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A behavioral approach to estimation in nD systems ⋆

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Abstract: In this paper we study the problem of estimation for multidimensional systems within the context of the behavioral approach. We consider the case where there are no disturbances as well as the case where the system dynamics is perturbed, and provide necessary conditions for the solvability of the corresponding estimation problems together with the construction of a solution, if it exists. Such solution is an estimator that is asymptotic, in the sense that the error trajectories are stable with respect to a pre-specified nD stability cone.

Keywords: Multidimensional behavioral systems, observer, stability, detectability subspace, estimation

1. INTRODUCTION

The construction of observers to estimate the state or a linear function thereof is an important issue in the classical theory of linear 1D state space systems and its applications (see, for instance, Basile and Marro [1969], Trentelman et al. [2001], and corresponding references). This question has also deserved the attention of researchers in the field of multidimensional systems, having given rise to a considerable body of results both for Roesser and Fornasini-Marchesini models, Conte and Perdon [1988], Ntogramatzidis and Cantoni [2012], Ntogramatzidis et al. [2008].

In this paper, we consider the more general case of multidimensional behavioral systems, and study the problem of estimation inspired by the behavioral theory of 1D observers, Valcher and Willems [1999], Trumpf et al. [2011], by the classical theory of observers and detectability subspaces (see Trentelman et al. [2001] and the references therein), and by our own results on observers and detectability subspaces for behavioral nD systems, Pereira and Rocha [2019], as well as for 2D behavioral systems, Bisiacco and Valcher [2008].

More concretely, we focus on discrete multidimensional behaviors with split variable (w, w_1), where w_1 is measured and w is not available for measurement, and consider the problem of estimating a linear function w_2 of the components of w and their shifts using a suitable asymptotic observer. This observer produces an estimate $\widehat{w}_2$ of w_2 such that the corresponding error $e := \widehat{w}_2 - w_2$ is stable with respect to a prespecified nD cone $\mathcal{S}$, i.e., $\lim_{\substack{\lambda \to +\infty \\ \lambda \in \mathbb{N}}} e(\lambda \overrightarrow{v}) = 0$ for every direction $\overrightarrow{v}$ contained in $\mathcal{S}$, Rocha [2008], Pillai and Shankar [1998]. Besides this problem, we also consider the case where an unknown disturbance is present.

The paper is organized as follows. Section 2 contains the necessary background material. Section 3 is devoted to the definition and solution (when possible) of two (deterministic) variable estimation problems: the first one without disturbances, and the second one with the presence of an unknown disturbance. Concluding remarks are left to Section 4.

2. PRELIMINARIES

As mentioned in the Introduction, we consider discrete multidimensional behaviors. More concretely, we assume that the corresponding admissible signals w are defined over $\mathbb{Z}^n$, take values on $\mathbb{R}^{\mathtt{w}}$, for some suitable positive integer $\mathtt{w}$, and are the solutions of a system of linear partial difference equations with constant coefficients that can be written in matrix form as:

$$H(\underline{\sigma}, \underline{\sigma}^{-1})w \equiv 0, \tag{1}$$

where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\underline{\sigma}^{-1} = (\sigma_1^{-1}, \ldots, \sigma_n^{-1})$, the σ_i's are the elementary nD shifts and $H(\underline{s}, \underline{s}^{-1})$ is an nD Laurent-polynomial matrix in the indeterminates $\underline{s} = (s_1, \ldots, s_n)$. The nD behavior $\mathcal{B}_w$ described by (1) coincides with $\ker H(\underline{\sigma}, \underline{\sigma}^{-1})$, when the operator $H(\underline{\sigma}, \underline{\sigma}^{-1})$ acts on the universe $\mathcal{U}_w := (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n}$ of the variable w.

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For the sake of simplicity, we sometimes omit the shift operators (indeterminates) and write H instead of $H(\underline{\sigma}, \underline{\sigma}^{-1})$ $(H(\underline{s}, \underline{s}^{-1}))$.

We next recall some definitions and results that will be used throughout the paper.

2.1 Behavior inclusion and quotient

Given two nD behaviors $\mathcal{B}_w^1 = \ker H_1$ and $\mathcal{B}_w^2 = \ker H_2$, $\mathcal{B}_w^1 \subset \mathcal{B}_w^2$ if and only if there exists an nD Laurent-polynomial matrix $E(\underline{s}, \underline{s}^{-1})$ such that $H_2(\underline{s}, \underline{s}^{-1}) = E(\underline{s}, \underline{s}^{-1}) H_1(\underline{s}, \underline{s}^{-1})$, Oberst [1990], Zerz [2000].

