GARCH-UGH: A bias-reduced approach for dynamic extreme Value-at-Risk estimation in financial time series

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Main findings

We construct a two-step bias-reduced estimation methodology called **GARCH-UGH (Unbiased Gomes-de Haan)**, whereby financial returns are first filtered using an AR-GARCH model, and then a bias-reduced estimator of extreme quantiles is applied to the standardized residuals to estimate one-step ahead dynamic extreme **Value-at-Risk (VaR)**. The GARCH-UGH estimates are more accurate than those obtained by combining conventional AR-GARCH filtering and extreme value estimates (e.g. **GARCH-EVT**) from the perspective of in-sample and out-of-sample backtestings of historical daily returns on several financial time series.

Background

The fluctuations of stock prices are relatively small and are often assumed to be normally distributed at longer time horizons. However, sometimes theses fluctuations can become catastrophic, especially when unforeseen large drops in prices are observed that could result in huge losses such as 2007-2008 crisis.

Two-fold Problem: **Stochastic volatility** and **heavy-tailedness** of the financial time series, which are known as the stylized facts.

- GARCH-type models assuming conditional normality are not well suited.
- Classical EVT models require i.i.d. assumption, which is violated.

Purpose: To evaluate potential losses occurring with extremely low probabilities, i.e., to estimate **Value-at-Risk (VaR)** which is just a quantile ($q_\tau = \inf\{x : F(x) \geq \tau\}, \quad \forall \tau \in (0,1)$) of the loss distribution with a small probability $p = 1 - \tau$. More precisely, **to introduce an alternative conditional VaR estimation** approach that copes with above listed problems.

Problem setting

Let $X_t = -\log(p_t/p_{t-1})$ be the negative daily log-return where p_t is a daily-recorded price for a stock, index or exchange rate at time t.
Assume the dynamics of X are governed by

$$X_t = \mu_t + \sigma_t Z_t \quad \text{(Today)}$$

where $\mu_t \in \mathbb{R}$, $\sigma_t > 0$ are (conditional) mean and standard deviation and Z_t are i.i.d. innovations.
We are principally interested in 1-step ahead behaviour,

$$X_{t+1} = \mu_{t+1} + \sigma_{t+1} Z_{t+1} \quad \text{(Tomorrow)}.$$

Target: 1-step ahead **conditional extreme quantile (or VaR)**

$$q_\tau(X_{t+1} \mid \mathscr{F}_t) = \mu_{t+1} + \sigma_{t+1} q_\tau(Z) \tag{1}$$

where we assume μ_{t+1} and σ_{t+1} are measurable w.r.t. the past $\mathscr{F}_t$, and Z has the same marginal distribution as of Z_t.

GARCH-UGH

McNeil and Frey (2000) first used a GARCH-type model for filtering and fit Generalized Pareto distribution (GPD) known as parametric Peaks-Over-threshold (POT) method to estimate an equation (1). The **GARCH-EVT** approach.

Our approach: We develop a new methodology called **GARCH-UGH** (standing for Unbiased Gomes-de Haan), which uses an asymptotically unbiased estimator of extreme quantile after filtering, instead of POT approach.

GARCH step (Target is $\hat{\mu}_{t+1}, \hat{\sigma}_{t+1}$):

1. Fit AR(1)-GARCH(1,1) model to the return data by the quasi-maximum likelihood (QML) approach.
   - Conditional variance of the mean-adjusted series $\epsilon_t = X_t - \mu_t$ is

   $$\sigma_t^2 = \lambda_0 + \lambda_1 \epsilon_{t-1}^2 + \lambda_2 \sigma_{t-1}^2, \quad (\lambda_0, \lambda_1, \lambda_2 > 0). \quad \text{(GARCH(1,1)}$$

   - Conditional mean is given by

   $$\mu_t = \phi X_{t-1}. \quad \text{(AR(1))}$$

   - The likelihood for GARCH(1,1) model with normal innovations is maximized to obtain $\hat{\theta} = (\hat{\phi}, \hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2)$.
2. **Residuals** are calculated as

   $$(\hat{Z}_{t-n+1}, ..., \hat{Z}_t) = \left(\frac{X_{t-n+1} - \hat{\mu}_{t-n+1}}{\hat{\sigma}_{t-n+1}}, ..., \frac{X_t - \hat{\mu}_t}{\hat{\sigma}_t} \right).$$

