \in \mathcal{A}(Z)$ is arbitrary. Then $P_0 V J_n \in \mathcal{A}(Y_n, X)$, so that for $m \neq n$ one gets that

$$\begin{aligned} 1 = dist(S_n, \mathcal{A}(Y_n, X)) &\le \|S_n - P_0 V J_n\| \\ &= \|P_0(T_n - T_m - V)J_n\| \le \|T_n - T_m - V\|, \end{aligned}$$

since $T_m J_n = J_0 S_m P_m J_n = 0$ above. This implies that (2) holds.

If condition (ii) holds, then there are compact operators $R_n \in \mathcal{K}(X, Y_n)$ so that $dist(R_n, \mathcal{A}(X, Y_n)) = 1$ for $n \in \mathbb{N}$. Let $U_n = J_n R_n P_0 \in \mathcal{K}(Z)$ for $n \in \mathbb{N}$. A straightforward modification of the estimate for (2) shows that also in this case $dist(U_n - U_m, \mathcal{A}(Z)) \geq 1$ for $n \neq m$. This completes the argument. □

Of course, the argument in Proposition 2.1 also shows that the quotient spaces $\mathcal{K}(Y, X)/\mathcal{A}(Y, X)$, respectively $\mathcal{K}(X, Y)/\mathcal{A}(X, Y)$, are infinite-dimensional for $Y = \big(\oplus_{n\in\mathbb{N}} Y_n\big)_{\ell^p}$. However, our focus is on the quotient algebras $\mathfrak{A}_X$, and it must be kept in mind that it is much more difficult to study $\mathfrak{A}_X$ for a *given* Banach space X. Recently Kürsten and Pietsch [22] produced various examples of compact non-approximable operators as the central part of the canonical factorisation $\ell^1 \to \ell^1/Ker(T) \to \overline{T(\ell^1)} \to c_0$ for certain operators $T \in \mathcal{L}(\ell^1, c_0)$, but such methods do not adapt easily to specific quotient algebras.

We will later apply the following variant of Proposition 2.1, which contains more precise information. In conditions (iii) and (iv) below the ℓ^p-sum is replaced by the supremum norm and a c_0-condition for $p = \infty$.

PROPOSITION 2.2. *Suppose that X is a Banach space having the following properties:*

(i) There is a uniformly bounded sequence $(P_n) \subset \mathcal{L}(X)$ of projections on X so that $P_n P_m = 0$ whenever $n \neq m$,

(ii) $\mathfrak{A}_{X_n} \neq \{0\}$ for every $n \in \mathbb{N}$, where $X_n = P_n X$.

Then $\mathfrak{A}_X$ is infinite-dimensional.

Suppose in addition that there is $1 \leq p \leq \infty$ and constants $C, D < \infty$ so that

(iii) $\|\sum_{n=1}^\infty x_n\| \leq C \cdot \big(\sum_{n=1}^\infty \|x_n\|^p\big)^{1/p}$

whenever $x_n \in P_n X$ for $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \|x_n\|^p < \infty$,

(iv) $\big(\sum_{n=1}^\infty \|P_n x\|^p\big)^{1/p} \leq D\|x\|$ for $x \in X$.

Then there is a linear embedding $c_0 \to \mathfrak{A}_X$, that is, $\mathfrak{A}_X$ contains a closed subspace which is linearly isomorphic to the sequence space c_0.

Note that in (iii) the condition implies that the series $\sum_{n=1}^\infty x_n$ converges in X. Moreover, (iii) is always valid with $C = 1$ for $p = 1$, and (iv) follows from (i) for $p = \infty$. The typical application of Proposition 2.2 is to direct ℓ^p-sums $X = \big(\oplus_{n\in\mathbb{N}} X_n\big)_{\ell^p}$, where $\mathfrak{A}_{X_n} \neq \{0\}$ for every $n \in \mathbb{N}$, see Corollary 2.3 below. However, it is not assumed here that the linear span of $\cup_{n\in\mathbb{N}} P_n X$ is dense in X, which will be relevant in Section 3.

PROOF. Suppose first that X satisfies conditions (i) and (ii), and put $M = \sup_n \|P_n\|$. From (ii) we find, after a normalisation and possibly by passing to $S_n - V_n$ for some $V_n \in \mathcal{A}(X_n)$, compact operators $S_n \in \mathcal{K}(X_n)$ so that

$$dist(S_n, \mathcal{A}(X_n)) = 1 \quad \text{and} \quad \|S_n\| \leq 2 \tag{3}$$

for $n \in \mathbb{N}$.

Let $J_n : X_n \to X$ be the inclusion map for $X_n = P_n X$ and $n \in \mathbb{N}$. Consider the compact operators $U_n = J_n S_n P_n \in \mathcal{K}(X)$ for $n \in \mathbb{N}$. By a straightforward modification of the estimates in Proposition 2.1 one obtains that

$$dist(U_n - U_m, \mathcal{A}(Z)) \geq 1/M$$

for any $n, m \in \mathbb{N}$ with $n \neq m$. For this estimate one uses the fact that $P_n P_m = 0$ implies that $P_n J_m = 0$, for $m \neq n$. Thus $\mathfrak{A}_X$ is infinite-dimensional.

Suppose next that X also satisfies conditions (iii) and (iv) for some $1 \leq p < \infty$. Let $(a_k) \in c_0$ be arbitrary. Observe that for any $m > n$ and $x \in X$ we get from (iii) and (iv) that

$$\begin{aligned}
\|\sum_{k=1}^{m} a_k U_k x - \sum_{k=1}^{n} a_k U_k x\| &= \|\sum_{k=n+1}^{m} a_k J_k S_k P_k x\| \\
&\leq C \cdot \Big(\sum_{k=n+1}^{m} |a_k|^p \cdot \|S_k P_k x\|^p \Big)^{1/p} \\
&\leq \ 2C \cdot \sup_{n+1 \leq k \leq m} |a_k| \cdot \Big(\sum_{k=n+1}^{m} \|P_k x\|^p \Big)^{1/p} \\
&\leq 2CD \cdot \sup_{n+1 \leq k \leq m} |a_k| \cdot \|x\|,
\end{aligned}$$

which converges uniformly to 0 as $n \to \infty$. We conclude that $(\sum_{k=1}^{n} a_k U_k)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}(X)$, so that $\sum_{k=1}^{\infty} a_k U_k \in \mathcal{K}(X)$ defines a compact operator for any $(a_k) \in c_0$. Moreover, for $x \in X$ and $(a_k) \in c_0$ we obtain, by taking above $n = 0$ and letting $m \to \infty$, that

$$\|\sum_{k=1}^{\infty} a_k U_k x\| \leq 2CD \cdot \sup_{k \in \mathbb{N}} |a_k| \cdot \|x\|,$$

so that

$$dist(\sum_{k=1}^{\infty} a_k U_k, \mathcal{A}(X)) \leq ||\sum_{k=1}^{\infty} a_k U_k|| \leq 2CD \cdot ||(a_k)||_{\infty}.$$

This means that $(a_k) \mapsto (\sum_{k=1}^{\infty} a_k U_k) + \mathcal{A}(X)$ defines a bounded linear map $\psi : c_0 \to \mathfrak{A}_X$. The argument for $p = \infty$ is similar.

We claim that ψ is bounded from below for $1 \leq p \leq \infty$, so that ψ defines a linear embedding of c_0 into $\mathfrak{A}_X$. Towards this, let $(a_k) \in c_0$ be arbitrary and pick $r \in \mathbb{N}$ such that $||(a_k)||_{\infty} = |a_r|$. Then for any $U \in \mathcal{A}(X)$ we get from $P_r J_k = 0$ for $r \neq k$, that

$$\begin{aligned}
M \cdot ||\sum_{k=1}^{\infty} a_k U_k - U|| \geq ||P_r(\sum_{k=1}^{\infty} a_k U_k - U)J_r|| &= ||P_r(\sum_{k=1}^{\infty} a_k J_k S_k P_k - U)J_r|| \\
&= ||a_r S_r - P_r U J_r|| \geq ||a_r S_r + \mathcal{A}(X_r)|| = |a_r|.
\end{aligned}$$

It follows that $||\psi(a_k)|| = dist(\sum_{k=1}^{\infty} a_k U_k, \mathcal{A}(X)) \geq (1/M)||(a_k)||_\infty$ for $(a_k) \in c_0$, which concludes the argument. $\square$

Note that in the above argument the (formal) series $\sum_{k=1}^{\infty} a_k U_k$ defines a bounded operator on X for any $(a_k) \in \ell^\infty$, but this operator will usually not be compact.