The quotient $\mathcal{B}_w^2/\mathcal{B}_w^1$ has the structure of a behavior and can be identified with $\ker M$ where $M := \begin{bmatrix} E^\top & L^\top \end{bmatrix}^\top$ and $L(\underline{s}, \underline{s}^{-1})$ is a minimal left annihilator (MLA) of H_1, i.e., $LH_1 = 0$ and if $Q(\underline{s}, \underline{s}^{-1})$ is such that $QH_1 = 0$ then $Q = ML$ for some nD Laurent-polynomial matrix $M(\underline{s}, \underline{s}^{-1})$, Rocha and Wood [2001].

2.2 Autonomy

Here we define an *autonomous* behavior $\mathcal{B}_w$ as a behavior where none of the components $w_i, i = 1, \ldots, \mathtt{w}$, of the variable w is free, i.e., no w_i can be arbitrarily assigned as a function from $\mathbb{Z}^n$ to $\mathbb{R}$. It turns out that $\mathcal{B}_w = \ker H$ is autonomous if and only if the nD Laurent-polynomial matrix $H(\underline{s}, \underline{s}^{-1})$ has full column rank (over the ring of nD Laurent-polynomials $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$), Zerz [2000].

2.3 Stability

The definition of stability used in this paper is the one introduced in Rocha [2008]. However, most of our results also apply to other stability notions, such as for instance the one introduced in Valcher [2000].

In order to formalize our stability property, we start by defining a *direction* in $\mathbb{Z}^n$ as an element $\overrightarrow{v} = (v_1, \ldots, v_n)$ with integer coprime components. A *stability cone* $\mathcal{S}$ in $\mathbb{Z}^n$ is the set of all positive integer linear combinations of n linearly independent directions.

Given a stability cone $\mathcal{S} \subset \mathbb{Z}^n$, a signal $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^n}$ is said to be $\mathcal{S}$-*stable* if $\lim\limits_{\substack{\lambda \to +\infty \\ \lambda \in \mathbb{N}}} w(\lambda \overrightarrow{v}) = 0$ for every direction $\overrightarrow{v} \in \mathcal{S}$. A behavior $\mathcal{B}_w$ is said to be S-*stable* if all the signals $w \in \mathcal{B}_w$ are $\mathcal{S}$-stable. According to what was shown in Rocha [2008] an nD behavior $\mathcal{B}_w = \ker H(\underline{\sigma}, \underline{\sigma}^{-1})$ is $\mathcal{S}$-stable if and only if $H(\underline{s}, \underline{s}^{-1})$ has full column rank $\mathtt{w}$ (over $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$) and, moreover, the following conditions hold:

$$\mathcal{V}(\mathcal{B}_w) := \left\{ \underline{\lambda} \in (\mathbb{C} \setminus \{0\})^n : \text{rank H}(\underline{\lambda}, \underline{\lambda}^{-1}) < \mathtt{w} \right\}$$

is finite; for all $\underline{\lambda} \in \mathcal{V}(\mathcal{B}_w)$ and every direction $\overrightarrow{v} = (k_1, \ldots, k_n) \in \mathcal{S}$, $\left| \lambda_1^{k_1} \ldots \lambda_n^{k_n} \right| < 1$. In this case, we also say that the Laurent-polynomial matrix $H(\underline{s}, \underline{s}^{-1})$ is $\mathcal{S}$-stable.

Note that, in particular, $\mathcal{S}$-stability implies autonomy and finite-dimensionality, i.e., *strong autonomy* Pillai and Shankar [1998].

2.4 Behaviors with split variables and variable elimination

Although in the behavioral approach the system variable is not *a priori* partitioned into inputs and outputs, there are situations where it is convenient to consider it to be split into a certain number of "sub-variables" (that do not necessarily correspond to inputs and outputs).

An nD behavior $\mathcal{B}_{(w_1,w_2)}$ with split variable (w_1, w_2), $w_1 \in (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}^n}$ and $w_2 \in (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$ can be described by a set of linear nD difference equations of the form:

$$H_2(\underline{\sigma}, \underline{\sigma}^{-1}) w_2 = H_1(\underline{\sigma}, \underline{\sigma}^{-1}) w_1, \tag{2}$$

where, $H_1(\underline{s}, \underline{s}^{-1})$ and $H_2(\underline{s}, \underline{s}^{-1})$ are nD Laurent-polynomial matrices with the same number of rows and, respectively, $\mathtt{w}_1$ and $\mathtt{w}_2$ columns.

