

**AN IMPULSIVE APPROACH FOR NUMERICAL INVESTIGATION OF HYBRID FUZZY  
DIFFERENTIAL EQUATIONS AND INTUITIONISTIC TREATMENT FOR FUZZY  
ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS**

**KAVIKUMAR JACOB  
MUSTAFA BIN MAMAT  
NOR SHAMSIDAH BINTI AMIR HAMZAH  
NOOR'ANI BINTI AHMAD  
LEE SIAW CHONG**

**GERAN FRGS  
NO. VOT 0552**

**UNIVERSITI TUN HUSSEIN ONN MALAYSIA  
2012**

## TABLE OF CONTENTS

<b>ACKNOWLEDGEMENTS</b>	<b>ii</b>
<b>DECLARATION</b>	<b>iii</b>
<b>TABLE OF CONTENTS</b>	<b>iv</b>
<b>LIST OF TABLES</b>	<b>xi</b>
<b>LIST OF FIGURES</b>	<b>xv</b>
<b>LIST OF PUBLICATIONS</b>	<b>xvii</b>
<b>ABSTRACT</b>	<b>xix</b>
<b>CHAPTER 1</b>	<b>INTRODUCTION</b>
1.1	Background of Research 1
1.2	Problem Statement 4
1.3	Research Objectives 6
1.4	Scope of the Research 7
1.5	Benefits and Applications 7
1.6	Preliminaries 8
1.6.1	Fuzzy Sets and Fuzzy Logic 9
1.6.2	Fuzzy Quantities 12
1.6.3	Fuzzy Set Valued Mappings 15
1.6.4	The Space $E^n$ 19
1.6.5	The Hausdorff Distance 20

	1.6.6 Hukuhara Difference	21
	1.6.7 Fuzzy Integration	23
	1.6.8 Simpson's 1/3 Rule	24
	1.6.9 Simpson's 3/8 Rule	25
	1.7 Organization of Thesis	25
<b>CHAPTER 2</b>	<b>LITERATURE REVIEW</b>	
	2.1 Introduction	29
	2.2 Fuzzy Differential Equations	29
	2.3 Generalized Differentiability	33
	2.4 Impulsive Differential Equations	35
	2.5 Impulsive Fuzzy Differential Equations	38
	2.6 Ant Colony Programming	39
	2.7 Conclusion	42
<b>CHAPTER 3</b>	<b>IMPULSIVE DIFFERENTIAL EQUATIONS</b>	
	3.1 Introduction	45
	3.2 Impulsive Differential Equations	46
	3.2.1 Properties of Solutions of Impulsive Differential Equations	50
	3.2.2 Algorithm of Solving Impulsive Differential Equations	53
	3.3 Motivating Examples of Real Life Problems	54
	3.3.1 Fish Breeding Pond	54
	3.3.2 Dynamic Market Model	55

	3.3.3	Optimal Control Problem	56
	3.4	Numerical Solutions of Impulsive Differential Equations	56
	3.5	Conclusion	75
<b>CHAPTER 4</b>		<b>IMPULSIVE FUZZY DIFFERENTIAL EQUATIONS</b>	
	4.1	Introduction	77
	4.2	Preliminaries	78
	4.3	Methodology	81
	4.4	Numerical Solutions of Impulsive Fuzzy Linear Differential Equations	83
	4.4.1	Numerical Examples	87
	4.5	Conclusion	97
<b>CHAPTER 5</b>		<b>IMPULSIVE FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY</b>	
	5.1	Introduction	99
	5.2	Generalized Differentiability of Impulsive Fuzzy Differential Equations	100
	5.3	Analytical Solutions of Impulsive Fuzzy Differential Equations	109
	5.4	Numerical Solutions of Impulsive Fuzzy Differential Equations	115
	5.5	Conclusion	120
<b>CHAPTER 6</b>		<b>STABILITY CRITERIA AND CONVERGENCE THEOREM OF IMPULSIVE FUZZY DIFFERENTIAL EQUATIONS</b>	
	6.1	Introduction	122

6.2	Global Existence	122
6.3	Stability Criteria	124
6.4	Generalized Characterization Theorem for Impulsive Fuzzy Differential Equations	125
6.5	Conclusion	130

**CHAPTER 7 FUZZY DIFFERENTIAL EQUATIONS AND MODIFIED ROMBERG'S METHOD**

7.1	Introduction	131
7.2	Fuzzy Initial Value Problem	131
7.3	Modified Two-step Simpson's Method	133
7.4	Modified Two-step Simpson's Method for Numerical Solution of Fuzzy Differential Equations	135
7.5	Modified Romberg's Method by using Modified Two-step Simpson's 1/3 Method	137
7.6	Modified Romberg's Method for Numerical Solution of Fuzzy Differential Equations	139
7.7	Numerical Results	141
7.8	Conclusion	149

**CHAPTER 8 FUZZY DIFFERENTIAL EQUATIONS AND MODIFIED TWO-STEP SIMPSON'S 3/8 METHOD**

8.1	Introduction	124
8.2	Modified Two-step Simpson's Method	150
8.3	Modified Two-step Simpson's 3/8 Method for Numerical Solution of Fuzzy Differential Equations	153

8.4	Modified Romberg's Method by using Modified Two-step Simpson's 3/8 Method	155
8.5	Modified Romberg's Method for Numerical Solution of Fuzzy Differential Equations	157
8.6	Numerical Example	160
8.7	Graphical Representation of Corresponding Errors of Example 8.7.1 by Runge-Kutta Method of Order Three and Modified Two-step Simpson's 3/8 Method	167
8.8	Application of Modified Romberg's Method	170
8.9	Conclusion	172

## CHAPTER 9

### **FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY BY ANT COLONY PROGRAMMING**

9.1	Introduction	173
9.2	Fuzzy Differential Equation under Generalized Differentiability	173
9.3	Ant Colony Programming (ACP)	175
9.4	Criteria of ACP	175
9.4.1	Choice of Terminals and Function	177
9.4.2	Construction of Graph	177
9.4.3	Defining Fitness Function	177
9.4.4	Defining Terminals Criteria	178
9.5	ACP Methodology	178
9.5.1	Elaboration	179

