A new convolution operator for the linear canonical transform with applications*

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Abstract

The linear canonical transform plays an important role in engineering and many applied fields, as it is the case of optics and signal processing. In this paper, a new convolution for the linear canonical transform is proposed and a corresponding product theorem is deduced. It is also proved a generalized Young's inequality for the introduced convolution operator. Moreover, necessary and sufficient conditions are obtained for the solvability of a class of convolution type integral equations associated with the linear canonical transform. Finally, the obtained results are implemented in multiplicative filters design, through the product in both the linear canonical transform domain and the time domain, where specific computations and comparisons are exposed.

Keywords Linear canonical transform, convolution, integral equations, filtering

Mathematics Subject Classification (2010) 44A15, 44A05, 44A35, 45E10, 94A12

1 Introduction

The linear canonical transform (LCT) (Healy et al., 2015, Moshinsky and Quesne, 1971) is a four-parameter family of linear integral transformations. The flexibility of choice on those parameters has several consequences and, in particular, allows us to recognize that many integral transforms are just special cases of the LCT. It is the case of the Fourier transform (FT), the fractional Fourier transform (FrFT), the Fresnel transform (FnT), and many other linear integral transforms used in signal processing and optics (Goel and Singh, 2016, Pei and Ding, 2001, Sharma and Joshi, 2006, Zhang, 2016a,b). Therefore, the study of the LCT, its properties, and associated operators are of great importance in view to obtain a unified analysis of the above-mentioned transforms.

*This is a post-peer-review, pre-copyedit version of an article published in Computational and Applied Mathematics. The final authenticated version is available online at: `https://doi.org/10.1007/s40314-021-01484-9`.

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The LCT was first introduced in the 1970s (Moshinsky and Quesne, 1971, Collins, 1970), where its significance in optics was recognized. Later, the interest in the fractional Fourier transform, during the 1990s, led to the emergence of a new interest in the LCT from new perspectives.

The four parameters in the LCT give extra degrees of freedom, which makes this transformation a very flexible one, and a powerful tool in the fields of signal processing, filter design, radar system analysis, pattern recognition, optics, solvability of integral and differential equations, as well as in many other areas of applied sciences (Anh et al., 2017, 2019, Barshan et al., 1997, Deng et al., 2006, Goel and Singh, 2013, 2016, Goel et al., 2016, Sharma and Joshi, 2006, Shi et al., 2012, 2014, Zhang, 2016a,b). Many properties of the LCT are currently well known (Sharma and Joshi, 2006, Xu and Li, 2013). In particular, this transform is unitary, invertible, and additive. The last property allows the LCT to be decomposed into series of simpler transforms. Some approaches to define discrete LCT (DLCT) have been proposed with several algorithms for its computation and a number of possible discretizations of LCT with different desirable properties (Pei and Ding, 2000, Healy et al., 2015, Koc et al., 2008, Koç et al., 2019, Pei and Lai, 2011, Hennelly and Sheridan, 2005). Those DLCTs are applied in the simulation of optical systems and in pure digital signal processing (Pei and Ding, 2000, Hennelly and Sheridan, 2005).

Convolutions and the so-called "convolution type operators" (Bogveradze and Castro, 2008, Castro and Saitoh, 2012, Castro and Speck, 2000, Castro et al., 2020) are very important mathematical objects which are used in the modeling of a great diversity of applied problems. As it is well known, the classical convolution operator "$*$" (Bracewell and Bracewell, 1986) is given by:

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d}\tau \,. \tag{1.1}$$

Its convolution theorem states that the Fourier transform of the convolution of two functions (in the appropriate spaces) is equal to the pointwise product of their Fourier transforms:

$$\mathcal{F}(f * g)(u) = (\mathcal{F}f)(u)\,(\mathcal{F}g)(u). \tag{1.2}$$

In the last years, many classical results in the Fourier transform domain have been extended to the LCT domain. By defining different forms of convolution operators, a variety of convolution theorems for LCT have been derived (Deng et al., 2006, Wei et al., 2009, 2012, Zhang, 2016a,b, Shi et al., 2014, Goel et al., 2016, Huo, 2019). Those convolutions, with applications in many theoretical and practical problems, can be considered as some extensions of the classical convolution operator for the LCT.

In this paper, we propose a new convolution operator associated with LCT, the corresponding convolution theorem, and we derive a Young's generalized inequality. Furthermore, we discuss the solvability of a class of convolution integral equations associated with the new convolution operator here introduced, and we present multiplicative filters design in the LCT and in the time domain.

We would like to stress that the convolution derived in this work differs from those obtained by (Pei and Ding, 2001, Deng et al., 2006, Shi et al., 2014, Wei et al., 2009, 2011, 2012, Zhang, 2016a,b, Huo, 2019). Although the convolution theorem proposed by Pei and Ding (Pei and Ding, 2001) is similar to the classical one, that convolution is more complicated than our, since it is defined by a triple integral. Additionally, Deng et al. (2006) obtained convolution operators and convolution theorems in the LCT and the time domains. Similarly to them, we cannot eliminate the chirp multiplier in the factorizations. However, with our convolution, we are able to derive a Young's type inequality and to study the solvability of a convolution type integral equation.

The convolution (and its associated theorem) proposed by Shi et al. (2014) has also a significant degree of complexity and, consequently, is not so useful in filter design. Wei et al. (2009, 2012) proposed a convolution theorem that differs from our, since it is based on a generalized translation. Huo (2019) introduced a convolution theorem which is a generalized version of Anh et al. (2017). Although the factorization property in those works is somehow similar to the one proposed in here, the structure of their operator is more complicated than ours, even when rewriting the operators using the help of the classical convolution. Due to this circumstance, there are also corresponding differences in the solvability of associated classes of integral equations.

In this paper, we also propose an equation with a weight function. Making use of the classical convolution, within the process of solving the integral equation, we are able to take profit of the use of the Fourier transform–which is easier to implement in concrete examples. Moreover, we can apply the obtained results in filter design in the LCT and in the time domains. The operator here introduced follows, in some sense, the method used by Zhang (2016a,b). Similarly to that one, our convolution theorem has a pertinent degree of simplicity, in both time and LCT domains, and it is easy to implement in the design of multiplicative filters. However, the disadvantage is the chirp multiplier. Although, theoretically, the chirp function seems to give some freedom to make adjustments, it may impose difficulties in real applications, given the difficulty in generating a chirp signal accurately in practical engineering.

Besides the present section, we organize the paper by having a Sect. 2 where the definition and some basic properties of the LCT are included. The new convolution operator and its associated convolution theorem are proposed in Sect. 3. In Sect. 4, it is derived a Young's type inequality and it is proposed the solvability of a class of convolution integral equations. Finally, Sect. 5 is devoted to a even more applied situation and analyzes filtering in the LCT and in the time domain, where concrete cases are exposed.