The w_2-behavior induced by (2) is defined as the projection of $\mathcal{B}_{(w_1,w_2)}$ into the universe of w_2, $(\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^n}$, i.e.,

$$\Pi_{w_2}(\mathcal{B}_{(w_1,w_2)}) = \\ = \{ w_2 \in (\mathbb{R}^{w_2})^{\mathbb{Z}^n} \mid \exists\, w_1 \in (\mathbb{R}^{w_1})^{\mathbb{Z}^n} : (w_1, w_2) \in \mathcal{B}_{(w_1,w_2)} \}$$

The *variable elimination property*, Oberst [1990], states that $\Pi_{w_2}(\mathcal{B}_{(w_1,w_2)})$ can be described as

$$\Pi_{w_2}(\mathcal{B}_{(w_1,w_2)}) = \ker L(\underline{\sigma}, \underline{\sigma}^{-1}) H_2(\underline{\sigma}, \underline{\sigma}^{-1}),$$

where the nD Laurent-polynomial matrix $L(\underline{s}, \underline{s}^{-1})$ is an MLA of $H_1(\underline{s}, \underline{s}^{-1})$.

Similar results also obviously hold when the roles of w_1 and w_2 are interchanged.

2.5 Asymptotic observers

Given an nD behavior $\mathcal{B}_{(w_1,w_2)}$ with measured variables w_1 and to-be-estimated variables w_2, an *observer for* w_2 *from* w_1 is a behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ where w_1 is the measured variable (shared with $\mathcal{B}_{(w_1,w_2)}$) and $\widehat{w}_2$ is an estimate of w_2. The *error* of this estimate is defined as $e := \widehat{w}_2 - w_2$, and the corresponding behavior, known as *error behavior*, is denoted by $\mathcal{B}_e$.

The observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is said to be an $\mathcal{S}$-*asymptotic observer* if $\mathcal{B}_e$ is $\mathcal{S}$-stable (where $\mathcal{S}$ is a given stability cone). If an $\mathcal{S}$-asymptotic observer for w_2 from w_1 exists, w_2 is said to be S-*detectable* from w_1.

Let $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ be the set of all the signals w_2 that are compatible with $w_1 \equiv 0$, i.e.,

$$\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \{ w_2 \mid (0, w_2) \in \mathcal{B}_{(w_1,w_2)} \}.$$

This set is known as the *hidden behavior*, Trumpf et al. [2011], Pereira and Rocha [2019].

If the behavior $\mathcal{B}_{(w_1,w_2)}$ is described by

$$R_2(\underline{\sigma},\underline{\sigma}^{-1})w_2 = R_1(\underline{\sigma},\underline{\sigma}^{-1})w_1, \quad (3)$$

then $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$.

As shown in Pereira and Rocha [2019], w_2 is S-detectable from w_1 (in $\mathcal{B}_{(w_1,w_2)}$) if and only if $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$ is S-stable.

3. BEHAVIORAL ESTIMATION

In this section we study the problem of estimating a linear combination of the components of a non-measured variable and corresponding nD shifts. First we assume that no disturbances are present, and later we consider the case where the system is perturbed by an unknown disturbance.

3.1 The behavior estimation (BE) problem

We define the *BE problem* as follows.

Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior described by (3), i.e.,

$$R_2(\underline{\sigma},\underline{\sigma}^{-1})w_2 = R_1(\underline{\sigma},\underline{\sigma}^{-1})w_1,$$

where the variable w_1 is measured and w_2 is not available for measurement. Let further

$$z = K(\underline{\sigma},\underline{\sigma}^{-1})w_2, \quad (4)$$

where $K(\underline{s},\underline{s}^{-1})$ is a full row rank Laurent-polynomial matrix, be a linear function of the components of w_2 and their shifts.

Given an nD stability cone $\mathcal{S}$, the behavioral estimation problem (BE problem) consists in designing (if possible) a (deterministic) estimator

$$N(\underline{\sigma},\underline{\sigma}^{-1})\widehat{z} = P(\underline{\sigma},\underline{\sigma}^{-1})w_1 \quad (5)$$

for z from w_1, such that the corresponding estimation error $e := \widehat{z} - z$ is $\mathcal{S}$-stable.

