Topological dihedral homology

齋藤 翔 統計的機械学習研究センター 特任助教

1. Introduction.

In this poster presentation I briefly describe my work on a topological generalization of the theory of dihedral homology.

The theory of cyclic homology, which has been an important tool in various fields including operator algebras, algebraic topology, arithmetic and algebraic geometry, studies three invariants arising from the canonical circle action on the Hochschild homology of an algebra. If the algebra in question has an anti-involution then the circle action can be lifted to an action by $O_2(\mathbf{R})$, which leads to a variant of cyclic homology called dihedral homology.

A topological generalization of cyclic homology was introduced by Bokstedt, Hsiang and Madsen [BHM] who defined the invariant called topological cyclic homology to refine the topological analogue of negative cyclic homology. Recent work by Nikolaus and Scholze [NS] showed that topological cyclic homology is the homotopy fibre of a certain map resulting from the cyclotomic structure on the topological variant of Hochschild homology.

The aim of my work is to define and study a dihedral analogue of topological cyclic homology, called topological dihedral homology. It is defined for $\mathbf{E}_1$-ring spectra equipped with an anti-involution, coincides with the homotopy fixed points of topological cyclic homology under the induced involution, and fits in a fibre sequence similar to Nikolaus and Scholze's.

2. Cyclic homology.

Let Spec(A) be an affine scheme, regarded as an ∞-functor from the ∞-category CdgR$^-$ of nonpositively graded commutative dg-rings to the ∞-category ∞Grpd of ∞-groupoids co-represented by A. By definition, its *free loop space* LSpec(A) is the mapping ∞-stack Map(B$\mathbf{Z}$, Spec(A)), where B$\mathbf{Z}$ denotes the classifying space of the group of integers $\mathbf{Z}$, or equivalently the ∞-stack associated to the ∞-prestack constant at S^1. It is easily shown that LSpec(A) is derived affine, co-represented by the commutative dg-ring HH(A) = $A\wedge_{A\wedge A}A$, where ^ denotes the derived tensor product, called the *Hochschild homology* of A.

The circle S^1 acts on the free loop space LSpec(A) by rotation of loops, and hence on its derived ring of functions HH(A).

[**Proposition.** The underlying complex of abelian groups with S^1-action of HH(A) is given by the geometric realization of the cyclic complex whose n-th term is $A^{\wedge(n+1)}$, where the cyclic group of order (n+1) acts by cyclic permutation of components.]

Note that the above cyclic complex makes sense even if A is noncommutative. In particular, HH(A) and the canonical S^1-action thereon can be defined in that generality.

[**Definition.** The *positive, negative* and *periodic cyclic homology complexes* of A, denoted by HC$^+$(A), HC$^-$(A) and HC$^{\mathrm{per}}$(A), are the homotopy quotient, homotopy fixed point and Tate fixed point complexes, respectively, taken in the ∞-category of complexes of abelian groups, of HH(A) with respect to the S^1-action. (Recall that, in general, the Tate fixed point object with respect to a group action is defined to be the homotopy cofibre of the canonical norm map from the homotopy quotient to homotopy fixed points.)]

[**Theorem** (Goodwillie, [G]). Let I be a nilpotent ideal in A. Then the homotopy fibre of the map of rationalized algebraic K-theory spectra $K(A)_{\mathbf{Q}} \to K(A/I)_{\mathbf{Q}}$ is equivalent to the homotopy fibre of the map of Eilenberg-MacLane spectra associated to the map of negative cyclic homology complexes HC$^-$($A_{\mathbf{Q}}$) $\to$ HC$^-$((A/I)$_{\mathbf{Q}}$), where $A_{\mathbf{Q}}$ and $(A/I)_{\mathbf{Q}}$ denote the A and A/I tensored by the field of rational numbers $\mathbf{Q}$, respectively.]

3. Topological cyclic homology.

Let R be an $\mathbf{E}_1$-ring spectrum. The *topological Hochschild homology* THH(R) of R is the spectrum with S^1-action given by the geometric realization of the cyclic spectrum whose n-th term is $R^{\wedge(n+1)}$, where ^ denotes the smash product of spectra and the cyclic group of order (n+1) acts by cyclic permutation of components. The *topological positive, negative* and *periodic cyclic homology spectra* of R, denoted by TC$^+$(R), TC$^-$(R) and TC$^{\mathrm{per}}$(R), are defined to be the homotopy quotient, homotopy fixed point and Tate fixed point spectra, respectively, of THH(R) with respect to the S^1-action.