In the sequel we will use the notation $Y \approx Z$ for linearly isomorphic spaces Y and Z. We first emphasise a special case of Proposition 2.2 for easy reference.

COROLLARY 2.3. *If there is $1 \leq p \leq \infty$ such that $X = \big(\oplus_{n\in\mathbb{N}} X_n \big)_{\ell^p}$, where $\mathfrak{A}_{X_n} \neq \{0\}$ for every $n \in \mathbb{N}$, then $\mathfrak{A}_X$ contains a linear isomorphic copy of c_0. In particular, this holds if X is a Banach space so that $\mathfrak{A}_X \neq \{0\}$ and $\big(\oplus_{\mathbb{N}} X)_{\ell^p} \approx X$.*

The basic examples are found in the following application.

COROLLARY 2.4. *Let $1 \leq p < \infty$ and $p \neq 2$. Then there is a closed linear subspace $X \subset \ell^p$ so that $\mathfrak{A}_X$ contains a linear isomorphic copy of c_0. Moreover, there is a closed linear subspace $X \subset c_0$, so that $\mathfrak{A}_X$ contains a copy of c_0.*

PROOF. There is a closed linear subspace $Z \subset \ell^p$ that fails the A.P. for any $1 \leq p < \infty$ and $p \neq 2$. This follows from the work of Enflo and Davie for $2 < p < \infty$, see [25, 2.d], and Szankowski [38] for $1 \leq p < 2$. By [4, Thm. 1] there is a closed linear subspace $Y \subset \ell^p$ such that $\mathcal{A}(Y \oplus Z) \subsetneq \mathcal{K}(Y \oplus Z)$. Let

$$X = \Big(\oplus_{\mathbb{N}} (Y \oplus Z) \Big)_{\ell^p} \subset \big(\oplus_{\mathbb{N}} \ell^p)_{\ell^p} \approx \ell^p.$$

Then Corollary 2.3 yields a linear embedding $c_0 \to \mathfrak{A}_X$. The argument for c_0 is similar. $\square$

Actually the copy of c_0 in $\mathfrak{A}_X$ produced by Corollary 2.4 is complemented in the quotient algebra for $p > 1$. This follows from the fact that $\mathcal{K}(X)$ is separable if and only if X^* is separable, together with Sobczyk's theorem. For this, let $K = (B_{X^{**}}, w^*) \times (B_{X^*}, w^*)$, where $B_{X^{**}}$ and B_{X^*} are the respective closed unit balls. It is known that the map

$$T \mapsto \chi(T)(x^{**}, x^*) = \langle x^{**}, T^* x^* \rangle, \quad (x^{**}, x^*) \in K,$$

defines a linear isometry χ from $\mathcal{K}(X)$ into $C(K)$, see e.g. [25, p. 40]. If X^* is separable, then K is a compact metrisable space, whence $C(K)$ (and hence also $\mathcal{K}(X)$) is separable, see e.g. [12, Lemma 3.102]. Conversely, X^* embeds isometrically into $\mathcal{K}(X)$. Altogether, these facts imply that in Corollary 2.4 the quotient algebra $\mathfrak{A}_X$ is separable for $p > 1$, whence any copy of c_0 is complemented in $\mathfrak{A}_X$ by Sobczyk's theorem, see [25, 2.f.5].

Remark. Suppose that E has the B.A.P., $E \oplus E \approx E$ and E contains a closed subspace X which fails the A.P. By [4, Thm. 1] there is a closed linear subspace $Y \subset E$ such that $\mathcal{A}(Y, X) \subsetneq \mathcal{K}(Y, X)$. Fix $n \in \mathbb{N}$ and consider

$$Z_n = X \oplus (Y \oplus \ldots \oplus Y),$$

with n copies of Y in the direct sum. Then Z_n is a closed subspace of E, up to linear isomorphism, since $E \oplus E \approx E$. It is not difficult to modify the argument in Propositions 2.1 and 2.2 to find a linearly independent family $\{T_j + \mathcal{A}(Z_n) : j = 1, \ldots, n\}$ in $\mathfrak{A}_{Z_n}$, whence $dim(\mathfrak{A}_{Z_n}) \geq n$.

However, we note that it is not clear whether in this setting from [4, Thm. 1] there always is a closed linear subspace $Z \subset E$ such that $\mathfrak{A}_Z$ is infinite-dimensional.

In the case of complex scalars the algebraic differences between c_0 and radical Banach algebras readily imply that the linear embedding $\psi : c_0 \to \mathfrak{A}_X$ in Proposition 2.2, or its applications, is not an algebra homomorphism. In fact, ψ cannot preserve much of the multiplicative structure.

Proposition 2.5. *Suppose that ψ is any linear embedding $c_0 \to \mathfrak{A}_X$, where X is a complex Banach space. Then for every non-zero $a \in c_0$ there is $b \in c_0$ such that $\psi(ab) \neq \psi(a)\psi(b)$.*

Proof. It is straightforward to check that

$$\mathcal{A} = \{a \in c_0 : \psi(ax) = \psi(a)\psi(x) = \psi(x)\psi(a) \text{ for all } x \in c_0\}$$

is a closed subalgebra of c_0. If $\mathcal{A} \neq \{0\}$, then the restriction $\psi_{|\mathcal{A}}$ defines an algebra isomorphism $\mathcal{A} \to \mathcal{B}$, where $\mathcal{B} = \psi(\mathcal{A})$ is a non-trivial closed subalgebra of $\mathfrak{A}_X$. In particular,

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(\psi(a)), \quad a \in \mathcal{A},$$

where $\sigma_{\mathcal{A}}(a)$ denotes the spectrum of $a \in \mathcal{A}$ computed in the subalgebra $\mathcal{A}$ (and analogously for $\sigma_{\mathcal{B}}(\psi(a))$). However, this is known to be impossible. In fact, $\mathcal{B}$ is a radical algebra as a subalgebra of the radical algebra $\mathfrak{A}_X$. On the other hand, non-zero elements $a \in \mathcal{A}$ are never quasi-nilpotent in $\mathcal{A}$, as $\sigma_{\mathcal{A}}(a)$ contains the spectrum $\sigma(a)$ of a in c_0. Thus $\mathcal{A} = \{0\}$, and the above claim follows. $\square$

For similar reasons the restriction of ψ to the closed ideal

$$\mathcal{A}_D = \{(x_n) \in c_0 : x_n = 0 \text{ for } n \notin D\}$$

of c_0 is not an algebra homomorphism for any subset $D \subset \mathbb{N}$. Nevertheless, we point out that the range $\psi(c_0)$ of the particular linear embedding ψ from Proposition 2.2 generates a commutative subalgebra in the quotient algebra.

Proposition 2.6. *Let $\psi : c_0 \to \mathfrak{A}_X$ be the linear embedding constructed in Proposition 2.2, and let $\mathcal{B}$ be the closed subalgebra of $\mathfrak{A}_X$ generated by the range $\psi(c_0)$. Then $\mathcal{B}$ is a commutative algebra.*

Proof. Note first that

$$\psi(a)\psi(b) = \psi(b)\psi(a) \quad \text{for} \quad a, b \in c_0, \tag{4}$$

so that $\psi(c_0)$ is a commuting subset of $\mathfrak{A}_X$. Namely, if $a = (a_k), b = (b_k) \in c_0$, then

$$\Big(\sum_{k=1}^{m} a_k U_k\Big)\Big(\sum_{r=1}^{n} b_r U_r\Big) = \sum_{k=1}^{m\wedge n} a_k b_k U_k^2 = \Big(\sum_{r=1}^{n} b_r U_r\Big)\Big(\sum_{k=1}^{m} a_k U_k\Big),$$

since by the construction in Proposition 2.2 one has $U_k U_r = 0$ for $k \neq r$. Above $m \wedge n$ denotes the smaller of m and n. Hence (4) follows by passing to the limit in the quotient norm of $\mathfrak{A}_X$.

