Polar Foliations on Positively Curved Manifolds

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Zusammenfassung

Wir untersuchen die geometrischen und topologischen Restriktionen, die eine polare Blätterung der Kodimension nicht kleiner als zwei einer positiv gekrümmten, einfach zusammenhängenden, kompakten riemannschen Mannigfaltigkeit auferlegt. Angelehnt an ein Resultat von Fang, Grove und Thorbergsson über polare Wirkungen auf ebensolchen Mannigfalitigkeiten assoziieren wir einen Kammerkomplex zur Blätterung und zeigen, dass dieser im Falle der Abwesenheit von Punktblättern, unter gewissen technischen Voraussetzungen, von einem sphärischen Gebäude überlagert wird, das isomorph ist zu dem Gebäude im Unendlichen eines nicht-kompakten symmetrischen Raumes. Dies impliziert, dass die gegebene Mannigfaltigkeit den Homöomorphietyp eines kompakten symmetrischen Raumes vom Rang eins hat. Dabei verwenden wir in entscheidender Weise Resultate zur lokalen Homogenität von singulären riemannschen Blätterungen. Im Fall von Punktblättern wenden wir direktere Methoden an um den Diffeomorphietyp bzw. den Kohomologiering der Mannigfaltigkeit zu bestimmen.

Abstract

We investigate the geometric and topological restrictions imposed by a polar foliation of codimension no less than two on a positively curved, simply connected, compact riemannian manifold. Inspired by a result of Fang, Grove and Thorbergsson on polar actions on such manifolds we associate to the foliation a chamber complex and show that in absence of point leaves this is covered by a spherical building, which under certain technical assumptions is isomorphic to the building at infinity of a non-compact symmetric space. This implies that the given manifold has the homeomorphism type of a compact symmetric space of rank one. We crucially employ results on the local homogeneity of singular riemannian foliations. In the case of point leaves we apply more direct methods to determine the diffeomorphism type or the cohomology ring of the manifold, respectively.

Contents

CHAPTER 1

Introduction

The exploitation of symmetries or, more generally, certain degrees of homogeneity, is a standard approach to investigating the geometry and topology of a given manifold. This is mathematically expressed by the concept of (isometric) group actions, with an important example being the isometric action of a Lie group on a riemannian manifold. Such an action is called *polar* if it admits a family of immersed submanifolds, the *sections*, such that these are everywhere perpendicular to the orbits and every point of the manifold is contained in one of them. Such actions have been shown to be closely related to symmetric spaces, i.e. riemannian manifolds whose isometry group contains geodesic reflections in every point (cf. [**Da**]). The terminology is supposedly inspired by the principal example provided by polar coordinates in euclidean space, which can be identified with an action by the rotations around the origin, yielding the azimuthal coordinates, together with a family of copies of the real line emanating radially from the origin, yielding the radial coordinate. The radial rays meet all rotation orbits perpendicularly and play the role of the sections for the action of the rotation group. Notice that the orbits have different dimension, in particular the orbit of the origin is only the origin itself and any neighbourhood of the origin contains the entire orbits of points close to it. Thus, local knowledge of the action around the origin yields global information on the structure of the orbits. This is a general phenomenon that we will investigate further below.
Given an action of a Lie group on a manifold, one can "forget" the homogeneity by considering the orbits of the action as a "smooth" partition of the manifold into submanifolds, called *leaves* or *foils*, which yields the concept of a *(singular) foliation*. The notion of smoothness for this will be made precise in Chapter 2. When a group acts isometrically then its orbits remain at constant distance from each other and geodesics once perpendicular to the orbits remain such for all times. This concept can be translated to the case of partitions by submanifolds, too, yielding the notion of a *transnormal system*.
The objects we will be considering will be a conjunction of these two concepts, called *singular riemannian foliations* that turn out to be smooth, locally equidistant partitions by submanifolds (whose dimension may vary, with those of lower than maximal dimension being called *singular*). If one additionally requires the existence of sections as for polar actions, one obtains the notion of singular riemannian foliations with sections or, as we will

call them, *polar foliations*.
One may ask what information on the geometry and topology of the manifold can be recovered from the purely geometric information provided by such a foliation, in contrast to the more algebraic information provided by a group action. It turns out that singular (riemannian) foliations, while not globally homogeneous, admit certain types of local or partial homogeneity. The idea to maintain is that, where an isometric group action yields general metric control along the orbits, a singular riemannian foliation yields such control only everywhere locally and only in the direction perpendicular to the leaves, called the *transverse* direction. The restriction of the riemannian metric of the ambient manifold to the distribution orthogonal to the tangent spaces of the leaves at every point is analogously called the *transverse metric* and in fact, as shown by Molino in [**Mo**], the structure of a singular riemannian foliation is, at least locally, fully determined by a given transverse metric, i.e. the foliation is "oblivious" to non-isometric deformation in the direction of the leaves. In the case of a polar foliation the transverse metric control means local metric control of the transverse submanifolds, the sections. Exploiting this will be crucial for our endeavours.
In [**FGT**] Fang, Grove and Thorbergsson proved the remarkable result that a simply connected, positively curved, compact manifold equipped with a polar action of cohomogeneity at least two is equivariantly diffeomorphic to a compact rank one symmetric space with a polar action. The symmetric spaces of rank one are the spheres $\mathbb{S}^n$, projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$, as well as the Cayley projective plane $\mathbb{O}P^2$.
The aim of this thesis will be to obtain a generalisation of this result, as far as possible, to the inhomogeneous case of a polar foliation of codimension at least two on a riemannian manifold with the same properties as above. The first step in this will be to establish local homogeneity along the leaves, using the module of vector fields everywhere tangent to the foliation, or the flows of such vector fields, respectively. It turns out that these vector fields are "too many" in such that multiplying such a vector field with any smooth function yields another such vector field. In order to avoid such unwanted deformations along the leaves we will *linearise* the foliation along each leaf and obtain a (possibly infinite dimensional) action on its normal bundle which via the exponential map will yield locally defined isometries between the sections along the leaf. Furthermore, considering only those vector fields that vanish in a given point, we obtain an in fact finite dimensional Lie group acting orthogonally on the normal space to the leaf of the point.
In the case of polar actions it is well known that the sections carry groups generated by reflections, i.e. isometries having a fixed point set of codimension one and acting as a geodesic reflection on its orthogonal complement. Crucial results by Alexandrino and Töben imply that in the case of polar foliations one obtains at least a pseudogroup generated by local reflections, and in many cases even a full group, acting on each section. We will endeavour to show that under our assumptions the latter is in fact the case,

which yields a cell decomposition, given by the connected components of the complement of the "mirrors" of the reflections. The prerequisite of codimension two allows us to exploit the positivity of the sectional curvature in the transverse direction. Together with the group of reflections this poses strong restrictions on the geometry of the sections, yielding

PROPOSITION. *Let $\mathcal{F}$ be a polar foliation on a compact, positively curved riemannian manifold M. Then any section Σ is diffeomorphic either to a sphere or a real projective space.*

This will be proved in Section 2.2.
It turns out that the reflection group of each section is (up to a lifting to its universal cover) a *Coxeter group* making its cell decomposition (covered by) a *Coxeter complex*. Such a group is characterised by a certain graph whose nodes represent the generators of the group and whose edges are non-trivial whenever two generators do not commute. If a reflection commutes with all other generators its action splits off trivially as acting on a one-dimensional factor and requiring there to be no isolated nodes in the group's graph (or *Coxeter diagram*) is thus equivalent to asking that the Coxeter complex be made up of components of dimension at least two, which will be a prerequisite for our result below.
The union of all such complexes from sections yields a chamber complex whose underlying point set is the entire manifold, which we will show to be covered by a highly regular kind of simplicial complex, a *(spherical Tits) building*, if the codimension is sufficiently high and there are no point leaves. This complex has a priori only a length space topology that "sees" only movement along sections, but can be equipped with a topology that is compatible with that of the foliated manifold. This will allow us to employ results of Burns and Spatzier ([**BSp**]) and their generalisations by Grundhöfer, Kramer, van Maldeghem and Weiss ([**GKMW**]) that establish a connection between the underlying point set of the building and the *sphere at infinity* of a non-compact symmetric space, which in turn will allow us to show:

THEOREM. *A compact, simply connected, positively curved riemannian manifold with a linearisable polar foliation of codimension at least three, whose Coxeter diagram has no isolated nodes, is homeomorphic to a compact rank one symmetric space equipped with an orbit equivalent polar foliation.*

Chapter 3 will be dedicated to this.
Finally we will investigate in Chapter 4 the – rather different – case when the foliation exhibits point leaves. Here we will be relying heavily on the aforementioned concept that a small neighbourhood of such a maximally singular point contains a large amount of information on the entire foliation. Exploiting this we show, under the additional assumption of a constantly curved section

THEOREM. *Let M be a compact, positively curved, simply connected manifold with a polar foliation $\mathcal{F}$ of codimension at least two. If the set of* *point leaves is non-empty and there exists a spherical section with constant curvature, then M is equifoliately diffeomorphic to a round sphere.*

In the case that a section is a projective space one obtains a certain fibration of the unit sphere at the point leaf, which will yield information on the cohomology structure of the ambient manifold:

THEOREM. *A simply connected, compact, positively curved riemannian manifold M, equipped with a polar foliation of codimension at least two admitting a projective section of constant curvature, has the cohomology of a projective space if there is a point leaf. More precisely: The unit sphere at the point leaf fibres over its cut locus. If the fibres are 1-spheres, then M is a $2m$-dimensional manifold with the cohomology of $\mathbb{C}P^n$, if the fibres are 3-spheres then M is a $4m$-dimensional manifold with the cohomology of $\mathbb{H}P^n$, and if the fibres are 7-spheres then M is a 16-dimensional manifold with the cohomology of the Cayley plane $\mathbb{O}P^2$.*

Chapter 2 will mostly be dedicated to establishing the technical tools used to study our given foliation. Precise definitions of all above mentioned concepts will be given throughout the thesis as they occur.

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CHAPTER 2

Praeliminaria

In this chapter we wish to introduce important concepts and examine general properties of singular riemannian foliations that will be essential tools in our subsequent endeavours.
Throughout this thesis manifolds will be considered to be smooth (C^∞), complete and connected.

2.1. Singular Riemannian Foliations

The fundamental concept of this thesis, that of a *singular riemannian foliation*, can be regarded as both a generalisation of two more special concepts and a conjunction of two more general concepts, all of which shed a certain light on it. The two more special concepts are that of a riemannian foliation, introduced by Reinhart in [**Re**] as "foliations with bundle-like metrics", and the decomposition of a riemannian manifold by the orbits of an isometric action. The former decomposes the manifold into submanifolds of constant dimension that are locally equidistant but not necessarily homogeneous. The latter yields a decomposition into equidistant, homogeneous submanifolds but not necessarily of the same dimension, as the dimension of the stabiliser subgroups of the action may differ.
The two more general concepts are that of a *singular foliation*, introduced independently by Sussmann ([**Su**]) and Stefan ([**St**]), and that of a *transnormal system*, due to Bolton ([**Bo**]), both of which we shall define below:

Definition 2.1 (Singular Foliation). A partition $\mathcal{F}$ of a differentiable manifold M into connected immersed submanifolds of not necessarily constant dimension, the *leaves*, is called a *singular foliation* if the set $\Xi_{\mathcal{F}} := \{X \in C^\infty(M, TM) | X_p \in T_p\mathcal{F}_p\}$ spans $T\mathcal{F}_p$, where $\mathcal{F}_p$ is the unique immersed submanifold passing through p.

Sussmann and Stefan proved that such partitions arise as integral manifolds from distributions if the vector fields in question form a locally finitely generated subalgebra of $\mathfrak{X}(M)$. The definition implies that the leaves can be realised as the unions of broken flow curves of the tangent vector fields, so we also refer to this property as *Transitivity*.

Definition 2.2 (Transnormal System). A partition $\mathcal{F}$ of a differentiable manifold M into connected submanifolds of not necessarily constant dimension, the *foils*, is called a *transnormal system* if M admits a riemannian metric g with respect to which any geodesic, that meets one leaf orthogonally, meets all leaves orthogonally.

This property is also referred to as *Transnormality*.

DEFINITION 2.3 (Singular Riemannian Foliation). Let (M,g) be a riemannian manifold, that is partitioned by a system $\mathcal{F}$ of connected immersed submanifolds of not necessarily constant dimension, the *leaves*, satisfying both Transitivity and Transnormality, then $(M,\mathcal{F},g)$ will be called a *singular riemannian foliation (SRF).*

We will mostly just refer to $\mathcal{F}$ as the singular riemannian foliation on M and as in Definition 2.1 denote the module of vector fields everywhere tangent to the leaves by $\Xi_{\mathcal{F}}$.
For the general theory of (singular) riemannian foliations we refer to [**Mo**], especially chapter six therein. It has been subject to discourse whether Transnormality already implies Transitivity, and even though the exposition in [**Mo**] claimed the contrary, no counterexamples have been found so far and the question remains open. While we shall be dealing exclusively with singular riemannian foliations it therefore still seems appropriate to examine the properties of the two concepts separately, where possible.
A geodesic that is at one point perpendicular to a leaf and hence by transnormality perpendicular to all leaves it meets will be called a *transverse geodesic.* In the same vein we call the restriction of the metric g to the normal bundle $\nu\mathcal{F}$ of the foliation the *transverse metric* of the foliation.
We refer to the leaves of maximal dimension as *regular leaves* and to those of lower dimension as *singular leaves.* The codimension of the regular leaves will be called the *codimension of the (singular) foliation* and denoted by $\operatorname{codim}(\mathcal{F})$. The points lying on regular or singular leaves are referred to as regular or singular points respectively. The set of points on regular leaves is denoted by M_r and called the *regular stratum.* Similarly we denote by M_{s_i} the sets of points on singular leaves of dimension s_i and call these the *singular strata of dimension* s_i. In order to etablish some more about this subdivision of M we need some preceding considerations first, though.

DEFINITION 2.4 (Plaques). Let $\mathcal{F}$ be a partition of a riemannian manifold into immersed submanifolds and p a point of the underlying manifold. For a neighbourhood U of p we call the connected components of the sets $\mathcal{F}_q \cap U$ for all $q \in U$ the *plaques* of $\mathcal{F}$ in U.

DEFINITION 2.5 (Local Equidistance). The foils of a partition $\mathcal{F}$ (of a riemannian manifold M) into immersed submanifolds are called *locally equidistant*, if for every $p \in M$ and every suitably small tubular neighbourhood $\mathcal{T}_P$ of a suitably small neighbourhood P of p in $\mathcal{F}_p$ the distance between every two plaques in $\mathcal{T}_P$ is constant along these.

