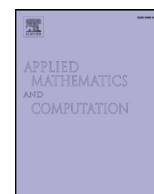


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## On travelling wave solutions of the diffusive Leslie–Gower model

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### ABSTRACT

We investigate the diffusive Leslie–Gower predator–prey model. Travelling wave solutions were found and a minimum wave speed relationship was derived. Linear stability analysis was performed in addition to full numerical simulation of the model. All travelling waves were found to be stable.

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### 1. Introduction

Models that describe predator–prey interactions are well established. These models incorporate competition, cooperation and predation terms to describe realistic dynamics. In these models, features such as extinction of one species or the other, persistence and oscillatory behaviour between species are commonly found. Different formulations of the above mentioned cases are an area of continuing research [2,14,16,21,26].

These models generally consist of two or more ordinary differential equations (ODEs) to represent the interactions of predator and prey. Multiplicity of solutions, limit cycles and chaos are among the features found in these type of systems. For example, Gilpin and Rosenzweig [9] have discussed the effects of enrichment to the predator and prey system that could lead to a limit cycle or periodic solution. Rinaldi et al. [25] studied a classical predator–prey model with a varying environment. They investigated several types of bifurcation properties and found chaos exists in the system. Harrison [10] conducted bacterial population experiments and validated the results using various modified predator–prey models.

Another predator–prey model commonly studied is the Leslie–Gower (LG) model

$$\frac{dX}{dt} = aX - \frac{bX^2}{Y}, \quad (1a)$$

$$\frac{dY}{dt} = cY - dY^2 - eXY, \quad (1b)$$

which is a type of ratio-dependent system. Introduced by Leslie [17] and Leslie and Gower [18], the model assumes that the carrying capacity of the predator  $X$  changes proportionally to the prey  $Y$ , and that the carrying capacity of the prey is limited by a fixed value  $c/d$ . In contrast, the classical models assume that the prey population grows without bound in the absence of the predators [2].

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System (1) has a unique coexistence equilibrium

$$X^* = \frac{ac}{ae + bd}, \quad Y^* = \frac{bc}{ae + bd}.$$

Korobeinikov [13] found a Lyapunov function for the LG model and showed that the equilibrium is globally stable, and a simpler case of LG model was considered by Safuan et al. [29].

Since the LG model has been introduced, numerous studies have modified and investigated the properties of system (1) to model various predator–prey populations. Collings [5] applied several functional forms to system (1) to investigate changes in the dynamical behaviour of mite predator–prey populations. Seo et al. [31] compared a LG-type model with another predator–prey model and found both models exhibit limit cycle solutions. A LG model with the effect of prey refuge was considered by Chen et al. [3]. They found that if the amount of prey refuge is increased, predator and prey densities can either increase or decrease. Chen and Chen [4] investigated a LG model with feedback controls and found that the feedback control variables have no effect on the global stability of the system, but only alter the location of the coexistence equilibria. Modified LG models for three species populations are also available, see references [1,15,27,28].

The aforementioned studies help biologists and ecologists to understand the dynamics that evolve between predators and preys temporarily without any spatial dimensions. To incorporate the distribution of populations over time and space domains, spatio-temporal models should be considered. System of partial differential equations (PDEs) will provide more information to explain the population distribution, the wave speed and the effects of diffusivity of each species over a space domain.

The LG model can be extended to include diffusive effects and takes the form

$$\frac{\partial U}{\partial t} = d_1 \frac{\partial^2 U}{\partial x^2} + aU - \frac{bU^2}{V}, \tag{2a}$$

$$\frac{\partial V}{\partial t} = d_2 \frac{\partial^2 V}{\partial x^2} + cV - dV^2 - eUV, \tag{2b}$$

where  $U$  is the predator and  $V$  is the prey population. Parameters  $d_1$  and  $d_2$  are the diffusion coefficients for the predator and prey, respectively. The predator grows logistically with growth rate  $a$  and is limited by the availability of prey with parameter  $b$ . The prey also grows logistically with growth rate  $c$ . The term  $eUV$  represents the effect of predation which reduces the prey’s per capita growth rate.

There have been several investigations into the diffusive LG model (2). Du and Hsu [6] investigated the behaviour of steady-state solutions of system (2) in environments that are homogenous and heterogeneous (where the constant parameter  $b$  in Eq. (2a) is replaced by a spatial-dependent function  $b(x)$ ). They found that the system has no non-constant positive solution in a homogeneous environment whereas in a heterogeneous environment, a non-constant solution can be obtained. Ko and Ryu [12] investigated non-constant positive steady-states of system (2) with general functional response and found in some conditions, there may have more than one non-constant positive steady-state. A diffusive LG model with a protection zone for the prey was studied by Du et al. [7]. They found results on the asymptotic profile of positive solutions of the model for large intrinsic predator growth rates. A cross-diffusion LG model was considered by Li and Zhang [19]. Depending on the natural and cross-diffusion coefficients, they showed the existence or non-existence of a non-constant positive solution of the system.

In this investigation, we are interested in studying travelling wave solutions within a homogeneous diffusive LG model. The wavefront solution provide information regarding how both populations disperse over space. It is a standard approach to consider travelling waves solution when investigating reaction-diffusion system. For instance, Dunbar [8] investigated a diffusive Lotka–Volterra model that gave rise to a travelling wavefront solution, and simulated the PDE using the method of lines. Huang and Weng [11] applied several numerical methods to study travelling waves for a diffusive predator–prey system with a general functional response. Our aim is to use a similar approach as used by Dunbar to determine and analyse the travelling wave solutions for the diffusive LG model in one spatial dimensional. We also numerically solve the PDEs with different initial conditions.