	9.5.2	ACP Algorithm	184
	9.6	Numerical Example	185
	9.6.1	Solution Obtained by using ACP	186
	9.7	Conclusion	190
<b>CHAPTER 10</b>		<b>SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY GENETIC PROGRAMMING</b>	
	10.1	Introduction	191
	10.2	Basic Concepts and Fuzzy Differential Equation	192
	10.3	Genetic Programming Method	192
	10.4	Numerical Example	194
	10.5	Conclusion	195
<b>CHAPTER 11</b>		<b>SIMULINK APPROACH TO SOLVE FUZZY DIFFERENTIAL EQUATIONS UNDER GENERALIZED DIFFERENTIABILITY</b>	
	11.1	Introduction	197
	11.2	Basic Concepts and Fuzzy Differential Equation	197
	11.3	Simulink Method	198
	11.4	Numerical Example	199
	11.5	Conclusion	199
<b>CHAPTER 12</b>		<b>CONCLUSIONS AND FUTURE RESEARCH</b>	
	12.1	Introduction	201

12.2	Discussions and Findings	202
12.3	Limitation of Research	214
12.4	Suggestions of Future Research	214
12.5	Ending Remarks	216

<b>REFERENCES</b>	<b>218</b>
-------------------	------------



## ABSTRACT

Many evolution processes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. It is assumed naturally, that those perturbations act instantaneously, in the form of impulses. The impulsive differential equations, by means differential equations involving impulse effects, are seen as a natural description of observed evolution phenomenon of several real world problems. For example, systems with impulse effect have applications in physics, biotechnology, industrial robotics, pharmacokinetics, population dynamics, ecology, optimal control production theory and many others. Therefore, it is beneficial to study the theory of impulsive differential equations as a well deserved discipline, due to the increase applications of impulsive differential equations in various fields in the future. However, in many mathematical modelling of the real world problems, fuzziness and impulsiveness occurs simultaneously. This problem could be better modelled by impulsive fuzzy differential equations. Therefore, this research applies the theory of impulsive fuzzy differential equations by combining the theories of impulsive differential equations and fuzzy differential equations. The numerical algorithms are developed and the solutions are verified by comparing the results with the analytical solutions.

The novel method for the first order linear impulsive fuzzy differential equations under generalized differentiability is also proposed analytically and numerically. The convergence theorem for the impulsive fuzzy differential equations (FDE) under generalized differentiability is defined.

In this study, Ant Colony Programming (ACP) was used to find the optimal solution of FDE. Results obtained show that the method is effective in solving fuzzy differential equation. The solution in this method is equivalent to the exact solution of the problem.

Modified Romberg's method and Modified Two-step Simpson's  $3/8$  method are used to solve FDE with fuzzy IVP has been successfully derived. The result has been shown that Modified Romberg's method gave smaller error than the Standard Euler's method. Therefore Modified Romberg's method can estimate the solution of fuzzy differential equation more effectively than the Euler's method in solving fuzzy differential equation.

Meanwhile, by using the Modified Two-step Simpson's  $3/8$  methods, it has been shown that the solution of FDE provide more accurate approximation to the exact solution and it also gives better results than the Runge-Kutta method. In other words, Modified Two-step Simpson's  $3/8$  method is an effective method to solve fuzzy differential equation compared to the Runge-Kutta method.

---

---

# CHAPTER 1

## Introduction

---

---

## CHAPTER 1

### INTRODUCTION

#### 1.1 Background of Research

The birth of calculus back in 1660s has initiated the study of differential equations. An early triumph of differential equations was Newton's demonstration that Kepler's empirical laws of planetary motion could be derived from Newton's law of motion using differential equations (Markley, 2004). Over the years, the subject of differential equations represents various applications in many disciplines. The theory of differential equations can be applied in chemical kinetics, economics, electronics, epidemiology, mechanics and population dynamics (Lynch, 2004). Therefore, the theory is of great use to engineers, scientists and also mathematicians especially in the study of dynamical systems which model the real life problems.

However, differential equations which model the evolutionary mechanisms of many real life problems in various fields of applications and disciplines are always pervaded by uncertainties and vagueness. These models are purely deterministic, so the application of this approach requires precise knowledge about the system under investigation (Hüllermeier, 1999). Unfortunately, the knowledge about parameter values, functional relationships or initial conditions will not be known precisely. As

for realistic models, these uncertainties and impreciseness have to be considered. In 1965, Zadeh published his first inspiring paper on how the nature of uncertainty in the behaviour of a given systems could be handled by the use of fuzzy sets (Klir and Yuan, 1996). According to Zadeh (2005), uncertainty is an attribute to information. Therefore, the knowledge of fuzzy logic which refers to the use of fuzzy sets is essential to handle these uncertainties. Fuzzy logic does not mean fuzzy. Fuzzy logic can be viewed as an attempt to formalize the capability of human to reason and make rational decisions in an environment of imperfect information and the capability of human to perform a variety of physical and mental tasks without any measurements and any computations (Zadeh, 2008).

Since then, the use of fuzzy sets and fuzzy logic becomes an effective tool in order to have a better understanding in handling the impreciseness information of the problem studied. As a result, the impreciseness due to uncertainty and vagueness initiates the introduction of the fuzzy differential equations theory. Kandel and Byatt (1980) applied the concept of fuzzy differential equations to the analysis of fuzzy dynamical systems. The term fuzzy differential equations refers to differential equations with fuzzy coefficients, or fuzzy initial values, or fuzzy boundary values or even differential equations which deal with functions on the space of fuzzy intervals (Ahmad and De Baets, 2009). Hence, the study of fuzzy differential equations forms a suitable setting to model dynamical system in which uncertainties or vagueness exists (Buckley and Feuring, 2001; Buckley and Eslami, 2002).

Impulsive differential equations is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics (Nenov, 1999; Stamova and Stamov, 2001; Corwin *et al.*, 2008; Wang *et al.*, 2010; El-Sayed and Ameen, 2011). There has been a significant development in the impulse theory especially on the qualitative behavior of solutions of the impulsive differential equations with fixed moments (Bainov and Dimitrova, 1999; Akhmet and Turan, 2006; Cabada and Liz, 1998; Guo *et al.*, 2007). However, the knowledge about the dynamic systems modeled by impulsive differential equations is often incomplete or vague thus contributing fuzziness due to the impossible way to characterize the whole set of system behaviors. Therefore, the fuzzy sets theory can be incorporated to handle the imprecise and fuzzy information of the modeled systems.