The solvability conditions and the construction of a suitable estimator, in case it exists, are given in the next theorem and its proof.

Theorem 1. Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior described by (3), where the only variable available for measurement is w_1. Let further z be defined by (4), and consider an nD stability cone $\mathcal{S}$. Then, the BE problem is solvable if and only if $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) \subset \ker QK$ for some $\mathcal{S}$-stable nD Laurent-polynomial matrix $Q(\underline{s},\underline{s}^{-1})$.

Proof.

"If part:" Recall that $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$. Moreover, the condition $\ker R_2 \subset \ker QK$ is equivalent to the existence of an nD Laurent polynomial matrix $T(\underline{s},\underline{s}^{-1})$ such that $QK = TR_2$.

Define the following estimator for w_2 from w_1:

$$TR_2\widehat{w}_2 = TR_1w_1. \quad (6)$$

Equation (3) implies that

$$TR_2w_2 = TR_1w_1, \quad (7)$$

which, together with (6) leads to

$$TR_2(\widehat{w}_2 - w_2) = 0. \quad (8)$$

Now, define

$$\widehat{z} := K\widehat{w}_2 \quad (9)$$

as an estimate for z. Since, by (4), $z = Kw_2$, this implies that the estimation error $e = \widehat{z} - z$ is such that

$$e = K(\widehat{w}_2 - w_2). \quad (10)$$

Because, by assumption $QK = TR_2$, we have by (8) and (10) that

$$Qe = QK(\widehat{w}_2 - w_2) = TR_2(\widehat{w}_2 - w_2) = 0. \quad (11)$$

Thus, the estimation error e belongs to $\ker Q$, and, since Q is $\mathcal{S}$-stable, the same applies to e.

In order to finish this part of the proof it is enough to note that the estimator

$$\begin{cases} TR_2\widehat{w}_2 = TR_1w_1 \\ \widehat{z} = K\widehat{w}_2 \end{cases} \quad (12)$$

gives rise to an estimator expressed only in terms of $\widehat{z}$ and w_1 by eliminating the variable $\widehat{w}_2$ from (12). This is achieved by writing (12) as:

$$\begin{bmatrix} TR_2 \\ K \end{bmatrix}\widehat{w}_2 = \begin{bmatrix} TR_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} w_1 \\ \widehat{z} \end{bmatrix} \quad (13)$$

and applying to both sides of (13) the operator $\left[-A(\underline{\sigma},\underline{\sigma}^{-1}) \ B(\underline{\sigma},\underline{\sigma}^{-1})\right]$ corresponding to an MLA $\left[-A(\underline{s},\underline{s}^{-1}) \ B(\underline{s},\underline{s}^{-1})\right]$ of $\begin{bmatrix} TR_2 \\ K \end{bmatrix}$. Taking into account that $TR_2 = QK$ and that K has full row rank, $[-A \ B] = [-I \ Q]$. This yields

$$[-I \ Q]\begin{bmatrix} TR_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} w_1 \\ \widehat{z} \end{bmatrix} = 0$$

$$\Leftrightarrow -TR_1w_1 + Q\widehat{z} = 0$$

$$\Leftrightarrow Q\widehat{z} = TR_1w_1,$$

which is the form (5) with $N = Q$ and $P = TR_1$.

"Only if part" Assume that the BE problem is solvable, i.e., that there exists an estimator for z from w_1 given by the (matrix) equations

$$N(\underline{\sigma},\underline{\sigma}^{-1})\widehat{z} = P(\underline{\sigma},\underline{\sigma}^{-1})w_1$$

such that the corresponding estimator error is stable. The behavior $\mathcal{B}_e$ of the estimation error $e := \widehat{z} - z$ can be obtained from the equations

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} e = \underbrace{\begin{bmatrix} R_2 & -R_1 & 0 & 0 \\ K & 0 & -I & 0 \\ 0 & P & 0 & -N \\ 0 & 0 & -I & I \end{bmatrix}}_{=:\,\overline{R}} \begin{bmatrix} w_2 \\ w_1 \\ z \\ \widehat{z} \end{bmatrix}. \quad (14)$$