## 2. Equilibrium and stability analysis

Introducing the transformations  $u = eU/c$ ,  $v = eV/c$ ,  $\tau = at$ , and  $\xi = \sqrt{a/d_2}x$ , we arrive at the non-dimensional version of system (2)

$$\frac{\partial u}{\partial \tau} = \delta \frac{\partial^2 u}{\partial \xi^2} + u \left( 1 - \frac{\alpha u}{v} \right), \tag{3a}$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} + \beta v (1 - \gamma v - u), \tag{3b}$$

where

$$\alpha = b/a, \beta = c/a, \gamma = d/e \text{ and } \delta = d_1/d_2.$$

Let the travelling wave solution have the form  $u(\xi, \tau) = u(\zeta)$ , and  $v(\xi, \tau) = v(\zeta)$  where  $\zeta = \xi - s\tau$  is a moving frame with speed  $s$ . Substitute these into the system (3) to give a system of second order ODEs which is the travelling wave system

$$\delta u'' + su' + u \left( 1 - \frac{\alpha u}{v} \right) = 0, \tag{4a}$$

$$v'' + sv' + \beta v(1 - \gamma v - u) = 0, \quad (4b)$$

where  $' = d/d\zeta$ . Next we will analyse the effects of the diffusivity parameter  $\delta$  on the system (4).

### 2.1. Sedentary predator and diffusing prey, $\delta = 0$

Most predators are fast-moving which is a crucial survival skill for some animals. For example, tigers have to run faster to catch a running deer and a large fish has to swim faster to catch a school of small fish. However, there are other cases in ecosystems where predators move slower than the prey. For example, a praying mantis remains still while waiting for butterflies, moths, bees and beetles to move towards it. Similarly the pacman frog which has a wide mouth that enables it to swallow prey that crosses its path uses a 'sit-and-wait' strategy. Another species that also applies the same strategy is the web-builders spider [24]. This type of spider stays and waits for the presence of any small insects on the sheet webs until it can make its move. Thus it is a relevant case to consider. In our model, we assume that the predator moves very slowly relative to the prey,  $d_1 \ll d_2$ , so we can neglect the ratio of the diffusion coefficients,  $\delta = 0$ , then system (4) becomes

$$su' + u\left(1 - \frac{\alpha u}{v}\right) = 0, \quad (5a)$$

$$v'' + sv' + \beta v(1 - \gamma v - u) = 0. \quad (5b)$$

From system (5) there is a pair of critical points. The first critical point is  $P_0 = (0, 1/\gamma)$  (extinction of predator) and the second critical point is  $P_1 = (u^*, v^*)$  (coexistence of predator and prey) where

$$u^* = \frac{1}{\alpha\gamma + 1}, \quad v^* = \frac{\alpha}{\alpha\gamma + 1}.$$

Our interest is to study system (5) which corresponds to orbits in phase space connecting one critical point to another. We can write the system (5) as a system of first order equations

$$u' = \frac{u}{s}\left(\frac{\alpha u}{v} - 1\right), \quad (6a)$$

$$v' = w, \quad (6b)$$

$$w' = \beta v(\gamma v + u - 1) - sw. \quad (6c)$$

Thus,  $P_0$  and  $P_1$  correspond to the critical points  $E_0 = (0, 1/\gamma, 0)$  and  $E_1 = (u^*, v^*, 0)$ .

The Jacobian matrix of system (6) takes the form

$$J = \begin{pmatrix} \frac{2\alpha u - v}{sv} & -\frac{\alpha u^2}{sv^2} & 0 \\ 0 & 0 & 1 \\ \beta v & \beta(2\gamma v + u - 1) & -s \end{pmatrix}, \quad (7)$$

and at equilibrium  $E_0 = (0, 1/\gamma, 0)$ , the matrix (7) becomes

$$J_{E_0} = \begin{pmatrix} -\frac{1}{s} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{\beta}{\gamma} & \beta & -s \end{pmatrix}. \quad (8)$$

The characteristic equation of (8) is

$$P_1(\lambda) = \lambda^3 + \left(\frac{s^2 + 1}{s}\right)\lambda^2 - (1 - \beta)\lambda - \frac{\beta}{s}, \quad (9)$$

which gives the eigenvalues of the linearisation of (7) at  $E_0$

$$\lambda_1 = -\frac{1}{s}, \quad \lambda_2 = -\frac{s}{2} + \frac{\sqrt{s^2 + 4\beta}}{2}, \quad \lambda_3 = -\frac{s}{2} - \frac{\sqrt{s^2 + 4\beta}}{2}. \quad (10)$$

The eigenvalues  $\lambda_1$  and  $\lambda_3$  are always negative while  $\lambda_2$  is always positive (see Fig. 1a). Thus, there is a one dimensional unstable manifold based at equilibrium  $E_0$ . Further, from (10), the wave speed of the system (5) can take any positive value,  $s > 0$ , and the eigenvalues  $\lambda_1 = \lambda_3$  if speed  $s_{c1} = \sqrt{1/(1 + \beta)}$ .