Recall next that the finite linear combinations of $\psi(x_{k_1}) \cdots \psi(x_{k_s})$, where $x_{k_1}, \ldots, x_{k_s} \in c_0$ and $s \in \mathbb{N}$, form a dense subset of $\mathcal{B}$. Since any two such finite linear combinations commute by (4), it follows by a standard approximation argument that $\mathcal{B} \subset \mathfrak{A}_X$ is a closed commutative subalgebra. $\square$

For completeness we include a couple of examples, where the quotient algebra $\mathfrak{A}_X$ is non-commutative. It will be convenient to denote operators $S \in \mathcal{L}(X \oplus Y)$ on the direct sum $X \oplus Y$ as 2×2-operator matrices

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

in the canonical way. This means that $S_{11} = P_1 S J_1$, $S_{12} = P_1 S J_2$, $S_{21} = P_2 S J_1$ and $S_{22} = P_2 S J_2$, where P_1 is the projection $X \oplus Y \to X$, J_1 is the inclusion $X \to X \oplus Y$, and P_2 and J_2 are the analogous operators for the summand Y.

Example 1. (i) Suppose that X is a Banach space for which there are $A, B \in \mathcal{K}(X)$ so that $AB \notin \mathcal{A}(X)$ or $BA \notin \mathcal{A}(X)$. Define $\widehat{A}, \widehat{B} \in \mathcal{K}(X \oplus X)$ by

$$\widehat{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix},$$

that is, $\widehat{A}(x, y) = (Ay, 0)$ and $\widehat{B}(x, y) = (0, Bx)$ for $(x, y) \in X \oplus X$. Then $\widehat{A}\widehat{B} - \widehat{B}\widehat{A} \notin \mathcal{A}(X \oplus X)$, since $\widehat{A}\widehat{B}(x, y) = (ABx, 0)$ and $\widehat{B}\widehat{A}(x, y) = (0, BAy)$ for $(x, y) \in X \oplus X$.

The above assumptions on X can be satisfied, since our subsequent Proposition 3.1 yields $A \in \mathcal{K}(X)$ such that $A^2 \notin \mathcal{A}(X)$ for a class of Banach spaces.

(ii) We claim that for any $1 \leq p < \infty$ with $p \neq 2$ there is a closed linear subspace $X \subset \ell^p$ such that $\mathfrak{A}_X$ is not a commutative algebra.

In fact, as explained at the start of the proof of Corollary 2.4, there is a closed subspace $Y \subset \ell^p$ and an associated operator $T \in \mathcal{K}(Y) \setminus \mathcal{A}(Y)$. By [4, Thm. 2'] there is a closed subspace $Z \subset \ell^p$ and a factorisation $T = VU$, where $U \in \mathcal{K}(Y, Z)$ and $V \in \mathcal{K}(Z, Y)$. Consider the compact operators

$$\widehat{U} = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}, \quad \widehat{V} = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$$

on $X = Y \oplus Z$, where X can be identified with a closed subspace of ℓ^p. One verifies as in part (i) that $\widehat{U}$ and $\widehat{V}$ do not commute modulo $\mathcal{A}(X)$.

3. $\mathfrak{A}_Z$ for the Willis spaces Z

The main result of this section demonstrates that c_0 embeds isomorphically into the quotient algebra $\mathfrak{A}_Z = \mathcal{K}(Z)/\mathcal{A}(Z)$, whenever Z belongs to a class of Banach spaces constructed by Willis [42].

Our starting point is the following observation, which points out classes of Banach spaces X for which $\mathfrak{A}_X$ is always non-trivial. Part (i) of the following proposition confirms the expectation from [10] that $\mathfrak{A}_Z$ is infinite-dimensional for the spaces Z from [42]. Part (ii) of the Remark following Proposition 3.1 indicates a different proof of this result suggested by Dales [10]. Recall that the Banach algebra $\mathcal{A}$ is nilpotent, if there is $m \in \mathbb{N}$ such that the products $x_1 \cdots x_m = 0$ for any $x_1, \ldots, x_m \in \mathcal{A}$.

Proposition 3.1. *(i) If the Banach space X has the B.C.A.P., but fails to have the A.P., then $\mathfrak{A}_X$ is infinite-dimensional. More precisely, there is $V \in \mathcal{K}(X)$ such that $V^n \notin \mathcal{A}(X)$ for every $n \in \mathbb{N}$, and the closed commutative subalgebra $\mathcal{B}$ generated by $V + \mathcal{A}(X)$ in $\mathfrak{A}_X$ is infinite-dimensional.*

(ii) If the Banach space X has the C.A.P., but fails to have the A.P., then $\mathfrak{A}_X \neq \{0\}$.

Proof. (i) Let $n \in \mathbb{N}$ be arbitrary and fix $M \geq 1$ such that X has the B.C.A.P. with constant M. Since X does not have the A.P. by assumption, there is a compact subset $K \subset X$ and a constant $c > 0$ so that

$$\sup_{x \in K} ||x - Vx|| \geq c$$

for every $V \in \mathcal{F}(X)$. On the other hand, since X has the B.C.A.P. with constant M, there is a compact operator $U \in \mathcal{K}(X)$, so that $||U|| \leq M$ and

$$\sup_{x \in K} ||x - Ux|| < \frac{c}{2M_n},$$

where $M_n = \sum_{k=0}^{n-1} M^k$. Hence we get that

$$||x - U^n x|| = ||(I_X + U + \ldots + U^{n-1})(x - Ux)|| \leq \big(\sum_{k=0}^{n-1} \|U\|^k\big) \cdot ||x - Ux||$$
$$\leq \big(\sum_{k=0}^{n-1} M^k\big) \cdot \frac{c}{2M_n} = c/2$$

for every $x \in K$. It follows that $U^n \in \mathcal{K}(X) \setminus \mathcal{A}(X)$. Note that here the operator U depends on n.

The preceding fact means that $\mathfrak{A}_X$ is a radical Banach algebra which is not nilpotent. It follows from general algebraic results about the Jacobson radical, see [9, Prop. 1.5.6.(iv)], that $\mathfrak{A}_X$ is infinite-dimensional. Moreover, by an application of the Baire theorem, see [16] or [33, Prop. 4.4.11.(b)], there is a compact operator $V \in \mathcal{K}(X)$ such that $V^n \notin \mathcal{A}(X)$ for every $n \in \mathbb{N}$. It follows as above that the closed commutative subalgebra $\mathcal{B}$ generated by $V + \mathcal{A}(X)$ in $\mathfrak{A}_X$ is infinite-dimensional. Note further that this argument is valid both for real and complex Banach spaces X.

(ii) If $K \subset X$ is a compact subset and $c > 0$ is a constant so that

$$\sup_{x \in K} ||x - Vx|| \geq c$$

whenever $V \in \mathcal{F}(X)$, it is enough to pick $U \in \mathcal{K}(X)$ with $\|x - Ux\| < c/2$ for any $x \in K$. □

Remark. (i) The first examples of Banach spaces, which have the metric C.A.P. but fail the A.P., were found by Willis [42]. In Proposition 3.1 the classes of Banach spaces satisfying condition (i), respectively (ii), are distinct. This fact is based on [42] in combination with known constructions and a simple observation, cf. [7, Prop. 8.2] or [31, Cor. 2.4].

For this let Y be a Banach space with the A.P. which fails to have the B.A.P., see e.g. [25, 1.e.18-1.e.20]. Observe that Y also fails the B.C.A.P., since $\mathcal{K}(Y) = \mathcal{A}(Y)$. Consider $X = Y \oplus Z$, where Z is a Banach space which has the metric C.A.P. but fails to have the A.P., as given by [42]. In this event $X = Y \oplus Z$ has the C.A.P., while X fails both the A.P. and the B.C.A.P., as these properties are inherited by complemented subspaces. (However, note that $\mathfrak{A}_X$ is still infinite-dimensional, since $\mathfrak{A}_Z$ embeds into $\mathfrak{A}_X$ by Proposition 4.2.)