Due to the benefits of fuzzy sets as a tool in handling problems modeled by differential equations in various fields, some researchers have extended their investigations of impulsive differential equations in the fuzzy environment (Lakshmikantham and McRae, 2001; Benchohra *et al.*, 2007; Rodriguez-Lopez, 2008b; Lan *et al.*, 2009). Fuzziness and impulsiveness occur in some real life problems such as the interest rate models in bond pricing, where the interest rate is unpredictable (Lakshmikantham and McRae, 2001). This problem can be modeled by impulsive fuzzy differential equations. Therefore, Lakshmikantham and McRae (2001) offer intelligent contributions in initiating the theory of impulsive fuzzy

differential equations by combining the theory of fuzzy differential equations and impulsive differential equations. Benchohra *et al.* (2007) investigated the existence of fuzzy solutions for the first and second order ordinary differential equations with impulsive effects by using the fixed point theorem for absolute retract. The results are then applied in the application models which have been established by several researchers (Guo *et al.*, 2003; Jafelice *et al.*, 2009).

## 1.2 Problem Statement

Knowledge about dynamical systems modeled by differential equations is often incomplete or vague (Novak, 2005). For example, for parametric quantities', functional relationships or initial conditions, the well known methods of solving differential equations analytically or numerically can only be used to find the selected behavior, such as by fixing the unknown parameters to some plausible values. However, by using this way it is not possible to characterize the whole set of system behaviors compatible with our partial knowledge. Then, the study of fuzzy differential equations has been carried out to form a suitable setting to mathematically model the real life problems which encounter uncertainties and vagueness (Abbasbandy *et al.*, 2004; Allahviranloo *et al.*, 2007; Nieto *et al.*, 2009). The alternative formulations of fuzzy initial value problems carried out by Lakshmikantham and Mohapatra (2003), suggested replacing them by a system of multivalued differential equations as well as by set of differential equations.

Impulsive differential equations models mathematical problems of real processes and phenomena which are subjected to short time disturbances. There has been a significant development in the impulse theory, particularly in the area of impulsive differential equations with fixed moments (Yao *et al.*, 2009). However, many impulsive differential equations cannot be solved in analytic expression or their solving is more complicated ( Liu *et al.*, 2007; Ran *et al.*, 2008 ). Meanwhile, huge numbers of practical problems do not need solution of impulsive differential equation in analytic form, but they need the numerical values of solution (Randelovich *et al.*, 2000). Furthermore, there are few articles referring to this domain (Liu *et al.*, 2007; Ran *et al.*, 2008; Hui *et al.*, 2011). Therefore, the solving of impulsive differential equations numerically is of great significance.

In this research, an approach, which is the incorporation of the numerical algorithm of fuzzy differential equations into impulsive differential equations so that one can overcome the problem of uncertainties and vagueness in some impulsive differential equations models is proposed. The numerical methods for solving ordinary differential equations for are going to be used in solving the impulsive fuzzy differential equations. The numerical methods used are the Euler method, the Second-order Taylor Series method, the Modified Euler method and the Runge-Kutta method. It seems that finally the accuracy of the results is better when better numerical method is used.

In real life, dynamical system often yields incomplete or vague knowledge (Kandel and Byatt, 1980; Ma *et al.*, 1999; Zadeh, 2005). It can occur during data collection,



the measurement process, as well as when determining the initial values. In order to deal with fuzzy dynamical systems under uncertainty, it is necessary to use FDE as one of the tools. First order linear FDE have no doubt to be the simplest FDE which can come up in various applications (Kandel and Byatt, 1980; Bede and Gal, 2004; Chalco-Cano and Roman-Flores, 2008; Nieto and Rodriguez-Lopez, 2009). This study will use three approaches to solve FDE namely Ant Colony Programming (ACP), modified Romberg's method and modified Two-step Simpson's 3/8 method. These three new techniques can serve as an alternative way to solve FDE. The previous methods in solving FDE are shown in Appendix A2. The numerical results obtained from these techniques are compared with previous methods based on the Standard Euler's method (Ma *et al.*, 1999) discusses in Chapter 8 and Runge-Kutta method (Duraisamy and Usha, 2004) discusses in Chapter 9 respectively.

### **1.3 Research Objectives**

The objectives of the research are :

1. to develop the numerical algorithm for linear impulsive differential equations as well as impulsive fuzzy differential equations,
2. to compare the numerical solutions of developed algorithm in objective (1) between chosen numerical methods and the analytical solutions,
3. to investigate the existence of analytical and numerical solutions of linear impulsive fuzzy differential equations under generalized differentiability,
4. to define the convergence theorem for impulsive fuzzy differential equations under generalized differentiability.

5. to solve fuzzy differential equations by using modified Romberg's method,
6. to solve fuzzy differential equations by using modified Two-step Simpson's 3/8 method,
7. to develop the ant colony programming and to find the numerical solution of fuzzy differential equations.
8. to develop the genetic programming and solve the numerical solution of fuzzy differential equations under generalized differentiability and,
9. to implement the simulink approach for solving differential equations generalized differentiability.

#### **1.4 Scope of the Research**

In this research, the numerical algorithm of fuzzy differential equations and impulsive method are combined into a single method and to be used to develop the solutions of the dynamical systems. The scope of the research is limited to the linear impulsive fuzzy dynamical systems, which covers fuzzy dynamical systems, impulsive and initial value problem in differential equations with fixed moments of impulses. Meanwhile in solving FDE, the study is limited to the linear first order fuzzy IVP.

#### **1.5 Benefits and Applications**

Some of the benefits and applications which can be referred to this research are given as follow.

1. The investigation is expected to be useful for the development of fuzzy numerical analysis in order to get the precise solutions.

2. The proposed method will minimize the computational complexity and achieve high detection performance compared to the other existing methods.
3. The accuracy of the results is better when using the better numerical method.
4. Research groups related to numerical analysis can use the results to improve the usage of the model.
5. Systems with impulse effect have wide applications in various disciplines such as in physics, biotechnology, industrial robotics, pharmacokinetics, population dynamics, ecology, optimal control, and production theory. Therefore, the research is of some significance and worthy of effort.
6. Impulses can make unstable systems stable, so it has important use in many fields which involve artificial intelligence. Rodriguez-Lopez (2008b) has illustrated the possibility to stabilize the solutions of fuzzy differential equations by using impulses.