LEMMA 2.6. *The foils of a transnormal system are locally equidistant.*

PROOF. Let p, q be points of M contained in different foils. Then any geodesic realising the distance between $p' \in P := \mathcal{F}_p \cap B_\epsilon(p)$ (for sufficiently small ϵ) and another plaque of $\mathcal{F}$ in $B_\epsilon(p)$ is perpendicular to the latter and thus also to $\mathcal{F}_p$, i.e. it realises an extremal length between the two foils. Thus all geodesics have locally extremal lengths and therefore locally constant lengths. It follows that locally all geodesics realising a distance between a point of one foil and a plaque of the other foil have the same length, which is the (local) distance between the two foils. $\square$

LEMMA 2.7. *For every relatively compact plaque P of a singular riemannian foliation $\mathcal{F}$ there exists a well-defined projection $\pi_P : \mathcal{T}_P \to P$ from a tubular neighbourhood of P onto P.*

PROOF. Choose the tubular neighbour $\mathcal{T}_P$ so small that for every point $q \in \mathcal{T}_P$ there is a unique transverse geodesic γ_q parametrised by arc length linking q to P. As P is relatively compact and leaves are locally equidistant we can find such a neighbourhood. Then $q \mapsto \gamma_q(d(q, P))$ defines the desired projection. As geodesics depend smoothly on their starting point and the distance to P is smooth around P this is a smooth map. $\square$

Consider the exponential map restricted to the normal ε-disc bundle $\nu^\varepsilon P$ of such a plaque. Then $q \mapsto \exp^{-1}_{\pi_P(q)}(q)$ defines a smooth inverse, mapping a tubular neighbourhood of radius ε back to the respective normal disc bundle. Hence we have obtained a diffeomorphism between the normal ε-disc bundle and a small tubular neighbourhood of the plaque, which we shall call the *plaque exponential* and denote it by $\exp_P$. Note that in the case of compact leaves one may choose the plaque to be the entire leaf.

DEFINITION 2.8 (Homothetic Transformations). Consider for any plaque P of a given singular riemannian foliation the linear map η_λ on νP given by multiplication with a non-zero scalar λ. Using the plaque exponential above we obtain a diffeomorphism h_λ of a suitably small tubular neighbourhood of P by setting $h_\lambda := \exp_P \circ \eta_\lambda \circ \exp_P^{-1}$. We call h_λ the *homothetic transformation with factor* λ, where we will often omit the reference to the factor for simplicity of expression.

The plaque exponential sends normal rays to geodesics starting perpendicular to the plaque and by transnormality these remain perpendicular to the plaques they meet. We can hence identify the distance tubes around a plaque with its normal sphere bundles of the respective radii.

LEMMA 2.9 (Homothetic Transformation Lemma). *The homothetic transformations associated to a singular riemannian foliation map plaques to plaques.*

PROOF (SEE [**Mo**], P. 193). We need to show that the plaque of the image of a point is the image of the plaque of that point. To that end, consider a plaque P, a point $y \in \mathcal{T}_P$ and λ sufficiently small so that h_λ is defined on $\mathcal{T}_P$. Let y_λ be the image of y under h_λ. As plaques are equidistant P_y, P

and P_{y_λ} remain at constant distance from each other. By construction the geodesic segments linking P, y and y_λ all lie on the same transverse geodesic, or equivalently stated, they satisfy equality in the triangle inequality.
Now let y' be any other point of P_y. Link it to P and P_{y_λ} by geodesics realising the distances between the plaques. Since these are perpendicular to the two plaques they are also perpendicular to P_y itself by transnormality. These two geodesics together with the one linking the endpoint of the second (and hence P_{y_λ}) to P satisfy, by equidistance, the same triangle inequality as those linking y, y_λ and P. But as seen above this is a strict equality and the broken geodesic formed by the two segments is in fact one normal geodesic starting on P and containing both y' and the endpoint of the geodesic linking it to P_{y_λ}. It follows that the latter endpoint is the homothetic image of y'. As homotheties are diffeomorphisms it follows that the plaque of the image of y is the image of the plaque of y. $\square$

As a consequence of the homothetic transformation lemma we have the following propositions by Molino (Proposition 6.3 and the following considerations in [**Mo**]):

PROPOSITION 2.10. *For every relatively compact plaque P of a singular riemannian foliation there is a small tubular neighbourhood, such that every geodesic emanating perpendicularly to P and tangent to a stratum of M remains in that stratum at least until it leaves the tubular neighbourhood.*

PROPOSITION 2.11. *The regular stratum of a singular riemannian foliation is an open and dense subset of the ambient manifold M and the singular strata have codimension at least 2.*

We will now examine how a given singular riemannian folation $(M, \mathcal{F})$ gives rise to a linear, homogenous foliations on the normal bundles of its relatively compact plaques P that corresponds to the foliation on a tubular neighbourhood of P via the plaque exponential. The procedure of linearising a vector field as shown in [**MR**] will be crucial for this.
For $X \in \Xi_{\mathcal{F}}$ let ϕ^t denote the associated flow. Then $d\phi^t$ defines a bundle automorphism of $TM|_P$. With π denoting the projection $TM|_P \to \nu P$ we have that $\pi \circ d\phi^t|_{\nu P}$ defines a bundle morphism of νP, which we shall call the *linearised flow* associated to X. As $\Xi_{\mathcal{F}}$ is involutive, $d\phi^t$ maps TP onto itself and since $d\phi^t$ is an isomorphism the image of νP must thus be transversal to TP, whence $\pi \circ d\phi^t$ restricts to an automorphism of νP.
Let us examine the vector field on νP associated to it. To that end consider a neighbourhood U of a point p in P chosen such that the tangential bundle restricted to U is trivial. We choose coordinates on a tubular neighbourhood of U as follows. Via the plaque exponential $\exp_P$ the tubular neighbourhood $\mathcal{T}U$ is diffeomorphic to a neighbourhood of the zero section in νU and on the latter we have a local trivialisation $\psi : \nu U \to U \times \mathbb{R}^k$ as well as coordinates $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_m)$ on U itself. Then $\tilde{x} \times \mathrm{Id}_{\mathbb{R}^k} \circ \psi \circ \exp_P^{-1} : \mathcal{T}U \to \mathbb{R}^m \times \mathbb{R}^k$ defines the desired coordinates in which U is mapped to an open subset of $\mathbb{R}^m \times \{0\}$

and the fibres of the tubular neighbourhood (seen as a metric disc bundle over U) to open subsets of $\{x\} \times \mathbb{R}^k$. We will denote the components of these coordinates as $(x, y) = (x_1, ..., x_m, y_1, ..., y_k)$ where the first m components coincide with the coordinates $\tilde{x}$ on U when restricted to it, i.e. when $y = 0$. In these coordinates write the given flow and vector field as

$$\begin{aligned}\phi^t &= (\phi_1^t(x,y), ..., \phi_{m+k}^t(x,y)) \\ X &= \sum_{i=1}^m a_i(x,y)\partial_{x_i} + \sum_{j=1}^k b_j(x,y)\partial_{y_j}.\end{aligned}$$

As X is tangent to $U \subset P$ we have $b_j(x, 0) = 0$, a fact that will be of use later in the discussion. Applying the defining equation for the flow we obtain by comparison of coefficients that for $i = 1, ..., m$ and $j = 1, ..., k$

$$\dot{\phi}_i^t = a_i \circ \phi^t \text{ and } \dot{\phi}_{m+j}^t = b_j \circ \phi^t,$$

where the dot denotes the time derivative. Let us calculate the vector field X^l on νP associated to $\phi'^t := \pi \circ d\phi^t|_{\nu P}$ now: The map $\tilde{x} \times \mathrm{Id}_{\mathbb{R}^k} \circ \psi$ yields coordinates $((\tilde{x}, 0), (0, v))$ on $\nu P|_U \subset TM$, where the first zero corresponds to the restriction to U and the second to the restriction to the normal bundle. These coordinates are compatible with the coordinates chosen for the tubular neighbourhood of U, i.e. we have $\exp_P((\tilde{x}, 0), (0, v)) = (x, y)$. With $x|_U = \tilde{x}$ it follows that

$$\begin{aligned}\pi \circ d\phi^t|_{\nu P}((\tilde{x},0),(0,v)) &= ((\phi_1^t(x,0),...,\phi_m^t(x,0),0,...,0),\pi(J_{\phi^t}(0,v))), \\ \text{where } \pi \circ J_{\phi^t}(0,v) &= \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ \frac{\partial\phi_{m+1}^t}{\partial x_1} & \cdots & \frac{\partial\phi_{m+1}^t}{\partial y_1} & \cdots & \frac{\partial\phi_{m+1}^t}{\partial y_k} \\ \vdots & & \vdots & \ddots & \vdots \\ \frac{\partial\phi_{m+k}^t}{\partial x_1} & \cdots & \frac{\partial\phi_{m+k}^t}{\partial y_1} & \cdots & \frac{\partial\phi_{m+k}^t}{\partial y_k} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_1 \\ \vdots \\ v_k \end{pmatrix} \\ &= \left(0,...,0,\sum_1^k \frac{\partial\phi_{m+1}^t}{\partial y_i}v_i,...,\sum_1^k \frac{\partial\phi_{m+k}^t}{\partial y_i}v_i\right)^T\end{aligned}$$

and hence

$$\begin{aligned}X^l \circ \phi'^t &= \sum_{i=1}^m \dot{\phi}_i^t(x,0)\,\partial_{\tilde{x}_i} + \sum_{i=1}^k\sum_{j=1}^k \frac{\partial\dot{\phi}_{m+i}^t}{\partial y_j}(x,0)\,v_j\,\partial_{v_i} \\ &= \sum_{i=1}^m a_i \circ \phi^t(x,0)\,\partial_{\tilde{x}_i} + \sum_{i=1}^k\sum_{j=1}^k \frac{\partial b_i \circ \phi^t}{\partial y_j}(x,0)\,v_j\,\partial_{v_i} \\ &= \sum_{i=1}^m a_i \circ \phi^t(x,0)\partial_{\tilde{x}_i} + \sum_{i=1}^k\sum_{j=1}^k \frac{\partial b_i}{\partial y_j}(\phi^t(x,0))\left(\sum_{l=1}^k \frac{\partial\phi_{m+j}^t}{\partial y_l}v_l\right)\partial_{v_i},\end{aligned}$$

where the last step follows by the chain rule and the fact that as $b_i(x,0)$ vanishes identically we have $\frac{\partial b_i}{\partial x_j}(x,0)=0$.
Consider on the other hand the vector field $\widetilde{X}$ on νP obtained by lifting X via the plaque exponential to the normal bundle:

$$\widetilde{X}=\sum_{i=1}^{m} a_i\circ \exp_P((\tilde{x},0),(0,v))\,\partial_{\tilde{x}_i}+\sum_{i=1}^{k} b_i\circ \exp_P((\tilde{x},0),(0,v))\,\partial_{v_i},$$

where we have used that $\partial_{x_i},\partial_{y_i}$ and $\partial_{\tilde{x}_i},\partial_{v_{m+i}}$ respectively are $\exp_P$-related. This vector field is tangent to the foliation $\exp_P^{-1}\mathcal{F}$, which by construction is invariant under the normal homothetic transformations η_λ. Using the fact that ∂_{x_i} is invariant under homotheties we obtain that the vector field

$$\begin{aligned} d(\eta_\lambda^{-1})\circ\widetilde{X}\circ\eta_\lambda &= \sum_{i=1}^{m} a_i\circ \exp_P((\tilde{x},0),(0,\lambda v))\,\partial_{\tilde{x}_i}\\ &\quad+\sum_{i=1}^{k} b_i\circ \exp_P((\tilde{x},0),(0,\lambda v))\,\frac{1}{\lambda}\partial_{v_i}\end{aligned}$$

is still tangent to the leaves of $\exp_p^{-1}\mathcal{F}$ and so is the limit $\widetilde{X}^0$ for $\lambda\to 0$ which yields to

$$\begin{aligned}\widetilde{X}^0 &= \sum_{i=1}^{m} a_i\circ \exp_P((\tilde{x},0),(0,0)\,\partial_{\tilde{x}_i}+\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{\partial b_i}{\partial y_j}\circ \exp_P((\tilde{x},0),(0,0))v_j\,\partial_{v_i}\\ &= \sum_{i=1}^{m} a_i(x,0)\,\partial_{\tilde{x}_i}+\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{\partial b_i}{\partial y_j}(x,0)v_j\,\partial_{v_i}.\end{aligned}$$

With the result for the coordinate representation of ϕ'^t we have obtained above it follows that

$$\begin{aligned}\widetilde{X}^0\circ\phi'^t &= \sum_{i=1}^{m} a_i\circ\phi^t(x,0)\,\partial_{\tilde{x}_i}+\sum_{i=1}^{k}\sum_{j=1}^{k}\frac{\partial b_i}{\partial y_j}\circ\phi^t(x,0)\left(\sum_{l=1}^{k}\frac{\partial\phi^t_{m+j}}{\partial y_l}v_l\right)\partial_{v_i}\\ &= X^l\circ\phi'^t.\end{aligned}$$

Thus by uniqueness $\widetilde{X}^0 = X^l$ and the latter is tangent to the leaves of $\exp_p^{-1}\mathcal{F}$.
As the leaves remain at constant distance to the zero section (or P respectively) the flow ϕ'^t preserves the norm on the normal bundle. By construction it is linear as the composition of a differential and a vector bundle projection and we can apply polarisation to obtain that it in fact leaves the transverse metric invariant. We have hence proven the following

PROPOSITION 2.12. *Let $(M,\mathcal{F})$ be a riemannian foliation with a relatively compact plaque P, then for each $X\in\Xi_{\mathcal{F}}$ the linearised flow ϕ'^t acts as a linear, orthogonal transformation of the normal bundle νP whose orbits are tangent to the leaves of* $\exp_P^{-1}\mathcal{F}$.

COROLLARY 2.13. *If in the above setting P is compact the set Φ' of linearised flows on νP is a subgroup of $O(\nu P)$.*

Denote by lin the operator that assigns to each $X \in \Xi_{\mathcal{F}}$ the field obtained from the associated linearised flow ϕ'^t and by I_P the set of smooth functions $M \to \mathbb{R}$ vanishing on P. The following proposition gives a description of the kernel of lin.

PROPOSITION 2.14. *The kernel of* lin *is the Lie-ideal generated by the set $I_P\Xi_{\mathcal{F}} = \{fX | X \in \Xi_{\mathcal{F}}, f|_P = 0\}$. In particular we have a Lie-algebra $\Xi_{\mathcal{F}}/I_P\Xi_{\mathcal{F}}$ exponentiating via the linearised flows to a group acting isometrically on νP and whose orbits are mapped into the leaves of $\mathcal{F}$ by the plaque exponential.*

PROOF. Consider a vector field of the form fX, where X is tangent to the foliation and f vanishes on P. Then in coordinates

$$fX(x,y) = \sum_{i=1}^{m} f(x,y)a_i(x,y)\partial_{x_i} + \sum_{j=1}^{k} f(x,y)b_j(x,y)\partial_{y_j}$$

and thus

$$\begin{aligned}
\operatorname{lin}(fX) &= \sum_{i=1}^{m} f(x,0)a_i(x,0)\partial_{\tilde{x}_i} \\
&\quad + \sum_{i=1}^{k}\sum_{j=1}^{k}\left(f(x,0)\frac{\partial b_i}{\partial y_j}(x,0) + b_i(x,0)\frac{\partial f}{\partial y_j}(x,0)\right)v_j\,\partial_{v_i} \\
&= 0,
\end{aligned}$$

since both f and the b_i vanish at $(x,0)$ for all x, or in other words, on the plaque P.
On the other hand an element of the kernel, written in coordinates as above must satisfy $a_i(x,0) = 0$ for all $i = 1, ..., m$ and hence can be written as a sum of tangent vector fields multiplied by functions vanishing on the plaque. The rest of the proposition follows by the observation that the quotient Lie-algebra acts effectively on the normal bundle via exponentiating to linearised flows and exponentiates to a subgroup of $O(\nu P)$. □

DEFINITION 2.15. We call the pseudogroup generated by linearised flows of $\Xi_{\mathcal{F}}$ along a leaf L the *normal holonomy pseudogroup* of L and denote it by $\eta(L)$.