(ii) Suppose that X has the B.C.A.P. It is a known consequence of the Cohen-Hewitt factorisation theorem, see e.g. [9, 2.9.26 and 2.9.37], that any operator $S \in \mathcal{K}(X)$ factors as $S = UV$ for suitable $U, V \in \mathcal{K}(X)$. Clearly this property passes to the quotient algebra $\mathfrak{A}_X$, so that $\mathfrak{A}_X$ cannot be a nilpotent algebra if $\mathfrak{A}_X \neq \{0\}$. This also implies that $\mathfrak{A}_X$ is infinite-dimensional in part (i) of Proposition 3.1, as pointed out by Dales [10], but the argument included above uses only elementary facts.

The argument for Proposition 3.1.(i) is algebraic in nature, and does not provide explicit information about $\mathfrak{A}_X$. The main purpose of this section is to sharpen the result for the class of Banach spaces found by Willis [42]. More precisely, for any Banach space X that fails the A.P., Willis obtains a Banach space Z_X that has the metric C.A.P., but fails to have the A.P. We begin by recalling the relevant details of the construction for our purposes. The original version in [42] produces real Banach spaces, and the minor modifications required in the case of complex scalars are found e.g. in [32]. Let X be a Banach space. We use $absconv(D)$ to denote the absolutely convex hull of a subset $D \subset X$. For complex scalars this means the balanced convex hull of D, so that $\alpha x + \beta y \in absconv(D)$ whenever $x, y \in D$ and $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha| + |\beta| \leq 1$.

Suppose that the Banach space X fails to have the A.P., and fix a compact subset $K \subset X$ and a constant $c > 0$ so that

$$\sup_{x \in K} \|x - Vx\| \geq c \tag{5}$$

for any finite rank operator $V \in \mathcal{F}(X)$. By a classical fact [25, Prop. 1.e.2] one may assume that $K = \overline{conv}\{x_n : n \in \mathbb{N}\}$, where $0 < \|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and $\|x_n\| \to 0$ as $n \to \infty$. For any $0 < t < 1$ consider the closed absolutely convex subset

$$U_t = \overline{absconv}\{\frac{x_n}{\|x_n\|^t} : n \in \mathbb{N}\}$$

of X, which is compact by Mazur's theorem. Let $Y_t \subset X$ be the linear span of U_t normed by the Minkowski functional

$$|x|_t = \inf\{\lambda > 0 : x \in \lambda U_t\}, \quad x \in Y_t.$$

We will require the following facts: $(Y_t, |\cdot|_t)$ is a Banach space for any $0 < t < 1$, U_t is the closed unit ball of Y_t and

$$|x_n|_t \leq ||x_n||^t, \quad n \in \mathbb{N}. \tag{6}$$

Moreover, for any $0 < s < t < 1$ one has $U_s \subset U_t \subset B_X$ and $Y_s \subset Y_t \subset X$, so that

$$\|x\| \leq |x|_t \leq |x|_s, \quad x \in Y_s. \tag{7}$$

In particular, the inclusions $Y_s \to Y_t$ and $Y_t \to X$ are bounded whenever $0 < s < t < 1$.

Let $\mathcal{Z}$ be the linear span of the simple functions $\{y\chi_{(s,t)} : 0 < s < t < 1, y \in Y_s\}$ on $(0,1)$ equipped with the norm

$$\|f\| = \int_0^1 |f(r)|_r dr = \sum_{k=1}^{n} \int_{u_k}^{v_k} |y_k|_r dr,$$

for $f = \sum_{k=1}^n y_k \chi_{(u_k,v_k)} \in \mathcal{Z}$, where $0 < u_1 < v_1 \le u_2 < \ldots \le u_k < v_k < 1$ and $y_k \in Y_{u_k}$ for $k = 1, \ldots, n$. Note that $f(r) \in Y_r$ for $f \in \mathcal{Z}$ and $0 < r < 1$, and that the above integral exists in the Riemann sense by the monotonicity of the map $r \mapsto |y_k|_r$ for each k. Finally, let $Z = Z_X$ be the completion of $(\mathcal{Z}, \| \cdot \|)$.

Willis showed [42, Prop. 1 and 2] that Z has the metric C.A.P., but fails the A.P. Note that Z is non-reflexive and Z^* is non-separable, since the closed subspace $L^1(0,1)x_1 \subset Z$ is isomorphic to $L^1(0,1)$, whence $\mathcal{A}(Z)$ is also non-separable. The following result is the main one of this paper.

Theorem 3.2. *Suppose that X fails to have the A.P. and let $Z = Z_X$ be the above Willis space associated to X. Then there is a linear isomorphic embedding $\psi : c_0 \to \mathfrak{A}_Z$.*

Recall from Proposition 2.5 that for complex scalars the linear embedding $\psi : c_0 \to \mathfrak{A}_Z$ is not an algebra homomorphism.

Proof. In the argument we will verify in the following steps that the conditions (i)-(iv) of Proposition 2.2 are satisfied with $p = 1$.

Step 1. Let $0 < s < t < 1$. We claim that the set U_s is compact in Y_t, so that the inclusion operator $J_{s,t} : Y_s \to Y_t$ is compact.

Indeed, towards this note that

$$|\frac{x_n}{\|x_n\|^s}|_t \le \|x_n\|^{t-s} \to 0$$

as $n \to \infty$. This yields that the $| \cdot |_t$-closure $\overline{absconv}^{|\cdot|_t}\{x_n/||x_n||^s : n \in \mathbb{N}\}$ is compact in Y_t by Mazur's theorem, so that $J_{s,t} : Y_s \to Y_t$ is a compact inclusion operator, because the unit ball $B_{Y_s} = \overline{absconv}\{x_n/||x_n||^s : n \in \mathbb{N}\}$.

Step 2. Claim: the inclusion map $R_s : Y_s \to X$ is not an approximable operator for any $0 < s < 1$, and $\mathcal{A}(Y_s, Y_t) \subsetneq \mathcal{K}(Y_s, Y_t)$ whenever $0 < s < t < 1$. In particular, Y_t does not have the A.P. for any $0 < t < 1$.

In fact, suppose to the contrary that $R_s \in \mathcal{A}(Y_s, X)$ and let $\varepsilon > 0$ be given. In view of the counter assumption there exists a finite rank operator

$$R = \sum_{k=1}^n f_k \otimes y_k \in Y_s^* \otimes X$$

such that $\|R_s - R\| < \varepsilon/2$, where $f_1, \ldots, f_n \in Y_s^*$ and $y_1, \ldots, y_n \in X$. In particular, note that $\|Rx - x\| < \varepsilon/2$ for $x \in K$, since $K \subset U_s$. Note further that $K \subset Y_s$ is a relatively compact subset, since (6) implies that $|x_n|_s \le \|x_n\|^s \to 0$ as $n \to \infty$.

Put $\delta = \varepsilon/(2\sum_{k=1}^n \|y_k\|) > 0$. According to the argument in [25, p. 33] one knows that the range $R_s^*(X^*)$ is dense in Y_s^* with respect to the topology of uniform convergence on the compact subsets of Y_s. Hence we may pick functionals $g_1, g_2, \dots, g_n \in X^*$ such that

$$\sup_{x\in K} |\langle R_s x, g_k\rangle - \langle x, f_k\rangle| = \sup_{x\in K} |\langle x, R_s^* g_k\rangle - \langle x, f_k\rangle| < \delta$$

for all $k = 1, 2, \dots, n$. Then the finite rank operator $S = \sum_{k=1}^n g_k \otimes y_k \in \mathcal{F}(X)$ satisfies

$$\begin{aligned}\|Sx - x\| &\le \|Sx - Rx\| + \|Rx - x\| \\ &\le \sum_{k=1}^n |\langle R_s x, g_k\rangle - \langle x, f_k\rangle| \cdot \|y_k\| + \|Rx - x\| < \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

for $x \in K$. This contradicts property (5) of the compact set $K \subset X$ once $0 < \varepsilon < c$.