2 The linear canonical transform

The linear canonical transform (LCT) with a set of real parameters $A = (a, b, c, d)$ of a signal $f(t)$ is defined as (Moshinsky and Quesne, 1971):

$$\begin{aligned} F_A(u) &= L_A[f(t)](u) \\ &= \begin{cases} \int_{-\infty}^{+\infty} f(t) h_A(t,u)\mathrm{dt} & b \neq 0 \\ \sqrt{d} e^{j\frac{cd}{2}u^2} f(du) & b = 0 \end{cases} \end{aligned} \tag{2.3}$$

where

$$h_A(t,u) = \sqrt{\frac{1}{j2\pi b}} e^{\frac{j}{2b}[at^2 - 2tu + du^2]},$$

j is the imaginary unit and $ad-bc=1$. The case $b=0$ is the limit of the integral in (2.3) for the case $b\neq 0$ and $|b|\to 0$. Without loss of generality, we assume $b\neq 0$ in the remaining part of the paper. The LCT includes many linear integral transforms as special cases. For instance, if $A=(0,1,-1,0)$, then LCT reduces to the Fourier transform; if $A=(\cos\theta,\sin\theta,-\sin\theta,\cos\theta)$, then LCT becomes the fractional Fourier transform; if $A=(1,b,0,1))$, the LCT becomes the Fresnel transform.

The inverse of the LCT with a set of parameters $A=(a,b,c,d)$ is given by an LCT with a set of parameters $A^{-1}=(d,-b,-c,a)$:

$$\begin{aligned} f(t) &= L_{A^{-1}}[F_A(u)](t) \\ &= \int_{-\infty}^{\infty} F_A(u)h_{A^{-1}}(u,t)\mathrm{du}. \end{aligned} \tag{2.4}$$

The LCT has many properties. In what follows, we recall the space shift and phase shift properties of LCT (Sharma and Joshi, 2006), which can also be directly verified using the definition (2.3):

(1) The space shift property:

$$L_A[f(t-\tau)](u)=F_A(u-a\tau)e^{-j\frac{ac\tau^2}{2}+jc\tau u}. \tag{2.5}$$

(2) The phase shift property:

$$L_A[f(t)e^{jvt}](u)=F_A(u-bv)e^{-j\frac{bdv^2}{2}+jdvu}. \tag{2.6}$$

(3) The space shift and phase shift properties:

$$L_A[f(t-\tau)e^{jvt}](u)=F_A(u-a\tau-bv)e^{-j\left(\frac{ac\tau^2+bdv^2}{2}-(c\tau+dv)u+bc\tau v\right)}. \tag{2.7}$$

In this paper, we define the Fourier transform of a signal $f(t)$ in the form

$$(\mathcal{F}f)(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-jut}\mathrm{dt}\,, \qquad u\in\mathbb{R}, \tag{2.8}$$

and its inverse as:

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\mathcal{F}f)(u)e^{jut}\mathrm{du}, \quad t\in\mathbb{R}.$$

We may observe that:

$$\sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} e^{\frac{j}{2b}(at^2-2tu+du^2)}f(t)\mathrm{dt} \;=\; \sqrt{\frac{1}{jb}}e^{\frac{j}{2b}du^2}\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-jt\frac{u}{b}}\cdot e^{j\frac{at^2}{2b}}f(t)\mathrm{dt}\,,$$

which allows us to recognize that the LCT of a signal $f(t)$ can be reduced to the standard Fourier transform. Namely, the LCT with a set of parameters $A=(a,b,c,d)$ of a signal $f(t)$ can be reduced to the Fourier transform (2.8) in the following way:

$$L_A[f(t)](u)=\frac{1}{\sqrt{jb}}e^{\frac{j}{2b}du^2}\mathcal{F}[\overline{f}(t)]\left(\frac{u}{b}\right),$$

where $\overline{f}(t)=e^{j\frac{at^2}{2b}}f(t)$.

3 New convolution and product theorem

In this section, we introduce a new convolution operator, and then, we study the corresponding convolution theorem associated with the LCT (based on its definition and properties). In addition, we will compare the proposed convolution theorem with the literature in terms of variable dependability, FT conversion, and hardware complexity (number of chirp functions).

3.1 Proposed convolution theorem

Let W be the subspace of all integrable functions with the property that $f(t) \in W$ if and only if $L_A[f(t)]$ is also in W.

Definition 1. *For any functions $f, g \in W$, we define a new convolution operator $f \otimes g$ for the LCT as follows:*

$$(f \otimes g)(t) = \sqrt{\frac{1}{j8\pi b}} \int_{\mathbb{R}} f(\tau) g(t-\tau) e^{j\frac{a}{4b}\left(-t^2+2\tau^2-2t\tau\right)} \mathrm{d}\tau \,. \tag{3.9}$$

Theorem 1. *Assume that $h(t) = (f \otimes g)(t)$ and that $H_A(u)$ denotes the LCT of $h(t)$, with a set of parameters $A = (a, b, c, d)$, and let $F_{\widetilde{A}}(u)$, $F_{\overline{A}}(u)$, $G_{\widetilde{A}}(u)$ and $G_{\overline{A}}(u)$ be the LCT of $f(t)$ and $g(t)$, respectively, with sets of parameters $\widetilde{A} = (a, 2b, c/2, d)$ and $\overline{A} = (a/2, b, c, 2d)$. Thus, the following identities hold:*

$$H_A(u) = e^{-\frac{j}{2b}3du^2} F_{\widetilde{A}}(2u) G_{\widetilde{A}}(2u) \tag{3.10}$$

and

$$H_A(u) = \frac{1}{2} e^{-\frac{j}{2b}3du^2} F_{\overline{A}}(u) G_{\overline{A}}(u). \tag{3.11}$$

Proof. Let us first observe that:

$$F_{\widetilde{A}}(2u) = \sqrt{\frac{1}{j4\pi b}} \int_{\mathbb{R}} e^{\frac{j}{4b}(a\tau^2-4tu+4du^2)} f(\tau) \mathrm{d}\tau \,. \tag{3.12}$$

From identities (2.3) and (2.7), we have that $L_A[f(t)](u)$ and $L_A[f(t-\tau)e^{jv\tau}](u)$ depend on the same parameter if we choose τ and v such that $a\tau + bv = 0$. Thus, we get:

$$v = -\frac{a}{b}\tau, \tag{3.13}$$

and by substituting (3.13) into (2.7), we obtain:

$$\begin{aligned} L_A\left[g(t-\tau)e^{-j\frac{a\tau}{b}t}\right](u) &= G_A(u) e^{-j\left(\frac{ac\tau^2}{2} + \frac{da^2\tau^2}{2b} - c\tau u + \frac{ad\tau}{b}u - ac\tau^2\right)} \\ &= G_A(u) e^{-j\left(a\left(\frac{ad-bc}{2b}\right)\tau^2 + \frac{ad-bc}{b}\tau u\right)}. \end{aligned}$$