At equilibrium  $E_1 = (u^*, v^*, 0)$ , the matrix (7) becomes

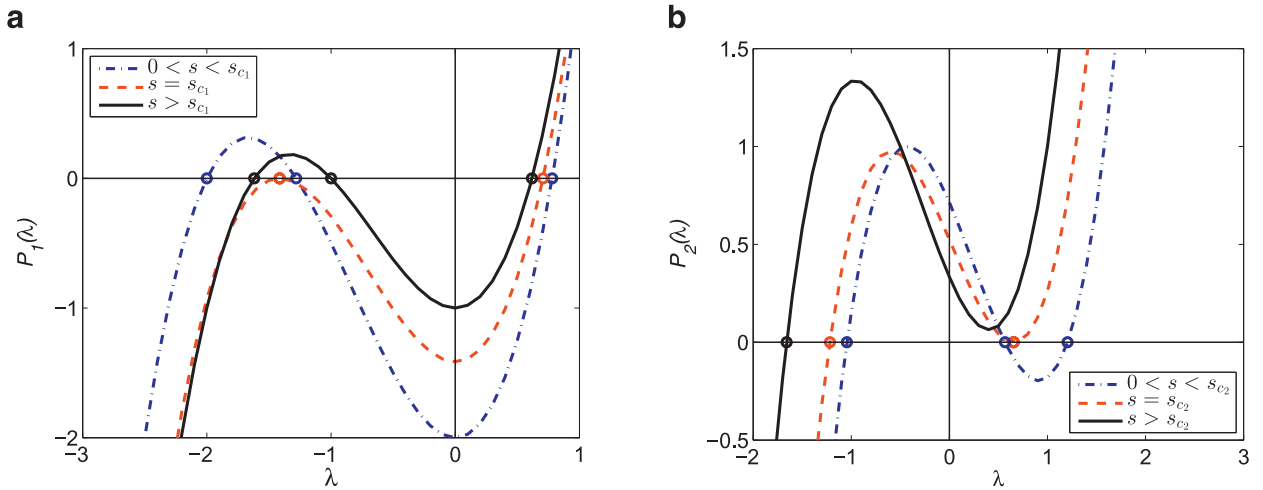


Fig. 1. Qualitative graphs of the characteristic polynomial with  $\delta = 0$  and varying  $s$  for (a)  $P_1(\lambda)$  in (9), and (b)  $P_2(\lambda)$  in (12).

$$J_{E_1} = \begin{pmatrix} \frac{1}{s} & -\frac{1}{\alpha s} & 0 \\ 0 & 0 & 1 \\ \frac{\alpha\beta}{\alpha\gamma+1} & \frac{\alpha\beta\gamma}{\alpha\gamma+1} & -s \end{pmatrix}. \tag{11}$$

It follows the characteristic equation of (11) at  $E_1$  is

$$P_2(\lambda) = \lambda^3 + \left(\frac{s^2 - 1}{s}\right)\lambda^2 - \left(1 + \frac{\alpha\beta\gamma}{\alpha\gamma + 1}\right)\lambda + \frac{\beta}{s}. \tag{12}$$

From  $P_2(\lambda)$ , there is exactly one negative root and two positive roots or a pair of complex roots with positive real parts (see Fig. 1b). Thus there is a two dimensional unstable manifold based at equilibrium  $E_1$ .

2.2. Predator and prey diffuse,  $\delta > 0$

Generally, predator and prey move at different rates. In a case where both of the species move, one could be moving a little faster than the other or both of them are moving at a similar rate. For example, equal diffusivity of both species ( $\delta = 1$ ) is due to spatial mixing from turbulence in a lake or in the sea. For such a case, one can assume that the magnitude of the diffusivity for the predator (zooplankton) and the prey (phytoplankton) is the same [22].

For general species, let's assume that the predator and prey diffuse at different rates. Then the ratio of the diffusion coefficients,  $\delta > 0$  in the system (4). Again we can write system (4) as a system of first order equations

$$u' = w, \tag{13a}$$

$$v' = z, \tag{13b}$$

$$w' = \frac{u}{\delta} \left( \frac{\alpha u}{v} - 1 \right) - \frac{sw}{\delta}, \tag{13c}$$

$$z' = \beta v(\gamma v + u - 1) - sz, \tag{13d}$$

where  $P_0$  and  $P_1$  correspond to the critical points  $F_0 = (0, 1/\gamma, 0, 0)$  and  $F_1 = (u^*, v^*, 0, 0)$ .

At equilibrium  $F_0 = (0, 1/\gamma, 0, 0)$ , we have the characteristic equation

$$P_1(v) = v^4 + \frac{s(1 + \delta)}{\delta} v^3 + \frac{1}{\delta}(s^2 + 1 - \beta\delta)v^2 + \frac{s}{\delta}(1 - \beta)v - \frac{\beta}{\delta}, \tag{14}$$

from which the eigenvalues are

$$\nu_1 = -\frac{s}{2} + \frac{\sqrt{s^2 + 4\beta}}{2}, \quad \nu_2 = -\frac{s}{2} - \frac{\sqrt{s^2 + 4\beta}}{2}, \tag{15}$$

$$\nu_3 = -\frac{s}{2\delta} + \frac{\sqrt{s^2 - 4\delta}}{2\delta}, \quad \nu_4 = -\frac{s}{2\delta} - \frac{\sqrt{s^2 - 4\delta}}{2\delta}. \tag{16}$$

From Eq. (15), the eigenvalue  $\nu_1$  is always positive while  $\nu_2$  is always negative. From (16), if  $s \geq 2\sqrt{\delta}$ ,  $\nu_3$  and  $\nu_4$  are a pair of real negative eigenvalues. If  $0 < s < 2\sqrt{\delta}$ ,  $\nu_3$  and  $\nu_4$  are a pair of complex conjugate eigenvalues with negative real part (see