## **1.6 Preliminaries**

The concept of fuzzy set is defined as a formal theory, which when maturing, becomes more sophisticated and specified and is enlarged by original ideas and concepts as well as by 'embracing' classical mathematical areas such as algebra, graph theory, topology and so on by generalizing (fuzzifying) them (Zimmerman, 1991). As a very powerful modeling language, the concept and theory of fuzzy have the ability to cope with a large fraction of uncertainties in real life situations (Zadeh, 2008). Because of its generality, it can be well adapted to different circumstances and contexts.

Therefore, fuzzy set theory can be used to model a whole variety of situations in which there is some degree of vagueness or uncertainty.

In this section, some of the main concepts and results which are useful for this research are presented.

### 1.6.1 Fuzzy Sets and Fuzzy Logic

The mathematical modeling of fuzzy concepts was first presented by Zadeh (1965). He claimed that meaning in our language could be expressed in a matter of degree. This means that membership in a fuzzy subsets should not be on two choices only for example either 0 or 1, black or white and true or false. The membership in fuzzy subsets is rather on a 0 to 1 scale, that is, the membership should be an element of the interval  $[0, 1]$ . The ordering induced by the membership degrees between elements is very important than the exact values of degrees of membership in a subset. Zadeh's idea was interpreted by Bojadziev and Bojadziev (1998) that fuzzy sets and fuzzy logic can be viewed as a broad conceptual framework enclosing the classical sets and logic.

Two basic notations for fuzzy sets are the letter  $A$  and  $\mu_A$ . For example, let  $A$  be the set of "old women". The notation  $A$  stands for the concept of "old" and  $\mu_A$  showed us the degree of oldness that has been assigned to each member of  $U$ . For a fuzzy set  $A : U \rightarrow [0, 1]$ , the function  $A$  is called the membership function and the value  $A(u)$  is called the degree of membership of  $u$  in the fuzzy set (Klir *et al.*, 1997). At this point,

we no longer have crisp sets but instead we have fuzzy sets. Infinite numbers of memberships are possible because the interval  $[0, 1]$  contains the infinity of numbers (Tsoukalas and Uhrig, 1997). In other words, a membership function maps every element of the universe of discourse  $X$  to the interval  $[0, 1]$ , and the mapping can be written as

$$\mu_A(x) : X \rightarrow [0,1] \quad (1.6.1)$$

According to Tsoukalas and Uhrig (1997), there are two commonly used ways of denoting fuzzy sets. Let  $X$  be a universe of discourse and  $x$  is a particular element of  $X$ , then a fuzzy set  $A$  defined on  $X$  can be written as a collection of ordered pairs

$$A = \{(x, \mu_A(x))\}, \quad x \in X \quad (1.6.2)$$

where each pair  $(x, \mu_A(x))$  is called a singleton. Another way of denoting fuzzy sets indicates a fuzzy set as the union of all  $\mu_A(x)/x$  singletons and can be expressed as

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i \quad (1.6.3)$$

For a continuous universe of discourse, the equation (1.6.3) can be written as

$$A = \int_X \mu_A(x) / x \quad (1.6.4)$$

Another important term in discussing the membership function is the  $\alpha$  - cut. We can associate a collection of crisp sets known as alpha cuts or the level sets of  $A$  to any fuzzy set  $A$ . Let  $A$  be a fuzzy subset of  $U$ , and let  $\alpha \in [0, 1]$ . The  $\alpha$  - cut of  $A$  is simply the set of those  $u \in U$  such that  $A(u) \geq \alpha$  that is,

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\} \quad (1.6.5)$$

$\alpha$  -cuts or the fuzzy level sets offer a method for resolving any fuzzy set in terms of constituent crisp sets and are indispensable in performing arithmetic operations with fuzzy sets that represent various qualities of numerical data. In some monographs, the symbol of  $\alpha$  is represented by  $r$ .

It can be seen that fuzzy sets deal with the type of uncertainty that arises when the boundaries of a class of objects are not sharply defined. Besides that, Nguyen and Walker (2000) stated that ambiguity is another kind of uncertainty. For example, if some parameter in a control system is only known to lie within a given interval, then there is uncertainty about any nominal value chosen from the particular interval for that parameter.

Meanwhile, the applications of fuzzy set models can be found in the areas of economics, operations research, information technologies, data knowledge bases, mathematical social science, and home appliances (Mukaidono, 2001; Klir and Yuan, 2001; Harris, 2004; Ross, 2006).

Fuzzy logic refers to the use of fuzzy sets in the representation and manipulation of imprecise information for the purpose of making decisions or taking actions to certain condition (Mc Neill and Freiberger, 1993). It leads to the invention of the fuzzy expert systems and the fuzzy controls. The fuzzy inference using the if-then rules is widely used in the usual expert systems and controls ( Driankov *et al.*, 1996; Reznik, 1997).

### 1.6.2 Fuzzy Quantities

Fuzzy quantities are represented by fuzzy numbers and fuzzy intervals. Suppose  $\mathfrak{R}$  denote the set of real numbers. The elements of  $F(\mathfrak{R})$  that is the fuzzy subsets of  $R$ , are fuzzy quantities. A fuzzy quantity  $A$  is convex if

- (i) its  $\alpha$  - cuts are convex, that is, if its  $\alpha$  - cuts are intervals.
- (ii)  $A(y) \geq A(x) \wedge A(z)$  whenever  $x \leq y \leq z$ .

A fuzzy quantity is upper semi continuous if its  $\alpha$  - cuts are closed.

Fuzzy numbers deal with applications where an explicit representation of the ambiguity and uncertainty found in numerical data is desirable. Fuzzy sets operations such as union and intersection, and also the  $\alpha$ -cuts are all applicable to fuzzy numbers. Fuzzy numbers have been successfully applied in the expert systems, fuzzy regressions, and fuzzy data analysis methodologies (Kauffman and Gupta, 1991). In other words, a fuzzy number is a fuzzy quantity  $A$  that represents a generalization of a real number,  $r$ .  $A(x)$  should be a measure of how well  $A(x)$  approximates  $r$ , and certainly one reasonable requirement is that  $A(r) = 1$  and that this holds only for  $r$ .

**Definition 1.6.1** (Nguyen and Walker, 2000)

A fuzzy number satisfies the following conditions :

- (i)  $A(x) = 1$  for exactly one value of  $x$ .
- (ii) The support  $\{x : A(x) > 0\}$  of  $A$  is bounded.
- (iii) The  $\alpha$  - cuts of  $A$  are closed intervals.