REMARK 2.16. As the local coordinate vector fields ∂_{x_i} tangent to a leaf of the foliation everywhere coincide with their linearisations along the given leaf the latter is always also a leaf of the exponential image of the linearised foliation. Hence the leaves of the exponentiated linearised foliation around a leaf are of no smaller dimension than the leaf itself.

Note that the dimension of the linearised leaves may be smaller than that of the leaves of the infinitesimal foliation on the normal bundle. Taking the suspension of an inhomogeneous foliation on a sphere by S^0 yields an example where the linearised foliation at either of the poles must be strictly finer than the infinitesimal foliation, as the former is homogeneous (see below) while the latter is isomorphic to the inhomogeneous foliation. This motivates the following

Definition 2.17. A foliation $\mathcal{F}$ on a manifold M is called *linearisable at* $p \in M$ if the linearised foliation on $\nu_p\mathcal{F}_p$ coincides with the pull-back of $\mathcal{F}$ via the normal exponential map. It is called *linearisable* if it is linearisable at every point in M.

Consider now for a point $p \in M$ the set $\Xi^p_{\mathcal{F}}$ of vector fields vanishing at p as well as the set I_p of smooth functions $M \to \mathbb{R}$ vanishing at p. From our previous considerations it is clear that $\mathrm{lin}(\Xi^p_{\mathcal{F}}) = \Xi^p_{\mathcal{F}}/I_p\Xi_{\mathcal{F}}$ acts effectively on $\nu_p\mathcal{F}_p$ as a subgroup of $O(\nu_p\mathcal{F}_p)$ via its flows. The orbits of this action are again tangent to the (normal) exponential preimages of the leaves of $\mathcal{F}$ around p, which implies that for regular points this action must be discrete and in fact trivial as the group acting is connected. We make the following

Definition 2.18. The action of $\Xi^p_{\mathcal{F}}/I_p\Xi_{\mathcal{F}}$ on $\nu_p\mathcal{F}_p$ will be called the *infinitesimal isotropy action* of $\mathcal{F}$ at p. We will denote $\Xi^p_{\mathcal{F}}/I_p\Xi_{\mathcal{F}}$ by F_p and call it the *infinitesimal isotropy group* at p.

This definition of ours represents the special case of the infinitesimal isotropy defined in [**AZ**] for a singular riemannian foliation.
Notice, however, that the stabiliser of a given point p under the action of the normal holonomy pseudogroup of its leaf may be larger than the infinitesimal isotropy: A smooth loop at p that lies within $\mathcal{F}_p$ which is not homotopic to the constant curve at p may induce an isometry of $\nu_p\mathcal{F}_p$ with discrete, non-trivial orbits. (In other words, the infinitesimal isotropy is the connected component of the identity in the stabiliser of a given point.) We call regular leaves with non-trivial stabiliser *exceptional* and *principal* otherwise. Every neighbourhood of an exceptional leaf contains a discrete non-trivial orbit, as the stabiliser acts linearly. Hence exceptional leaves are isolated in the regular stratum. The stabiliser of a singular point p acts linearly and orthogonally with orbits of positive dimension on the normal space $\nu_p\mathcal{F}_p$. For it not to be contained in $SO(\nu_p\mathcal{F}_p)$ it would have to contain a reflection, but that reflection would fix a hypersurface, which would imply that the (uncountably many) regular orbits traced in this corresponded to exceptional leaves. As exceptional leaves are isolated this cannot happen and thus the stabiliser of a singular point always coincides with its infinitesimal isotropy.

2.2. Polar Foliations

A singular riemannian foliation $\mathcal{F}$ is said to have *sections* if for every regular point p the set $\exp_p(\nu_p\mathcal{F}_p)$ is a complete, totally geodesic immersed submanifold. The intersection of a convex set around p with a section through p will be called a *local section* at p. The intersection of a tubular neighbourhood of $\mathcal{F}_p$ with $\exp_p(\nu_p\mathcal{F}_p)$ is called the *slice* at p. It can be written as the union of all local sections containing p (cf. Proposition 2.1 in [**Al2**]). The dimension of a section coincides with the codimension of the singular foliation and is thus the same for all sections.
An important class of examples for this are the orbit decompositions of polar actions of a Lie group on a riemannian manifold and in particular the adjoint action on a compact Lie group by a compact subgroup, where in the latter case the sections are the maximal tori of the group. Another class of examples are *isoparametric foliations* of euclidean space, which arise as the level sets of isoparametric maps, where a smooth map $f = (f_1, ..., f_m) : \mathbb{R}^{n+m} \to \mathbb{R}$ is called isoparametric if

- f has a regular value
- the "first and second differential parameters", $\langle \nabla f_i, \nabla f_j \rangle$ and Δf_k, are constant on the level sets of f for all i, j, k (hence the name),
- $[\nabla f_i, \nabla f_j]$ is a linear combination of $\nabla f_1, ..., \nabla f_m$ where the coefficients are constant on the level sets of f for all i, j.

In [**Al2**] Alexandrino proved a remarkable "Slice Theorem" showing that isoparametric foliations are a universal local model for singular riemannian foliations with sections:

Theorem (Theorem 2.10 in [**Al2**]). *Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold M and $\mathcal{S}_p$ the slice at a point $p \in M$. Then the restriction of $\mathcal{F}$ to $\mathcal{S}_p$ is diffeomorphic to an isoparametric foliation on an open set of $\mathbb{R}^n$, where n is the dimension of $\mathcal{S}_p$.*

In analogy to the homogeneous case we also refer to singular riemannian foliations with sections (sometimes abbreviated as s.r.f.s.) as *polar (singular) foliations*, where we will in general omit the mention of "singular". (If needed we will explicitly refer to a foliation as "regular" if it has no singular leaves.)

From now on we will fix our manifold M to be compact, simply connected and positively curved. Furthermore the singular riemannian foliation $\mathcal{F}$ will be assumed to have sections and be of codimension at least two. This has several consequences which we shall endeavour to exhibit below.

As a first easy observation we note that the curvature of M is bounded away from 0 due to compactness. The sections as totally geodesic submanifolds of dimension ≥ 2 inherit this curvature bound and are thus compact by

the theorem of Bonnet-Myers. We also obtain immediately that the flows of the elements of $\Xi_{\mathcal{F}}$ are globally defined and thus the elements of the normal holonomy pseudogroup of a leaf L are in fact defined on all of L, yielding a normal holonomy group as a subgroup of $O(\nu L)$.
Consider the strata of M as traced in a section Σ. Then any geodesic anywhere tangent to the intersection of a stratum with Σ is everywhere contained in Σ due to total geodesicity and thus perpendicular to the leaves of the singular points where it touches the singular stratum. It is hence by Proposition 2.10 traced entirely in that stratum in a neighbourhood of the singular leaves it meets. The connected components of the singular stratum as traced in a section are thus totally geodesic. It follows in particular that the singular points on a transverse geodesic are either isolated or all of its points. As a corollary of the Slice Theorem in [**Al2**] one moreover obtains that the singular stratum as traced in a local section consists of a finite union of hypersurfaces. The union of all local sections contained in a given section Σ constitutes an open cover which by compactness has a finite subcover consisting of local sections σ_i, each containining finitely many (open subsets of the) totally geodesic hypersurfaces forming the singular stratum. Hence there can be only finitely many such hypersurfaces in total and we have proven:

LEMMA 2.19. *For any section Σ of a polar foliation $\mathcal{F}$ of codimension at least two on a compact, positively curved manifold M the intersection of Σ with the singular strata is a finite union of totally geodesic, compact hypersurfaces.*

We furthermore have that the regular stratum as traced in a section is an open and dense subset of that section if the ambient manifold is complete (see [**Al3**]) and the following:

PROPOSITION 2.20 (Theorem 1.2 in [**Ly**]). *If $\mathcal{F}$ is a polar foliation on a simply connected manifold then all leaves are closed.*

Since M is compact this implies that the leaves are compact themselves and we can apply

PROPOSITION 2.21 (Theorem 1.5 in [**AT**]). *If $\mathcal{F}$ is a polar foliation on a simply connected manifold such that the leaves are compact then:*

- *The quotient $M/\mathcal{F}$, i.e. the space of leaves, is a Coxeter orbifold.*
- *Any connected component Ω of the intersection of the regular stratum with a section is homeomorphic to the quotient $M_r/\mathcal{F}$ and its closure in the section is homeomorphic to $M/\mathcal{F}$, both via the canonical projections.*
- *Any such Ω is a convex subset of M.*

The second item implies that the intersection of every regular leaf with any such Ω is unique and thus there can be no exceptional leaves:

COROLLARY 2.22. *Let $\mathcal{F}$ be a polar foliation on a simply connected, compact riemannian manifold M, then all regular leaves are principal leaves, i.e. they have trivial normal holonomy.*

We now wish to introduce a notion that is somewhat analogous that of an equivariant normal field in the case of a group action. Recall that such a field is defined to be a normal field ξ along an orbit of the action such that for any group element g we have $\xi_{g(p)} = dg_p(\xi_p)$.

DEFINITION 2.23. A vector field ξ which is normal to the foliation is called an *equifoliate normal field* if for any $X \in \Xi_{\mathcal{F}}$ with flow ϕ^t it satisfies the condition

$$\xi_{\phi^t(p)} = \pi \circ d\phi^t_p(\xi_p),$$

where π is the projection to the horizontal distribution $\nu\mathcal{F}$.

PROPOSITION 2.24. *Let L be a principal leaf of a singular riemannian foliation $(M, \mathcal{F})$ and $p \in L$. Then on any sufficiently small neighbourhood U of p in L there exist* $\dim \nu_p L$ *linearly independent equifoliate normal fields.*

PROOF. Let $v_1, ..., v_k$ be a basis of $\nu_p L$. We choose U so small that every point $q \in U$ can be written as $\phi^t(p)$ for some flow of a vector field $X \in \Xi_{\mathcal{F}}$. Define ξ^i on U by $\xi^i_{\phi^t(p)} = \pi \circ d\phi^t_p(\xi^i_p)$ and $\xi^i_p = v_i$. It remains to prove that ξ^i is well-defined. Assume $\phi^t_1(p) = \phi^t_2(p)$ holds for two flows ϕ^t_1, ϕ^t_2. Then $\phi^{-t}_1 \circ \phi^t_2(p) = p$. and thus the linearised flow $\pi \circ d\phi^{-t}_1 \circ d\phi^t_2$ acts on $\nu_p L$ as an element of the infinitesimal isotropy group. But the latter is trivial as L is a principal leaf and so $\pi \circ d\phi^{-t}_1 \circ d\phi^t_2 = \mathrm{Id}_{\nu_p L}$ which implies $\pi \circ d\phi^t_1 = \pi \circ d\phi^t_2$ and thus ξ^i is well-defined. □

LEMMA 2.25. *Let ξ be an equifoliate normal field along a principal leaf L of a polar foliation $(M, \mathcal{F})$. Then ξ is parallel with respect to the normal connection $\nabla^\perp$.*

PROOF. Consider an equifoliate normal field ξ along a principal leaf L, p a point in a plaque $P \subset L$. Extend ξ_p to vector field on a small neighbourhood of p in the respective local section and then extend this vector field equifoliately along the foliation to a vector field on a small tubular neighbourhood of L which by abuse of notation we will also denote by ξ. This is well-defined by iterating the proof of Proposition 2.24 over sufficiently small neighbourhoods covering L. Let furthermore $X \in \Xi_{\mathcal{F}}$ be chosen such that its pull-back via the plaque exponential $\exp^\perp$ coincides with its linearisation X^* along L. Such fields can easily be constructed by pushing forward the linearisation of any other $X' \in \Xi_{\mathcal{F}}$ via $\exp^\perp$ and by Remark 2.16 they still everywhere span TL. Hence considering such fields will be sufficient for proving parallelity of ξ along L. Considering X as a normal field along any appropriate section Σ we have the Weingarten formula

$$\nabla_\xi X = -A^\Sigma_X \xi + \nabla^{\nu\Sigma}_\xi X,$$

where the Weingarten tensor A_X vanishes as the sections are totally geodesic, and hence $\nabla_\xi X$ is identical to its Σ-normal part, i.e. tangent to the foliation. Let furthermore ξ^* be the pull back of ξ via the plaque exponential. This is a vector field on νL which is everywhere tangent to the fibres. The flows of the linearised vector fields can be written as $\Phi^t = \pi \circ d\phi^t$ where ϕ^t is the flow of an appropriately chosen $X \in \Xi_{\mathcal{F}}$. Since they are linear isometries their differentials are obtained by composition with $d\exp^\perp$ and its inverse from the left and right respectively. In toto we have

$$d\Phi^t = (d\exp_P)^{-1} \circ \pi \circ d\phi^t \circ d\exp_P .$$

With $q = \exp_P(\Phi^t(0_p)) = \phi^t(p)$ it follows that

$$\begin{aligned} d\Phi^{-t}_{0_{\phi^t(p)}}(\xi^*_{0_{\phi^t(p)}}) &= (d\exp_P)^{-1} \circ \pi \circ d\phi^{-t} \\ &\quad \circ d\exp_P((d\exp_P)^{-1} \circ \xi \circ \exp_P(0_q)) \\ &= (d\exp_P)^{-1} \circ \pi \circ d\phi^{-t}((\xi \circ \exp_P(0_q)) \\ &= (d\exp_P)^{-1} \circ \pi \circ d\phi^{-t}(\xi_{\phi^t(p)}) \\ &= (d\exp_P)^{-1}(\xi_p), \text{ as } \xi \text{ is equifoliate,} \\ &= \xi^*_{0_p}. \end{aligned}$$

This allows us to compute

$$[X^*, \xi^*]_{0_p} = \lim_{t\to 0} \frac{1}{t}(d\Phi^{-t}_{0_{\phi^t(p)}}(\xi^*_{0_{\phi^t(p)}}) - \xi^*_{0_p}) = \lim_{t\to 0} \frac{1}{t}(\xi^*_{0_p} - \xi^*_{0_p}) = 0.$$

Since X^*, ξ^* and X, ξ respectively are $\exp_P$-related, we have

$$[X, \xi] = (\exp_P)_*[X^*, \xi^*] = 0 \text{ along the leaf L,}$$

and so $\nabla_X \xi = \nabla_\xi X - [X, \xi] = \nabla_\xi X$, which is tangent to the foliation by our above argument. It follows that $\nabla^\perp_X \xi = 0$ for all $X \in \Xi_{\mathcal{F}}$ as above, which readily implies $\frac{\nabla}{dt}^\perp \xi = 0$ along any curve contained in L. □

REMARK 2.26. Let p be a point on a principal leaf and σ a local section containing p. Any smooth curve $c : I \to \mathcal{F}_p$ with $c(0) = p$ can locally be written as the flow curve of a vector field tangent to the foliation by transitivity. As the leaves are compact we obtain a linear isometry $\phi_c : \nu_p\mathcal{F}_p \to \nu_{c(1)}\mathcal{F}_p$ as the composition of finitely many elements of the normal holonomy group of $\mathcal{F}_p$. This can be exponentiated to a diffeomorphism $\varphi_c : \sigma \to \operatorname{Im}\varphi_c$, where the image $\operatorname{Im}\varphi_c = \exp_{c(1)}(\phi_c(\exp_p^{-1}(\sigma)))$ is again a local section by construction. Since ϕ_c respects the leaves of the infinitesimal foliation on the normal bundle, so does φ_c with the leaves of the foliation around $\mathcal{F}_p$. The distance between two points $\varphi_c(q), \varphi_c(q') \in \sigma'$ is realised by a transverse geodesic and hence equal to the distance between their respective plaques, but the same holds for the distance between $q, q' \in \sigma$, which implies that φ_c preserves distances and is hence an isometry of the local sections along $\mathcal{F}_p$.