Applying to both sides of (14) the operator

$$\left[W(\underline{\sigma},\underline{\sigma}^{-1}) \ X(\underline{\sigma},\underline{\sigma}^{-1}) \ Y(\underline{\sigma},\underline{\sigma}^{-1}) \ Z(\underline{\sigma},\underline{\sigma}^{-1})\right],$$

where

$$\left[W(\underline{s},\underline{s}^{-1}) \ X(\underline{s},\underline{s}^{-1}) \ Y(\underline{s},\underline{s}^{-1}) \ Z(\underline{s},\underline{s}^{-1})\right]$$

is an MLA of $\overline{R}(\underline{s},\underline{s}^{-1})$, the variables w_2, w_1, z and $\widehat{z}$ are eliminated from (14), yielding the equation:

$$Z(\underline{\sigma},\underline{\sigma}^{-1})e = 0, \quad (15)$$

that describes the behavior $\mathcal{B}_e$. Because $[W\ X\ Y\ Z]$ is an MLA of $\overline{R}$, the following equalities hold:

$$\begin{cases} WR_2 + XK = 0 & (16) \\ -WR_1 + YP = 0 & (17) \\ -X - Z = 0 & (18) \\ -YN + Z = 0. & (19) \end{cases}$$

In particular, it follows from (19) that (15) is equivalent to:

$$Y(\underline{\sigma}, \underline{\sigma}^{-1})N(\underline{\sigma}, \underline{\sigma}^{-1})e = 0$$

and hence $\mathcal{B}_e = \ker YN$. Since $\mathcal{B}_e$ is, by assumption, $\mathcal{S}$-stable, the matrix YN must be $\mathcal{S}$-stable.

Now, from equations (16) and (18)-(19) we respectively get:

$$\begin{cases} WR_2 = -XK \\ -X = YN \end{cases}$$

which implies that

$$WR_2 = YNK.$$

This allows to conclude that

$$\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2 \subset \ker QK$$

with $Q = YN$ $\mathcal{S}$-stable.

Remark 2. As already mentioned, the condition $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2 \subset \ker QK$, for some $\mathcal{S}$-stable Laurent-polynomial matrix, is equivalent to the existence of a Laurent-polynomial matrix $T(\underline{s}, \underline{s}^{-1})$ such that $QK = TR_2$. Writing this equality as

$$[-T\ Q]\begin{bmatrix} R_2 \\ K \end{bmatrix} = 0$$

we conclude that $[-T\ Q]$ is an annihilator of $\begin{bmatrix} R_2 \\ K \end{bmatrix}$. Therefore there exists a Laurent-polynomial matrix $G(\underline{s}, \underline{s}^{-1})$ such that

$$[-T\ Q] = G\,[-F\ E]\,, \qquad (20)$$

where $[-F\ E]$ is an MLA of $\begin{bmatrix} R_2 \\ K \end{bmatrix}$. Clearly, (20) implies that

$$Q = GE.$$

Consequently, $\ker E(\underline{\sigma}, \underline{\sigma}^{-1}) \subset \ker Q(\underline{\sigma}, \underline{\sigma}^{-1})$ and, since $\ker Q$ is $\mathcal{S}$-stable, the same must hold for $\ker E$, meaning that $E(\underline{s}, \underline{s}^{-1})$ is an $\mathcal{S}$-stable matrix. $\diamond$

The previous remark leads to the following corollary of Theorem 1.

Corollary 3. Let $\mathcal{B}_{(w_1,w_2)}$ be an nD behavior described by (3), where the only variable available for measurement is w_1. Let further z be defined as in (4), and consider an nD stability cone $\mathcal{S}$. Then, the BE problem is solvable if and only if every MLA $[-F\ E]$ of $\begin{bmatrix} R_2 \\ K \end{bmatrix}$ is such that $E(\underline{s}, \underline{s}^{-1})$ is an $\mathcal{S}$-stable nD Laurent-polynomial matrix. In this case, an $\mathcal{S}$-asymptotic estimator for z from w_1 is given by $E\widehat{z} = FR_1w_1$.

Proof.

"Only if part:" See Remark 2.

"If part:" Let $[-F\ E]$ be an MLA of $\begin{bmatrix} R_2 \\ K \end{bmatrix}$. Define the following estimator for w_2 from w_1:

$$FR_2\widehat{w}_2 = FR_1w_1,$$

and set $\widehat{z} := K\widehat{w}_2$. Similar to what happens in the proof of Theorem 1, the estimation error $e := \widehat{z} - z$ is such that $e = K(\widehat{w}_2 - w_2)$. Thus,

$$\begin{aligned} Ee = EK(\widehat{w}_2 - w_2) &= FR_2(\widehat{w}_2 - w_2) \\ &= FR_1w_1 - FR_1w_1 = 0, \end{aligned}$$

and therefore $e \in \ker E$. Since, by assumption, $E(\underline{s}, \underline{s}^{-1})$ is $\mathcal{S}$-stable, the error e is also $\mathcal{S}$-stable.