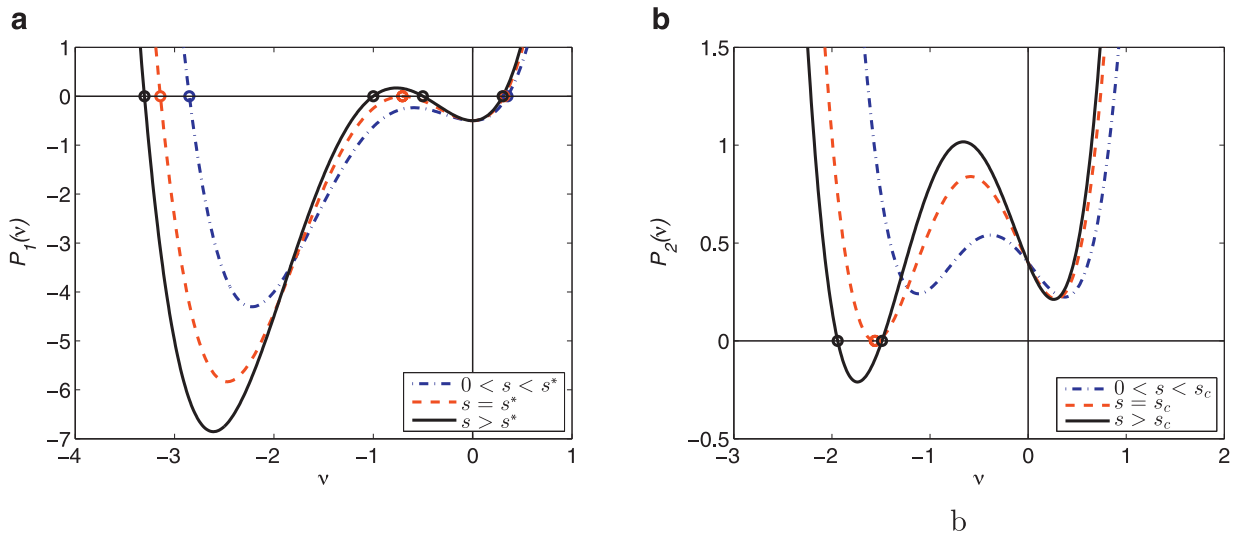


Fig. 2. Qualitative graphs of the characteristic polynomial with  $\delta > 0$  and varying  $s$  for (a)  $P_1(v)$  in (14), and (b)  $P_2(v)$  in (18).

Fig. 2a). In our example, we use  $\delta = 2$  to indicate predator’s diffusion rate is twice than the prey’s. There is a one dimensional unstable manifold and a two dimensional stable manifold based at equilibrium  $F_0$ . For  $0 < s < 2\sqrt{\delta}$ , the trivial equilibrium  $F_0$  is a spiral point on the stable manifold. Therefore, if the travelling wave solution exists, the possible minimum speed of system (13) for biologically relevant solution with non-negative  $u$  and  $v$  is

$$s \geq 2\sqrt{\delta}. \tag{17}$$

With  $s$  satisfying condition (17), a realistic solution may exist which tends to  $u = 0$  and  $v = 1/\gamma$  as  $\zeta \rightarrow +\infty$ . At the non-trivial equilibrium  $F_1 = (u^*, v^*, 0, 0)$ , we have the following characteristic equation:

$$P_2(v) = v^4 + \frac{s(1 + \delta)}{\delta} v^3 + \frac{1}{\delta} \left( s^2 - 1 - \frac{\alpha\beta\gamma\delta}{\alpha\gamma + 1} \right) v^2 - \frac{s}{\delta} \left( 1 + \frac{\alpha\beta\gamma}{\alpha\gamma + 1} \right) v + \frac{\beta}{\delta}. \tag{18}$$

From  $P_2(v)$ , there is exactly two negative roots or a pair of complex roots with negative real parts and two positive roots or a pair of complex roots with positive real parts (see Fig. 2b). Thus there is a two dimensional unstable manifold based at equilibrium  $F_1$ .

### 3. Numerical results

In this section, we describe how we solve a two-point boundary value problem given by system (4) and we simulate the full PDE problem (3) for both  $\delta = 0$  and  $\delta = 2$ .

#### 3.1. Travelling wave profiles

We use a relaxation or collocation method [23] to solve the travelling wave system (4) together with the following boundary conditions:

$$u(-\infty) = \frac{1}{\alpha\gamma + 1}, \quad u(+\infty) = 0, \tag{19a}$$

$$v(-\infty) = \frac{\alpha}{\alpha\gamma + 1}, \quad v(+\infty) = \frac{1}{\gamma}. \tag{19b}$$

In the process of finding travelling wave profiles, we use the following initial trial functions:

$$u(\zeta) = \begin{cases} \frac{1}{\alpha\gamma + 1}, & \zeta < \zeta_0, \\ 0, & \zeta > \zeta_0, \end{cases} \quad v(\zeta) = \begin{cases} \frac{\alpha}{\alpha\gamma + 1}, & \zeta < \zeta_0, \\ \frac{1}{\gamma}, & \zeta > \zeta_0. \end{cases} \tag{20}$$