From the LCT definition, $ad - bc = 1$. Thus:

$$L_A\left[g(t-\tau)e^{-j\frac{a\tau}{b}t}\right](u) = G_A(u) e^{-j\left(\frac{a}{2b}\tau^2 + \frac{\tau u}{b}\right)}.$$

Considering now $\widetilde{A}$ instead of A and, consequently, substituting b by $2b$, we get:

$$L_{\widetilde{A}}\left[g(t-\tau)e^{-j\frac{a\tau}{2b}t}\right](u) = G_{\widetilde{A}}(u)e^{-j\left(\frac{a}{4b}\tau^2+\frac{\tau u}{2b}\right)},$$

and thus:

$$L_{\widetilde{A}}\left[g(t-\tau)e^{-j\frac{a\tau}{2b}t}\right](2u) = G_{\widetilde{A}}(2u)e^{-j\frac{a\tau^2}{4b}+j\frac{\tau}{b}u}. \tag{3.14}$$

Using the definition of LCT with a set of parameters $\widetilde{A}$, the LCT of $h(t)$ can be expressed as:

$$\begin{aligned} H_A(u) &= \sqrt{\frac{1}{j2\pi b}}\int_{\mathbb{R}} e^{\frac{j}{2b}(at^2-2tu+du^2)} \\ &\quad \cdot\left(\sqrt{\frac{1}{j8\pi b}}\int_{\mathbb{R}} f(\tau)g(t-\tau)e^{j\frac{a}{4b}\left(-t^2+2\tau^2-2t\tau\right)}\mathrm{d}\tau\right)\mathrm{dt} \\ &= \left(\sqrt{\frac{1}{j4\pi b}}\right)^2\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\frac{j}{4b}(at^2+2a\tau^2-4tu+2du^2-2at\tau)}f(\tau)g(t-\tau)\mathrm{d}\tau\,\mathrm{dt}\,. \end{aligned}$$

Making use of (3.12) and (3.14), we obtain:

$$\begin{aligned} H_A(u) &= \sqrt{\frac{1}{j4\pi b}}\int_{\mathbb{R}} e^{\frac{j}{4b}(a\tau^2-2du^2)}f(\tau) \\ &\quad \cdot\left(e^{\frac{ja\tau^2}{4b}}\sqrt{\frac{1}{j4\pi b}}\int_{\mathbb{R}} g(t-\tau)e^{-j\frac{a\tau}{2b}t}\cdot e^{\frac{j}{4b}(at^2-4tu+4du^2)}\mathrm{dt}\right)\mathrm{d}\tau \\ &= \sqrt{\frac{1}{j4\pi b}}\int_{\mathbb{R}} e^{\frac{j}{4b}(a\tau^2-2du^2)}f(\tau)\cdot G_{\widetilde{A}}(2u)e^{-j\frac{\tau}{b}u}\mathrm{d}\tau \\ &= e^{-\frac{j}{4b}6du^2}\cdot\sqrt{\frac{1}{j4\pi b}}\int_{\mathbb{R}} e^{\frac{j}{4b}(a\tau^2-4tu+4du^2)}f(\tau)\mathrm{d}\tau\;\cdot G_{\widetilde{A}}(2u) \\ &= e^{-\frac{j}{2b}3du^2}F_{\widetilde{A}}(2u)G_{\widetilde{A}}(2u), \end{aligned}$$

and the proof of (3.10) is achieved.

Let us now observe that:

$$\begin{aligned} F_{\widetilde{A}}(2u) &= \sqrt{\frac{1}{j4\pi b}}\int_{\mathbb{R}} e^{\frac{j}{4b}(a\tau^2-4tu+4du^2)}f(\tau)\mathrm{d}\tau \\ &= \sqrt{\frac{1}{j4\pi b}}\int_{\mathbb{R}} e^{\frac{j}{2b}(\frac{a}{2}\tau^2-2tu+2du^2)}f(\tau)\mathrm{d}\tau \\ &= \frac{1}{\sqrt{2}}\sqrt{\frac{1}{j2\pi b}}\int_{\mathbb{R}} e^{\frac{j}{2b}(\frac{a}{2}\tau^2-2tu+2du^2)}f(\tau)\mathrm{d}\tau \\ &= \frac{1}{\sqrt{2}}F_{\overline{A}}(u), \end{aligned}$$

where $\overline{A} = \left(\frac{a}{2}, b, c, 2d\right)$. In the same way, $G_{\widetilde{A}}(2u) = \frac{1}{\sqrt{2}}G_{\overline{A}}(u)$, Thus, we obtain the factorization (3.11) and the proof of the theorem is complete. □

If we consider the Fractional Fourier Transform (FrFT) as a special case of LCT, when $(a,b,c,d) = (cos\,\theta, sin\,\theta, -sin\,\theta, cos\,\theta)$, the identity (3.11) becomes:

$$\begin{aligned} H_{(cos\,\theta,sin\,\theta,-sin\,\theta,cos\,\theta)}\,(u) &= \frac{1}{2}e^{-\frac{j\,3\times cot\,\theta\times u^2}{2}}F_{\left(\frac{cos\,\theta}{2},sin\,\theta,-sin\,\theta,2\times cos\,\theta\right)}(u) \\ &\cdot G_{\left(\frac{cos\,\theta}{2},sin\,\theta,-sin\,\theta,2\times cos\,\theta\right)}(u), \end{aligned} \tag{3.15}$$

where $\theta = \tilde{a}\pi/2$. Similarly, considering the Fourier Transform (FT) as a special case of LCT, when $(a,b,c,d) = (0,1,-1,0)$, we realize that the identity (3.11) becomes:

$$H_{(0,1,-1,0)}\,(u) = \frac{1}{2}F_{(0,1,-1,0)}(u)G_{(0,1,-1,0)}(u). \tag{3.16}$$

Formulas (3.15) and (3.16) are special cases of factorizations for the LCT.