**Propositions 1.6.1** (Nguyen and Walker, 2000)

The following hold :

- (i) Real numbers are fuzzy numbers.
- (ii) A fuzzy number is a convex fuzzy quantity.
- (iii) A fuzzy number is upper semi continuous.
- (iv) If  $A$  is a fuzzy number with  $A(r) = 1$ , then  $A$  is non-decreasing on  $(-\infty, r]$  and non increasing on  $[r, \infty)$ .

**Definition 1.6.2** (Nguyen and Walker, 2000)

A triangular fuzzy number is a fuzzy quantity fully determined by the triplet  $(a, b, c)$  of crisp numbers with  $a < b < c$ , and its membership function is given by the following formula,

$$\mu(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{x-c}{b-c}, & b < x \leq c \\ 0, & c < x \end{cases} \quad (1.6.6)$$

for some  $a \leq b \leq c$ .

For example if  $a = 2$ ,  $b = 3$  and  $c = 4$ , the picture is given in Figure 1.1.



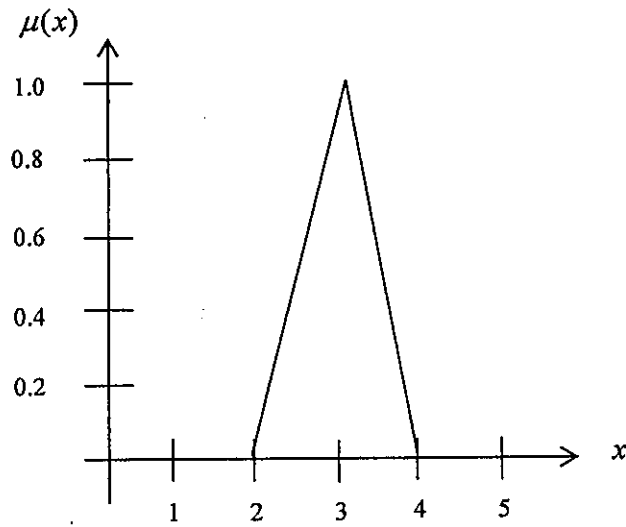


Figure 1.1 Membership function of triangular (2, 3, 4)

**Theorem 1.6.1** (Nguyen and Walker, 2000)

For triangular numbers,

$$(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$$

Using  $((a, b, c) + (d, e, f))_\alpha = (a, b, c)_\alpha + (d, e, f)_\alpha$ , it follows that the support of the sum is the interval  $(a + d, c + f)$  and that 1 is assumed exactly at  $b + e$ . Suppose that  $\alpha > 0$ , the left endpoint of  $\alpha$ -cut of  $(a, b, c)$  is  $u$  and that of  $(d, e, f)$  is  $v$ . Then

$$a \leq u \leq b, \quad d \leq v \leq e \quad \text{and} \quad \alpha = \frac{u - a}{b - a} = \frac{v - d}{e - d}.$$

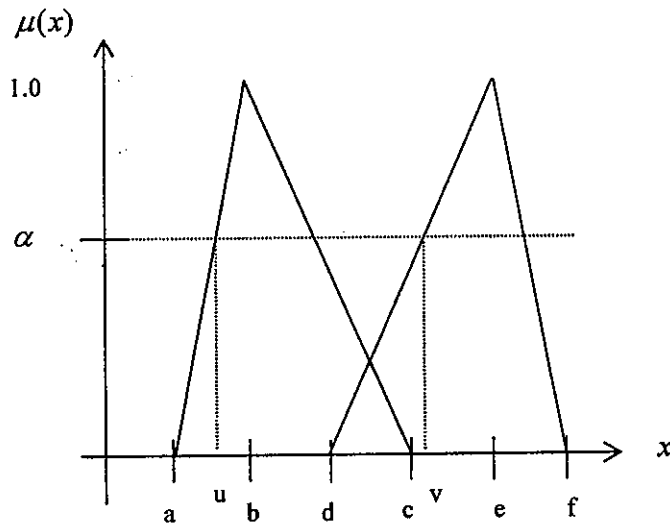


Figure 1.2 The  $\alpha$  - cut

An easy calculation shows that

$$\alpha = \frac{u + v - (a + d)}{b + e - (a + d)}$$

so it means that  $u + v$  is the left endpoint of the  $\alpha$  - cut of  $(a + d, b + e, c + f)$ . However, it is known that the left endpoint of the  $\alpha$  - cut of  $(a, b, c) + (d, e, f) = u + v$ . The same goes for the right endpoint. Thus, adding two triangular numbers is as the same as adding two triples of real numbers following their coordinates.

### 1.6.3 Fuzzy Set Valued Mappings

Let  $T$  be a mapping from  $U$  to a space  $V$ . Let  $F$  be a fuzzy set in  $V$  according to the membership function  $F(v)$ . Then, the inverse mapping  $T^{-1}$  induces a fuzzy set  $G$  in  $U$ . The membership function is defined by  $G(u) = F(v)$ ,  $v \in V$  for all  $u \in U$  which are mapped by  $T$  into  $v$ . More generally, let  $U = U_1 \times \dots \times U_r$ , and  $F_1, \dots, F_r$  be  $r$  fuzzy

sets in  $U_1, \dots, U_r$ , respectively, and let  $f$  maps  $U = U_1 \times \dots \times U_r$  to  $V$  such that  $v = f(u_1, \dots, u_r)$ .

**Theorem 1.6.2** (*Extension Principle of Zadeh*)

Let  $\xi_1, \xi_2, \dots, \xi_n$  be the fuzzy quantities with membership functions  $\mu_1, \mu_2, \dots, \mu_n$ , respectively, and  $f: \mathfrak{R}^n \rightarrow \mathfrak{R}$  is a function. Then, the membership function  $\mu$  of  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is derived from the membership function  $\mu_1, \mu_2, \dots, \mu_n$  by

$$\mu(x) = \sup_{x=f(x_1, x_2, \dots, x_n)} \min_{1 \leq i \leq n} \mu_i(x_i)$$

Therefore, by using the extension principle the fuzzy set  $G$  is induced from the  $r$  fuzzy sets  $F_i$  on  $U$  through  $f$  such that

$$G(v) = \sup_{v=f(u_1, u_2, \dots, u_r)} \min_{u_1, \dots, u_r} (F_1(u_1), \dots, F_r(u_r))$$

The extension principle provides a general rule for the application of fuzzy sets to any mathematical domain (Lakshmikantham and Mohapatra, 2003). The main idea is to replace the concept that a variable has a value with the fuzzy concept that a variable has a degree of membership to each possible value (Dubois and Prade, 1980; Dubois and Prade, 1982a).