In this way we conclude that the BE problem is solvable by means of the estimator obtained by eliminating the variable w_2 from the description

$$\begin{cases} FR_2\widehat{w}_2 = FR_1w_1 \\ \qquad \widehat{z} = K\widehat{w}_2, \end{cases}$$

as was done in the proof of Theorem 1. Indeed, taking into account that $FR_2 = EK$, and that K has full row rank, we conclude that the $(\widehat{z}, w_1)$ behavior corresponding to the previous equations is given by

$$E\widehat{z} = FR_1w_1.$$

Remark 4. Note that, when the BE problem is solvable, it is possible to construct an $\mathcal{S}$-asymptotic observer for w_2 from w_1 ($FR_2\widehat{w}_2 = FR_1w_1$) with corresponding error behavior $\mathcal{E} = \ker FR_2 = \ker EK$ such that:

- $\mathcal{N} := \mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2 \subset \mathcal{E}$ and
- $\mathcal{E}/\mathcal{N}$ is $\mathcal{S}$-stable (as it is isomorphic to a behavior contained in the $\mathcal{S}$-stable behavior $\ker E$).

This means that the behavior $\mathcal{N}$ is an $\mathcal{S}$-detectability subspace, according to [Pereira and Rocha 2019, Definition 7] $\diamond$

Example 5. Consider the 2D behavior $\mathcal{B}_{(w_1,w_2)}$ described by $R_2(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w_2 = R_1(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w_1$ with

$$R_2(s_1, s_2, s_1^{-1}, s_2^{-1}) = \begin{bmatrix} s_1 + \dfrac{1}{2} \\ (s_2 + \dfrac{1}{2})(s_1 + s_2 + 1) \end{bmatrix}$$

and

$$R_1(s_1, s_2, s_1^{-1}, s_2^{-1}) = \begin{bmatrix} s_1 + s_2 + 2 \\ s_1^2 + s_2^2 \end{bmatrix}.$$

Assume that the only variable available for measurement is w_1 and, moreover, that one wishes to construct an estimator for $z = K(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w_2$, where

$$K(s_1, s_2, s_1^{-1}, s_2^{-1}) = s_2 + \tfrac{1}{2},$$

with respect to the 2D stability cone $\mathcal{S}$ corresponding to the first quadrant of $\mathbb{Z}^2$ (which is generated by the directions $\overrightarrow{v}_1 = (1, 0)$ and $\overrightarrow{v}_2 = (0, 1)$). It is not difficult to check that:

$$\begin{aligned}&[-F(s_1,s_2,s_1^{-1},s_2^{-1})\,|\,E(s_1,s_2,s_1^{-1},s_2^{-1})]\\ &=\left[\begin{array}{cc|c}-(s_2+\frac{1}{2}) & 0 & s_1+\frac{1}{2}\\ 0 & -1 & s_1+s_2+1\end{array}\right]\end{aligned}$$

is an MLA of

$$\begin{bmatrix}R_2\\ K\end{bmatrix}=\begin{bmatrix}s_1+\frac{1}{2}\\ (s_2+\frac{1}{2})(s_1+s_2+1)\\ s_2+\frac{1}{2}\end{bmatrix}.$$

Moreover, the matrix $E(s_1,s_2,s_1^{-1},s_2^{-1})=\begin{bmatrix}s_1+\frac{1}{2}\\ s_1+s_2+1\end{bmatrix}$ has full column rank over $\mathbb{R}[s_1,s_2,s_1^{-1},s_2^{-1}]$ and $E(\lambda_1,\lambda_2,\lambda_1^{-1},\lambda_2^{-1})$ only loses rank for $(\lambda_1,\lambda_2)=(-\frac{1}{2},-\frac{1}{2})$.