Let $L^p(\mathbb{R}_+)$, $1 \leq p < \infty$, be the space of all (Lebesgue) measurable complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ with the finite norm:

$$\|f\|_p := \left(\int_0^\infty |f(u)|^p du\right)^{\frac{1}{p}}.$$

Theorem 2. *If f and $g \in L^1(\mathbb{R})$, then $f \otimes g \in L^1(\mathbb{R})$.*

Proof. Let f and $g \in L^1(\mathbb{R})$. By the definition of $\otimes$ convolution, taking $s = t - \tau$, we obtain

$$\begin{aligned} \|f \otimes g\|_1 &= \int_{-\infty}^{\infty} |(f \otimes g)(t)|\mathrm{dt} \\ &\leq \frac{1}{2\sqrt{|b|}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(\tau)g(t-\tau)|\mathrm{d}\tau\,\mathrm{dt} \\ &= \frac{1}{2\sqrt{|b|}}\int_{-\infty}^{+\infty}|f(\tau)|\mathrm{d}\tau\int_{-\infty}^{+\infty}|g(s)|\mathrm{d}s \\ &= \frac{1}{2\sqrt{|b|}}\|f\|_1\|g\|_1 < \infty, \end{aligned}$$

and the proof is completed. □

The new proposed convolution can be rewritten into a different form according to the classical convolution operator $*$ (cf. (1.1)). For instance:

$$\begin{aligned} (f \otimes g)(t) &= \sqrt{\frac{1}{j8\pi b}}\int_{\mathbb{R}} f(\tau)g(t-\tau)e^{j\frac{a}{4b}\left(-t^2+2\tau^2-2t\tau\right)}\mathrm{d}\tau \\ &= \sqrt{\frac{1}{j8\pi b}}e^{-j\frac{at^2}{2b}}\int_{\mathbb{R}} e^{j\frac{a\tau^2}{4b}}f(\tau)\cdot e^{j\frac{a(t-\tau)^2}{4b}}g(t-\tau)\mathrm{d}\tau \\ &= \sqrt{\frac{1}{j4b}}e^{-j\frac{at^2}{2b}}\left(\left(e^{j\frac{as^2}{4b}}f(s)\right) * \left(e^{j\frac{as^2}{4b}}g(s)\right)\right)(t). \end{aligned}$$

Therefore, using the notation:

$$\widetilde{f}(s) = e^{j\frac{as^2}{4b}}f(s), \qquad \widetilde{g}(s) = e^{j\frac{as^2}{4b}}g(s), \tag{3.17}$$

we conclude that:

$$(f \otimes g)(t) = \sqrt{\frac{1}{j4b}}e^{-j\frac{at^2}{2b}}\left(\widetilde{f} * \widetilde{g}\right)(t). \tag{3.18}$$

Table 1: Comparative analysis of different convolution theorems for the LCT

Parameter	Deng et. al 2006		Wei et. al 2009		Wei et. al 2011		Wei et. al 2012		Goel et. al 2013		Zhang et. al 2016a, b		Huo 2019		Proposed	
Variable Dependability	Yes		No		No		Yes		Yes		Yes		Yes		Yes	
FT Conversion	Yes		Yes		Yes		Yes		Yes		Yes		Yes		Yes	
Number of chirp multiplications	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS
	3	7	7	6	2	5	2	10	2	7	3	6	4	7	3	5

3.2 Comparative analysis

A comparative analysis of the proposed convolution theorem with some of the other relevant ones in the literature is given in Table 1. Variable dependability means that the left-hand side (LHS) and the right-hand side (RHS) of the convolution theorem are expressed in terms of their respective domain variables only. 'YES', in Table 1, means that the variable dependability is satisfied. The satisfaction of variable dependability is necessary for chip-level realization. FT conversion means LCT as a generalization of FT; the derived convolution theorem should converge to the classical convolution theorem for the FT when $(a, b, c, d) = (0, 1, -1, 0)$. For hardware complexity, in terms of chirp multiplications, LHS represents the chirp multiplications in the time domain, whereas RHS represents the chirp multiplications in the transform domain. The proposed convolution theorem in the time domain contains three chirp multiplications, and in the transform domain, it contains seven chirp multiplications; hence, the total chirp multiplications are eight. The convolution theorem given by (Wei et al., 2011) has seven chirp multiplications in total that is less than the proposed convolution theorem and, at the same time, it does not satisfy the variable dependability parameter which makes this method practically unrealisable.

4 Applications of the new convolution operator

4.1 Generalized Young's inequality

In this subsection, we introduce a generalized Young's inequality for the new convolution operator $\otimes$. Let us recall the classical Young's inequality (Stein and Weiss, 2016):

Theorem 3. *Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $\frac{1}{r} + \frac{1}{r'} = 1$. Then:*

$$\|f * g\|_r \leq A_p A_q A_{r'} \|f\|_p \|g\|_q,$$

where $A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Making use of the classical Young's inequality, the next theorem states that the introduced convolution operator $\otimes$ also satisfies the Young's inequality.

Theorem 4. *Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$, $\frac{1}{r}+\frac{1}{r'}=1$. Then:*

$$\|f \otimes g\|_r \leq \frac{1}{2\sqrt{|b|}} A_p A_q A_{r'} \|f\|_p \|g\|_q,$$

where $A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}$, $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. Having into consideration the relation (3.18), we have that:

$$\begin{aligned}
\|f \otimes g\|_r &= \left(\int_{\mathbb{R}} \left|\sqrt{\frac{1}{j8\pi b}} e^{-j\frac{at^2}{2b}} \left(\left(e^{j\frac{as^2}{4b}} f(s)\right) * \left(e^{j\frac{as^2}{4b}} g(s)\right)\right)(t)\right|^r \mathrm{dt}\right)^{1/r} \\
&= \frac{1}{2\sqrt{|b|}} \left(\int_{\mathbb{R}} \left|\left(\left(e^{j\frac{as^2}{4b}} f(s)\right) * \left(e^{j\frac{as^2}{4b}} g(s)\right)\right)(t)\right|^r \mathrm{dt}\right)^{1/r} \\
&= \frac{1}{2\sqrt{|b|}} \| \left(\left(e^{j\frac{a(\cdot)^2}{4b}} f(\cdot)\right) * \left(e^{j\frac{a(\cdot)^2}{4b}} g(\cdot)\right)\right)(t)\|_r \\
&= \frac{1}{2\sqrt{|b|}} \|\widetilde{f} * \widetilde{g}\|_r,
\end{aligned}$$

with $\widetilde{f}$ and $\widetilde{g}$ as defined in (3.17). Since $\widetilde{f} \in L^p(\mathbb{R})$ and $\widetilde{f} \in L^q(\mathbb{R})$, we can apply Young's inequality for the functions $\widetilde{f}$ and $\widetilde{g}$ and obtain:

$$\|\widetilde{f} * \widetilde{g}\|_r \leq A_p A_q A_{r'} \|\widetilde{f}\|_p \|\widetilde{g}\|_q.$$

Finally, since $\|\widetilde{f}\|_p = \|f\|_p$ and $\|\widetilde{g}\|_q = \|g\|_q$, we have:

$$\begin{aligned}
\|f \otimes g\|_r &\leq \frac{1}{2\sqrt{|b|}} A_p A_q A_{r'} \|\widetilde{f}\|_p \|\widetilde{g}\|_q \\
&= \frac{1}{2\sqrt{|b|}} A_p A_q A_{r'} \|f\|_p \|g\|_q,
\end{aligned}$$

and the proof is completed. □

4.2 Solvability for a class of convolution integral equations

In this subsection, we will discuss the solvability of a class of convolution integral equations. Namely, we will consider convolution equations of the form:

$$\lambda\phi(t) + e^{j\frac{at^2}{4b}} (g \otimes \phi)(t) = f(t), \tag{4.19}$$

where $\lambda \in \mathbb{C}$, $f, g \in L^1(\mathbb{R})$ are given, and ϕ is the unknown function to be determined.