Moreover, since $R_s = R_t \circ J_{s,t}$, where $R_s \notin \mathcal{A}(Y_s, X)$, it follows from the result in Step 1 that $J_{s,t} \in \mathcal{K}(Y_s, Y_t) \setminus \mathcal{A}(Y_s, Y_t)$.

Step 3. For any fixed $s \in (0,1)$ define the linear map $T_s : Y_s \to Z$ by

$$T_s(y) = \frac{1}{1-s} y\chi_{(s,1)}, \quad y \in Y_s.$$

We claim that T_s is a compact operator. (This fact was used in [42] for $s = 1/2$.)

To verify the claim, note that for any $n \in \mathbb{N}$ one has

$$\begin{aligned}\|T_s(x_n)\| &= \frac{1}{1-s}\int_0^1 |x_n\chi_{(s,1)}(r)|_r dr = \frac{1}{1-s}\int_s^1 |x_n|_r dr \\ &\le \frac{1}{1-s}\int_s^1 \|x_n\|^r dr = \frac{1}{(1-s)}\frac{(\|x_n\| - \|x_n\|^s)}{\ln \|x_n\|} \\ &\le \frac{1}{(1-s)}\frac{\|x_n\|^s}{|\ln \|x_n\||},\end{aligned}$$

so that

$$\|T_s(x_n/\|x_n\|^s)\| \le \frac{1}{(1-s)}\frac{1}{|\ln \|x_n\||}.$$

Since $x_n \to 0$ in X as $n \to \infty$, we conclude that $\|T_s(x_n/\|x_n\|^s)\| \to 0$ as $n \to \infty$. By Mazur's theorem the set

$$\overline{absconv}\{T_s(x_n/\|x_n\|^s) : n \in \mathbb{N}\}$$

is compact in Z, and since $T_s U_s \subset \overline{absconv}\{T_s(x_n/\|x_n\|^s) : n \in \mathbb{N}\}$, we get that T_s is a compact operator $Y_s \to Z$.

For the remainder of the argument we fix intertwining sequences (s_n) and (t_n) such that $0 < s_1 < t_1 < s_2 < t_2 < \ldots < 1$, where $s_n \to 1$ as

$n \to \infty$. Then $f \mapsto P_n f = f\chi_{(s_n,t_n)}$ is a well-defined norm-1 projection of Z onto the closed subspace $Z_n = P_n Z \subset Z$ for any $n \in \mathbb{N}$, since

$$||P_n f|| = \int_{s_n}^{t_n} |f(r)|_r dr \leq \int_0^1 |f(r)|_r dr = \|f\|$$

for any $f \in Z$. Moreover, we may define a bounded linear operator $J : Z \to X$ by

$$J(f) = \int_0^1 f(r)dr, \quad f \in Z. \tag{8}$$

In fact, for a simple function $f = \sum_{k=1}^n y_k \chi_{(u_k,v_k)} \in \mathcal{Z}$, where $0 < u_1 < v_1 \leq u_2 < \ldots \leq u_k < v_k < 1$ and $y_k \in Y_{u_k}$ for each $k = 1, \ldots, n$, put

$$J(f) = \sum_{k=1}^n (v_k - u_k) y_k = \int_0^1 f(r)dr.$$

Observe that by (7) one has

$$\|J(f)\| \leq \sum_{k=1}^n (v_k - u_k)\|y_k\| \leq \sum_{k=1}^n \int_{u_k}^{v_k} |y_k|_r dr = \|f\|$$

for such simple functions $f \in \mathcal{Z}$, so that the above map admits by density a bounded linear extension J to Z that satisfies (8).

The following step verifies the crucial condition (ii) from Proposition 2.2.

Step 4. Claim: $\mathfrak{A}_{Z_n} \neq \{0\}$ for all $n \in \mathbb{N}$.

Towards this claim let $J_n : Z_n \to Z$ denote the inclusion map for $n \in \mathbb{N}$. It is easy to check that the inclusion $R_{s_n} : Y_{s_n} \to X$ factors as

$$R_{s_n} = c_n J J_n P_n T_{s_n}, \tag{9}$$

where $c_n = \frac{1-s_n}{t_n-s_n}$ and the operators $T_{s_n} : Y_{s_n} \to Z$ and $J : Z \to X$ are defined in Step 3, respectively in (8). Since R_{s_n} is not an approximable operator by Step 2, it follows from (9) that $P_n T_{s_n}$ is not approximable $Y_{s_n} \to Z_n$. Recall from Step 3 that T_{s_n} is a compact operator, whence

$$P_n T_{s_n} \in \mathcal{K}(Y_{s_n}, Z_n) \setminus \mathcal{A}(Y_{s_n}, Z_n).$$

This means that Z_n does not have the A.P. On the other hand, Z_n is a 1-complemented subspace of Z, where Z has the metric C.A.P. by [42, Prop. 2], so Z_n also has the metric C.A.P. Consequently $\mathfrak{A}_{Z_n} \neq \{0\}$ in view of Proposition 3.1.

We finally check the remaining conditions of Proposition 2.2 with $p = 1$ for Z and the sequence $(P_n) \subset \mathcal{L}(Z)$ of norm-1 projections onto the subspaces Z_n. Condition (iii) is obvious for $p = 1$. Moreover, in view of the disjoint supports on $(0,1)$ of the functions in the subspaces Z_n, the integration norm in Z satisfies

$$\sum_{k=1}^{\infty} \|P_k f\| = \sum_{k=1}^{\infty} \int_{s_k}^{t_k} |f(r)|_r dr \leq \int_0^1 |f(r)|_r dr = \|f\|$$

for any $f \in Z$. Hence condition (iv) is also satisfied, and the proof of the theorem is completed by an application of Proposition 2.2. $\square$

Remark. The proof in [42, Prop. 2] that Z has the metric C.A.P. uses a sequence $(T_n) \subset \mathcal{K}(Z)$ of vector-valued convolution operators on Z. It is conceivable that basic subsequence techniques applied to $(T_n + \mathcal{A}(Z))$ in $\mathfrak{A}_Z$ might produce isomorphic copies of c_0, or even of other spaces. However, for this approach to be feasible one is likely to need descriptions of the dual spaces Z^*, Z^{**} and $\mathcal{K}(Z)^*$.

Recall from the proof of Corollary 2.4 that there are closed linear subspaces $X \subset \ell^p$ that fail the A.P. for any $1 < p < \infty$ and $p \neq 2$. Willis [42, Prop. 3 and 4] includes a modified construction, where these subspaces X lead to a separable reflexive space $Z_p^\sharp$, so that again $Z_p^\sharp$ has the metric C.A.P. but fails to have the A.P. The space $Z_p^\sharp$ is a quotient of a closed subspace of $L^p(0,1)$, see [42, Prop. 3]. We observe next for completeness that Theorem 3.2 remains valid for $Z_p^\sharp$, but we will not reproduce all the technical details.

Proposition 3.3. *There is a linear isomorphic embedding $\psi : c_0 \to \mathfrak{A}_{Z_p^\sharp}$ for $1 < p < \infty$ and $p \neq 2$, where the range $\psi(c_0)$ is complemented in $\mathfrak{A}_{Z_p^\sharp}$.*

Proof. The construction in [42, p. 103] is based on certain modified compact absolutely convex sets $V_t \subset X$ for $0 < t < 1$ that also depend on p. As before one introduces associated Banach spaces $(W_t, |\cdot|_t)$ such that $W_s \subset W_t \subset X$ for $0 < s < t < 1$, where W_t is the linear span of V_t in X. The space $Z_p^\sharp$ is the completion of the linear span of the simple functions $\{y\chi_{(s,t)} : 0 < s < t < 1, y \in W_s\}$ on $(0,1)$ equipped with the norm

$$\|f\|_p = \Big(\int_0^1 |f(r)|_r^p dr\Big)^{1/p}.$$

Fix once more sequences (s_n) and (t_n) such that $0 < s_1 < t_1 < s_2 < t_2 < \ldots < 1$, where $s_n \to 1$ as $n \to \infty$, and let P_n be the norm-1 projection of $Z_p^\sharp$ onto the closed subspace $U_n := P_n(Z_p^\sharp) \subset Z_p^\sharp$ consisting of the functions supported on (s_n, t_n) for each n.