In **[Al2]** Alexandrino constructs on every local section a *pseudogroup of singular holonomy* consisting of foliate isometries. The construction of the isometries, as seen in the proof of Proposition 3.1 ibidem, is done by parallel transport along regular leaves. As we have shown above the local isometries of a given section obtained from exponentiating the isometries of the normal bundle where the leaves intersect the section can be written as such a parallel transport and are hence contained in the pseudogroup constructed by Alexandrino whenever there is a local section containing both the domain and image.

DEFINITION 2.27. We shall write W_Σ for the pseudogroup generated by all partial isometries φ_c obtained from curves c tangent to the regular leaves of the foliation with $c(0), c(1) \in \Sigma$ for a given section Σ and call it the *polar pseudogroup* of Σ.

ANNOTATION. As we will see below this coincides with the *generalised Weyl pseudogroup* referred to by Alexandrino and Töben (cf. **[AT]**, **[Tö]**).

By construction all elements of W_Σ are *foliate*, i.e. they preserve the leaves of the foliation. According to Proposition 3.3. in **[Al2]** this pseudogroup contains the (local) reflections in the singular hypersurfaces. Here by *reflection* we mean an isometry that fixes a set of codimension one and whose differential acts as multiplication by -1 on the normal bundle of the fixed point set. Since the singular hypersurfaces are compact there exists a small tube of constant radius around each hypersurface Λ within which every point is linked to Λ by a unique shortest geodesic, which thus meets Λ perpendicularly. We can therefore extend any reflection to this tube around its corresponding hypersurface by mapping each of the short normal geodesic segments starting on Λ to the one with initial velocity multiplied by -1. As they are the fixed point sets of reflections we thus also call the singular hypersurfaces the *mirrors* of Σ.

Consider now the connected components C of the regular stratum traced in a given section Σ, which we will suggestively call *chambers*. By Proposition 2.21 they are convex and intersect each regular leaf in a unique point. If it is not already defined there, the convexity allows us to extend any element φ of W_Σ at least to the chambers intersecting its domain.

LEMMA 2.28. *An element φ of W_Σ that maps a regular point p into its own chamber is the identity on its domain.*

PROOF. As the regular stratum is open φ must map a small open neighbourhood containing only regular points into the chamber containing p. The intersections of regular leaves with chambers are unique, though, and thus φ must map this open neighbourhood identically onto itself. Hence its differential at p is the identity, but φ being an isometry this causes it to be the identity on all of its domain. □

LEMMA 2.29. *Outside of their intersections the mirrors separate two locally unique chambers.*

PROOF. It is clear that a globally compact hypersurface locally separates its ambient space into at most two (global) components. It thus remains to be seen that for a sufficiently small open neighbourhood V of a singular point lying on a unique mirror H the components of $V \setminus H$ do not lie in the same chamber. Assume the contrary, then the local reflection in V maps an open set of that chamber back into the chamber itself. However, by Lemma 2.28 this means that the reflection is the identity, which is a contradiction. $\square$

PROPOSITION 2.30. *The pseudogroup W_Σ is generated by the reflections in the singular hypersurfaces.*

PROOF. Consider $\varphi \in W_\Sigma$ and an arbitrary regular point $p \in \Sigma$. Link p to $\varphi(p)$ by a smooth curve γ in Σ, which will meet the singular stratum only in isolated points, as it starts tangent to the regular stratum. Since the intersections of the mirrors are of codimension at least two in Σ we can perturb γ such that it only meets the singular stratum outside of the intersections of the mirrors. Let $r_1, ..., r_l$ denote the reflections in the mirrors crossed by γ when moving from p to $\varphi(p)$ and write C_i inductively for the chamber obtained from C_{i-1} by applying r_i, starting with C_0 the chamber containing p. Choose $0 = t_0 < t_1 < ... < t_l = 1$ such that $\gamma(t_i) \in C_i$. With Lemma 2.29 it then follows inductively that $r_i \circ ... \circ r_1(p)$ lies in the same chamber as $\gamma(t_i)$ and hence $r_l \circ ... \circ r_1(p) \in C_l$ which also contains $\varphi(p)$. The composition $\varphi \circ r_1 \circ ... \circ r_l$ thus maps C_0 to itself and is the identity by Lemma 2.28, which completes the proof. $\square$

We call the intersections of the mirrors with the closure $\overline{C}$ of a chamber the *(codim 1-)faces* of C. Their mutual intersections traced in $\overline{C}$ are then referred to as the *faces* of subsequently higher codimensions. We can in particular label the faces of $\overline{C}$ by the reflections that fix them, where the interior C is labelled by the empty set as the "codim 0-face". This yields a cell decomposition of Σ where the open chambers are the cells of maximal dimension and thus in particular have the same dimension. The faces of a chamber (or more generally any cell) are then all sub-cells contained in it. Thus we have the following

PROPOSITION 2.31. *The partition by the mirrors equips a given section Σ with the structure of a labellable chamber complex and the pseudogroup W_Σ acts transitively on the set of chambers of this complex.*

We will denote this complex by $\mathcal{C}(\Sigma, W_\Sigma)$.

LEMMA 2.32. *All open chambers of the complexes $\mathcal{C}(\Sigma, W_\Sigma)$ for all sections $\Sigma \subset M$ are isometric to each other. Furthermore their closures are isometric as metric spaces.*

PROOF. Open chambers are convex by Proposition 2.21 and thus any element of a given pseudogroup W_Σ can be extended at least on the open chambers having non-empty intersection with its domain. We know thus by the above proposition that all chambers in a given complex $\mathcal{C}(\Sigma, W_\Sigma)$ are isometric. It now suffices to note that a regular leaf meets every section and the holonomy pseudogroup of any regular leaf therefore provides an isometry between any fixed chamber and at least one chamber in every section. Since being isometric to each other is an equivalence relation this completes the argument for the open chambers. For the statement about the closures it suffices to note that the singular strata are totally geodesic and thus the closed chambers remain convex as metric spaces. $\square$

In order to see that these complexes are not trivial it suffices to prove

LEMMA 2.33. *A polar foliation on a complete, positively curved manifold must have singular leaves.*

PROOF. Assume the contrary, then the foliation is regular and by Theorem 1.3 in [**Wa**] the leaves must be totally geodesic. Since the foliation is polar and sections can only intersect in singular leaves this means that the sections form another totally geodesic foliation, orthognal to the original foliation. It follows that the manifold everywhere locally splits as a metric product, which implies that the sectional curvature on planes spanned by a vector tangential to the original foliation and a vector orthogonal to it must vanish. This contradicts the positive curvature. $\square$

The question arises whether the pseudogroup W_Σ can be extended to a group of isometries, generated by reflections and acting on Σ. Töben proved in [**Tö**] (cf. p. 21f. ibidem) that W_Σ is covered by a group $\widetilde{W_\Sigma}$ containing the reflections in the lifts of the mirrors to the universal cover of Σ. This lifted group contains the deck transformations of the covering $\widetilde{\Sigma} \to \Sigma$ and the obstruction to pushing this group action down to Σ is the non-normality of the deck transformation group in $\widetilde{W_\Sigma}$. We will make active use of this in the proof of the following

PROPOSITION 2.34. *Let $\mathcal{F}$ be a polar foliation on a compact, positively curved riemannian manifold M. Then any section Σ is diffeomorphic either to a sphere or a real projective space and the polar pseudogroup W_Σ extends to a group of foliate isometries generated by reflections, acting on Σ.*

PROOF. We know by Lemma 2.33 that $\mathcal{F}$ has singular leaves. Hence there exists at least one singular hypersurface $\Lambda \subset \Sigma$ with an associated reflection r acting on a small tubular neighbourhood $\mathcal{T}_\varepsilon\Lambda$ which we may conceive as the exponential image of the normal ε-disc bundle. As curvature is strictly positive the distance function to Λ on Σ is strictly concave (cf. [**Wu**]). Thus the connected components of $\Sigma \setminus \mathcal{T}_\varepsilon\Lambda$ are locally convex and contain a unique point at maximal distance from Λ (sometimes referred to as the "soul point"). By the Soul Lemma (Corollary 1.10 in [**Gr**]) the components are diffeomorphic to the normal bundle of these soul points and thus are diffeomorphic to open balls.

Consider first the case where $\Sigma \setminus \Lambda$ has two components. Then Σ is diffeomorphic to the gluing of two open balls along Λ and thus homeomorphic to a sphere. (In general it might still be only a twisted sphere.) Thus we know that Σ is simply connected and hence its own universal cover. By the work of Töben mentioned above we then know that W_Σ extends to a group of foliate isometries generated by reflections. In particular the reflection r can be extended globally onto Σ, allowing us to repeat the argument given for the homogeneous case (section of a polar action) in the proof of Proposition 2.3 in [**FGT**]: Choose a diffeomorphism ψ from the upper hemisphere S_+ of the standard sphere S to one of the connected components of $\Sigma \setminus \Lambda$, say Σ_+, as provided by the Soul Lemma. This diffeomorphism is constructed using the flow lines of a gradient-like vector field on Σ_+ induced by the distance function to the soul point, and thus may be chosen such that the north pole is mapped to the soul point and the flow lines are the images of radial geodesics around the north pole in a neighbourhood of the soul point and of normal geodesics in a tubular neighbourhood of Λ. Let ρ denote the reflection in the equator of the standard sphere, bounding S_+. Then we define a diffeomorphism $\Psi : S \to \Sigma$ by $\Psi(x) = \psi(x)$ if $x \in S_+$ and $\Psi(x) = r\psi\rho(x)$ if $x \in S_-$.
Now consider the case where the complement of Λ has only one component. It follows that the boundary of the tubular neighbourhood $\mathcal{T}_\varepsilon\Lambda$ is diffeomorphic to the boundary of an open ball, i.e. a sphere. The local reflection r acts freely and isometrically on this sphere and the metric projection $\pi : \partial(\mathcal{T}_\varepsilon\Lambda) \to \Lambda$ is thus a covering of Λ with involutive deck transformation r. Thus Λ and by Seifert-van Kampen Σ have fundamental group $\mathbb{Z}_2$. By the work of Töben we can lift W_Σ to a reflection group on the two-fold universal cover $\widetilde{\Sigma}$, which by the argument from the previous paragraph is diffeomorphic to a standard sphere. We note now, that any local reflection in W_Σ has two lifts into $\widetilde{W}_\Sigma$, one which is a reflection in a lifted mirror, and one which acts as the deck transformation a on the mirror and as a rotation on the connected components of its complement. Each of these lifts preserves the mirror and hence a commutes with any lifted reflection, whence the deck transformation group $\langle a \rangle$ is normal in the lifted group. It follows that we can push down $\widetilde{W}_\Sigma$ to obtain an extension of W_Σ to a group acting on the section. Again, this allows us to repeat the argument given in [**FGT**] similar to the above paragraph, yielding a diffeomorphism from the standard sphere to the universal cover of Σ which is now in particular equivariant with respect to the antipodal map on the standard sphere and the deck transformation on $\widetilde{\Sigma}$. It thus descends to a diffeomorphism $\mathbb{R}P^k \to \Sigma$. $\square$

As the polar group preserves leaves the reflections must map mirrors to mirrors. Since there are only finitely many mirrors the polar group must in particular be finite.

LEMMA 2.35. *The quotients Σ/W_Σ and $M/\mathcal{F}$ coincide and any two reflections generate a dihedral group.*

PROOF. The first part is a simple consequence of the second bullet in Proposition 2.21 and Proposition 2.31. By Theorem 1 in [**Fr**] the mirrors of any two reflections must intersect at some point, as the ambient manifold Σ is positively curved. The second part now follows from the first bullet in Proposition 2.21 and that the mirrors project to the boundary of the Coxeter orbifold $M/\mathcal{F}$. □

This yields directly the following result obtained by Fang, Grove and Thorbergsson in [**FGT**] in subsequence of the analogon of Proposition 2.34, where the respective proof uses the homogeneity of the foliation only to obtain the dihedral group we have obtained in the previous lemma and otherwise relies entirely on the geometric properties of the mirrors and positive curvature:

LEMMA 2.36 (Lemma 2.5 in [**FGT**]). *If Σ is a sphere of dimension k with group W_Σ, then*

- *the intersections of mirrors are spheres.*
- *there are at most $k+1$ chamber faces and the intersection of all of them is* $\mathrm{Fix}(W_\Sigma)$.
- *if there are $k+1$ chamber faces then the closure of any chamber is a k-simplex and the fixed point set of W_Σ is empty.*
- *if there are $l+1 < k+1$ chamber faces, then the closure of any chamber is a join of an l-simplex with* $\mathrm{Fix}(W_\Sigma)$.

