Eigenvalue Distributions of Wigner and Wishart Ensembles of Sparse Vinberg Models

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Abstract

We study limiting eigenvalue distributions related to Vinberg matrices, which are matrix spaces corresponding to daizy graphs. This is a first step toward studying eigenvalue distributions of Wigner and Wishart Ensembles on matrix spaces related to growing graphical models. Since the space of Vinberg matrices is endowed with transitive group actions, covariance matrices are defined naturally. This is a joint work with Piotr Graczyk (Université d'Angers).

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Theorem 1 (Wigner Ensembles)

Limiting eigenvalue distributions μ of rescaled Wigner matrices $\frac{1}{\sqrt{n}}U_n \in \mathcal{U}_n$ is given as

$$\mu(t) = \frac{\sqrt[3]{R_c^+(t)} - \sqrt[3]{R_c^-(t)}}{2\sqrt{3}\,\pi\,t}\chi_{[\alpha_c,\beta_c]}(t^2) + [1-2c]_+\delta_0(t),$$

Note that the diagonal part in the right bottom does not affect the result.

Assumptions:
- u_{ij} is i.i.d., 0 mean and 1 variance whenever $u_{ij} \neq 0$,
- $\lim_{n\to+\infty} \frac{a_n}{n} = c$.

Notation:
- $\delta_0(t)$ is the Dirac delta function at $t=0$,
- $[a]_+ := \max(a, 0)$ for $a \in \mathbb{R}$,

- $R_c^{\pm}(x) = x^6 - 3(c+1)x^4 + \frac{3}{2}(5c^2-2c+2)x^2 + (2c-1)^3 \pm 3c\sqrt{3-3c}\cdot x\sqrt{(x^2-\alpha_c)(\beta_c - x^2)}$,
- $\alpha_c = \dfrac{8+4c-13c^2-\sqrt{c(8-7c)^3}}{8(1-c)}, \quad \beta_c = \dfrac{8+4c-13c^2+\sqrt{c(8-7c)^3}}{8(1-c)}$.

Theorem 2 (Wishart Ensembles)

Stieltjes transforms with respect to rescaled Wishart matrices $\frac{1}{n}X_n$ of $\mathcal{U}_n$ is given as, by using a generalized Lambert function $W_{\kappa,\gamma}$,

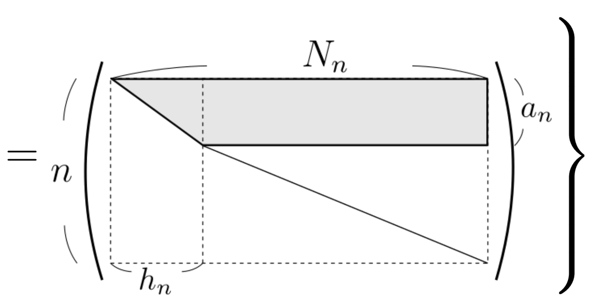
$$S(z) = -1 - \frac{c}{zW_{\kappa,\gamma}(-\frac{c}{z})} - \frac{1-c\beta}{z} \quad (z \in \mathbb{C}^+).$$

Assumptions:
- $\lim_n \frac{a_n}{n} = c$, $\lim_n \frac{h_n}{a_n} = \alpha$ and $\lim_n \frac{N_n}{a_n} = \beta$,
- $\kappa = \frac{1}{1-\alpha}$ $(\alpha \neq 1)$, $\kappa = +\infty$ $(\alpha = 1)$ and $\gamma = 1-\beta$.

Terminologies:

- A Wishart matrix is a Vinberg matrix $X_n \in \mathcal{U}_n$ of the form $X_n = \xi_n\xi_n^\top$, where $\xi_n = (\xi_{ij})$ is a matrix in E defined by

$$E = \left\{ \xi_n = \right.$$



Each entries ξ_{ij} is i.i.d., centered and variance 1 whenever $\xi_{ij} \neq 0$. The space E appears naturally in the theory of homogeneous cones.

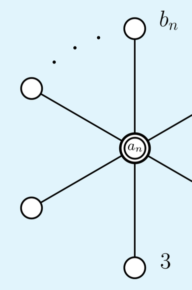
- A generalized Lambert function $W_{\kappa,\gamma}$ is the main branch of the inverse function of a function $f_{\kappa,\gamma}$ defined by

$$f_{\kappa,\gamma}(z) = \frac{z}{1+\gamma z}\left(1+\frac{z}{\kappa}\right)^{\kappa} \quad \text{where } 1+\frac{z}{\kappa} \in \mathbb{C} \text{ is not negative reals.}$$

Here, κ and γ are real numbers such that $\frac{1}{\kappa} - \gamma > 0$.