In what follows, we will use the notation:

$$\mathcal{H}(u) := \lambda + \mathcal{F}[\widetilde{g}(t)](u)$$

with $\widetilde{g}$ as defined in (3.17).

The next proposition is an auxiliary result to obtain the main theorem.

Proposition 5. *(i) If $\lambda \neq 0$, $\mathcal{H}(u) \neq 0$ for every $|u| > C$;*

(ii) If $\mathcal{H}(u) \neq 0$ for every $u \in \mathbb{R}$, then $\frac{1}{\mathcal{H}(u)}$ is continuous and bounded on $\mathbb{R}$.

Proof. (i) By the Riemann–Lebesgue lemma, the function $\mathcal{H}(u)$ is continuous on $\mathbb{R}$ and $\lim_{|u|\to\infty} \mathcal{H}(u) = \lambda \neq 0$. Thus, there exists an $C > 0$, such that $\mathcal{H}(u) \neq 0$ for every $|u| > C$.

(ii) Since $\mathcal{H}(u)$ is continuous and $\lim_{|u|\to\infty} \mathcal{H}(u) = \lambda$, there exist $R > 0$ and $\epsilon_1 > 0$, such that $\inf_{|u|>R} |\mathcal{H}(u)| > \epsilon_1$. Because $\mathcal{H}(u)$ does not vanish on the compact set $\{u \in \mathbb{R} : |u| \leq R\}$, there exists $\epsilon_2 > 0$, such that $\inf_{|u|\leq R} |\mathcal{H}(u)| > \epsilon_2$. Thus, $\frac{1}{|\mathcal{H}(u)|} \leq \max\{\frac{1}{\epsilon_1}, \frac{1}{\epsilon_2}\} < \infty$, which implies that $\frac{1}{|\mathcal{H}(u)|}$ is continuous and bounded on $\mathbb{R}$.

□

In the next theorem, using the previous auxiliary result, we obtain necessary and sufficient conditions for the solvability of the integral equation (4.19).

Theorem 6. *Let $\mathcal{H}(u) \neq 0$ for all $u \in \mathbb{R}$. Suppose that one of the following conditions holds:*

(i) $\lambda \neq 0$ and $\mathcal{F}\widetilde{f} \in L^1(\mathbb{R})$,

(ii) $\lambda = 0$ and $\frac{\mathcal{F}\widetilde{f}}{\mathcal{F}\widetilde{g}} \in L^1(\mathbb{R})$,

with $\widetilde{f}(t)$ and $\widetilde{g}(t)$ as defined in (3.17). Then, Eq. (4.19) has a solution in $L^1(\mathbb{R})$ if and only if $\mathcal{F}^{-1}\left(\frac{\mathcal{F}\widetilde{f}}{\mathcal{H}}\right) \in L^1(\mathbb{R})$. Furthermore, the solution has the form:

$$\phi(t) = e^{j\frac{at^2}{4b}} \mathcal{F}^{-1}\left(\frac{\mathcal{F}[\widetilde{f}(t)](u)}{\mathcal{H}(u)}\right)(t).$$

Proof. Let us consider the condition (i). Suppose that Eq. (4.19) has a solution $\phi \in L^1(\mathbb{R})$. Multiplying $e^{j\frac{at^2}{4b}}$ to both members of the equation, applying the Fourier transform to both members, and using (3.18), we obtain:

$$\mathcal{H}(u)\mathcal{F}[\widetilde{\phi}(t)](u) = \mathcal{F}[\widetilde{f}(t)](u),$$

where $\widetilde{\phi}(t) = e^{j\frac{at^2}{4b}}\phi(t)$. Since $\mathcal{H}(u) \neq 0$, for every $u \in \mathbb{R}$, we have:

$$\mathcal{F}[\widetilde{\phi}(t)](u) = \frac{\mathcal{F}[\widetilde{f}(t)](u)}{\mathcal{H}(u)}.$$

As $\frac{1}{\mathcal{H}}$ is bounded and continuous on $\mathbb{R}$ and $\mathcal{F}\widetilde{f} \in L^1(\mathbb{R})$, we have that $\frac{\mathcal{F}\widetilde{f}}{\mathcal{H}} \in L^1(\mathbb{R})$. Applying the inverse of Fourier transform, we obtained the solution as stated in the theorem.

Consider now the function $\phi(t) = e^{j\frac{at^2}{4b}} \mathcal{F}^{-1}\left(\frac{\mathcal{F}[\widetilde{f}(t)](u)}{\mathcal{H}(u)}\right)(t) \in L^1(\mathbb{R})$. Hence:

$$\mathcal{F}[\widetilde{\phi}(t)](u) = \frac{\mathcal{F}[\widetilde{f}(t)](u)}{\mathcal{H}(u)},$$

and consequently:

$$\mathcal{F}[\widetilde{\phi}(t)](u)\mathcal{H}(u) = \mathcal{F}[\widetilde{f}(t)](u).$$

Thus, it follows that:

$$\mathcal{F}[\lambda\widetilde{\phi}(t) + e^{j\frac{at^2}{2b}} \cdot e^{-j\frac{at^2}{2b}}(\widetilde{g} * \widetilde{\phi})(t)](u)) = \mathcal{F}[\widetilde{f}(t)](u).$$

By the uniqueness theorem of the Fourier transform, we obtain that:

$$\lambda e^{j\frac{at^2}{4b}}\phi(t) + e^{j\frac{at^2}{2b}}(g \otimes \phi)(t) = e^{j\frac{at^2}{4b}}f(t),$$

which is equivalent to:

$$\lambda\phi(t) + e^{j\frac{at^2}{4b}}(g \otimes \phi)(t) = f(t), \qquad t \in \mathbb{R}.$$

Thus, ϕ fulfills Eq. (4.19).

Item (ii) can be proved similarly to that of item (i), and thus, the proof is concluded. □

5 Filter design in the LCT and time domains

In this section, we discuss an application of the new convolution to the design of multiplicative filters in the LCT domain and in the time domain.