One obtains from this an analogous description of the chambers for the case where Σ is a projective space by considering the universal cover $\widetilde{\Sigma}$ equipped with the group generated by all those lifts of reflections $r \in W_\Sigma$ that are reflections in the lifted mirrors. It then suffices to note that the chambers in Σ are isometric to those in $\widetilde{\Sigma}$ and that the former are obtained from the latter by identifying the orbits of the decktransformation in the boundary of the closed chambers.
Notice that the lifted reflection group described above will in general be smaller than the group $\widetilde{W}_\Sigma$ considered by Töben, as the latter always contains the deck transformation while the former may not. We will denote the former by $\overline{W}_\Sigma$ for distinction.
One furthermore has the following result from [**FGT**], which complements the description of the fixed point set of the action of W_Σ given in the previous lemma for the spherical case by the one for the projective case:

LEMMA 2.37 (Proposition 2.8 in [**FGT**]). *Let M be a positively curved, simply connected, compact manifold with a polar foliation and Σ a section of it. If Σ is a projective space, then the chambers are simplices, the group of reflections in the lifted mirrors in $\widetilde{\Sigma}$ contains the deck transformation and the fixed point set is a subset of the vertices of any (and thus every) chamber.*

The above results imply that the polar group of a spherical section and the lifted polar group on the universal cover of a projective section are reflection groups on simply connected spaces and thus Coxeter groups, i.e. they are generated by reflections, any two reflections form a dihedral group and there are no other relations between the generators. Any such group can be presented in the form $\langle r_1, ..., r_l | (r_i r_j)^{m_{ij}} \rangle$. For finite Coxeter groups the orders m_{ij} of the dihedral groups form a positive definite matrix and the matrix uniquely determines the Coxeter group. Furthermore we can represent the group by a (weighted) graph, where each node corresponds to a generator and any two nodes r_i, r_j are linked by an edge, weighted with m_{ij}, if $m_{ij} \geq 3$, and not linked by an edge if the order of their dihedral group is 2. A Coxeter group together with a vector space (or, equivalently, its unit sphere) on which it acts linearly is called a *Coxeter system*. Such a system is called *irreducible* if it cannot be written as a product of proper subgroups acting each on proper subspaces (or sub-spheres), and *reducible* otherwise. It is irreducible if and only if its associated graph is connected. The graph is also referred to as the *Coxeter diagram* of the complex. We refer to the book by Davis ([**Da**]) for these results and a general treatise of Coxeter groups.

We have the following theorem due to Fang, Grove and Thorbergsson:

PROPOSITION 2.38 (Theorem 2.11 in [**FGT**]). *For a simply connected, compact, positively curved manifold with a polar foliation the action of each polar group W_Σ is differentiably equivalent to a linear action on a round sphere or real projective space and each section admits a metric of constant curvature invariant under the polar group.*
In particular the chamber complex $\mathcal{C}(\Sigma, W_\Sigma)$ (for the spherical case) respectively $\mathcal{C}(\widetilde{\Sigma}, \overline{W}_\Sigma)$ (for the projective case) is a Coxeter complex.

REMARK 2.39. In the homogeneous case described in [**FGT**] Fang, Grove and Thorbergsson furthermore prove that the entire manifold admits a metric adapted to the foliation by the orbits of the polar action $G \curvearrowright M$ such that the sections are all positively curved. This is achieved by employing a result of Mendes ([**Me**]) that allows to lift a W_Σ-invariant metric on a section Σ to a G-invariant metric on M. No such result for the inhomogeneous case is known to us. In the light of the recent interesting results on smooth basic functions obtained by Mendes and Radeschi in [**MR**] it may seem reasonable to hope, though, that a similar extension result for metrics can be obtained for polar foliations in general.

As a final point in this chapter consider the isotropy action at a point p of a polar foliation $\mathcal{F}$. Recall from Definition 2.18 that this is the action of the infinitesimal isotropy F_p, consisting of linearised flows of those vector fields vanishing at p, on the normal space $\nu_p \mathcal{F}_p$.

PROPOSITION 2.40. *The isotropy action of a linearisable polar foliation $\mathcal{F}$ at a point p on a compact, simply connected manifold M is a polar action by a compact, connected Lie-Group.*

PROOF. By Proposition 2.20 and the compactness of M we know that the leaves are compact. Since they are traced in the distance tubes around

$\mathcal{F}_p$ and their exponential preimages coincide with the orbits of the isotropy action by linearisability the latter are compact. As the infinitesimal isotropy F_p is connected and acts effectively it is compact, too, and by its linear, isometric action a subgroup of the Lie group $O(\nu_p\mathcal{F}_p)$. Thus it is a closed subgroup and a Lie group itself.
It remains to be seen that the action is polar. Let Σ be a section through p and X any element of $\Xi_{\mathcal{F}}$ and ϕ^t its flow. As the sections are totally geodesic and X is everywhere orthogonal to Σ we know that the shape operator A_X, i.e. the Σ-tangential part of the endomorphism $\nabla X : T_pM \to T_pM$, vanishes. We wish to see that the linearised flow curves $\pi \circ d\phi^t(v)$ are orthogonal to $T_p\Sigma$ for every $v \in \nu_p\mathcal{F}_p$. For that it suffices to examine $d\phi^t(v)$ as the $T_p\mathcal{F}_p$-component of it is orthogonal to $T_p\Sigma$ by the polarity of $\mathcal{F}$. Consider therefore a curve γ in M with $\dot\gamma(0) = v$ and:

$$\begin{aligned} d_t|_0 d\phi^t(v) &= d_t|_0 d_s|_0 \phi^t(\gamma(s)) \\ &= \frac{\nabla}{dt}|_0 \partial_s|_0 \phi^t(\gamma(s)) \\ &= \frac{\nabla}{ds}|_0 \partial_t|_0 \phi^t(\gamma(s)) \\ &= \frac{\nabla}{ds}|_0 X_{\gamma(s)} = \nabla_v X \perp T_p\Sigma, \end{aligned}$$

which shows, that for any section Σ the tangent space $T_p\Sigma$ is a section for the isometric action of the infinitesimal isotropy F_p. $\square$

Remark 2.41. Notice that, since the leaves of points in the stratum of p are of the same dimension as $\mathcal{F}_p$, the orbits of their exponential preimages under the isotropy action at p must again be of the same dimension as the orbit of the origin in $\nu_p\mathcal{F}_p$, which is zero. It follows that F_p acts trivially on the tangential space to the stratum of p. As the action is linear and isometric it is furthermore uniquely determined by the orbits on the unit sphere in $\nu_p\mathcal{F}_p$. Together this yields that the isotropy action is completely determined by the action of F_p on $S_p^\perp$, the unit sphere normal to the stratum of p in $\nu_p\mathcal{F}_p$, which is a polar action with sections $T_p\Sigma \cap S_p^\perp$.

CHAPTER 3

Chamber Systems and Buildings

In this chapter we will study a combinatorial structure on the manifold M induced by the actions of the reflection groups on the sections of the polar foliation and show how this places strong restrictions on the geometry of M. As before we will be considering the positively curved, simply connected, compact manifold M with a polar foliation $\mathcal{F}$ of codimension at least two.

3.1. Chamber Systems

Consider the chamber systems $\mathcal{C}(\Sigma, W_\Sigma)$ for every section of the polar foliation and define $\mathcal{C}(M, \mathcal{F})$ as their union. Since every point of M lies in a section this yields a cell decomposition of M, inheriting the chamber complex structure from the complexes on the sections. Curves in this chamber complex are all broken transverse curves in M, i.e. all piecewise smooth curves such that each smooth segment lies in a section (or, more generally, is perpendicular to any leaf it meets). In **[Wl]** Wilking defines the *dual foliation* of a singular riemannian foliation by the sets of all points that can be connected by broken transverse curves and then proceeds to show

Proposition 3.1 (Theorem 1 in **[Wl]**). *For any singular riemannian foliation on a complete, positively curved manifold the dual foliation has only one leaf.*

Since all chambers are isometric as length metric spaces by Lemma 2.32 this induces a length metric space structure on $\mathcal{C}(M, \mathcal{F})$, which by the above result by Wilking is path-connected and whose geodesics (in the metric sense) are broken transverse geodesics (of M). The topology induced by this length metric will be called the *thin* topology on $\mathcal{C}(M, \mathcal{F})$. It will in the course of our considerations be complemented by a *thick* topology to recover the topology of M and identify M with its chamber complex.
If k is the codimension of the foliation then all open chambers are subsets of M of dimension k. We say that two chambers in $\mathcal{C}(M, \mathcal{F})$ are *adjacent* if they share a common $(k-1)$-face, as defined in Section 2.2.

Remark 3.2. Notice that for every $p \in M$ the action of the infinitesimal isotropy F_p on $S_p^\perp$ is polar (see Proposition 2.40) and the sphere normal to the stratum is compact, simply connected and, as a round sphere, positively

curved. Hence all results obtained for $\mathcal{C}(M,\mathcal{F})$ also apply to the chamber complex $\mathcal{C}(S_p^\perp, F_p)$.
This in particular implies that F_p acts transitively on the set of chambers of $\mathcal{C}(S_p^\perp, F_p)$, as F_p is transitive on the leaves of its orbit foliation and each leaf intersects each chamber. Moreover, the exponential map carries (suitably scaled) chambers of $\mathcal{C}(S_p^\perp, F_p)$ to the intersection of all chambers containing p with a small distance sphere around p. It follows that the chambers containing p are in one-to-one correspondence with the chambers of its infinitesimal isotropy action.

As a first result on $\mathcal{C}(M,\mathcal{F})$ we have

Lemma 3.3. *The complex* $\mathcal{C}(M,\mathcal{F}) := \bigcup_\Sigma \mathcal{C}(\Sigma, W)$ *is labellable.*

Proof. We choose a labelling on a given (closed) chamber C as follows: Choose an indexing $I = \{0,1,2,...,l\}$ of the mirrors in the section Σ containing C and assign to a face the label $i_1,...,i_m$ if and only if it is the intersection of all the thusly indexed mirrors with C, with the chamber itself being labelled by the empty set. As the intersection of each leaf with any chamber is unique by Proposition 2.21 we can define a retraction of chamber complexes from $\mathcal{C}(M,\mathcal{F})$ to C by

$$p \mapsto \mathcal{F}_p \cap C.$$

By construction this is an adjacency preserving chamber map that is the identity on C. Following Appendix C of Chapter 1 in [**Br**] we can pull back the labelling on C via this retraction to a labelling on all of $\mathcal{C}(M,\mathcal{F})$. $\square$

The label of any sub-cell of $\mathcal{C}(M,\mathcal{F})$ is also called its *type* and the map $typ : \mathcal{C}(M,\mathcal{F}) \to I$ that assigns to every point the label of the smallest sub-cell containing it is called the *type map*. Having chosen a labelling by a set I we then also refer to a sub-cell as being of *cotype* J for a $J \subset I$ if its type equals $I \setminus J$.
By a *gallery* we mean any sequence of chambers $C_1,...,C_m$ such that C_j and C_{j-1} are adjacent. If their common face has type i_j we also specify them to be i_j-adjacent and refer to the gallery as being *of type* $i_1 \cdots i_m$ or *of type* J, where $J = \{i_1,...,i_m\}$. The connected components of the set of chambers that can be connected by galleries of type J, $J \subset I$, are called the J*-residues* of a given complex. The complex is called *connected* if it has only one I-residue. For any point p we denote by $\mathrm{res}(p)$ the residue of type $typ(p)$ containing p and more generally we speak of an m-residue, or a residue of rank m, if the cardinality of the indexing set J is m. Remark 3.2 thus implies that (up to identification under $\exp_p$) the infinitesimal isotropy F_p acts transitively on $\mathrm{res}(p)$.
We have the following important result:

Proposition 3.4. *If M is a simply connected, positively curved, compact riemannian manifold with a linearisable polar foliation $\mathcal{F}$, then the chamber system $\mathcal{C}(M,\mathcal{F})$ is connected.*

The proof of this proposition is completely parallel to that in the homogeneous case shown in [**FGT**] (Theorem 3.1 ibidem), where our infinitesimal isotropy group F_p will replace the isotropy of the polar group action in the homogeneous case.

PROOF. We will proceed inductively by the dimension of the chambers in $\mathcal{C}(M,\mathcal{F})$.
For any two chambers C, C' in $\mathcal{C}(M,\mathcal{F})$ choose points p, p' in their interiors. By Proposition 3.1 we can find a broken transverse geodesic γ linking them. As the regular stratum as traced in each section is open and dense we can choose γ such that its intersections with the singular stratum are isolated on it, and if necessary deform it such that it meets the singular stratum perpendicularly (though still with velocity 1). Let $\{t_i\}_{i=1,\ldots,m}$ denote the times at which $\gamma(t_i)$ is singular. It follows that at any such point the one-sided derivatives $\dot{\gamma}^+(t_i), -\dot{\gamma}^-(t_i)$ are contained in the unit sphere $S^\perp_{\gamma(t_i)}$ normal to the stratum of $\gamma(t_i)$. They are hence interior points of two chambers in the chamber complex $\mathcal{C}(S^\perp_{\gamma(t_i)}, F_{\gamma(t_i)})$ generated by the polar action of the infinitesimal isotropy $F_{\gamma(t_i)}$, whose chambers are of strictly lower dimension than those of $\mathcal{C}(M,\mathcal{F})$. By assumption these lower dimensional complexes contain galleries linking the chambers containing the two derivatives of γ. Exponentiating these yields a gallery in $\mathcal{C}(M,\mathcal{F})$ linking the chambers containing $\gamma(t_i-\varepsilon)$ and $\gamma(t_i+\varepsilon)$ (for sufficiently small ε) and thus in toto a gallery from C to C'.
If the chambers are of dimension one, then any broken transverse geodesic yields the desired gallery as its isolated singular points are identical with the zero-dimensional faces separating the chambers traversed by it, and which are hence subsequently adjacent. This provides the induction anchor. $\square$

Conjoining the above result with Remark 3.2 we can prove the following

PROPOSITION 3.5. *Let M be a positively curved, simply connected, compact manifold with a linearisable polar foliation $\mathcal{F}$ of codimension k, then (the exponential images of) the orbits of the isotropy actions of any collection of points representing the (open) $(k-1)$-faces of the chambers of $\mathcal{C}(M,\mathcal{F})$ are the leaves of the foliation.*

PROOF. Let p be any point in M and consider an arbitrary point $q \in \mathcal{F}_p$. Let C, C' be chambers containing p and q respectively. By Proposition 3.4 we can find a gallery joining C to C'. The elements C_j of the gallery are subsequently adjacent along $(k-1)$-faces. Choose in every $(k-1)$-face a point p_j, such that p_1 is contained in a face of C and p_m in a face of C'. Note that the (exponentiated) action of each F_{p_j} can by convexity be extended to all open chambers incident to the face of p_j and can by continuity be extended to its closure. Since F_{p_j} acts transitively on $\operatorname{res}(p_j)$ for each j we can find an element of F_{p_1} mapping p to a point in C_1 and proceeding inductively we obtain a sequence of elements of the F_{p_j} whose consecutive application to p maps p into C' and by uniqueness of the leaf-chamber intersection thus to q. Since we already know that the orbits of the infinitesimal isotropy are mapped into the leaves of $\mathcal{F}$ by the exponential maps at the respective points this completes the proof. $\square$

REMARK 3.6. Combining Proposition 3.4 and the proof of Proposition 3.5 above one sees that every gallery can be described as consecutively "folding" chambers onto the following chamber by an element of the infinitesimal isotropy of a point in their common face.

As a further application of Proposition 3.4 we can now describe the relation between the various fixed point sets of the polar groups W_Σ and the set of point leaves of the foliation $\mathcal{F}$.