Let p be a prime and write $C(p)$ for the group of p-th roots of unity in the complex plane, considered as a subgroup of the circle group S^1 of complex numbers of norm 1. Note that the Tate fixed point spectrum THH$(R)^{tC(p)}$ of THH(R) with respect to the restricted action by $C(p)$ has the induced $S^1/C(p)$-action. By identifying S^1 with $S^1/C(p)$ via the p-th power map, an S^1-action on THH$(R)^{tC(p)}$ is obtained. It is known that there is a canonical S^1-equivariant map Φ_p: THH(R) $\to$ THH$(R)^{tC(p)}$.

[**Remark.** There is no counterpart of this map for HH(A) in the ∞-category of complexes of abelian groups.]

[**Definition.** A *cyclotomic spectrum* is a spectrum with S^1-action X, together with an S^1-equivariant map $X \to X^{tC(p)}$ for each prime p. The ∞-category of cyclotomic spectra is denoted by CycSp.]

[**Definition.** The *topological cyclic homology* TC(R) of the $\mathbf{E}_1$-ring spectrum R is the mapping spectrum map$_{\mathrm{CycSp}}$($\mathbf{S}$, THH(R)), where $\mathbf{S}$ denotes the sphere spectrum with the canonical structure of a cyclotomic spectrum.]

[**Theorem** (Nikolaus-Scholze, [NS]). There is a homotopy fibre sequence of spectra TC(R) $\to$ TC$^-$(R) $\to$ TC$^{\mathrm{per}}$(R)$^\wedge$, where (-)$^\wedge$ denotes the profinite completion and the second map is the canonical map minus the map induced by the cyclotomic structure maps Φ_p's.]

[**Theorem** (Dundas-Goodwillie-McCarthy, [DGM]). Let A be a ring and I a nilpotent ideal. Then the homotopy fibre of the map of algebraic K-theory spectra $K(A) \to K(A/I)$ is equivalent to the homotopy fibre of the map of topological cyclic homology spectra TC(A) $\to$ TC(A/I).]

4. Topological dihedral homology.

Suppose the $\mathbf{E}_1$-ring spectrum R is equipped with an anti-involution α, i.e. an equivalence of R with its opposite ring spectrum R^{op}. Then the S^1-action on the topological Hochschild homology spectrum THH(R) lifts to an $O_2(\mathbf{R})$-action, as the action of the cyclic group of order (n+1) on $R^{\wedge(n+1)}$ lifts to an action of the dihedral group of order 2(n+1) and thus THH(R) is the geometric realization of a dihedral spectrum. Moreover, the cyclotomic structure maps Φ_p: THH(R) $\to$ THH$(R)^{tC(p)}$ are $O_2(\mathbf{R})$-equivariant.

[**Definition.** A *dihedro-cyclotomic spectrum* is a spectrum with $O_2(\mathbf{R})$-action X, together with an $O_2(\mathbf{R})$-equivariant map $X \to X^{tC(p)}$ for each prime p. The ∞-category of dihedro-cyclotomic spectra is denoted by DihCycSp.]

[**Definition.** The *topological dihedral homology* TD(R, α) of the $\mathbf{E}_1$-ring spectrum R with anti-involution α is the mapping spectrum map$_{\mathrm{DihCycSp}}$($\mathbf{S}$, THH(R)), where the sphere spectrum $\mathbf{S}$ is equipped with the canonical structure of a dihedro-cyclotomic spectrum.]

Note that if ρ: $S^1 \to$ Aut(X) is an S^1-action on some object X, then the composition of ρ with the automorphism of S^1 given by inversion of elements is again an action. This operation gives rise to an involutive auto-equivalence of the ∞-category CycSp, i.e. an ∞-functor from the classifying ∞-groupoid of the group of two elements to the ∞-category Cat$_\infty$ of ∞-categories that takes the unique object to CycSp. The homotopy limit of this ∞-functor viewed as a diagram in Cat$_\infty$ is denoted by Dual(CycSp).

[**Proposition.** There is an equivalence of ∞-categories between DihCycSp and Dual(CycSp).]

[**Corollary.** The topological dihedral homology spectrum TD(R, α) is equivalent to the homotopy fixed point spectrum of the topological cyclic homology spectrum TC(R) with respect to the involution induced by conjugation from the $O_2(\mathbf{R})$-actions on $\mathbf{S}$ and THH(R).]

The *topological positive, negative* and *periodic dihedral homology spectra*, denoted by TD$^+$(R, α), TD$^-$(R, α) and TD$^{\mathrm{per}}$(R, α), are defined to be the homotopy quotient, homotopy fixed point and Tate fixed point spectra, respectively, of THH(R) with respect to the $O_2(\mathbf{R})$-action.

[**Theorem.** There is a homotopy fibre sequence of spectra TD(R, α) $\to$ TD$^-$(R, α) $\to$ TD$^{\mathrm{per}}$(R, α)$^\wedge$, where (-)$^\wedge$ denotes the profinite completion and the second map is the canonical map minus the map induced by the dihedro-cyclotomic structure maps Φ_p's.]

Reference.

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