Let $R_s : Y_s \to X$ be the inclusion map from Theorem 3.2, and let $T_{s_n}^\sharp : Y_{s_n} \to Z_p^\sharp$ and $J^\sharp : Z_p^\sharp \to X$ be analogous operators to those of Step 3 of that argument. One verifies as in Step 3 that $T_{s_n}^\sharp \in \mathcal{K}(Y_{s_n}, Z_p^\sharp)$ for each n. Moreover,

$$R_{s_n} = c_n J^\sharp J_n P_n T_{s_n}^\sharp,$$

where $c_n = \frac{1-s_n}{t_n-s_n}$. Thus $P_n T^\sharp_{s_n}$ is a compact non-approximable operator $Y_{s_n} \to U_n$, since R_{s_n} is non-approximable by Step 2 of Theorem 3.2. Hence U_n has the metric C.A.P. but fails the A.P. for all $n \in \mathbb{N}$, so that $\mathfrak{A}_{U_n} \neq \{0\}$ by Proposition 3.1.

The rest of the argument is similar to that of Theorem 3.2, since conditions (iii) and (iv) from Proposition 2.2 are satisfied for $Z^\sharp_p$ with the exponent $p \in (1, \infty)$. Namely, by the disjointness of the supports one gets that

$$\| \sum_{n=1}^{\infty} f_n \|_p = \big(\sum_{n=1}^{\infty} \|f_n\|_p^p \big)^{1/p}$$

whenever $f_n \in P_n(Z^\sharp_p)$ for $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|f_n\|^p$ is finite.

The fact that the range $\psi(c_0)$ is complemented in $\mathfrak{A}_{Z^\sharp_p}$ follows from Sobczyk's theorem as explained after Corollary 2.4. □

We note that various other results and applications which involve the Willis spaces Z_X can be found e.g. in [41], [8], [24] and [32].

4. Further examples on universal spaces

In this section we discuss explicit examples of universal Banach spaces X, where the quotient algebra $\mathfrak{A}_X$ is large. The examples include the complementably universal conjugate space C_1^* of Johnson, and some universal compact factorisation spaces that originate in the work of Johnson and Figiel.

We first recall from [25, 1.e.7] that X has the A.P. whenever the dual space X^* has the A.P., and that the converse fails in general. We next show that analogous facts hold for the respective quotient algebras.

Proposition 4.1. *Let X be any Banach space and define $\theta : \mathfrak{A}_X \to \mathfrak{A}_{X^*}$ by*

$$\theta(S + \mathcal{A}(X)) = S^* + \mathcal{A}(X^*), \quad S \in \mathcal{K}(X).$$

Then θ is an isometry $\mathfrak{A}_X \to \mathfrak{A}_{X^*}$, which is an anti-homomorphism.

The above means that θ is linear and reverses products, that is,

$$\theta(ST + \mathcal{A}(X)) = \theta(T + \mathcal{A}(X)) \cdot \theta(S + \mathcal{A}(X))$$

for $S, T \in \mathcal{K}(X)$. This property of θ is obvious from $(ST)^* = T^* S^*$.

Proof. Note first that $U \in \mathcal{A}(X)$ implies that $U^* \in \mathcal{A}(X^*)$, so θ is a well-defined map. Moreover, θ is an isometry since the principle of local reflexivity implies that

$$dist(S^*, \mathcal{A}(X^*)) = dist(S, \mathcal{A}(X))$$

for any X and any compact operator $S \in \mathcal{K}(X)$, see e.g. [6, Prop. 2.5.2]. Note also that if the space X is reflexive, then the preceding identity can be seen just by passing to adjoints. $\square$

Remark. Let X be any Banach space. The map $S \mapsto S^*$ also induces well-defined quotient maps $\psi : \mathcal{L}(X)/\mathcal{A}(X) \to \mathcal{L}(X^*)/\mathcal{A}(X^*)$ and $\chi : \mathcal{L}(X)/\mathcal{K}(X) \to \mathcal{L}(X^*)/\mathcal{K}(X^*)$ as above, but ψ and χ behave differently from θ.

Namely, by the principle of local reflexivity one also has

$$dist(S, \mathcal{A}(X)) \leq 5 \cdot dist(S^*, \mathcal{A}(X^*)) = 5 \cdot \|\psi(S + \mathcal{A}(X))\|$$

for any bounded operator $S \in \mathcal{L}(X)$ and any Banach space X, see e.g. [6, Prop. 2.5.4]. Moreover, ψ is not always an isometry, see (2.5.7) and (2.5.9) in [6].

Finally, by [41, Example 2.5] there is a Banach space X such that χ is not even bounded from below as a map $\mathcal{L}(X)/\mathcal{K}(X) \to \mathcal{L}(X^*)/\mathcal{K}(X^*)$ in the respective quotient norms. Above X does not have the B.C.A.P. by [41, Thm. 2.4].

We next provide examples where the difference between $\mathfrak{A}_X$ and $\mathfrak{A}_{X^*}$ is large. For this purpose, as well as for later use, we require the following more precise version of Proposition 2.2 for finite direct sums. It will again be convenient to represent operators $S \in \mathcal{L}(X \oplus Y)$ by 2×2-operator matrices as explained before Example 1.

Proposition 4.2. *Let X and Y be Banach spaces, where $\mathfrak{A}_X \neq \{0\}$. Then $\mathfrak{A}_X$ is algebra isomorphic to the complemented subalgebra*

$$\mathcal{M} = \left\{ \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}(X \oplus Y) : S_{11} \in \mathcal{K}(X) \right\}$$

of $\mathfrak{A}_{X \oplus Y}$.

Proof. We first verify that the map Φ given by

$$\Phi(S + \mathcal{A}(X \oplus Y)) = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}(X \oplus Y), \quad S \in \mathcal{K}(X \oplus Y),$$

defines a bounded linear projection $\mathfrak{A}_{X \oplus Y} \to \mathcal{M}$. Towards this fact we consider the map $\Psi : \mathcal{K}(X \oplus Y) \to \mathcal{K}(X \oplus Y)$ defined by

$$\Psi(S) = \begin{pmatrix} S_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathcal{K}(X \oplus Y).$$

Observe that Ψ is a bounded linear operator, $\|\Psi\| = 1$, since we may write

$$\Psi(S) = J_1(P_1 S J_1)P_1, \quad S \in \mathcal{K}(X \oplus Y).$$

Clearly $\Psi(\mathcal{A}(X \oplus Y)) \subset \mathcal{A}(X \oplus Y)$, so that Ψ induces the above map Φ on the quotient space $\mathfrak{A}_{X\oplus Y}$, whence $\|\Phi\| \leq \|\Psi\| = 1$.

Next define $\alpha : \mathfrak{A}_X \to \mathfrak{A}_{X\oplus Y}$ by

$$\alpha(U + \mathcal{A}(X)) = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}(X \oplus Y), \quad U \in \mathcal{K}(X).$$

It is not difficult to check that α is a well-defined algebra homomorphism, and by arguing as above, one gets that $\|\alpha\| = 1$. Moreover, if $U \in \mathcal{K}(X)$ and $A \in \mathcal{A}(X \oplus Y)$, then

$$\| \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} - A\| \geq \|P_1(\begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} - A)J_1\| = \|U - A_{11}\| \geq dist(U, \mathcal{A}(X)).$$

We conclude that $\|\alpha(U + \mathcal{A}(X))\| = \|U + \mathcal{A}(X)\|$ holds for all $U \in \mathcal{K}(X)$, so that α defines an algebra isomorphism $\mathfrak{A}_X \to \mathcal{M}$. $\square$

Fix $1 \leq p \leq \infty$ and let $C_p = (\oplus_{m\in\mathbb{N}} G_m)_{\ell^p}$ be the universal spaces constructed by Johnson [18]. For the definition fix a sequence $(E_n)_{n\in\mathbb{N}}$ of finite-dimensional spaces which is dense in the Banach-Mazur distance d_{BM} in the class of all finite-dimensional spaces, and repeat each E_n infinitely often to obtain the listing $(G_m)_{m\in\mathbb{N}}$. Above

$$d_{BM}(E, F) = \inf\{\|T\| \cdot \|T^{-1}\| : T \text{ linear isomorphism } E \to F\}.$$

For unity of notation $p = \infty$ corresponds again to a direct c_0-sum. Note that with this convention C_p has the A.P. for any $1 \leq p \leq \infty$.