For the sedentary predator and moving prey ( $\delta = 0$ ), we set  $\alpha = 0.5, \beta = 0.9, \gamma = 1, \delta = 0, s = 2$  in system (5) with the trial functions (20), we plot the travelling wave solution for  $u$  and  $v$  in Fig. 3. The parameters are chosen to simulate the shape of the travelling wave solutions. The travelling wave solutions join equilibrium  $(1/(\alpha\gamma + 1), \alpha/(\alpha\gamma + 1))$  to  $(0, 1/\gamma)$ . From the figure, we can see that initially at the right boundary, only prey exists. After the introduction of the predator at the right boundary, the sedentary predator consumes the large amount of moving prey. The predator increases in numbers after consuming the prey,

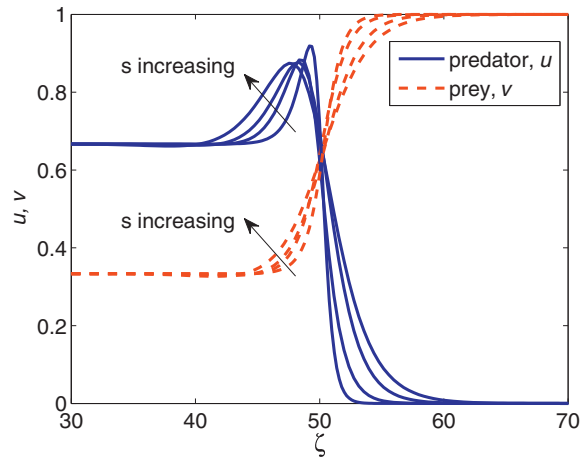


Fig. 3. Numerical solution of the travelling wave system (5) obtained from relaxation method with  $\alpha = 0.5, \beta = 0.9, \gamma = 1, \delta = 0, \zeta_0 = 50$ , with various speeds  $s = 0.5, 1, 1.5, 2$ .

and in contrast prey numbers decrease due the predation activity. After passing the interaction regime, both predator and prey stabilise at the coexistence equilibrium. Fig. 3 also shows the solution of the system (5) with various wave speeds. As we found in (10),  $s$  can be any value greater than zero. As  $s$  increases, we observe that the peak in predator  $u$  is slightly decreasing.

In system (5), each parameter  $\alpha, \beta, \gamma$  has different effects to the properties of the travelling wave solution. Different values of  $\alpha$  and  $\gamma$  affect the location of the coexistence equilibrium  $(1/(\alpha\gamma + 1), \alpha/(\alpha\gamma + 1))$  and the predator extinction equilibrium  $(0, 1/\gamma)$ , respectively. The variable  $\beta$  does not have any significant impact to the travelling wave of the prey  $v$ , but the density of the predator  $u$ , decreases as  $\beta$  increases. This is the result of the predator's carrying capacity is dependent on the availability of the prey. Thus, any slight change in the system will affect the numbers of the predators. Also, this behaviour is caused by the nature of the system (5) where  $\beta$  is written as the factor for the growth terms in Eq. (5b).

For the case where both species diffuse ( $\delta > 0$ ), we plot the region where the travelling wave of system (4) is possible from the speed condition (17). Fig. 4a shows the minimum speed curve  $s = 2\sqrt{\delta} = s^*$  that divides the parameter space into two regions. The shaded region represents parameter space where  $s \geq s^*$  and the unshaded region represents  $s < s^*$ . We set the ratio of diffusion coefficients,  $\delta = 2$ , which represents the diffusion parameters  $d_1$  and  $d_2$  in the dimensional system (2) and other parameters remain the same as before for all our travelling wave profiles. With  $\delta = 2$ , the minimum wave speed is  $s^* = 2\sqrt{2}$ . Using this set of parameters (in the shaded region), we plot the travelling wave profile as shown in Fig. 4b. However, if we plot the wave profile using parameter set in the unshaded region, we will have negative population density which is not biologically sensible thus it is not shown here. Fig. 4c shows the predator's profile with different values of diffusivity parameter,  $\delta$ . The character of the travelling wave remains the same, but the region of solution expands as  $\delta$  increases due to the numerical computations to relax the solution at the left and right boundaries.