The term fuzzy function refers to mappings between sets that involve fuzzy sets which generalizes ordinary functions. A fuzzy function is interpreted according to where the fuzziness appears.

**Definition 1.6.3** (Somasundaram, 2006)

A normed space  $N$  is a vector space or linear space with a norm defined on it. A norm on  $N$  is a function  $\| \cdot \| : N \rightarrow R$  such that for all  $x, y \in N$  and all  $\alpha \in R$  has the properties :

- (i)  $\|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x = 0$
- (ii)  $\|x + y\| \leq \|x\| + \|y\|$
- (iii)  $\|\alpha x\| = |\alpha| \|x\|$

A linear space with a norm  $\| \cdot \|$  defined as above is called a normed linear space.

**Definition 1.6.4** (Somasundaram, 2006)

A sequence  $\{x_n\}$  in a normed linear space,  $N$ , is called a Cauchy sequence in  $N$ , if for every  $\varepsilon > 0$ , there exists a positive integer  $n_0$  such that

$$\|x_m - x_n\| < \varepsilon \text{ for all } m, n \geq n_0$$

Therefore, if  $\{x_n\}$  is a Cauchy sequence in  $N$ , then  $\|x_m - x_n\| \rightarrow 0$  as  $m, n \rightarrow \infty$ . Here, it is noted that if  $N$  is a normed linear space, every convergent sequence is a Cauchy sequence.

**Definition 1.6.5** (Somasundaram, 2006)

A normed linear space,  $N$  is complete if every Cauchy sequence in  $N$  converges to an element of  $N$ . It means there exists  $x \in N$ , such that  $\|x_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Definition 1.6.6** (Somasundaram, 2006)

A complete normed linear space is called a Banach space.

It can be concluded that a Banach space completeness means that  $\|x_m - x_n\| \rightarrow 0$  as  $m, n \rightarrow \infty$ , where  $\{x_n\} \subset N$ , then there exists  $x \in N$  such that  $\|x_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Definition 1.6.7** (Granas and Dugundji, 2003)

A fixed point of a transformation  $F : X \rightarrow X$  is a point  $x \in X$  such that  $F(x) = x$ .

**Theorem 1.6.3** (Granas and Dugundji, 2003)

If  $R$  is a closed subset of a Banach space  $X$  and  $T : R \rightarrow R$  is a contraction on  $R$ , then  $T$  has a unique fixed point  $x$  in  $R$ . This is the Banach contraction theorem.

**Theorem 1.6.4** (Granas and Dugundji, 2003)

If  $B$  is a convex, compact subset of a Banach space  $X$ , where  $f : B \rightarrow B$  is continuous, then  $f$  has a fixed point in  $B$ . This is called as Schauder fixed point theorem.

**Definition 1.6.8** (Butcher, 2008)

A function  $f : C \times D \rightarrow \mathbb{R}^n$ ,  $C \subset \mathbb{R}$ ,  $D \subset \mathbb{R}^n$  is called Lipschitz continuous in  $x$ , if for all  $(t, x), (t, y) \in C \times D$ ,  $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$ ;  $L > 0$  is called the Lipschitz constant.

#### 1.6.4 The space $E^n$

Let consider the following three spaces of nonempty subsets of  $\mathfrak{R}^n$  :

- (i)  $C^n$  : all nonempty closed subsets of  $\mathfrak{R}^n$  ,
- (ii)  $K^n$  : all nonempty compact (closed and bounded) subsets of  $\mathfrak{R}^n$  ,
- (iii)  $K^n_c$  : all nonempty compact convex subsets of  $\mathfrak{R}^n$  .

Let  $U$  be a subset of real  $n$ -dimensional Euclidean space. A fuzzy set  $F$  defined on  $U$  is convex if and only if its  $\alpha$ -level sets are convex. An equivalent definition is for all  $u_1, u_2 \in U$ , and for all  $\lambda \in [0, 1]$ ,  $F(\lambda u_1 + (1 - \lambda)u_2) \geq \min\{F(u_1), F(u_2)\}$ .

Let the family of all non-empty compact convex subsets of  $\mathfrak{R}^n$  is denoted by  $K^n_c$ .

For  $\alpha, \beta \in \mathfrak{R}$  and  $A, B \in K^n_c$ ,

- (i)  $\alpha(A + B) = \alpha A + \alpha B$
- (ii)  $\alpha(\beta A) = (\alpha\beta)A$
- (iii)  $1 \cdot A = A$
- (iv)  $(\alpha + \beta)A = \alpha A + \beta A, \quad \alpha, \beta \geq 0$

A subset  $\mathcal{A}$  of  $K^n_c$  is uniformly bounded if there exists a finite constant  $c(\mathcal{A})$  such that

$$\|A\| \leq c(\mathcal{A}) \text{ for all } A \in \mathcal{A} \text{ where } \|A\| = \sup\{\|a\| : a \in A\}.$$

**Proposition 1.6.2** (Dubois and Prade, 1980).

A non-empty subset  $A$  of the metric space  $(K^n, d_H)$  is compact if and only if it is closed and uniformly bounded.

### 1.6.5 The Hausdorff Distance

Let  $x$  be a point in  $\mathfrak{R}^n$  and  $A$  a nonempty subset of  $\mathfrak{R}^n$ . The distance  $d(x, A)$  from  $x$  to  $A$  is defined by

$$d(x, A) = \inf \{ \|x - a\| : a \in A \} \quad (1.6.7)$$

Now, let  $A$  and  $B$  be the nonempty subsets of  $\mathfrak{R}^n$ . The Hausdorff separation of  $B$  from  $A$  is defined by

$$d_H(B, A) = \sup \{ d(b, A) : b \in B \} \quad (1.6.8)$$

Also, the triangle inequality

$$d_H(B, A) \leq d_H(B, C) + d_H(C, A)$$

holds for all nonempty subsets  $A, B$  and  $C$  of  $\mathfrak{R}^n$ . However, generally

$$d_H(A, B) \neq d_H(B, A)$$

Therefore, we define the Hausdorff distance between nonempty subsets  $A$  and  $B$  of  $\mathfrak{R}^n$  by

$$d_H(A, B) = \max \{ d_H(A, B), d_H(B, A) \} \quad (1.6.9)$$

Consequently, these are true for any nonempty subsets of  $A, B$  and  $C$  of  $\mathfrak{R}^n$ .