PROPOSITION 3.7. *Let M be a simply connected, compact, positively curved manifold with a linearisable polar foliation $\mathcal{F}$ of codimension at least two. Then for any section Σ the set $M^{\mathcal{F}}$ of point leaves is a subset of the fixed point set of $W_\Sigma \curvearrowright \Sigma$ with equality if there exists a spherical section.*

PROOF. Since the polar groups W_Σ preserve leaves the inclusion follows immediately. Now suppose there is a section $\Sigma \cong S^k$ and $W_\Sigma \curvearrowright \Sigma$ has non-empty fixed point set. By Lemma 2.36 this is a subsphere $S \subset \Sigma$ and contained in every chamber of $\mathcal{C}(\Sigma, W_\Sigma)$. Let without loss of generality the complex $\mathcal{C}(M, \mathcal{F})$ be labelled by the set I indexing the reflections acting on Σ. The type map thus sends all points of S to I. Let C' be any chamber adjacent to a chamber C of $\mathcal{C}(\Sigma, W_\Sigma)$, then they share a common (closed) face, which must contain S as it is in particular fixed by the reflection in W_Σ corresponding to the mirror containing the face. Hence $S \subset C'$ and since its type is of the same cardinality as the number of reflections in the chamber faces it must be fixed by the polar group acting on the section containing C'. By Proposition 3.4 we can proceed inductively along galleries emanating from C to see that all sections contain S as their fixed point set. (In particular all sections are spheres, since projective sections do not admit spherical fixed point sets.) Now any leaf of $\mathcal{F}$ passing through an element of S must intersect any other chamber in a point with the same type, but the set of type I in any chamber is S itself and so the leaf must be a point, implying $\mathrm{Fix}(W_\Sigma) = M^{\mathcal{F}}$ for every section Σ. $\square$

COROLLARY 3.8. *It follows that if $M^{\mathcal{F}} = \emptyset$ then all chambers of $\mathcal{C}(M, \mathcal{F})$ are simplices.*

We shall from now on consider the case where there are no point leaves. The contrary case will be dealt with in Chapter 4.

3.2. Buildings

The aim of this section will be to see how the chamber complex $\mathcal{C}(M, \mathcal{F})$ gives rise to a spherical building, a concept we shall define below, which in turn will yield information about the homeomorphism type of M.

DEFINITION 3.9 ((Spherical) Building). A simplicial complex Δ together with a family $\{\delta_\alpha\}_{\alpha \in A}$ of subcomplexes, the *apartments*, is called a *(spherical) building* if

- every subcomplex δ_α is a (spherical) Coxeter complex,
- every two simplices of Δ are contained in at least one common apartment δ_α, and
- for every two simplices a, b in Δ both contained in the two apartments $\delta_\alpha, \delta_\beta$ there is an isomorphism of chamber complexes $\delta_\alpha \to \delta_\beta$ fixing a and b.

Notice that in particular any Coxeter complex is a building in its own right, even though a "trivial" one. The books by Brown ([**Br**]) and Ronan ([**Ro**]) may serve as further reference on buildings, where we note that in our above definition we follow the exposition in [**Br**] which gives more of a geometric intuition. It was shown in [**Ti2**] that every spherical building has a unique maximal system of apartments, so we may omit the reference to it henceforth. There is an equivalent, combinatorial-group theoretic definition as given in [**Ro**], which may prove more convenient for technical considerations:

DEFINITION 3.10 (Alternative Definition of (Spherical) Building). Let W be a (finite) Coxeter group with generating reflection r_i indexed over a set I. Then a chamber system Δ over I is called a *building* if every proper face is contained in at least two chambers and there is a *W-distance function* $\delta : \Delta \times \Delta \to W$ such that $\delta(x, y) = r_{i_1} \cdots r_{i_j}$ if and only x and y can be connected by a gallery of type $i_1, ..., i_j$, where $r_{i_1} \cdots r_{i_j}$ cannot be expressed as a shorter combination of reflections.

The underlying simplicial complex of a building is also referred to as its *geometric realisation*.

REMARK 3.11. For a riemannian, simply connected, semisimple symmetric space G/K the action of the isotropy group K on $T_{eK}G/K$ by differentials is called an *s-representation*. This can be identified with the adjoint representation of K on $\mathfrak{p} \cong T_{eK}G/K$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of the Lie algebra of G corresponding to the symmetric pair (G, K). This representation, endowed with the metric given by the negative of the Killing form, is polar, with sections the maximal abelian subalgebras of $\mathfrak{p}$. Hence we can consider its restriction to the unit sphere $S(\mathfrak{p})$. This gives rise to a chamber complex which admits the intersections of the maximal abelian subalgebras with the unit sphere as a system of apartments, making it into a spherical building (see [**Fn**], 3.4.2). By work of Dadok ([**Da**]) every polar representation of a compact Lie group without fixed points, restricted to the unit sphere, is orbit equivalent to an s-representation and thus its associated chamber complex on the unit sphere is in fact a spherical building.

The following example of a building will be of particular use to us:

EXAMPLE 3.12. Consider now a symmetric space N specifically of non-compact type, i.e. it is the quotient of a non-compact, semisimple, connected real Lie group U by a subgroup K that up to a central factor is a maximal compact subgroup. As the centre is a normal subgroup of K (see [**He**]) we may w.l.o.g. assume the centre to be trivial and K to be compact without changing N. There is a U-invariant metric on N such that it may be identified with the identity component of the isometry group of N for that metric. One defines an equivalence relation on all geodesics in N by saying that two geodesics are equivalent if they have finite Hausdorff distance form each other. One can show that each equivalence class of geodesics has a unique representative meeting a given point and so the set S_∞ of equivalence classes of geodesics is homeomorphic to the unit sphere at any point in N, the action by U extends to it in a continuous way and S_∞ is called the *sphere at infinity* of the non-compact symmetric space N. The action of the isotropy groups of U at every point on the tangent spaces is again polar and the sections project to the sphere at infinity as apartments of a building equivalent to that associated with the action of the isotropy group of a point on its unit tangent sphere (see above and [**Ji**] for further reference).

An *m-covering* of chamber complexes is defined to be a morphism of chamber complexes (i.e. a map preserving the chamber structure) which is an isomorphism on every J-residue with $|J| \leq m$.
As in the case of topological coverings there is the notion of a universal m-covering, whose uniqueness follows from the usual universal property that it covers all other m-coverings. It's existence follows by a similar construction as the set of homotopy classes of galleries, where by a *homotopy of galleries* we mean a finite sequence of so-called *elementary homotopies*. The latter are defined as the passage from a gallery $\Gamma = \Gamma_0\Gamma_1\Gamma_2$ to a gallery $\Gamma' = \Gamma_0\Gamma_1'\Gamma_2$ such that Γ_1, Γ_1' lie in the same residue of rank 2 and have the same initial and final chambers. Notice that in buildings any two galleries are homotopic and hence they are simply connected in the sense of combinatorial coverings (cf. Theorem 4.3 in [**Ro**]).
We have the following remarkable result by Tits:

PROPOSITION 3.13 (Corollary 3 in [**Ti1**]). *The universal 2-covering of a connected chamber system, labelled over a finite set, is a building if and only if all 3-residues are 2-covered by buildings.*

COROLLARY 3.14. *If M is a positively curved, compact, simply connected manifold with a linearisable polar foliation of codimension at least three without point leaves, then the universal cover $\widetilde{\mathcal{C}}(M, \mathcal{F})$ of the chamber complex $\mathcal{C}(M, \mathcal{F})$ is a spherical building.*

PROOF. Since the foliation has codimension three and no point leaves the chambers are 3-simplices and hence $\mathcal{C}(M, \mathcal{F})$ is of rank four. From Remark 3.2 and our identification of the residue of a point with the sphere normal to its stratum it is clear that all proper residues $\mathrm{res}(p)$ are isomorphic to the chamber complexes $\mathcal{C}(S_p^{\perp}, F_p)$ which are buildings by Remark 3.11. Since this includes the residues of rank three and $\mathcal{C}(M, \mathcal{F})$ is connected by Proposition 3.4 this completes the proof. □

We will denote the universal covering by $p : \widetilde{\mathcal{C}}(M, \mathcal{F}) \to \mathcal{C}(M, \mathcal{F})$ and for simplicity of notation we may omit the reference to M and $\mathcal{F}$. Using the combinatorial definition we have an associated Coxetergroup $W = W(M)$ with its diagram. (Alternatively, note that by the geometric definition all Coxeter complexes are isomorphic and hence "the" Coxeter group of the building is well-defined up to isomorphism.) We write $\mathcal{S}$ for the simplicial complex underlying the building. Furthermore we will denote by V_i the set of vertices of cotype i and by $\mathrm{Vert}(\widetilde{\mathcal{C}}(M, \mathcal{F}))$ the union over all V_i, $i \in I$. Notice that the cardinality of I is one higher than the codimension of the foliation $(M, \mathcal{F})$ as by our premise all chambers are simplices.
It will be our aim to invoke a result of Burns and Spatzier in **[BSp]** and its generalisation by Grundhöfer, Kramer, van Maldeghem and Weiss (see [**GKMW**]) linking certain types of buildings to Lie groups. In particular we will need the following

Definition 3.15 (Topological Building, see [**BSp**]). A (spherical) building $\widetilde{\mathcal{C}}$ is called a *topological building* if it is equipped with a Hausdorff topology on $\mathrm{Vert}(\widetilde{\mathcal{C}})$ such that the set $\widetilde{\mathcal{C}}_{i_1,...,i_j}$ of simplices of type $\{i_1, ..., i_j\}$ is closed in $V_{i_1} \times \cdots \times V_{i_j}$ for any collected of indices. Such a building is then furthermore called *compact, connected, locally connected or infinite* if the set $\widetilde{\mathcal{C}}_{1,...,m}$, is such in the thusly induced topology, with m being the rank of the building. Its *topological automorphism group* is defined to be the group of all chamber system isomorphisms that restrict to homeomorphisms on the each $\widetilde{\mathcal{C}}_{i_1,...,i_j}$.

Our subsequent endeavours will now be focussed on establishing the appropriate topology on $\widetilde{\mathcal{C}}$. As a first step we define the *chamber topology* on $\widetilde{\mathcal{C}}$ as follows:
Note first that since M is compact so are the (closed) chambers of $\mathcal{C}$ and they thus form a metric space with the Hausdorff-metric.
Choose now a chamber $\widetilde{C}_0$ in $\widetilde{\mathcal{C}}$ once and for all. Then for $k \geq \frac{1}{2}|W|$, $\varepsilon > 0$, and a chamber $\widetilde{C} \in \widetilde{\mathcal{C}}$ define $B_{\varepsilon,k}(\widetilde{C})$ to be set of all chambers $\widetilde{C}'$ such that there are galleries Γ, Γ' starting at $\widetilde{C}_0$ and ending in $\widetilde{C}, \widetilde{C}'$ respectively, of length at most k and whose projections to M lie at Hausdorff distance no more than ϵ from each other.
The chamber topology on $\widetilde{\mathcal{C}}$ is now defined to be the topology generated by the sets $B_{\varepsilon,k}(\widetilde{C})$.
The following result relies only on basic properties of buildings in general:

Lemma 3.16 (Lemma 4.5 in [**FGT**]). *The chamber topology is independent of the choice of* $\widetilde{C}_0$ *or* k.

Proposition 3.17 (cf. Proposition 4.4 in **[FGT]**). *Equipped with the chamber topology the space $\widetilde{\mathcal{C}}$ is compact, separable and metrizable.*

Proof. The proof of Proposition 4.4. in **[FGT]** for compactness and metrizability relies entirely on the compactness of M and basic properties of buildings and thus carries over identically. For separability consider for every infinitesimal isotropy group F_p the subgroup F_p' consisting only of the rational points (thinking of F_p as being appropriately represented as a subgroup of the real matrix group $O(\nu_p\mathcal{F}_p)$). Then the set of all chambers reachable from $\widetilde{C}_0$ by galleries obtained from folding (cf. Remark 3.6) with elements of the F_p' is dense and as countable union of countable sets itself countable. □

The next step will be to see how the chamber topology induces a topology on the set of vertices. Denote by $\pi_i : \widetilde{\mathcal{C}} \to V_i$ the projection that assigns to each chamber its vertex of cotype i and topologise V_i with the quotient topology induced by this, making π_i into a continuous map.

Lemma 3.18 (cf. Lemma 4.6 in **[FGT]**). *The projection π_i is an open map for every i, and V_i is compact and Hausdorff. For $x \in V_i$ the fibre under π_i is the residue res(x). This is compact and the restriction of the covering $p : \widetilde{\mathcal{C}} \to \mathcal{C}$ to it is a homeomorphism onto the residue res($p(x)$) in $\mathcal{C}$.*

Our proof will closely follow that in **[FGT]**, where we will make use of the infinitesimal isotropy groups and the normal holonomy groups of principal to circumvent the use of the polar group action in **[FGT]**. In fact, the first two parts of the proof carry over without modification to the arguments and we recount them here for completeness' sake only.

Proof. Any residue in a building is again a building by Theorem 3.5 in **[Ro]** and thus the universal covering of its projection under p. Any proper residue in $\mathcal{C}$ is, however, a building itself and thus the restricted covering must be an isomorphism. For the last claim it thus suffices to see that p is continuous and maps a compact into a Hausdorff space. Both $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are compact and Hausdorff as topological spaces, the former inheriting this as a length metric space from M, the latter by Proposition 3.17. Hence it remains to be seen that $\operatorname{res}(x)$ is a closed set. Let $(\widetilde{C}_n)$ be a sequence of chambers in $\operatorname{res}(x)$ converging in $\widetilde{\mathcal{C}}$. Choose a gallery from a fixed chamber $\widetilde{C}_0$ to $\widetilde{C}_1$ and furthermore galleries linking $\widetilde{C}_1$ to every subsequent $\widetilde{C}_n$, where the latter can be chosen as minimal galleries within $\operatorname{res}(x)$, again, because this is a building itself. Project the concatenations of the fixed initial gallery with the minimal ones in $\operatorname{res}(x)$ via p to the compact space $\mathcal{C}$ with its Hausdorff metric. There a subsequence of these projected galleries will converge to a gallery Γ which lifts uniquely to a gallery whose final chamber is the limit of the $\widetilde{C}_n$. The projections of those parts of the galleries that lie in $\operatorname{res}(x)$ lie in $\operatorname{res}(p(x))$ and so must their limit in the Hausdorff metric. Hence the final chamber of the lift of the limit gallery must thus lie in $\operatorname{res}(x)$.