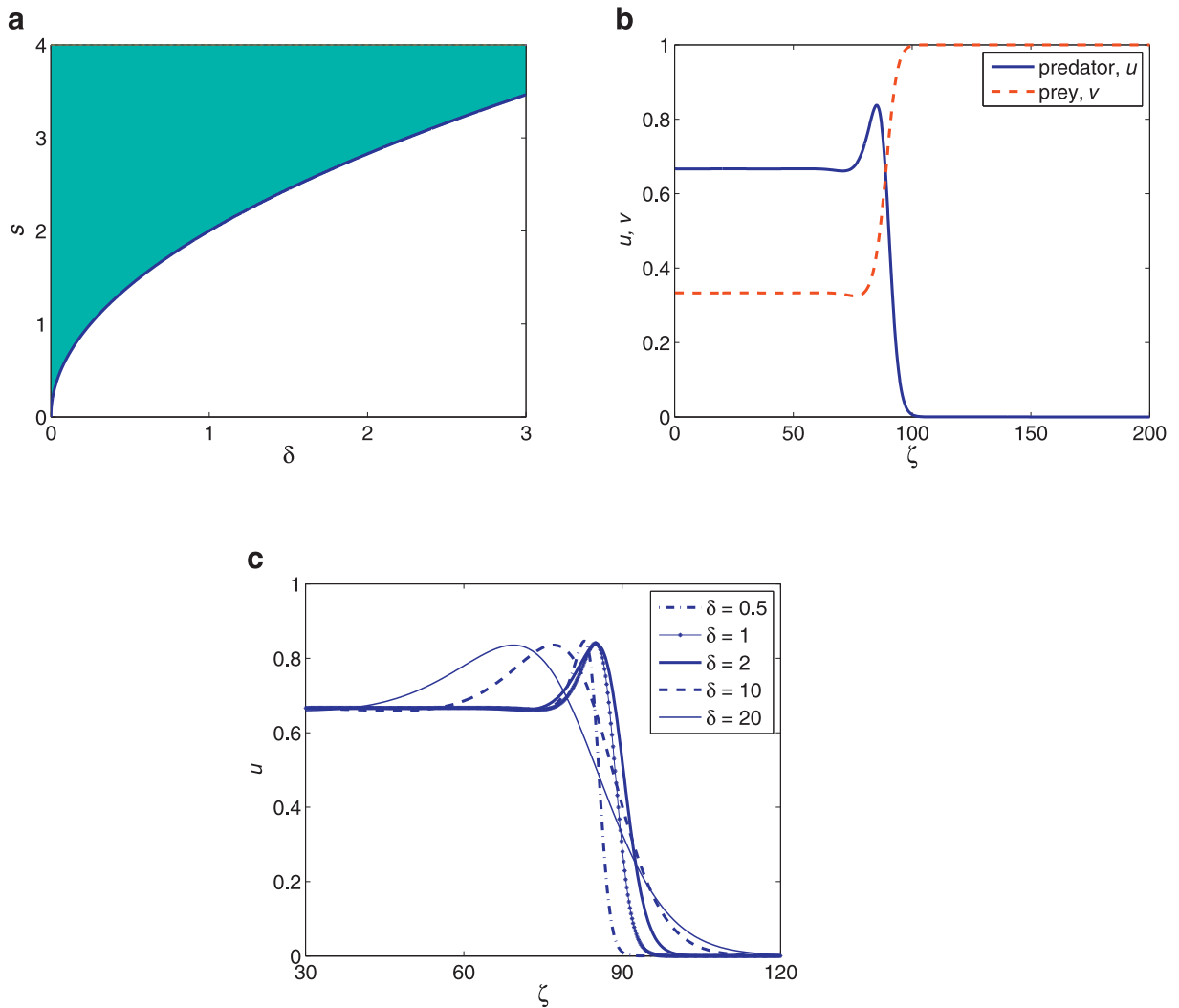
### 3.2. Simulations

We numerically simulate the PDE system (3) via method of lines [30] with the initial conditions obtained from the relaxation method for  $\delta = 0$  with fixed speed  $s = 2$  are shown in Fig. 5. The distribution of each species is shown across the domain for time intervals of size 100 and we discretise one dimensional space into equal size of  $n = 2000$  in a finite interval  $[0, 200]$ . In all simulations, we use Neumann boundary condition  $\partial u/\partial \mathbf{n} = \partial v/\partial \mathbf{n} = 0$ , which reflects no population can escape across the boundary.

Fig. 5 shows that the fixed wave profile moving to the right with constant speed  $s$ . We can observe that both equilibria are connected by the travelling wave solution – similar results as the relaxation method found in the previous section. The presence of predator at the right boundary induces the prey to increase and move. Once the predator starts consuming the prey, the prey numbers are reduced. The prey are depicted by a tanh-shaped kink solution whereas the predator shows a one-hump travelling wave as a result of consumption. Both predator and prey coexist at the left boundary. The behaviour of the travelling wave solution can be viewed similar relative to a dispersion of a soliton. Solitons are known to originate from the nonlinearity effects in the governing equations and have been shown to have a single hump-shaped that moves along the  $x$ -axis direction. In our case, the peak in the predator and a slight dip in the prey are induced by the nonlinear interaction terms found in system (5).

We also plot the solutions for  $\delta = 2$  with minimum speed  $s^* = 2\sqrt{2}$  in Fig. 6. Compared to the stationary case  $\delta = 0$  in Fig. 5, the waves travel further for the case with  $\delta = 2$ . This shift is driven due to the effect of the diffusion parameter  $\delta$ . The greater value of  $\delta$ , the further the populations move over space. In this case, the predator moves twice as fast compared to the prey.

After analysing the travelling wave solution of the Leslie–Gower model with the effects of the diffusion parameter  $\delta$ , we are interested to observe the dynamics of population distribution for different types of initial conditions. For example, Medvinsky



**Fig. 4.** (a) The curve  $s = 2\sqrt{\delta} = s^*$  separates the shaded and the unshaded regions in the  $(s, \delta)$  parameter space. Shaded region represents parameter space where  $s \geq 2\sqrt{\delta}$ , and the unshaded region represents parameter space  $s < 2\sqrt{\delta}$ . (b) Wave profile of system (4) obtained from a relaxation method with  $\alpha = 0.5, \beta = 0.9, \gamma = 1, \delta = 2, \zeta_0 = 100,$  and  $s = s^* = 2\sqrt{2}$ . (c)  $u$  profile with different values of diffusivity parameter,  $\delta$ .

et al. [20] study a predator–prey system that depends on the choice of initial conditions. They found that for some small perturbation in an otherwise homogeneous distribution gives rise to a heterogeneous spatial distribution of species. However, they did not discuss travelling wave solution for the PDE problem. Dunbar [8] found travelling wave solution in a predator–prey system using a particular step-function form of initial conditions.