   - regarded as i.i.d. and are used to estimate $q_\tau(Z)$ in equation (1).
3. Estimate 1-step ahead conditional mean and volatility by

$$\hat{\mu}_{t+1} = \hat{\phi} X_t, \quad \hat{\sigma}_{t+1} = \sqrt{\hat{\lambda}_0 + \hat{\lambda}_1 \hat{\epsilon}_t^2 + \hat{\lambda}_2 \hat{\sigma}_t^2}.$$

GARCH-UGH Cont.

UGH step (Target is $\hat{q}_\tau(Z)$):
The residuals $\hat{Z}_{t-j}, 0 \leq j \leq n-1$, approximate the true unobservable Z_{t-j}.
Use **top k order statistics** above the random threshold $\hat{Z}_{n-k,n}$, which need to be an upper intermediate (i.e. $k = k_n$ with $k \to \infty$ and $k/n \to 0$ as $n \to \infty$).

1. Let $\hat{Z}_{1,n} \leq \cdots \leq \hat{Z}_{n,n}$ denote the order values of $(\hat{Z}_{t-n+1}, ..., \hat{Z}_t)$.
2. Let m the number of positive observations in the sample n.
3. For each sample fraction k satisfying $k \leq \min(m-1, \frac{2m}{\log\log m})$, and $\alpha = 1, ..., 4$, calculate the statistics and estimators given below;

$$M_k^{(\alpha)} = \frac{1}{k} \sum_{i=1}^{k} (\log Z_{n-i+1,n} - \log Z_{n-k,n})^\alpha,$$

$$S_k^{(2)} = \frac{3}{4} \frac{(M_k^{(4)} - 24(M_k^{(1)})^4)(M_k^{(2)} - 2(M_k^{(1)})^2)}{M_k^{(3)} - 6(M_k^{(1)})^3},$$

$$\hat{\rho}_k = \frac{-4 + 6S_k^{(2)} + \sqrt{3S_k^{(2)} - 2}}{4S_k^{(2)} - 3}, \quad \text{provided} \quad S_k^{(2)} \in \left(\frac{2}{3}, \frac{3}{4}\right),$$

$$\hat{\gamma}_{k,k_\rho} = \hat{\gamma}_k^H - \frac{M_k^{(2)} - 2(\hat{\gamma}_k^H)^2}{2\hat{\gamma}_k^H \hat{\rho}_{k_\rho}(1 - \hat{\rho}_{k_\rho})^{-1}} \quad \text{given} \quad M_k^{(1)} = \hat{\gamma}_k^H \quad \text{is Hill estimator},$$

$$\hat{q}_\tau(Z) = \hat{Z}_{n-k,n} \left(\frac{k}{np}\right)^{\hat{\gamma}_{k,k_\rho}} \left(1 - \frac{[M_k^{(2)} - 2(\hat{\gamma}_k^H)^2][1 - \hat{\rho}_{k_\rho}]^2}{2\hat{\gamma}_k^H \hat{\rho}_{k_\rho}^2} \left(1 - \left(\frac{k}{np}\right)^{\hat{\rho}_{k_\rho}}\right)\right),$$

where γ is the **extreme value index** (i.e. shape parameter), ρ is the second-order parameter and $\hat{\rho}_{k_\rho}$ is the one optimal $\hat{\rho}_k$ selected following the recommendation given in de Haan *et al.* (2016).