Let now $\overrightarrow{v}=(k_1,k_2)$, with k_1 and k_2 positive integers, be an arbitrary direction in $\mathcal{S}$. Clearly:

$$\left|\lambda_1^{k_1}\lambda_2^{k_2}\right|=\left|\left(-\tfrac{1}{2}\right)^{k_1}\left(-\tfrac{1}{2}\right)^{k_2}\right|=\frac{1}{2^{k_1+k_2}}<1,$$

meaning that E is an $\mathcal{S}$-stable matrix. Thus, by Corollary 3 and its proof, the given BE problem is solvable by the estimator that results from eliminating the variable w_2 in the equations:

$$\begin{cases}\begin{bmatrix}-(\sigma_2+\frac{1}{2}) & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\sigma_1+\frac{1}{2}\\ (\sigma_2+\frac{1}{2})(\sigma_1+\sigma_2+1)\end{bmatrix}\widehat{w}_2=\begin{bmatrix}-(\sigma_2+\frac{1}{2}) & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\sigma_1+\sigma_2+2\\ \sigma_1^2+\sigma_2^2\end{bmatrix}w_1\\ \widehat{z}=(\sigma_2+\frac{1}{2})\widehat{w}_2.\end{cases}$$

After the elimination procedure one finally obtains the estimator

$$\begin{bmatrix}\sigma_1+\frac{1}{2}\\ \sigma_1+\sigma_2+1\end{bmatrix}\widehat{z}=\begin{bmatrix}(\sigma_2+\frac{1}{2})(\sigma_1+\sigma_2+2)\\ \sigma_1^2+\sigma_2^2\end{bmatrix}w_1$$

$\diamond$

3.2 The behavior estimation problem with disturbances (BED problem)

We define the *BED problem* as follows.

Let $\mathcal{B}_{(w_1,w_2,d)}$ be an nD behavior described by

$$R_2(\underline{\sigma},\underline{\sigma}^{-1})w_2=R_1(\underline{\sigma},\underline{\sigma}^{-1})w_1+D(\underline{\sigma},\underline{\sigma}^{-1})d, \qquad (21)$$

where the variable w_1 is measured, w_2 is not available for measurement and d is an unknown disturbance. $R_2(\underline{s},\underline{s}^{-1})$, $R_1(\underline{s},\underline{s}^{-1})$ and $D(\underline{s},\underline{s}^{-1})$ are nD Laurent-polynomial matrices and $D(\underline{s},\underline{s}^{-1})$ is assumed not to have full row rank. Let further

$$z=K(\underline{\sigma},\underline{\sigma}^{-1})w_2 \qquad (22)$$

be a linear function of the components of w_2 and their shifts.

Given an nD stability cone $\mathcal{S}$, the behavioral estimation problem with disturbances (BED problem) consists in designing (if possible) a (deterministic) estimator

$$N(\underline{\sigma},\underline{\sigma}^{-1})\widehat{z}=P(\underline{\sigma},\underline{\sigma}^{-1})w_1 \qquad (23)$$

for z from w_1, such that the corresponding estimation error $e:=\widehat{z}-z$ is $\mathcal{S}$-stable independently from the disturbance d.

Remark 6. The assumption that $D(\underline{s},\underline{s}^{-1})$ does not have full row rank implies that the pair (w_1,w_2) is not arbitrary. If this were the case, the estimation problem would not make sense.

Note that the BED problem can be transformed into a BE problem (without disturbances) by eliminating the disturbance d from (21). This is achieved by applying to both sides of (21) an operator $L(\underline{\sigma},\underline{\sigma}^{-1})$, where $L(\underline{s},\underline{s}^{-1})$ is an MLA of $D(\underline{s},\underline{s}^{-1})$. In this way one obtains the equation:

$$L(\underline{\sigma},\underline{\sigma}^{-1})R_2(\underline{\sigma},\underline{\sigma}^{-1})w_2=L(\underline{\sigma},\underline{\sigma}^{-1})R_1(\underline{\sigma},\underline{\sigma}^{-1})w_1 \qquad (24)$$

that describes the (w_1,w_2)-behavior corresponding to (21).

Now, the results for the BE problem can be used to derive the solvability conditions of the BED problem, yielding the following theorem.