5.1 Multiplicative filters through the product in the LCT domain

We consider an input signal $r_{in}(t)$ that comprises the desired chirp signal $f(t)$ and the noise $n(t)$ such that:

$$r_{in}(t) = f(t) + n(t).$$

According to Theorem 1, the output signal of LCT can be obtained in the following way:

$$r_{out}(t) = L_{A^{-1}}\left\{\frac{1}{2}e^{-\frac{j}{2b}3du^2}R_{\overline{A}}(u)G_{\overline{A}}(u)\right\}(t), \tag{5.20}$$

where $R_{\overline{A}}(u)$ is the LCT of $r_{in}(t)$ with a set of parameters $\overline{A} = (a/2, b, c, 2d)$. Let us denote by $F_{\overline{A}}(u)$ and $N_{\overline{A}}(u)$ the LCT components of desired signal and noise signal, respectively. We admit that $F_{\overline{A}}(u)$ and $N_{\overline{A}}(u)$ have no overlapping or minimal overlapping, which will permit that the desired signal can be recovered and the noise can be discarded using a multiplicative filter in the LCT domain. The relationship between the LCT and the time–frequency distribution depicts that there should be a greater overlap of the desired signal and noise in time domain, whereas a lesser overlap in the LCT domain.

There are many possible types of canonical filters such as low pass, high pass, band pass, and pass-stopband. The design method of $G_{\overline{A}}(u)$ will decide the type of the filter. According to (5.20), we can choose:

$$G(u) = \frac{1}{2}e^{-\frac{j}{2b}3du^2}G_{\overline{A}}(u). \tag{5.21}$$

as the transfer function of the multiplicative filter. Since we are interested in the frequency spectrum of the LCT in the region $[u_1, u_2]$ of the desired signal $f(t)$, we can select a filter impulse function $g(t)$, such that the transfer function $G(u)$ is constant over $[u_1, u_2]$ and zero or with a rapid decay outside that region. The simplest filter that can be used is the pass–stopband filter with the transfer function given by:

$$G\left(u\right) = \prod\left(\left(u - u_0\right)/B\ \right) \tag{5.22}$$

Pei and Ding (2001), that is

$$G\left(u\right) = \begin{cases} 1 & \text{for } \ u_0 - B/2 < u < u_0 + B/2 \\ 0 & \text{otherwise,} \end{cases} \tag{5.23}$$

where the LCT parameters can be calculated as below (see Fig. 2):

$$\frac{a}{b} = \frac{u_1}{t_1} \text{ and } u_0 = a(t_2 + t_1)/2,\ \ B = \left|a\left(t_1 - t_2\right)\right|.$$

Thus, we have:

$$r_{out}(t) = L_{A^{-1}}\left\{R_{\overline{A}}(u)\right\}(t),\quad u \in \left[u_0 - B/2, u_0 + B/2\right]. \tag{5.24}$$

The design model of multiplicative filter is shown in Fig. 1. Alternatively, we can also choose $G_{\overline{A}}(u)$ to be equal to a constant over $[u_0 - B/2, u_0 + B/2]$ and zero outside that region. In this case, we get:

$$r_{out}(t) = L_{A^{-1}}\left\{\frac{1}{2}e^{-\frac{j}{2b}3du^2}R_{\overline{A}}(u)\right\}(t),\quad u \in \left[u_0 - B/2, u_0 + B/2\right]. \tag{5.25}$$

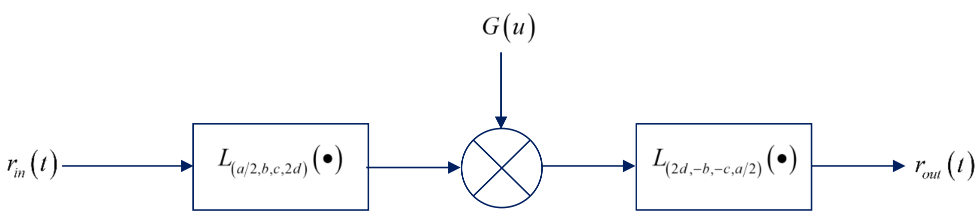

Figure 1: Multiplicative Filter in LCT domain

To validate the proposed model of multiplicative filtering as shown in Figure 1, let us consider one example in which the received signal consists of the desired signal with the noise signals $n_1(t)$ and $n_2(t)$. The time–frequency distribution of the received signal is shown in Fig. 2. The undesired signal can be filtered out completely from the received signal and keeping the desired signal undisturbed through two consecutive multiplicative filters in the LCT domain having different slopes; i.e., $(a_1/b_1 = u_1/t_1,\ \ a_2/b_2 = u_2/t_2)$.

Hence, the noise can be discarded to a large extent and the signal-to-noise ratio (SNR) can be increased through a multiplicative filter in the LCT domain when the LCT components of the noise and the desired signal have no overlapping or minimal overlapping.

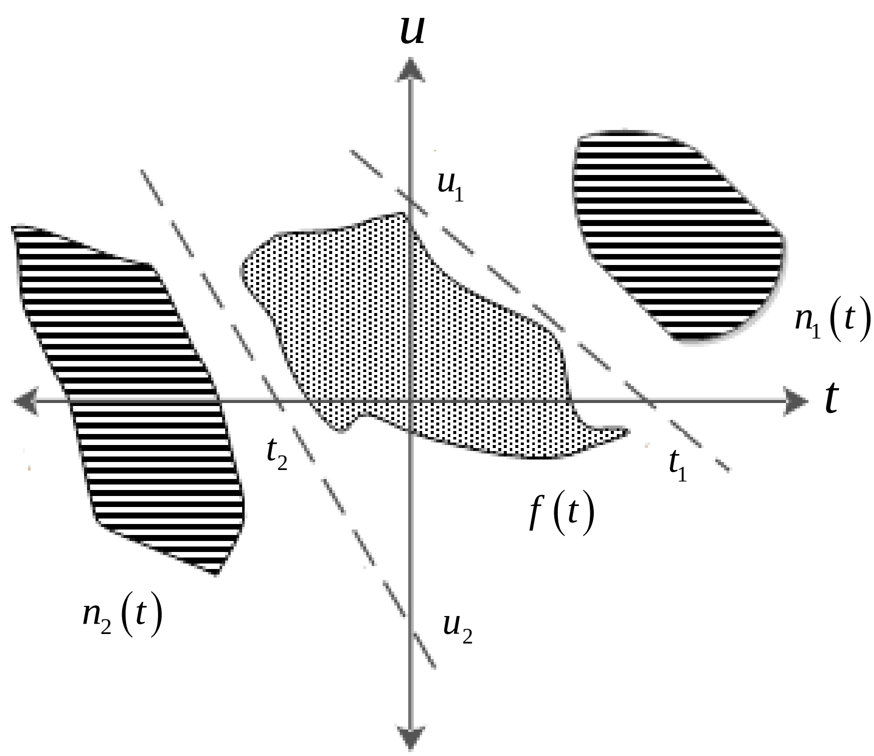

Figure 2: Time–frequency distribution of the received signal

Let the original chirp signal $f(t)$ be given by:

$$f(t) = 2 \times \exp\left(\frac{-(t-1)^2}{18} - j1.9\,t^2\right). \tag{5.26}$$

The real part and imaginary part of the original chirp signal are shown in Fig. 3a. The time–frequency representation of this signal is obtained by taking the Wigner Ville Distribution (WVD) (Boashash and Black, 1987, Claasen and Mecklenbrauker, 1980) and is shown in Fig. 3b. The original signal is corrupted by Additive White Gaussian Noise (AWGN) of 6dB SNR, as shown in Fig. 3c. The WVD of the corrupted signal is shown in Figure 3d. Finally, as an operation of LCT domain filtering, a comparison of the real part and imaginary part of the recovered signals and original signals are shown in Fig. 3e and Fig. 3f, respectively.