Real data analysis

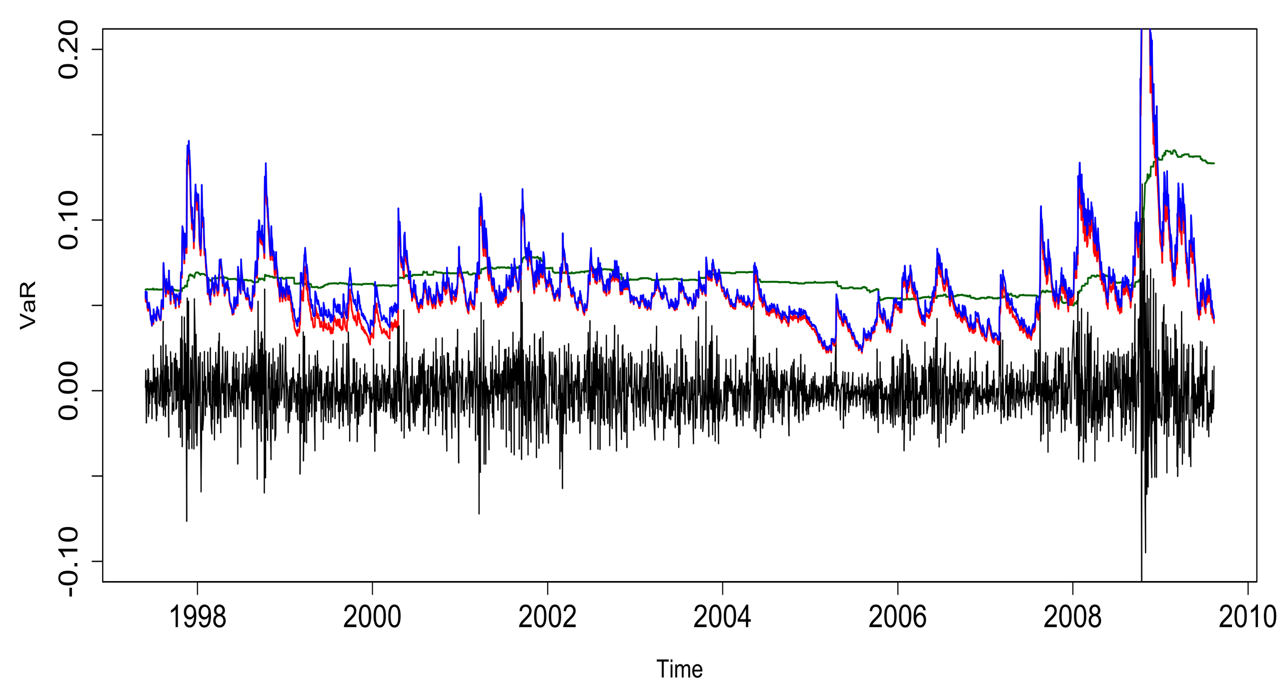

Figure 1: **Out-of-sample backtesting** of the **NIKKEI** index from 1997-05-29 to 2009-08-12, and 99.9%-VaR estimates calculated using rolling estimation windows made of 1000 observations (historical period 1993-05-14 to 1997-05-28), with k corresponding to the top 10% of observations from this window. GARCH-UGH, (blue line), GARCH-EVT, (red line) and UGH (green line) are superimposed on the negative log-returns (black line).

Testing window	3000				
Estimation window	1000				
% of top obs. used	5%	10%	15%	20%	25%
0.999 Quantile					
Expected	3	3	3	3	3
UGH	7	6	6	5	**5**
	(0.049, 0.142)	(0.128, 0.310)	(0.128, 0.310)	(0.292, 0.569)	(0.292, 0.569)
GARCH-UGH	**4**	**3**	**2**	**2**	**1**
	(0.583, 0.855)	(1.000, 0.997)	(0.538, 0.826)	(0.538, 0.826)	(0.179, 0.406)
GARCH-EVT	5	**4**	6	6	6
	(0.292, 0.569)	(0.583, 0.855)	(0.128, 0.310)	(0.128, 0.310)	(0.128, 0.310)

Table 1: The Closest numbers of **VaR violations** (i.e. observations > VaR) to theoretically expected ones are highlighted in bold. The p-values for the unconditional coverage test by Kupiec (1995) and conditional coverage test by Christoffersen (1998) at 5% significance level are given in brackets in order.

McNeil, A.J. & Frey, R. (2000). Journal of Empirical Finance.
de Haan, L., Mercadier, C. & Zhou, C. (2016). Finance and Stochastics.