Theorem 7. Let $\mathcal{B}_{(w_1,w_2,d)}$ be an nD behavior described by (21), where the only variable available for measurement is w_1 and d is an unknown disturbance. Let further z be defined by (22), and $L(\underline{s},\underline{s}^{-1})$ be an MLA of $D(\underline{s},\underline{s}^{-1})$. Given an nD stability cone $\mathcal{S}$, the corresponding BED problem is solvable if and only if every MLA $[-F\ E]$ of $\begin{bmatrix}LR_2\\ K\end{bmatrix}$ is such that $E(\underline{s},\underline{s}^{-1})$ is an $\mathcal{S}$-stable nD Laurent-polynomial matrix. In this case, an $\mathcal{S}$-asymptotic observer for z from w_1 is given by:

$$E\widehat{z}=FLR_1w_1.$$

4. CONCLUSIONS

In this paper we considered the problem of estimation for discrete nD systems following the behavioral approach. More concretely, given a behavior with partitioned system variable $w=(w_1,w_2)$, where only w_1 is available for measurement, we studied the problem of finding a deterministic asymptotic estimator for a variable z with components given by linear combinations of the components of w_2 and their nD shifts. We derived necessary conditions for the solvability of this problem and, in case those conditions were fulfilled, provided a suitable estimator. Such estimator is asymptotic in the sense that the corresponding error trajectories are stable with respect to a pre-specified nD stability cone.

Besides this (unperturbed) estimation problem, we also considered the case where unknown disturbances are present, affecting the relationship between w_2 and w_1. This case can easily be transformed into the previous one, using the elimination principle to remove the disturbances.

Future work includes the application of our approach to classical nD state-space models.

REFERENCES

Basile, G. and Marro, G. (1969). Controlled and conditioned invariant subspaces in linear system theory.

Journal of Optimization Theory and Applications, 3(5), 306–315. doi:10.1007/BF00931370.

Bisiacco, M. and Valcher, M. (2008). Dead-beat estimation problems for 2d behaviors. *Multidimensional Systems And Signal Processing*, 19(3), 287–306. doi: 10.1007/s11045-007-0045-9.

Bose, N.K. (1995). *Multidimensional Systems Theory and Applications*. Kluwer Academic Publishers, Springer-Verlag.

Conte, G. and Perdon, A. (1988). A geometric approach to the theory of 2-d systems. *IEEE Transactions on Automatic Control*, 33(10), 946–950. doi:10.1109/9.7251.

Ntogramatzidis, L. and Cantoni, M. (2012). Detectability subspaces and observer synthesis for two-dimensional systems. *Multidimensional Systems and Signal Processing*, 23(1), 79–96. doi:10.1007/s11045-010-0118-z. URL `https://doi.org/10.1007/s11045-010-0118-z`.

Ntogramatzidis, L., Cantoni, M., and Yang, R. (2008). A geometric theory for 2-d systems including notions of stabilisability. *Multidimensional Systems and Signal Processing*, 19(3), 449–475. doi:10.1007/s11045-007-0046-8. URL `https://doi.org/10.1007/s11045-007-0046-8`.

Oberst, U. (1990). Multidimensional constant linear systems. *Acta Appl. Math.*, 20(1-2), 1–175.

Pereira, R. and Rocha, P. (2019). Towards a geometric theory for nd behaviors:conditioned invariance and detectability subspaces. *IFAC-PapersOnLine*, 52(17), 77 – 81. doi:https://doi.org/10.1016/j.ifacol.2019.11.030. 7th IFAC Symposium on System Structure and Control SSSC 2019.

Pillai, H. and Shankar, S. (1998). A behavioral approach to control of distributed systems. *SIAM J. Control Optim.*, 37(2), 388–408.

Rocha, P. (2008). Stabilization of multidimensional behaviors. *in Multidimens. Systems Signal Process.*, 19, 273–286.

Rocha, P. and Wood, J. (2001). Trajectory control and interconnection of 1D and nD systems. *SIAM J. Control Optim.*, 40(1), 107–134.

Trentelman, H.L., Stoorvogel, A.A., and Hautus, M. (2001). *Control theory for linear systems.* London: Springer.

Trumpf, J., Trentelman, H.L., and Willems, J.C. (2011). An internal model principle for observers. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, 3992–3999. doi: 10.1109/CDC.2011.6160666.

Valcher, M.E. and Willems, J.C. (1999). Observer synthesis in the behavioral approach. *IEEE Trans. Autom. Control*, 44(12), 2297–2307. doi:10.1109/9.811213.

Valcher, M. (2000). Characteristic cones and stability properties of two-dimensional autonomous behaviors. *IEEE Trans. on Circuits and Systems, Part I*, CAS-47(no.3), pp.290–302.

Zerz, E. (2000). *Topics in Multidimensional Linear Systems Theory*, volume 256. Lecture Notes in Control and Information Sciences, Springer-Verlag London.