In our study, we simulate the PDE system (3) with several initial conditions: constant-gradient, step-function, and sinusoidal function. For all simulations, we solve (3) with a Neumann boundary condition and  $\delta = 2$ . The minimum wave speed according to the condition (17) is  $s^* = 2\sqrt{\delta} = 2\sqrt{2}$ .

We begin with a simple step function for predator  $u$  and constant distribution for prey  $v$

$$u(x, \tau) = \begin{cases} \frac{1}{\alpha\gamma+1}, & x < x_0, \\ 0, & x > x_0, \end{cases} \quad v(x, \tau) = \frac{\alpha}{\alpha\gamma + 1}, \tag{IC1}$$

and then consider step functions for both species

$$u(x, \tau) = \begin{cases} \frac{1}{\alpha\gamma+1}, & x < x_0, \\ 0, & x > x_0, \end{cases} \quad v(x, \tau) = \begin{cases} \frac{\alpha}{\alpha\gamma+1}, & x < x_0, \\ \frac{1}{\gamma}, & x > x_0, \end{cases} \tag{IC2}$$

and finally investigate a sinusoidal distribution in prey  $v$

$$u(x, \tau) = \begin{cases} \frac{1}{\alpha\gamma+1}, & x < x_0, \\ 0, & x > x_0, \end{cases} \quad v(x, \tau) = \frac{\alpha}{\alpha\gamma + 1} + A \sin\left(\frac{2\pi(x - x_0)}{S}\right), \tag{IC3}$$

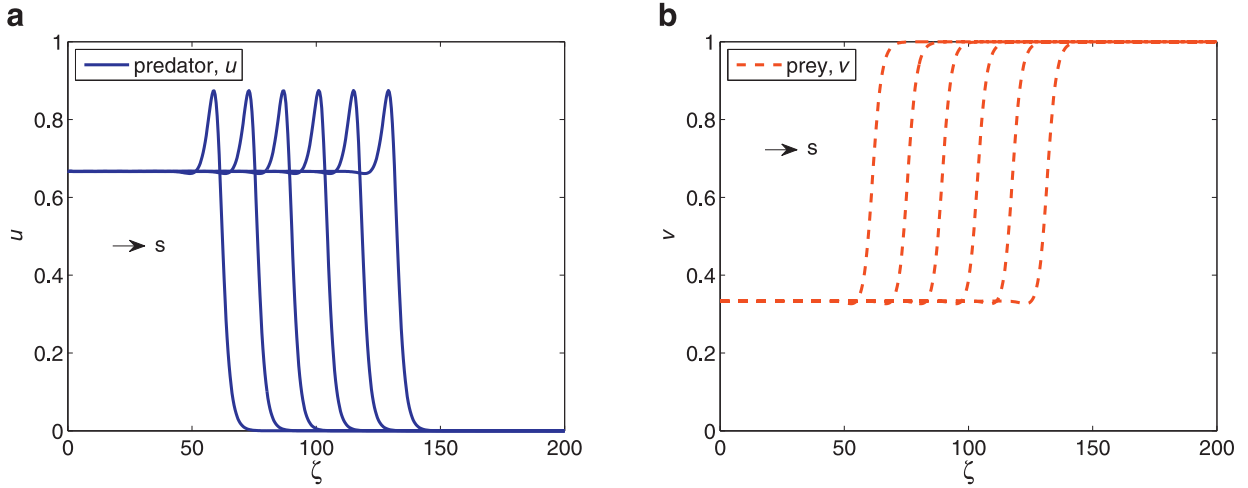


Fig. 5. Solutions of system (3) via method of lines with the initial conditions obtained from the relaxation method. Solutions are plotted on the same axes at  $\tau = 5.7, 12.7, 19.6, 26.7, 33.7, 40.7$ . Parameters are  $\alpha = 0.5, \beta = 0.9, \gamma = 1, \delta = 0$  and fixed speed  $s = 2$ .