Following the procedure developed by Pei and Ding (2001), the optimal filtering domain for the LCT domain filtering is found to be $(a/2, b, c, 2d) = (0.3022, 1, -1, 0)$, as special cases of the LCT; the optimal FrFT domain filtering is $(a/2, b, c, 2d) = (0.5177, 0.8556, -0.8556, 0.5177)$ and the optimal frequency domain filtering is $(a/2, b, c, 2d) = (0, 1, -1, 0)$.

As a comparison of the LCT domain filtering, fractional domain filtering, and frequency domain filtering, the values of the Mean Square Error (MSE) for different values of AWGN SNR are tabulated in Table 2 and the comparison plot is shown in Fig. 4. The most relevant difference takes account of how these transformations preserve the original support area of the chirp signal. Basically, LCT preserves affine properties, but introduces some type of deformation, while FrFT is spatially invariant. As result, the elliptical distribution of the

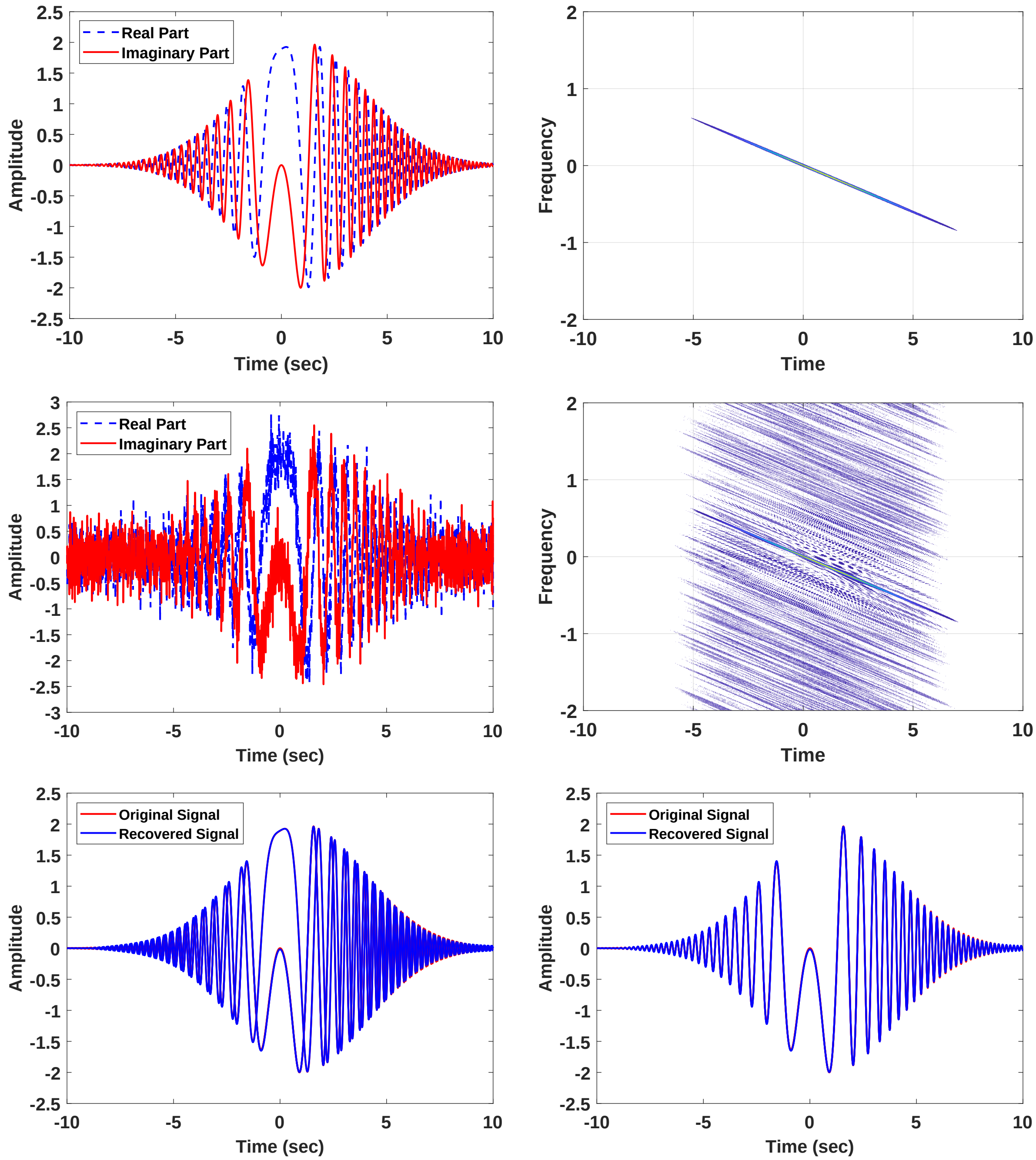

Figure 3: Optimal filtering in the LCT domain: **a** real and imaginary parts of the original signal; **b** WVD of the original signal; **c** corrupted signal (AWGN, SNR 6dB); **d** WVD of the corrupted signal; **e** imaginary part of the recovered signal; **f** real part of the recovered signal.

LCT is less eccentric than the FrFT distribution. Due to a less eccentric distribution of LCT, as compared to the FrFT distribution, the noise overlapping area of the LCT distribution is less and it also results in a minimal MSE as compared to FrFT. Moreover, the additional exponential component (i.e., $e^{-\frac{j}{2b}3du^2}$) in the convolution result diminishes to 1 in case of the LCT, when $d = 0$, whereas in the FrFT case, it has a certain value and results in a higher MSE as compared to the LCT.