Proposition 4.3. *The quotient $\mathfrak{A}_{C_1^*}$ contains a linear isomorphic copy of c_0, but $\mathfrak{A}_{C_1} = \{0\}$.*

Proof. By Corollary 2.4 there is, for $1 < p < \infty$ and $p \neq 2$, a separable reflexive subspace $X \subset \ell^p$ such that $\mathfrak{A}_X$ contains a linear isomorphic copy of c_0. Proposition 4.1 implies that $\mathfrak{A}_{X^*}$ contains an isomorphic copy of c_0. (The space $X = Z_p^\sharp$ from Proposition 3.3 can also be used instead of Corollary 2.4, but Theorem 3.2 and Proposition 3.3 depend on longer arguments.)

Johnson [19, Thm. 1] showed that C_1^* is complementably universal for separable spaces. This result says that X^* is isometric to a 1-complemented subspace of C_1^*, that is,

$$C_1^* = W \oplus M,$$

where W is isometric to X^* and W is complemented in C_1^* by a norm-1 projection. Finally, by applying Proposition 4.2 to $C_1^* = W \oplus M$, we obtain that $\mathfrak{A}_{C_1^*}$ contains a linear isomorphic copy of c_0. $\square$

The universal conjugate space C_1^* is not separable, but we note that there are spaces Y with Y^* separable, which display the same duality behaviour as $X = C_1$ in Proposition 4.3. For this we use another known construction.

Example 2. Let X be a separable reflexive Banach space for which $\mathfrak{A}_{X^*}$ contains a linear isomorphic copy of c_0, as in Proposition 4.3. A construction due to James and Lindenstrauss, see [25, 1.d.3], provides a Banach space Z (which depends on X) such that Z^{**} has a Schauder basis and $Z^{**}/Z \approx X$. It follows that

$$Z^{***} \approx Z^* \oplus X^*,$$

so that Z^{***} is separable. Proposition 4.2 implies that $\mathfrak{A}_{Z^{***}}$ contains a linear isomorphic copy of c_0, while $\mathfrak{A}_{Z^{**}} = \{0\}$.

Actually, in this example one may explicitly identify $\mathfrak{A}_{Z^{***}}$ with $\mathfrak{A}_{X^*}$, since the component operators with respect to the decomposition $Z^{***} \approx Z^* \oplus X^*$ satisfy $\mathcal{K}(Z^*) = \mathcal{A}(Z^*)$, $\mathcal{K}(X^*, Z^*) = \mathcal{A}(X^*, Z^*)$ and $\mathcal{K}(Z^*, X^*) = \mathcal{A}(Z^*, X^*)$. Here one uses the fact that Z^{**} as well as Z^* have the A.P.

Johnson [18, Thm. 1] and Figiel [13, Prop. 3.1] showed that for any $1 \le p \le \infty$ the spaces C_p have the following compact factorisation property: given any Banach spaces X, Y and a compact operator $T \in \mathcal{K}(X,Y)$, there is a closed infinite-dimensional subspace $W \subset C_p$ as well as compact operators $A_0 \in \mathcal{K}(X,W)$, $B_0 \in \mathcal{K}(W,Y)$ so that $T = B_0A_0$. Following Aron et al. [3] and [27] we consider the direct sum

$$Z_{FJ}^p = \big(\oplus_W W\big)_{\ell^p},$$

where W runs through all the closed infinite-dimensional subspaces of C_p in the summation. The case $p = \infty$ is interpreted as a vector-valued c_0-type direct sum (see below).

The space Z_{FJ}^p is a Figiel-Johnson universal compact factorisation space for $1 \le p \le \infty$, since Z_{FJ}^p has the following universal property: for any Banach spaces X, Y and $T \in \mathcal{K}(X,Y)$ there is $A \in \mathcal{K}(X, Z_{FJ}^p)$ and $B \in \mathcal{K}(Z_{FJ}^p, Y)$ so that $T = BA$. In fact, if $T = B_0A_0$ factors compactly through some $W \subset C_p$ as above, put $A = J_WA_0$ and $B = B_0P_W$, where $J_W : W \to Z_{FJ}^p$ is the inclusion map into the W:th component, and $P_W : Z_{FJ}^p \to W$ is the corresponding canonical projection.

Note further that Z_{FJ}^p and Z_{FJ}^q are non-isomorphic spaces for $p \ne q$. This follows from the fact that the space Z_{FJ}^p is ℓ^p-saturated, that is, any closed infinite-dimensional subspace $M \subset Z_{FJ}^p$ contains an isomorphic copy of ℓ^p. The non-isomorphism result will not be required here, and we only refer to the analogous argument for countable direct sums [23, Thm. 2] (cf. also the proof of [30, Prop. 2.2]).

Let Γ be an uncountable set. Recall that $(x_\gamma) \in c_0(\Gamma)$ if the set $\{\gamma \in \Gamma : |x_\gamma| > \varepsilon\}$ is finite for all $\varepsilon > 0$. It is well known that $c_0(\Gamma)$ is a non-separable Banach space equipped with the supremum norm $\|\cdot\|_\infty$. We next show that the universal factorisation spaces Z_{FJ}^p have a non-separable quotient algebra $\mathfrak{A}_{Z_{FJ}^p}$ for all $1 \le p \le \infty$.

THEOREM 4.4. *There is an uncountable set Γ so that $c_0(\Gamma)$ embeds isomorphically as a linear subspace into $\mathfrak{A}_{Z^p_{FJ}}$ for any $1 \leq p \leq \infty$.*

PROOF. We fix $p \in [1, \infty]$ for the duration of the argument. The main novelty of the argument is contained in the following claim.

Claim. There is an uncountable family $\{Z_\gamma : \gamma \in \Gamma\}$ of distinct closed subspaces of C_p such that $\mathfrak{A}_{Z_\gamma} \neq \{0\}$ for all $\gamma \in \Gamma$.

Observe first that there is a closed subspace $Z \subset C_p$ such that $\mathfrak{A}_Z \neq \{0\}$. In fact, consider Banach spaces X and Y such that $\mathcal{A}(X,Y) \subsetneq \mathcal{K}(X,Y)$. The preceding Johnson-Figiel factorisation applied to $A \in \mathcal{K}(X,Y) \setminus \mathcal{A}(X,Y)$ gives a closed subspace $W \subset C_p$, as well as $B \in K(W,Y)$ and $C \in K(X,W)$, so that $A = BC$. Here B cannot be an approximable operator, since A is not approximable. Thus we may also factor $B \in K(W,Y)$ compactly through a closed subspace $V \subset C_p$. This produces a compact non-approximable operator $S : W \to V$, so that $T = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \notin \mathcal{A}(Z)$, where

$$Z := W \oplus V \subset C_p \oplus C_p \cong C_p.$$

Here $T(x,y) = (0, Sx)$ for $(x,y) \in Z$. Consequently $T \in \mathcal{K}(Z) \setminus \mathcal{A}(Z)$.