- (i)  $d_H(A, B) \geq 0$
- (ii)  $d_H(A, B) = 0$  if and only if  $A = B$

$$(iii) \quad d_H(A, B) \leq d_H(A, C) + d_H(C, B)$$

Referring to the nonempty subsets of  $\mathfrak{R}^n$ , we see that the Hausdorff distance is a metric. This means that  $(C^n, d_H)$  is a metric space.

**Proposition 1.6.3** (Diamond and Kloeden, 1994)

$(C^n, d_H)$  is a complete separable metric space in which  $K^n$  and  $K^n_C$  are closed subsets. Hence,  $(K^n, d_H)$  and  $(K^n_C, d_H)$  are also complete separable metric spaces.

Consider the fuzzy set  $F$  that maps  $\mathfrak{R}^n$  onto the set  $u = [0, 1]$ , in which the fuzzy convex  $F$  with level sets are bounded on  $\mathfrak{R}^n$ . Therefore, the Hausdorff distance between level sets is a metric on  $E^n$  and is used to define the supremum metric  $D$  on  $E^n$ .

$$D(F, G) = \sup\{d_H(F_\alpha, G_\alpha) \text{ for } \alpha \in U \text{ where } F, G \in E^n\}.$$

In that case, the space  $E^n$  is complete.

### 1.6.6 Hukuhara Difference

A useful definition of the differentiation of a fuzzy set valued mapping of a single variable is based on the quotient of Hukuhara difference (Diamond and Kloeden, 1994). A mapping  $F : T \rightarrow E^n$  is Hukuhara differentiable at  $t_0 \in T \subseteq \mathfrak{R}^1$  for some  $h_0 > 0$  the Hukuhara differences

$$F(t_0 + \Delta t) -_h F(t_0), \quad F(t_0) -_h F(t_0 - \Delta t) \quad (1.6.10)$$



exist in  $E^n$  for all  $0 < \Delta t < h_0$  and if there exists an  $F'(t_0) \in E^n$  such that

$$\lim_{\Delta t \rightarrow 0^+} \left[ \frac{d_\infty(F(t_0 + \Delta t) -_h F(t_0))}{\Delta t}, F'(t_0) \right] = 0 \quad (1.6.11)$$

and

$$\lim_{\Delta t \rightarrow 0^+} \left[ \frac{d_\infty(F(t_0) -_h (F(t_0 + \Delta t)))}{\Delta t}, F'(t_0) \right] = 0 \quad (1.6.12)$$

In this case,  $F'(t_0)$  is called the Hukuhara derivatives of  $F$  at  $t_0$ .

Note that,  $x -_h y = z \in E^n$  is defined on level sets, that is as  $[x -_h y]^\alpha = [x]^\alpha - [y]^\alpha = z^\alpha$  for all  $\alpha \in I$ . Due to the definition of the metric  $d_\infty$ , all the level set mappings  $[F(\cdot)]^\alpha$  are Hukuhara differentiable at  $t_0$  with Hukuhara derivatives  $[F'(t_0)]^\alpha$  for each  $\alpha \in I$  when  $F : T \rightarrow E^n$  is Hukuhara differentiable at  $t_0$  with Hukuhara derivatives  $F'(t_0)$ .

**Proposition 1.6.4** (Diamond and Kloeden, 1994)

If  $F : T \rightarrow E^n$  is Hukuhara differentiable at  $t_0 \in T \subseteq \mathfrak{R}$ , then

- (i) its derivative  $F'(t_0)$  is unique,
- (ii) it is continuous at  $t_0$ .

**Proposition 1.6.5** (Diamond and Kloeden, 1994)

If  $F, G: T \rightarrow E^n$  are Hukuhara differentiable at  $t_0 \in T \subseteq \mathfrak{R}$ , then  $F + G$ , and  $cF$ , for all  $c \in \mathfrak{R}$ , are Hukuhara differentiable at  $t_0$  and

$$(F + G)'(t_0) = F'(t_0) + G'(t_0), \quad (cF)'(t_0) = cF'(t_0)$$

**Theorem 1.6.5** (Kaleva, 1987)

If  $u \in E^n$  then,

- (1)  $[u]^\alpha \in P_{\mathcal{K}}(\mathfrak{R}^n)$  for all  $0 \leq \alpha \leq 1$ ,
- (2)  $[u]^{\alpha_2} \subset [u]^{\alpha_1}$  for  $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ ,
- (3) if  $(\alpha_k)$  is a nondecreasing sequence converging to  $\alpha > 0$  then  $[u]^\alpha = \bigcap_{k \geq 1} [u]^{\alpha_k}$ .

Conversely, if  $\{A^\alpha \mid 0 \leq \alpha \leq 1\}$  is a family of subsets of  $\mathfrak{R}^n$  satisfying (1) – (3) then

there exists a  $u \in E^n$  such that  $[u]^\alpha = A^\alpha$  for  $0 < \alpha \leq 1$  and  $[u]^0 = \overline{\bigcap_{0 < \alpha \leq 1} A^\alpha} \subset A^0$ .

### 1.6.7 Fuzzy Integration

The integration of a fuzzy set valued mapping of a single variable is defined level set wise in terms of the Aumann integrals of its level sets mappings (Lakshmikantham and Mohapatra, 2003). The mappings are defined on the unit interval  $[0, 1]$ , where  $F: [0, 1] \rightarrow E^n$ . A mapping  $F: [0, 1] \rightarrow E^n$  is integrably bounded if there exists an integrable function  $h: [0, 1] \rightarrow \mathfrak{R}$  such that

$$\|F(t)\| \leq h(t) \text{ for all } t \in [0, 1]. \quad (1.6.13)$$

Notice that  $\|F(t)\| = \|[F(t)^0]\| \geq \|[F(t)^\alpha]\|$  for all  $t \in [0, 1]$ . Therefore, each level set valued mapping  $[F(\cdot)]^\alpha : [0, 1] \rightarrow K^n_c$  is integrably bounded.