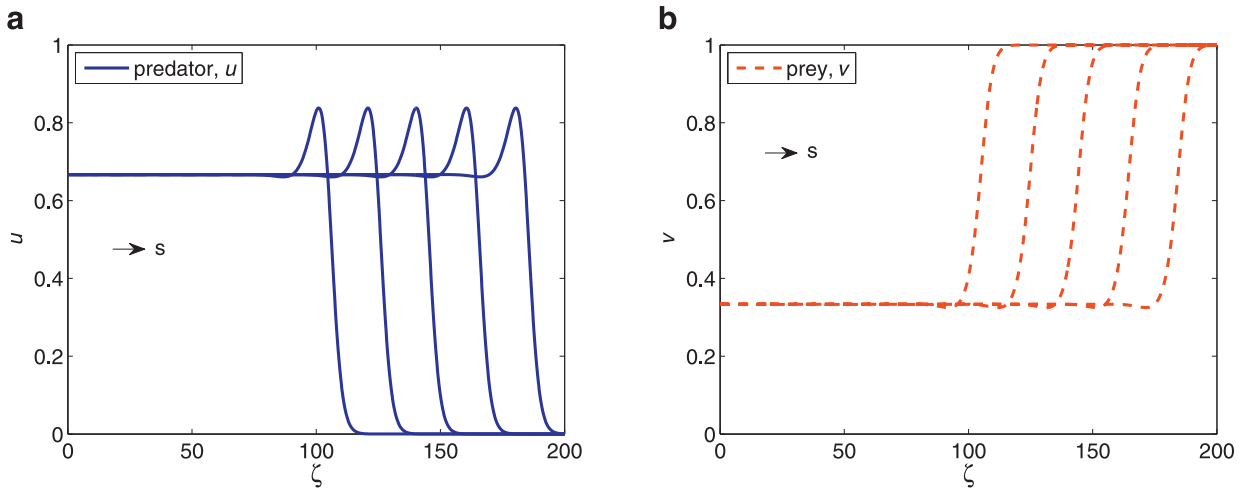


Fig. 6. Solutions of system (3) via method of lines with the initial conditions obtained from the relaxation method. Solutions are plotted on the same axes at  $\tau = 5.7, 12.7, 19.7, 26.7, 33.7$ . Parameters are  $\alpha = 0.5, \beta = 0.9, \gamma = 1, \delta = 2$  and minimum speed  $s^* = 2\sqrt{2}$ .

where  $A$  and  $S$  are constants with  $A < \frac{\alpha}{\alpha\gamma+1}$ .

The simulations of the travelling waves with different initial conditions are given in Fig. 7. For all three types of initial conditions, after some initial transients (see Fig. 7a (ii), b(ii), and c(ii)), the travelling wave profiles are similar to those found by the relaxation method in Section 3.1. These simulations show that a wide range of initial conditions for the PDE system will relax to a travelling wave. Hence, it demonstrates that the travelling wave solutions found in our work are attractors in the PDE system and are nonlinearly stable.

#### 4. Conclusions

We have studied a spatio-temporal reaction-diffusion system modelling Leslie–Gower predator–prey population with diffusion. We showed that system (4) gives rise to a travelling wave solution that connects one critical point to the other. Using stability analysis, a minimum wave speed condition was derived for the case with diffusion parameter  $\delta > 0$ . The relaxation method is used to find the travelling wave profile for both cases  $\delta = 0$  and  $\delta > 0$ .

Our results for the diffusive Leslie–Gower model of predator–prey interaction show that both a “sit and wait” strategy ( $\delta = 0$ ) and a small hunting ground approach ( $\delta = 2$ ) exist as travelling wave solutions. Furthermore, these travelling waves solutions are stable implying the important biological quality of persistence. Further, our numerical simulations show that a variety of initial conditions will resolve to the above mentioned travelling wave solution. We may interpret this to mean the coexistence of the



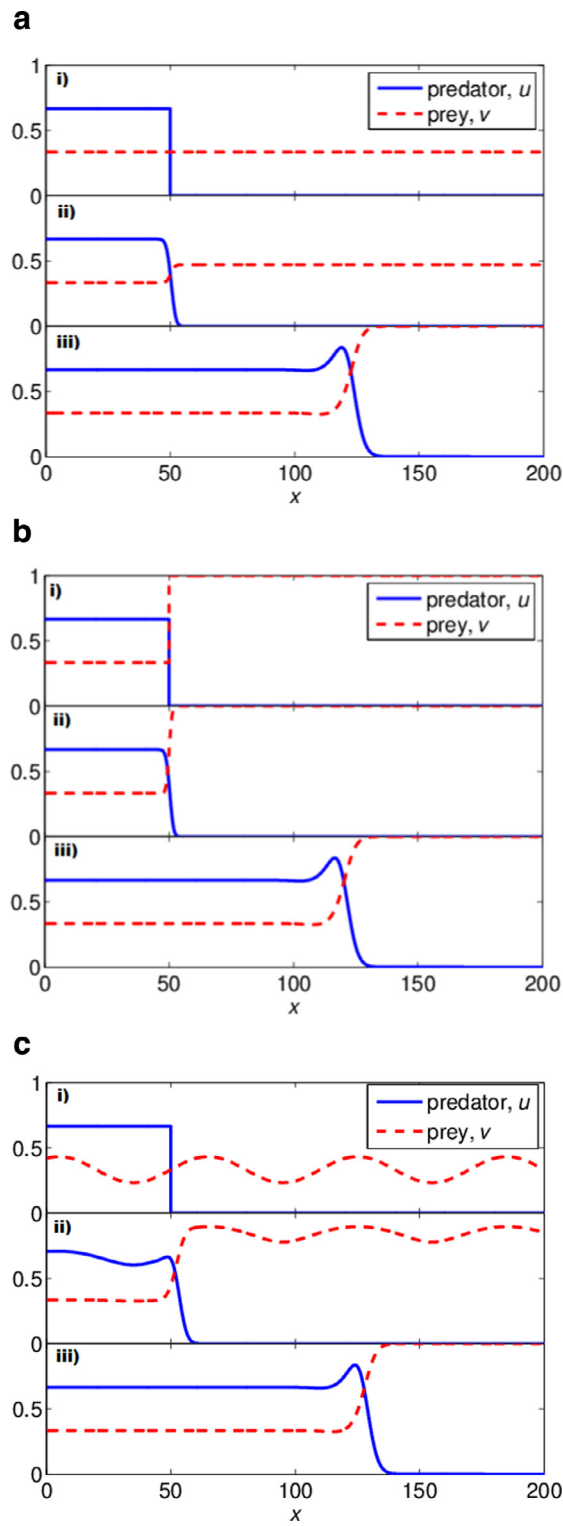


Fig. 7. Simulations of the system (3) with initial conditions (a) IC1, (b) IC2, and (c) IC3 with  $x_0 = 50$ ,  $A = 0.1$ ,  $S = 60$ .

predator and prey is robust to either how the two species initially encounter one another or external shocks to the population numbers.

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