Next we find a sequence of distinct closed subspaces Z_n, where $Z \subset Z_n \subset C_p$, as well as operators $T_n \in \mathcal{K}(Z_n) \setminus \mathcal{A}(Z_n)$ for $n \in \mathbb{N}$. In fact, since the subspace $Z \subset C_p$ does not have the A.P., the quotient space C_p/Z is infinite-dimensional, and we may pick a normalised basic sequence $(x_n + Z)_{n \in \mathbb{N}}$ in C_p/Z, see e.g. [25, Thm. 1.a.5]. Consider the closed linear subspace

$$Z_n = Z + [x_k : 1 \leq k \leq n] \subset C_p, \quad n \in \mathbb{N},$$

where we use $[D]$ to denote the closed linear span in C_p of any subset $D \subset C_p$. It is clear that $Z_n \neq Z_m$ whenever $n \neq m$, since the sequence (x_k) is independent modulo Z. Moreover, vectors $x \in Z_n$ have a unique representation $x = z + \sum_{k=1}^n c_k x_k$, where $z \in Z$. Hence there is for any $n \in \mathbb{N}$ a bounded projection $P_n : Z_n \to Z$, which is defined by $P_n x = z$ for $x = z + \sum_{k=1}^n c_k x_k \in Z_n$. We next define the linear map T_n in Z_n by

$$T_n x = Tz + \sum_{k=1}^n c_k x_k, \quad x = z + \sum_{k=1}^n c_k x_k \in Z_n,$$

for $n \in \mathbb{N}$. One may identify $T_n = \begin{pmatrix} T & 0 \\ 0 & I_n \end{pmatrix}$ with respect to the direct sum decomposition $Z_n = Z \oplus [x_k : 1 \leq k \leq n]$, where I_n is the associated identity map. It follows that $T_n : Z_n \to Z_n$ is a compact non-approximable operator, since $T \in \mathcal{K}(Z) \setminus \mathcal{A}(Z)$. Hence $\mathfrak{A}_{Z_n} \neq \{0\}$ for each $n \in \mathbb{N}$.

Recall from [18, p. 341] that $C_p \cong \big(\bigoplus_{k=1}^{\infty} C_p\big)_{\ell^p}$, so that the above construction can be applied coordinatewise in each summand. This gives an uncountable family $\{Z_\gamma : \gamma \in \Gamma\}$ of closed subspaces of C_p, where

$$Z_\gamma = \big(\oplus_{j\in\mathbb{N}} (Z^{(j)})_{n_j} \big)_{\ell^p} \subset \big(\bigoplus_{k=1}^{\infty} C_p \big)_{\ell^p} \cong C_p$$

and $\gamma = (n_j)_{j\in\mathbb{N}} \in \Gamma := \prod_{j\in\mathbb{N}} \mathbb{N}$. Here $Z^{(j)}$ denotes an isomorphic copy of Z in the j:th summand C_p in the above direct sum. To check that $Z_\gamma \neq Z_{\gamma'}$ whenever $\gamma \neq \gamma'$, note that if $\gamma \neq \gamma'$, where $\gamma = (n_j)$ and $\gamma' = (m_j)$, then $n_j \neq m_j$ for some $j \in \mathbb{N}$. Thus the respective j:th component spaces satisfy $(Z^{(j)})_{n_j} \neq (Z^{(j)})_{m_j}$, whence $Z_\gamma \neq Z_{\gamma'}$. Moreover, $(Z^{(1)})_{n_1} \subset Z_\gamma$ is a complemented subspace, so it follows from Proposition 4.2 that $\mathfrak{A}_{Z_\gamma} \neq \{0\}$ for any $\gamma \in \Gamma$. This finishes the verification of the Claim.

To complete the argument we consider the complemented subspace $Y = \big(\oplus_{\gamma\in\Gamma} Z_\gamma\big)_{\ell^p}$ of Z^p_{FJ}. By Proposition 4.2 it suffices to find a linear isomorphic embedding $c_0(\Gamma) \to \mathfrak{A}_Y$. This step is a modification of the corresponding argument in Proposition 2.2. Firstly, for each $\gamma \in \Gamma$ pick a compact operator $T_\gamma \in \mathcal{K}(Z_\gamma)$ so that

$$dist(T_\gamma, \mathcal{A}(Z_\gamma)) = 1, \quad \|T_\gamma\| < 2.$$

Let $U_\gamma = J_\gamma T_\gamma P_\gamma$, where $J_\gamma : Z_\gamma \to Y$ is the natural inclusion map and $P_\gamma : Y \to Z_\gamma$ the natural projection for $\gamma \in \Gamma$. Then the series $\sum_{\gamma\in\Gamma} a_\gamma U_\gamma$ defines a compact operator $Y \to Y$, and

$$\|\sum_{\gamma\in\Gamma} a_\gamma U_\gamma\| \leq 2 \cdot \sup_\gamma |a_\gamma|, \tag{10}$$

for any $(a_\gamma) \in c_0(\Gamma)$. In fact, consider the finite set $\Gamma_r = \{\gamma \in \Gamma : |a_\gamma| > 1/r\}$ and the associated compact operator $V_r = \sum_{\gamma\in\Gamma_r} a_\gamma U_\gamma$ on Y for $r \in \mathbb{N}$. It follows that $(V_r) \subset \mathcal{K}(Y)$ is a Cauchy sequence, since

$$\|V_{r+s} - V_r\| = \|\sum_{\gamma\in\Gamma_{r+s}\setminus\Gamma_r} a_\gamma U_\gamma\| \leq 2/r$$

for each $r, s \in \mathbb{N}$. Thus $\sum_{\gamma\in\Gamma} a_\gamma U_\gamma = \lim_{r\to\infty} V_r$ defines a compact operator on Y. To check (10), put $\Gamma_0 = \cup_{r\in\mathbb{N}}\Gamma_r$ and let $x = (x_\gamma) \in Y$. It follows that

$$\|\sum_{\gamma\in\Gamma} a_\gamma U_\gamma x\|_{\ell^p} = \|\sum_{\gamma\in\Gamma_0} a_\gamma J_\gamma T_\gamma x_\gamma\|_{\ell^p} \leq 2 \cdot \sup_\gamma |a_\gamma| \cdot \|x\|_{\ell^p}.$$

Finally, suppose that $(a_\gamma) \in c_0(\Gamma)$ and $|a_\beta| = \|(a_\gamma)\|_\infty$ for some $\beta \in \Gamma$. If $V \in \mathcal{A}(Y)$ is arbitrary, then

$$\|\sum_{\gamma\in\Gamma} a_\gamma U_\gamma - V\| \geq \|P_\beta\big(\sum_{\gamma\in\Gamma} a_\gamma U_\gamma - V\big)J_\beta\| = \|a_\beta T_\beta - P_\beta V J_\beta\| \geq |a_\beta|,$$

since $U_\gamma J_\beta = 0$ for $\gamma \neq \beta$ by construction.

This completes the proof that $c_0(\Gamma)$ embeds into $\mathfrak{A}_Y$, and consequently of the theorem, after an application of Proposition 4.2. □

5. Concluding remarks

We recall that it is unknown whether $\mathfrak{A}_P \neq \{0\}$, where P belongs to the family of spaces constructed by Pisier [35], [36, section 10]. The spaces P fail the A.P. and $\mathcal{A}(P) = \mathcal{N}(P)$, where $\mathcal{N}(P)$ denotes the space of nuclear operators $P \to P$. Moreover, $\mathcal{K}(P, P^*) = \mathcal{A}(P, P^*) = \mathcal{N}(P, P^*)$ by [17].

Dales [10] also points out that it remains unclear whether there is a Banach space X failing the A.P., such that $\mathfrak{A}_X \neq \{0\}$ is a finite-dimensional (radical) algebra. Note that if such a space X exists, then X cannot have the B.C.A.P. by Proposition 3.1.(i). We recall in comparison that Argyros and Haydon [2] constructed Banach spaces X_{AH} having a Schauder basis such that the Calkin algebra $\mathcal{L}(X_{AH})/\mathcal{K}(X_{AH})$ is one-dimensional. Moreover, let $\mathcal{S}(X)$ be the closed ideal of $\mathcal{L}(X)$ consisting of the strictly singular operators. The quotient algebra $\mathcal{S}(X)/\mathcal{K}(X)$ is also a radical algebra, whose properties are better understood for classical Banach spaces X. Here it may happen that $\mathcal{S}(X)/\mathcal{K}(X) = \{0\}$ (this is the case if $X = \ell^p$ for $p \in [1, \infty)$, see e.g. [34, 5.1-5.2]). Moreover, Tarbard [40] constructed for each natural number $k \geq 2$ a Banach space X_k such that $dim\big(\mathcal{S}(X_k)/\mathcal{K}(X_k)\big) = k - 1$ (and $dim\big(\mathcal{L}(X_k)/\mathcal{K}(X_k)\big) = k$).

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