Daisy graphs and Vinberg Matrices

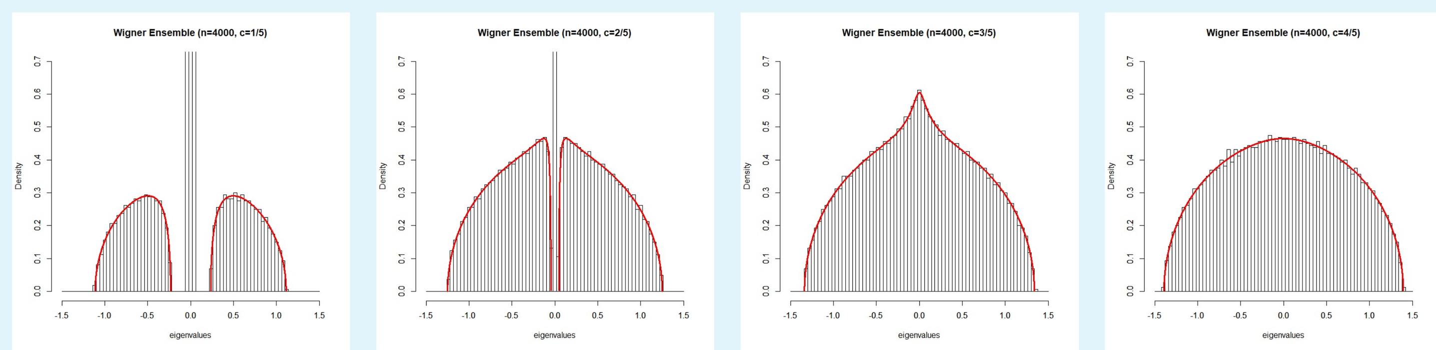
We want to know eigenvalue distributions related to the daizy graph (the double circle indicates the complete graph). The corresponding matrix spaces $\mathcal{U}_n$ is the space of **Vinberg matrices**.



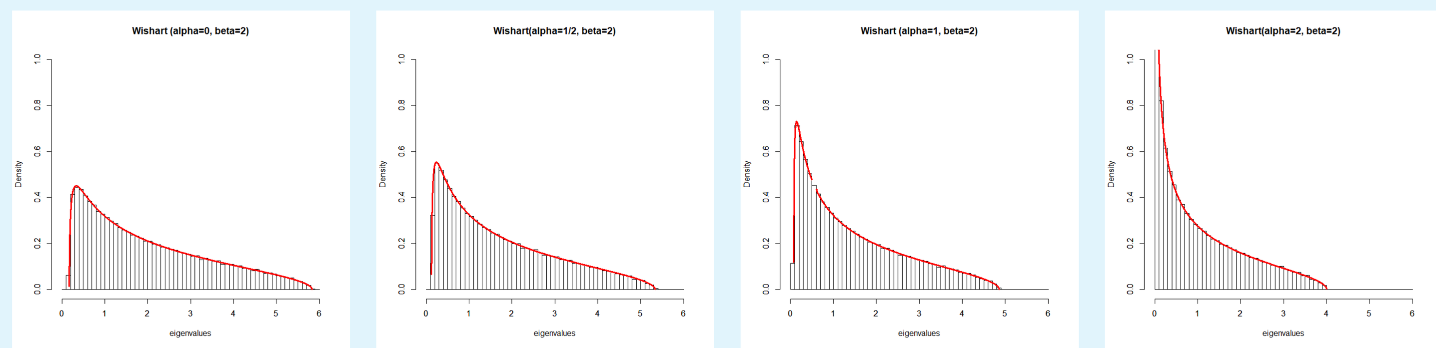
$$\mathcal{U}_n = \left\{ U_n = \begin{pmatrix} x_{11} & \cdots & x_{1a_n} & y_{11} & \cdots & \cdots & y_{1b_n} \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ x_{1a_n} & \cdots & x_{a_na_n} & y_{a_n1} & \cdots & \cdots & y_{a_nb_n} \\ y_{11} & \cdots & y_{a_n1} & d_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & 0 & d_2 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ y_{1b_n} & \cdots & y_{a_nb_n} & 0 & \cdots & 0 & d_{b_n} \end{pmatrix} ; \begin{array}{l} x_{ij} \in \mathbb{R}, \\ y_{ij} \in \mathbb{R}, \\ d_j \in \mathbb{R} \end{array} \right\}$$

Simulations

- **Simulations for Wigner Ensembles**
  - $c = 0.2$ • $c = 0.4$ • $c = 0.6$ • $c = 0.8$



- **Simulations for Wishart Ensembles**
  - $\alpha = 0$ • $\alpha = 0.5$ • $\alpha = 1$ • $\alpha = 2$

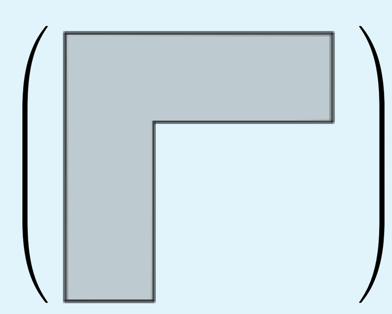


Red lines in these figures indicate the limiting eigenvalue distributions obtained theoretically.

Main tool for results

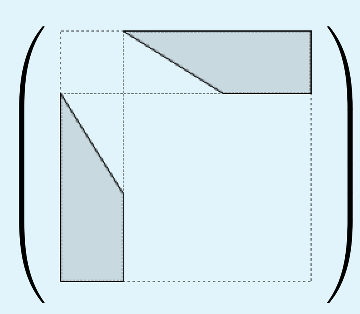
- For the proofs, we use the variance profile method (cf. Bordenave).
- Roughly speaking, if we know a "variance profile" of random matrices, then we obtain an integral. The corresponding Stieltjes transform can be given by using its solution.
  - A variance profile for Wigner case.
  - A variance profile for Wishart case.

$\sigma =$



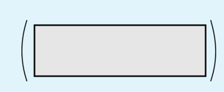
Solve a cubic equation.

$\sigma =$



Solve a differential equation.

Remarks

- Our results include the classical cases; the Wigner's semicircle law for Wigner cases and the Marchenko-Pastur law for Wishart cases.
- If $\gamma = 0$, then the generalized Lambert function $W_{\kappa,0}$ converges to the original Lambert function as $\kappa \to \infty$.
- Quadratic construction includes the following matrix spaces.
  - Spaces of rectangular matrices (Wishart), i.e. $E =$ 
  - Spaces of upper triangular matrices, i.e. $E =$