It means for each  $\alpha \in I$ ,

$$\left[ \int_0^1 F(t) dt \right]^\alpha = \int_0^1 [F(t)]^\alpha dt \quad (1.6.14)$$

**Theorem 1.6.6** (Diamond and Kloeden, 1994)

If  $F : [0, 1] \rightarrow E^n$  is integrably bounded, then it is integrable over  $[a, b]$  for any  $[a, b] \subseteq [0, 1]$  with  $a < b$ .

### 1.6.8 Simpson's 1/3 Rule (Nakamura, 1993)

Simpson's rule is based on quadratic (second-order) polynomials. A domain  $[a, b]$  which is divided into three points  $a = x_0$  to  $b = x_2$ , it is written as

$$I = \int_a^b f(x) dx = \frac{h}{3} [f_0 + 4f_1 + f_2] + E \quad (2.10)$$

where  $h = \frac{(b-a)}{2}$ ,  $f_i = f(x_i) = f(a+ih)$  and  $E$  represent the error term. The error

term is given by  $E = \mathfrak{S} - \frac{1}{90} h^5 f^{(4)}(\bar{x})$  where  $\bar{x} = \frac{a+b}{2}$  and  $\mathfrak{S} = [a, b]$ .

## REFERENCES

- Abbasbándy, S. and Allahviranloo, T. 2002. Numerical Solution of Fuzzy Differential Equations by Taylor Method. *Computational Methods in Applied Mathematics* 2 (2) : 113-124.
- Abbasbándy, S. and Allahviranloo, T. 2004. Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method. *Nonlinear Studies* 11(1) : 117-129.
- Abbasbándy, S., Allahviranloo, T., Lopez-Pouso, O., and Nieto, J. J. 2004. Numerical methods for Fuzzy Differential Inclusions. *Computers and Mathematics with Applications* 48 : 1633-1641.
- Abbasbandy, S., Nieto, J.J. and Alavi, M. 2005. Tuning of reachable set in one dimensional fuzzy differential inclusions. *Chaos, Solitons and Fractals* 26 : 1337-1341.
- Ahmad, M. Z. and De Baets, B.D. 2009. *A Predator-Prey Model with Fuzzy Initial Populations*. Proceeding of IFSA-EUSFLAT 2009 : 1311-1314.
- Ahmad, M.Z and Hasan, M.K. 2010. A new approach to incorporate uncertainty into Euler's method. *Applied Mathematical Sciences* 4(51): 2509-2520.
- Akhmetov, M. U., and Zafer, A. 2000. Successive approximation method for quasilinear impulsive differential equations with control. *Applied Mathematics Letters* 13(5) : 99 – 105.
- Akhmet, M.U. and Turan, M. 2006. The differential equations on time scales through impulsive differential equations. *Nonlinear Analysis* 65 : 2043-2060.
- Akhmet, M.U. 2009. The complex dynamics of the cardiovascular system. *Nonlinear Analysis: Theory, Methods & Applications* 71(12) : 1922 – 1931.
- Akhmet, M.U. and Bekmukhambetova, G.A. 2010. A prototype compartmental model of blood pressure distribution. *Nonlinear Analysis : Real World Applications* 11(3) : 1294-1257.
- Allahviranloo, T. and Salahshour, S. 2011. Euler method for solving hybrid fuzzy differential equation. *Soft Computing* 15 : 1247 – 1253.
- Allahviranloo, T., Kiani, N.A. and Motamedi, N. 2009a. Solving fuzzy differential equations by differential transformation method. *Information Sciences* 179 : 956-966.
- Allahviranloo, T., Kiani, N.A. and Barkhordari, M. 2009b. Toward the existence and uniqueness of solutions of second-order fuzzy differential equations. *Information Sciences* 179 : 1207-1215.

- Allahviranloo, T., Ahmady, E. and Ahmady, N. 2008a.  $N$  th-order fuzzy linear differential equations. *Information Sciences* 178 : 1309 – 1324.
- Allahviranloo, T., Ahmady, N. and Ahmady, E. 2008b. Erratum to “Numerical solution of fuzzy differential equations by predictor-corrector method”. *Information Sciences* 178 : 1780-1782.
- Allahviranloo, T., Ahmady, N. and Ahmady, E. 2007. Numerical solution of fuzzy differential equations by predictor-corrector method. *Information Sciences* 177 : 1633-1647.
- Allahviranloo, T., Ahmady, N. and Ahmady, E. 2006. Two Step Method for Fuzzy Differential Equations. *International Mathematical Forum* 1(17) : 823 – 832.
- Angelova, J. and Dishliev, A. 2000. Optimization problem for one impulsive models from population dynamics. *Nonlinear Analysis : Theory, Methods & Applications* 39(4) : 483 – 497.
- Bainov, D.D. and Dimitrova, M.B. 1999. Sufficient Conditions for the Oscillation of Bounded Solutions of a Class of Impulsive Differential Equations of Second Order with a Constant Delay. *Georgian Mathematical Journal* 6(2) : 99-106.
- Bainov, D.D. and Stamova, I.M. 1995. Lipschitz Stability of Linear Impulsive Differential-difference Equations. *Note di Matematica* 15(2) : 137-142.
- Bainov, D. D., Kostadinov, S. I., van Minh, N., and Zabreiko, P. P. 1994. A topological classification of differential equations with impulse effect. *Tamkang Journal of Mathematics* 25 : 15-27.
- Bainov, D. D., and Simenov, P. S. 1989. *Systems with Impulse Effect Stability Theory and Applications*. Ellis Horwood Limited, Chichester.
- Bainov, D. D., and Kulev, G. 1988. Application of Lyapunov’s functions to the investigation of global stability of solutions of systems with impulses. *Applied Analysis* 26(1) : 255 – 270.
- Bede, B. 2008. Note on “Numerical solutions of fuzzy differential equations by predictor-corrector method”. *Information Sciences* 178 : 1917-1922.
- Bede, B., Rudas, I., and Bencsik, A. 2007. First order linear fuzzy differential equations under generalized differentiability. *Information Sciences* 177 : 1648-1662.
- Bede, B. 2006. A note on “two-point boundary value problems associated with non-linear fuzzy differential equations”. *Fuzzy Sets and Systems* 157 : 986-989.
- Bede, B., and Gal, S.G. 2005. Generalized of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and System* 151 : 581-599.