Now let $(\widetilde{C}_n, \widetilde{C}'_n)$ be a sequence in $\widetilde{\mathcal{C}} \times \widetilde{\mathcal{C}}$ converging to $(\widetilde{C}, \widetilde{C}')$ such that $\pi_i(\widetilde{C}_n) = \pi_i(\widetilde{C}'_n)$. It follows that they share a vertex of cotype i and their projections (C_n, C'_n) converge to (C, C') in the topology on $\mathcal{C} \times \mathcal{C}$ induced from the Hausdorff topology on M. Proceeding similarly to the previous step in the proof we choose minimal galleries between $\widetilde{C}_n$ and $\widetilde{C}'_n$ within the cotype-i-residue the two chambers are contained in, such that (a subsequence of) their projection converges to a minimal gallery linking C to C'. It follows that the limits C, C' have a common vertex of cotype i and thus so do $\widetilde{C}, \widetilde{C}'$. This shows that having a common cotype i vertex is a closed relation on the set of chambers and in order to prove that V_i is Hausdorff it thus suffices to see that π_i is an open map:
We need to show that for any open $U \subset \widetilde{\mathcal{C}}$ the image $\pi_i(U)$ is open, which thus is equivalent to $\pi_i^{-1}(\pi_i(U))$ being open in the chamber topology of $\widetilde{\mathcal{C}}$. Since unions carry through the mapping we only need to consider the case where U is a finite intersection of $B_{\epsilon_j,k}(\widetilde{C}_j)$'s.
Let thus $\widetilde{D}$ be an element of $\pi_i^{-1}(\pi_i(U))$. There exists a $\widetilde{C} \in U$ such that $\pi_i(\widetilde{C}) = \pi_i(\widetilde{D})$, i.e. $\widetilde{C}$ and $\widetilde{D}$ share a common cotype-i-vertex x and lie in the residue $\operatorname{Res}(x)$. We can thus construct a gallery Γ linking $D = p(\widetilde{D})$ and $C = p(\widetilde{C})$ by folding within the residue in M, using the infinitesimal isotropy groups in faces containing x. Let V be a neighbourhood of C in $p(U)$ which we can assume to be a projection of a finite intersection of $B_{\epsilon_j,k}(\widetilde{C}_j)$'s. Hence all elements of V have Hausdorff distance less than or equal to $\varepsilon := \min\{\varepsilon_j\}$ from C. Every chamber Hausdorff close to C can, however, be written as φC for a φ in the normal holonomy group of a principal leaf L sufficiently close to the leaf of $p(x)$, that is ε-close to the identity, i.e. $\varphi = \exp \circ \Phi^\varepsilon$ for some linearised flow Φ^t acting on νL. (By compactness the global convexity radius of M is positive and we can in fact choose L so close to $p(x)$ that a small convex ball around $L \cap C$ meets all chambers in $\operatorname{res}(p(x))$. Since the images of chambers are locally uniquely determined by the images of small open sets this allows us to define the action of φ on the entire residue.)
For any such φ the gallery $\varphi\Gamma$ links a chamber in V to a chamber ε-close to D and conversely any such chamber of the latter kind can be reached by a gallery $\varphi\Gamma$ by applying the above argument to a neighbourhood of D. Since the normal holonomy groups, the infinitesimal isotropies and the projection p preserve types each gallery $\varphi\Gamma$ lies entirely in the cotype-i-residue of $\varphi p(x)$. We can now lift these galleries via p. The endpoints of the lifted galleries constitute neighbourhoods of $\widetilde{C}$ and $\widetilde{D}$ respectively which are in one-one correspondence via them. As π_i is constant along these galleries we obtain that for every $\widetilde{D}'$ in the neighbourhood of $\widetilde{D}$ there exists a $\widetilde{C}'$ close to $\widetilde{C}$ such that $\pi_i(\widetilde{D}') = \pi_i(\widetilde{C}')$, i.e. $\widetilde{D}' \in \pi_i^{-1}(\pi_i(U))$ which proves the claim. $\square$

Thus $\operatorname{Vert}(\widetilde{\mathcal{C}})$ is a Hausdorff topological space. Since every chamber is uniquely determined by its vertices in the simplicial complex $\widetilde{\mathcal{S}}$ this in turn induces a topology on the set of chambers, called the *thick* topology on $\widetilde{\mathcal{C}}$. Notice now that the product of maps $\pi_{i_1} \times \cdots \times \pi_{i_l}$ is continuous and thus the image of $\widetilde{\mathcal{C}}_{i_1,\dots,i_l}$ in $V_{i_1} \times \cdots \times V_{i_l}$ is closed, i.e. $\widetilde{\mathcal{C}}$ with the chamber and thick topologies is a topological building and by Proposition 3.17 compact, metrizable and separable. We thus obtain:

THEOREM 3.19. *Let $\widetilde{\mathcal{C}}$ be the building covering the chamber complex associated to a linearisable polar foliation $\mathcal{F}$ of codimension at least three (i.e. $\widetilde{\mathcal{C}}$ has rank at least four) on a positively curved, simply connected, compact manifold M, and assume its Coxeter diagram has no isolated nodes. Then it is the building at infinity of a product of irreducible symmetric spaces of noncompact type of rank at least two. The topological automorphism group $Aut_{top}(\widetilde{\mathcal{C}})$ of $\widetilde{\mathcal{C}}$ is a real noncompact semisimple Lie group with finitely many connected components and its identity component is isomorphic to the identity component of the isometry group of the product of symmetric spaces.*

PROOF. All that we need to note is that by Theorem 1.2 in [**GKMW**] $\widetilde{\mathcal{C}}$ is the building at infinity of the claimed product of symmetric spaces with a totally disconnected building. However, the residues of $\widetilde{\mathcal{C}}$ are all connected as we have seen, our building is locally connected and thus the totally disconnected factor must be trivial. The Main Theorem from [**BSp**] implies the statements about $\mathrm{Aut}_{top}(\widetilde{\mathcal{C}})$. $\square$

PROPOSITION 3.20. *The deck transformation group π of the covering $p : \widetilde{\mathcal{C}} \to \mathcal{C}$, i.e. the group of all lifts of the identity, equipped with the compact open topology, is a compact subgroup of $Aut_{top}(\widetilde{\mathcal{C}})$.*

PROOF. The $\alpha \in \pi$ are combinatorial automorphisms of $\widetilde{\mathcal{C}}$, i.e. chamber structure preserving bijections. As α^{-1} is again an element of π for every α, we only need to show that they are open. With a fixed chamber $\widetilde{C}_0$ chosen consider an elementary open set $B_{\varepsilon,k}(\widetilde{C})$ for some chamber $\widetilde{C}$. The elements of $B_{\varepsilon,k}(\widetilde{C})$ are the final chambers of galleries of length no more than k, starting at $\widetilde{C}_0$, such that their projections under $p : \widetilde{\mathcal{C}} \to \mathcal{C}$ lie at Hausdorff distance less than ε to the projection of a gallery from $\widetilde{C}_0$ to $\widetilde{C}$. The images of all these galleries under $\alpha \in \pi$ project by definition of π to the very same galleries in $\mathcal{C}$. It follows that $\alpha(B_{\varepsilon,k}(\widetilde{C}))$ is the set of all chambers that can be reached by galleries of length at most k starting at $\alpha\widetilde{C}_0$ whose projections are no further away than ε from those of galleries linking $\alpha\widetilde{C}_0$ to $\alpha\widetilde{C}$, i.e. it is an elementary open set in the chamber topology with respect to $\alpha\widetilde{C}_0$. As by Lemma 3.16 the chamber topology is independent of the choice of base chamber, this implies that the image is open. Since the thick topology is induced by the chamber topology the $\alpha \in \pi$ are homeomorphisms and thus π is a subgroup of $\mathrm{Aut}_{top}(\widetilde{\mathcal{C}})$.
Consider then a sequence (α_n) in π converging to some $\alpha \in \mathrm{Aut}_{top}(\widetilde{\mathcal{C}})$ in the compact open topology. It follows that $\alpha_n(\widetilde{C})$ is a convergent sequence in $\widetilde{\mathcal{C}}$

with limit $\alpha(\widetilde{C})$ for any chamber $\widetilde{C}$. Since the sequence $p(\alpha_n(\widetilde{C})) = p(\widetilde{C})$ is constant it converges to $p(\alpha(\widetilde{C})) = p(\widetilde{C})$ which implies $\alpha \in \pi$. As π acts freely and has closed, hence compact, orbits it is itself compact. $\square$

THEOREM 3.21. *A positively curved, simply connected, compact manifold M with a linearisable polar foliation $\mathcal{F}$ of codimension at least three such that the Coxeter diagram of its associated building has no isolated nodes is homeomorphic to a compact rank one symmetric space with a polar foliation orbit equivalent to that of M.*

PROOF. By Theorem 3.19 the simplicial complex $\mathcal{S}$ underlying the associated building $\widetilde{\mathcal{C}}$ equipped with the thick and thin topologies is the sphere at infinity of a noncompact symmetric space N. Furthermore the deck transformation group is a compact subgroup of the Lie group $\mathrm{Aut}_{top}(\widetilde{\mathcal{C}})$ and hence a Lie group itself. Let π_0 denote the identity component of π, which again acts freely on $\mathcal{S}$. If π is disconnected the covering $\mathcal{S}/\pi_0 \to \mathcal{S}/\pi \cong M$ would be non-trivial which is impossible as M is simply connected. As π acts freely on the sphere $\mathcal{S}$ it therefore follows from Theorem 8.5 in [**Bd2**] that $\pi \cong \{1\}, S^1$ or S^3. As it is compact and connected π is contained in some maximal compact subgroup K of the identity component G of $\mathrm{Aut}_{top}(\widetilde{\mathcal{C}})$. Let N denote the symmetric space from Theorem 3.19. By Theorem 2.1 in Chapter VI of [**He**] K must fix a point n in N. The unit tangent sphere at n can be identified with the sphere at infinity of N, which is $\mathcal{S}$, and the action of K on the unit tangent sphere at n is topologically equivalent to the action of K on $\mathcal{S}$, which consists of combinatorial automorphisms and thus respects the chamber structure. Similarly the linear action of π on the round sphere $\mathbb{S} := T_n^1(N)$ is conjugate to the action of π on $\mathcal{S}$ via the projection to the sphere at infinity. The action of $K \supset \pi$ on the unit tangent sphere at n is polar and hence so is the linear action of π on this round sphere $\mathbb{S} := T_n^1(N)$. It follows that the fibres of π are equidistant and thus by a result of Grove and Gromoll (Corollary 5.4 in [**GG**]) its action is congruent to a Hopf fibration, as $\pi \neq S^7$ (the only case excluded in the result of Grove and Gromoll). It follows that the quotient $M \cong \mathcal{S}/\pi$ is homeomorphic to the rank one symmetric space $\mathbb{S}/\pi$. The orbit foliation induced by the polar action on $\mathbb{S}$ contains the orbits of π, but not not necessarily normally. Hence it projects to an in general inhomogeneous polar foliation on $\mathbb{S}/\pi$ whose chambers are isomorphic to those of $\mathcal{S}$ and hence to those on M. As Proposition 2.21 thus applies to $\mathbb{S}/\pi$, too, this implies that the orbit spaces are isomorphic, too, which completes the proof. $\square$

Notice that we needed the foliation to be of codimension at least three for the above theorem, as we only have information on the building structure of proper residues of the chamber complex $\mathcal{C}(M, \mathcal{F})$ and the statement of the theorem of Tits is trivial for connected chamber complexes of rank three, which are their own and only residues of this rank. If we further exclude isolated nodes of the Coxeter diagram from our consideration this yields that the diagrams of Coxeter complexes of rank 3 without isolated nodes must be irreducible. By work of Münzner ([**Mü1**] and [**Mü2**]) in conjunction with Alexandrino's slice theorem (see [**Al2**]) the possible Coxeter groups must be *crystallographic*, i.e. the dihedral groups formed by any two of their generating reflections must be of orders 1, 2, 3, 4 or 6. There are according to the classification of finite Coxeter groups (see [**BG**]) only two such groups of rank 3, denoted by A_3 and C_3. Chamber complexes based on Coxeter groups of type A_3 have been shown to be naturally related to buildings by work of Tits ([**Ti1**]) while residues of type C_3 play the role of an obstruction to being a building. Thus, even though our so far established methods break down for the case of codimension 2 one might hope to at least obtain a result if the associated Coxeter group is of type A_3. This may be subject to future considerations.

CHAPTER 4

The Point Leaf Case

In the previous chapter we assumed the polar foliation $\mathcal{F}$ to contain no point leaves in order to ensure that the chambers of $\mathcal{C}(M, \mathcal{F})$ are simplices. We shall now investigate the contrary case. Throughout this chapter we will assume that one (and hence any) section is constantly curved. As by equidistance all leaves are traced in distance spheres around a point leaf this assumption will allow us to recover information on the geometry (or topology in the case of projective sections) of M from its foliation.
In light of Remark 2.39 the work by Mendes in [**Me**] and the recent generalisations of some of its prerequisite considerations to the inhomogeneous case in [**MR**] may indicate that this is in fact no additional restriction on M. We call a map between manifolds equipped with (singular) foliations *equifoliate* if the image of a leaf is the leaf of any image of a point in that leaf.

THEOREM 4.1. *Let M be a compact, positively curved, simply connected manifold with a polar foliation $\mathcal{F}$ of codimension at least two. If the set of point leaves is non-empty and there exists a spherical section with constant curvature, then M is equifoliately diffeomorphic to a round sphere.*

PROOF. Let Σ be a spherical section with constant curvature and polar group W. Then consider the fixed point sphere $S := \Sigma^W$ and its dual sphere T at the maximal distance d in Σ. Then Σ decomposes as a spherical join $S * T$ and the underlying manifold $M = \Phi\Sigma$ as $\Phi(S * T) = S * \Phi T$, where Φ is the pseudogroup generated by all the flows ϕ_t^X of the vector fields $X \in \Xi_{\mathcal{F}}$, since the orbits under Φ are perpendicular to the joining transverse geodesics and we have used that by Proposition 3.7 S consists of point leaves. In particular ΦT is the submanifold at maximal distance from $\mathbb{S}$ in M, since Φ leaves transverse distances invariant, the sections are totally geodesic and S lives in every section. From the proof of Proposition 3.7 it also follows that all sections must be spheres and we can hence sensibly refer to W as the polar group *of the foliation* (up to isomorphism).
Consider, for an arbitrary fixed $p \in S$, the map $\psi : \nu_p^1 S \to \Phi T; v \mapsto \exp_p(dv)$. We wish to see that ψ is an equifoliate diffeomorphism w.r.t. the linearised foliation on $\nu_p^1 S$ and the restriction of $\mathcal{F}$ to the Φ-invariant submanifold ΦT. Let q be the antipodal point of p in $\mathcal{S}$. Then $\{p, q\} * \Phi T$ is a polarly foliated manifold with section $\{p, q\} * T$, the same polar group W and two isolated fixed points p, q. Hence the unit tangent sphere U of any of the two fixed points is diffeomorphic to ΦT via the map $v \mapsto \exp_p(dv)$ and by Remark

2.16 this map is equifoliate.
But the unit tangent sphere U is just the unit normal sphere to S at (w.l.o.g.) p, i.e. $\nu_p^1 S$. Since, as stated before, the joining transverse geodesics are orthogonal to the leaves, we obtain an equifoliate diffeomorphism $Id_S * \psi : S^n = S * U \to S * \Phi T = M; [p, q, t] \mapsto [p, \psi(q), 2d/\pi \cdot t]$. $\square$

Consider now a projective section Σ with constant curvature. By assumption we have $\Sigma \cong \mathbb{R}P^n$ and there is a polar group W acting on it, generated by reflections. From our considerations in Chapter 2 we know:
The group W lifts to a Coxeter group $\overline{W}$ acting on the universal cover $\widetilde{\Sigma} \cong S^n$. Furthermore $\mathrm{Fix}(W)$ is discrete and consists of the vertices of the cell complex $\mathcal{C}(\Sigma)$, containing itself the set of point leaves of the foliation $\mathcal{F}$. We let a denote the deck transformation of the universal cover of Σ.
Under our assumption there is a point leaf $\{p\}$ of $\mathcal{F}$. Then p is a vertex of the cell complex on Σ and by constant positive curvature any (closed) cell C containing p contains a codimension one face opposite p. This in turn is contained in a mirror Λ which is the cut locus of p in Σ and whose reflection r_Λ fixes p.
Set $B = \Phi\Lambda$, the union of leaves of $\mathcal{F}$ intersecting Λ. Since $\{p\}$ is a point leaf the distance between p and points of B is always realised by transverse geodesics. As the flows in Φ leave transverse distance invariant B is at constant maximal distance l from p in M. Hence M decomposes as the n-disc $\exp_p(D_l(p))$ attached to B via the attaching map $\exp_p|_{S_l(p)} \to B$.