Table 2: Comparison plot of MSE against AWGN SNR (dB) for LCT domain filtering, fractional domain filtering and frequency domain filtering

Filtering domain	**MSE at different values of AWGN SNR (dB)**								
	1	6	11	16	21	26	31	36	41
LCT domain filtering	0.0012	2.7754×10^{-4}	1.1464×10^{-4}	4.5626×10^{-5}	2.4449×10^{-5}	9.0857×10^{-6}	4.0453×10^{-6}	1.2764×10^{-6}	1.1937×10^{-7}
Fract. domain filtering	0.0014	4.6116×10^{-4}	2.7865×10^{-4}	2.3033×10^{-4}	2.0038×10^{-4}	0.9483×10^{-4}	6.8797×10^{-5}	1.8620×10^{-5}	0.8574×10^{-5}
Freq. domain filtering	0.0418	0.0161	0.0051	0.0022	8.0018×10^{-4}	4.5203×10^{-4}	8.3041×10^{-5}	2.4809×10^{-5}	1.0854×10^{-5}

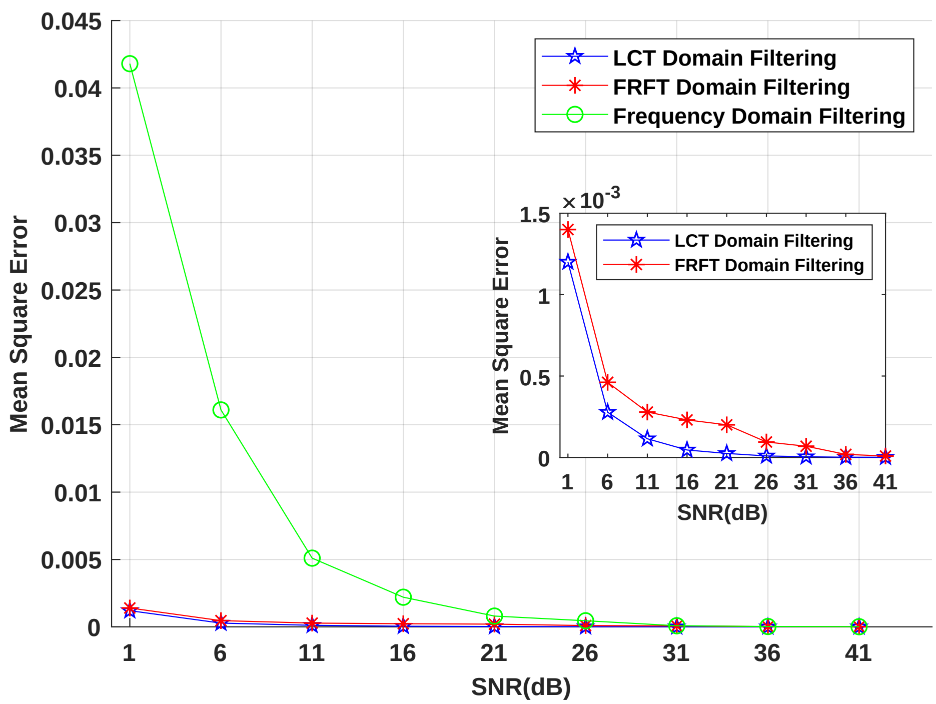

Figure 4: Comparison plot of MSE against AWGN SNR (dB) for LCT domain filtering, fractional domain filtering and frequency domain filtering

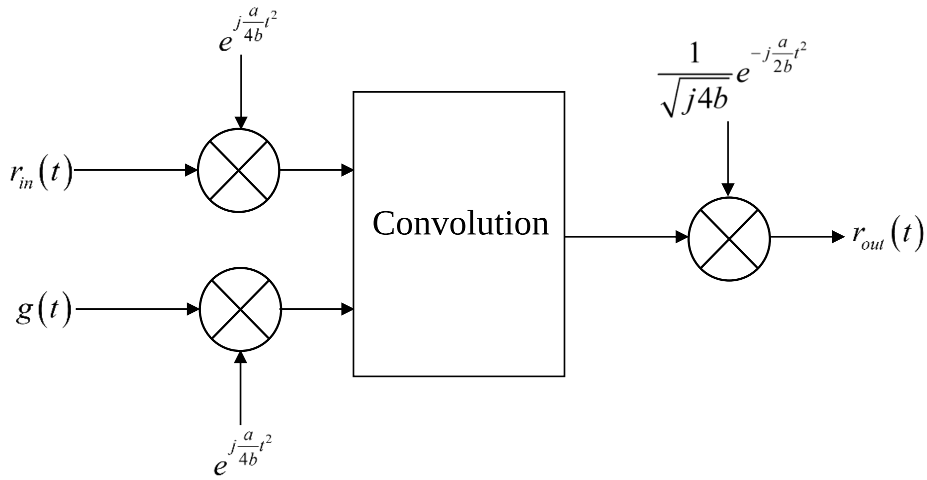

Figure 5: Multiplicative filter in LCT domain with respect to the new convolution in the time domain.

5.2 Designing of multiplicative filters in the time domain

In this subsection, using the introduced convolution theorem, we obtain a multiplicative filter in the LCT domain with respect to the new convolution in the time domain. The visualization of this multiplicative filter is shown in Fig. 5. According to (3.18) and (1.2):

$$(r_{in} \otimes g)(t) = \sqrt{\frac{1}{j4b}} e^{-j\frac{at^2}{2b}} \mathcal{F}^{-1}[\mathcal{F}[\widetilde{r_{in}}(t)](u)\mathcal{F}[\widetilde{g}(t)](u)](t), \tag{5.27}$$

where we recall that $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, $\widetilde{r_{in}}(t) = e^{j\frac{at^2}{4b}} r_{in}(t)$ and $\widetilde{g}(t) = e^{j\frac{at^2}{4b}} g(t)$. This shows that the computational complexity of the multiplicative filter in the LCT domain, with respect to the new convolution in the time domain as shown in Fig. 5, for N number of samples, is equivalent to the computational complexity of the fast Fourier transform, i.e., $O(N \log_2 N)$.

6 Conclusion

In this paper, we defined a new convolution operator and studied some of their consequences. For this new operator, we derived a product theorem and a Young's type inequality. Moreover, a relation between this convolution and the classical one was also presented. In addition, we investigated and characterized the solvability of a class of convolution equations associated with the introduced convolution operator.

Finally, as examples of possible applications of the results derived in this paper, a simulation comparison of multiplicative filters in the LCT domain, fractional domain, and frequency domain was carried out. In particular, it has been shown that the computational complexity of multiplicative filter in the LCT domain with respect to the new convolution in the time domain is equivalent to the computational complexity of the fast Fourier transform.

Acknowledgements

This work was supported by Fundação para a Ciência e a Tecnologia (FCT), within project UIDB/04106/2020 (CIDMA) and by national funds (OE), through FCT, I.P., in the scope of the framework contract foreseen in the numbers 4, 5, and 6 of the article 23, of the Decree-Law 57/2016, of August 29, changed by Law 57/2017, of July 19.

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