Proposition 4.2. *With the above assumptions M is Blaschke at p, i.e. the set of tangents of geodesics running from p to any point in $Cut(p)$ form a great sphere in the tangent space of that point.*

Proof. By Thm. 5.43 in [**Be**] we know that being Blaschke at p is equivalent to the tangential cut locus $cut(p)$ being spherical. In order to prove the latter we will show that in fact the submanifold B is the cut locus of p and hence that the component of minimal radius of its preimage under the exponential map, $S_l(p)$, is the tangential cut locus.
To that end consider a geodesic γ starting at p. As p is a point leaf any geodesic starting at p is a transverse geodesic, which hence lies within a section. By the corollary above we know that all sections are constantly curved and hence that the cut point on γ occurs precisely at time l. However, any point at distance l from p lies in B by its construction and the proof is complete. $\square$

In fact, as B is a submanifold $d\exp_p$ has constant rank along $S_l(p) \subset T_pM$ and $S_l(p)$ is compact. Hence by Ehresmann's Lemma $\exp_p$ restricted to $S_l(p)$ is a (locally trivial) fibration. By work of Browder (cf. [**Bw**], Thm. 5.1) the fibres are homotopically equivalent to 1-, 3- or 7-spheres. By the proposition above M is Blaschke at p. Hence we know that the tangents to geodesics running from p to any point q in $B = Cut(p)$ form a (great) sphere in $T_q^1 M$. The image of this sphere under the time-l (or time-$(-l)$) map of the geodesic flow is (up to homothety) the fibre above q and hence this fibre is diffeomorphic to a sphere itself. We have thus proven the following

PROPOSITION 4.3. *Let M be a compact, simply connected, positively curved manifold with a polar foliation. If there is a point leaf and a projective section of constant curvature then the tangential cut locus $cut(p)$ is the (appropriately scaled) tangential sphere at p, the cut locus $Cut(p)$ is a submanifold of M and $cut(p)$ fibres over $Cut(p)$ via the exponential map with fibres diffeomorphic to 1-, 3- or 7-spheres.*

We wish to study this fibre bundle further in order to obtain information about the cohomology of M. We start with a remark on the fundamental group of the base.

REMARK 4.4. The manifold M decomposes as the union of an open ball around the point leaf and a tubular neighbourhood around the base $B = Cut(p)$. Their intersection is homeomorphic via $\exp_p$ to a thickening of $S_p^1 M$ by an open interval and thus homotopy equivalent to it. The ball $D(p)$ is homotopically trivial and $\pi_1(S_p^1 M)$ is nonzero only when $\dim M = 2$, but then the foliation is trivial and M is its own simply connected section, i.e. a 2-sphere, a case covered by Theorem 4.1. We can hence assume that $\dim M \geq 3$ and thus that the intersection of the two open sets has trivial fundamental group. With the Theorem of Seifert-van Kampen it then follows that $\pi_1(B) = \pi_1(M) = \{0\}$.

THEOREM 4.5. *A simply connected, compact, positively curved riemannian manifold M, equipped with a polar foliation of codimension at least two admitting a projective section of constant curvature, has the cohomology of a projective space if there is a point leaf. More precisely: If the fibres are 1-spheres, then M is a $2m$-dimensional manifold with the cohomology of $\mathbb{C}P^n$, if the fibres are 3-spheres then M is a $4m$-dimensional manifold with the cohomology of $\mathbb{H}P^n$, and if the fibres are 7-spheres then M is a 16-dimensional manifold with the cohomology of the Cayley plane $\mathbb{O}P^2$.*

REMARK 4.6. Since differentiable manifolds are triangulable (cf. [**Mk**], Ch. II) they in particular carry a CW-structure. By Satz 2.13 in [**Ma**] cellular and singular homology coincide and by Poincaré duality we can thus study the cohomology ring via the cellular decomposition, which we shall do in the subsequent proof.

PROOF. Recall that by p we denoted the point leaf whose cut locus is B. Since B is simply connected the bundle $S^{k-1} \hookrightarrow S_pM \twoheadrightarrow B$ is orientable (see [**Ha**], p. 442) and we can apply the Gysin sequence to it (see [**Bd1**], Theorem 13.2):

$$\cdots \to H^{l+k-1}(S_pM) \to H^l(B) \to H^{l+k}(B) \to H^{l+k}(S_pM) \to \cdots .$$

For $l \neq n-k, n-k-1$ the spherical cohomology groups vanish on both sides of the above part of the sequence and we obtain the short exact sequence

$$0 \to H^l(B) \to H^{l+k}(B) \to 0,$$

i.e. $H^l(B) \cong H^{l+k}(B)$.
As B is compact and simply connected, hence orientable, we have $H^{n-k}(B) \cong \mathbb{Z}$ and $H^l(B) = 0$ for $l \geq n-k+1$. Simple connectedness furthermore implies $H_1(B) = 0$ and thus by Poincaré duality $H^{n-k-1}(B) = 0$. We conclude:

$$H^l(B) = \begin{cases} \mathbb{Z}, \text{if } 0 \leq l \leq n-k \text{ is divisible by k} \\ 0, \text{otherwise.} \end{cases}$$

Since M is obtained from B by attaching an n-cell centred at p via $\exp_p$, the cohomology of M coincides with that of B up to dimension $n-k$, there is no cohomology in dimensions $n-k+1, ..., n-1$ and hence just a free cyclic group in dimension n, yielding

$$H^l(M) = \begin{cases} \mathbb{Z}, \text{if } 0 \leq l \leq n \text{ is divisible by k} \\ 0, \text{otherwise.} \end{cases}$$

From the Gysin sequence we obtain that the isomorphism $H^l(M) \to H^{l+k}(M)$ for $l = 0, ..., n-2k$ is given by the cup product with the Euler class χ of the sphere-bundle over B. Hence $H^l(M)$ is generated by χ^l for $l = 0, ..., n-k$, which gives us the ring structure up to that dimension. In order to determine the full ring structure it thus remains to find the generator of $H^n(M)$. To that end consider a point $q \in B$ and the set $F := \{\exp_p(tv) | v \in \exp_p^{-1}(q), t \in [0,1]\}$. The only possible non-manifold points of this are p and q. Since by Proposition 4.2 above the differential image of the fibre in the normal bundle to B is a great sphere, F is smooth at q and for the same reason the fibre is invariant under $-\mathrm{Id}_{S_pM}$, so F is smooth at p, too. Geometrically F is the suspension of an exponential image of the fibre sphere by the embedded $S^0 = \{p, q\}$ or the mapping cone of $\exp_p$ restricted to the fibre. It intersects B in q and in q only. We will employ the geometric interpretation of the cup product: The cup product of the Poincaré duals of the fundamental classes of two (transversal) submanifolds is the Poincaré dual of the fundamental class of their intersection. (We will denote the Poincaré dual by an asterisk.) Furthermore the fundamental class of a point generates $H_0(M)$ and hence its dual ω generates $H^n(M)$. As F is a k-dimensional manifold the dual of its fundamental class is an element of $H^{n-k}(M)$ and thus a multiple of χ^{m-1} where $km = n$. The dual of $[B]$ is simply χ itself by definition (see [**Bd1**], p. 390). Hence we have that

$$\omega = [\{q\}]^* = [B \cap F]^* = [B]^* \smile [F]^* = \chi \smile r\chi^{m-1} = r\chi^m,$$

for some $r \in \mathbb{Z}$ which must be nonzero as $r\chi^m$ is the generator of $H^n(M)$. It follows that the cup product $H^k(M) \times H^{n-k}(M) \to H^n(M)$ is non-degenerate and thus $\chi^m = \chi \smile \chi^{m-1} \in H^n(M)$. This, however, implies $r = \pm 1$, as otherwise ω could not generate χ^m over $\mathbb{Z}$. Since F is constructed via $\exp_p$, which is the attaching map in obtaining M from B and an n-ball, the intersection of F and B is positively oriented which finally yields $r = 1$ and thus $H^*(M) = \mathbb{Z}[\chi]/\chi^{m+1}$ as asserted.

It remains to be shown that in the case $k = 7$ the dimension of M must be 16. Assume that M has the cohomology ring $\mathbb{Z}[\chi]/\chi^{m+1}$ with $\deg(\chi) = 8$ and $m \geq 3$. We can thus consider the 24-skeleton M_{24} of M. By Satz 2.10 (first equality below) and Satz 2.13 (second equality below) in [**Ma**] we have for $0 \leq l \leq 24$ that

$$H^l(M_{24}) \cong H^l(M_{8m}) \cong H^l(M).$$

Furthermore by Satz 2.12 ibidem $H^l(M_{24}) = 0$ for $l > 24$ and up to that dimension the ring structure carries over by the isomorphisms sending generator to generators. Hence M_{24} has the cohomology ring $\mathbb{Z}[\chi]/\chi^4$ with $\deg(\chi) = 8$. The Ext functor in the universal coefficient theorem for cohomology (Corollary 7.2 in [**Bd1**]) vanishes if we pass from integral to mod-2 cohomology and the short exact sequence from the universal coefficient theorem yields a simple isomorphism between the cohomologies that carries over the ring structure. Therefore M_{24} also has the mod-2 cohomology ring $\mathbb{Z}_2[\chi]/\chi^4$ with $\deg(\chi) = 8$, but that cohomology ring does not exist according to Theorem 6.3 in Appendix B in [**Hu**], a contradiction. Hence m is at most 2. It is also at least 2 since B is non-trivial. Thus the claim follows, which completes the proof. $\square$

Notice that the reflection group W can have fixed points not arising from point leaves of the foliation. In fact, consider the following

EXAMPLE 4.7. Consider the standard action of $U(1) \times U(1) \times U(2)$ on $\mathbb{C}^4 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2$. The principal orbits are embedded copies of $S^1 \times S^1 \times S^3$ and the action is polar with sections embedded copies of $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, each $\mathbb{R}$-component a radial line in the $\mathbb{C}^k$-factors, $k = 1, 2$. This action is equivalent (by homothety) to its restriction to $S^7 = S(\mathbb{C}^4)$ with sections copies of S^2 and the same principal orbits. The polar group of a given section is of type $A_1 \times A_1 \times A_1$. The Hopf action on S^7 is contained in the acting group as the diagonal subgroup $\{(e^{i\phi}, e^{i\phi}, \mathrm{diag}(e^{i\phi}, e^{i\phi})) \mid \phi \in [0, 2\pi]\}$.
Hence the action projects to an action on $\mathbb{C}P^3$, inducing an orbit foliation, which has as principal leaves embedded copies of $S^1 \times S^3$, is polar with section $\mathbb{R}P^2$ and a reflection group generated by two commuting reflections (which in this case coincidentally is a Coxeter group itself). The action of this reflection group has three fixed points which are the vertices of the chamber, a right angled triangle. The group action itself has two fixed points on $\mathbb{C}P^3$ originating from the two $\mathbb{C}$-components of the action above. The singular leaf of the third fixed point of the reflection group consists of points of the form $(0 : 0 : z : w)$ which form a projective line $\mathbb{C}P^1 = S^2 \subset \mathbb{C}P^n$.

The following lemma relates isolated nodes of the Coxeter diagram to fixed points of the reflection group.

LEMMA 4.8. *The fixed points of the action of a reflection group W on a real projective k-space Σ correspond exactly to the isolated nodes in the Coxeter diagram of the lifted group $\overline{W}$ acting on the covering sphere $\widetilde{\Sigma}$.*

PROOF. First, let $r \in \overline{W}$ represent an isolated node of the Coxeter diagram, i.e. r commutes with all other reflections. Its mirror Λ separates $\widetilde{\Sigma}$ into two convex halves, each with a unique point at maximal distance from the mirror, that must hence be interchanged by r and a. For any other reflection s and $p, \mathsf{a}p$ the two points at maximal distance we then get $s\{p, \mathsf{a}p\} = s\,r\{p, \mathsf{a}p\} = r\,s\{p, \mathsf{a}p\}$. Since the deck transformation commutes with all reflections this implies that $s(p), \mathsf{a}s(p)$ are two antipodal points interchanged by r and by uniqueness s must leave $\{p, \mathsf{a}p\}$ invariant. Hence $\overline{p} = \pi(p)$ is the isolated fixed point of the reflection $\overline{r}$ below r which is left fixed by all other reflections as their lifts leave the antipodal orbit above invariant.

Now consider the converse: Let $\overline{p}$ be a fixed point of W. Let Λ, r be the mirror and reflection corresponding to the antipodal orbit $\{p, \mathsf{a}p\}$ above $\overline{p}$. We wish to show that any given mirror Λ' which is distinct from Λ meets the latter orthogonally, i.e. that they meet and their normals at any intersection point enclose an angle of $\pi/2$.
That they meet follows from Theorem 1 in **[Fr]** since the mirrors are compact totally geodesic hypersurfaces. For the second part note that each reflection acts on the normal space at any fixed point (and trivially on the respective tangential space) by its differential. Hence, at any intersection point $q \in \Lambda \cap \Lambda'$ the differentials of the two corresponding reflections r, r' form a dihedral group acting on $\nu_q(\Lambda \cap \Lambda') = \nu_q\Lambda \oplus \nu_q\Lambda'$, where the sum is direct because the normal spaces are each one dimensional and uniquely determine their distinct corresponding mirrors. This dihedral group is a linear representation of and hence isomorphic to the dihedral group formed by the two reflections themselves and is hence independent of the particular intersection point. However, this order also directly corresponds to the angle between the two normals, which span the normal space of the intersection. Hence, if Λ, Λ' are orthogonal to each other at one intersection point they are so at all. Now choose a geodesic γ from p to $\Lambda \cap \Lambda'$ within Λ'. Since the latter is totally geodesic this is also a geodesic in the ambient space. By the second variational formula γ meets $\Lambda \cap \Lambda'$ orthogonally in its endpoint q. Thus we obtain for its derivative n at q that $n \in \nu_q\Lambda \oplus \nu_q\Lambda'$. Additionally $n \in T_q\Lambda'$ by construction of γ, so the $\nu_q\Lambda'$-component of n must vanish. It follows that $n \in \nu_q\Lambda$ and thus that n is a non-zero multiple of the unit normal to Λ at q since the normal space is one-dimensional, which completes the proof. $\square$

In light of this and the fact that the prerequisites of the theorem proved in Chapter 3, which exclude isolated nodes in the Coxeter diagram of the associated building, there is a "remaining" case when there are no point leaves but the Coxeter diagram of the lifted reflection group has isolated nodes. It may be subject to future investigations what can be said about the non-trivial singular submanifolds represented by the such isolated nodes.

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