

Improved Richardson's Extrapolation Spreadsheet Calculator for Numerical Differentiation

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Abstract. In this paper, we have improved the limitations of our previous Richardson's extrapolation spreadsheet calculator for computing differentiations numerically. These limitations are the value of $D(0,0)$ keyed in by users using 3-point central difference formula, and the fact that the previous spreadsheet calculator can only calculate the approximate definite differentiation up to level 4×4 . If the function to be differentiated is complicated, calculating $D(0,0)$ using 3-point central difference formula can be tedious as parentheses should be put in a proper order when writing the calculation command. Otherwise, the calculation command may lead to a wrong answer. In this improved Richardson's extrapolation spreadsheet calculator, we redesigned the Richardson's extrapolation spreadsheet calculator, where users are only required to give the value of x , the function to be differentiated $f(x)$, and the step size h value without writing the command to obtain $D(0,0)$. Consequently, the calculations will be done automatically to approximate the definite differentiation up to level 10×10 .

Keywords: spreadsheet calculator, Richardson's extrapolation, numerical differentiation.

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INTRODUCTION

In real-world problems, applications of the rate of change have been well-defined. For examples, flow rate in a tank, productivity of an industrial process and marginal revenue of a manufacturing company. In fact, calculating the rate of change requires the knowledge of differentiation, which is an interesting topic in numerical analysis. Since the physical problems are complex, the exact differentiation is impossible to obtain. In such situations, an approximation of the definite differentiation is important to provide a practical solution. Richardson's extrapolation [1], [2], which is one of the powerful approaches in numerical differentiation, improves the rate of convergence of a sequence in numerical differentiation.

Implementation of Richardson's extrapolation via mathematical software or any computational tool leads to a desired definite differentiation. According to the works of [3-11], Excel spreadsheet shows the ability of calculating for a wide range of numerical methods. In addition, the application of Excel spreadsheet in implementing Richardson's extrapolation for numerical differentiation has been developed in [12] recently. However, previous Richardson's extrapolation Excel spreadsheet calculator is limited to level four in calculating the numerical differentiation and the initial value $D(0, 0)$ of approximation to the derivative of a function is required to be keyed in by users using 3-point central difference formula.

RICHARDSON'S EXTRAPOLATION

There are two ways to improve derivative estimates when employing finite divided differences; decreasing the step size or using a higher-order formula that employs more points. A third approach, based on Richardson's extrapolation which uses two derivatives, estimates a more accurate approximation. [13]

For a given function of f , approximations D , to the derivative of f at a specified value of x , can be computed for a chosen value of h , using Richardson's extrapolation method as follows: [14]

$$D(J, K) = \frac{4^K D(J, K-1) - D(J-1, K-1)}{4^K - 1} + O(h^{2K+2}), \quad J=1, 2, \dots, \quad K=1, 2, \dots \quad (1)$$

$$D(J, 0) = \frac{f(x + h_J) - f(x - h_J)}{2h_J}, J = 0, 1, 2, \dots \quad (2)$$

$$h_0 = h, h_1 = \frac{h}{2}, h_2 = \frac{h}{4}, \dots, h_J = \frac{h}{2^J} \quad (3)$$

The iteration process (1) is repeated until $|D(J, J) - D(J, J-1)| < \varepsilon$ or $|D(J, J) - D(J-1, J-1)| < \varepsilon$ for a specified value of ε . The value of $D(J, J)$ then approximates $f'(x)$ at this level.

Richardson's extrapolation table for K from 1 to 4 is shown in Table (1).

Table (1). Richardson's Extrapolation table for level K values from 1 to 4.

J	h_j	K				
		0	1	2	3	4
		$D(J, 0)$	$D(J, 1)$	$D(J, 2)$	$D(J, 3)$	$D(J, 4)$
0	h_0	$D(0, 0)$				
1	h_1	$D(1, 0)$	$D(1, 1)$			
2	h_2	$D(2, 0)$	$D(2, 1)$	$D(2, 2)$		
3	h_3	$D(3, 0)$	$D(3, 1)$	$D(3, 2)$	$D(3, 3)$	
4	h_4	$D(4, 0)$	$D(4, 1)$	$D(4, 2)$	$D(4, 3)$	$D(4, 4)$

NUMERICAL EXAMPLE

In this section, a numerical example to be solved by Richardson's extrapolation is provided.

Question

The velocity of a falling parachutist is given by $v(t) = \frac{gm}{c}(1 - e^{-\frac{c}{m}t})$,

where $g = 9.81 \text{ m/s}^2$, $m = 80 \text{ kg}$ and $c = 10 \text{ kg/s}$. At time $t = 0 \text{ s}$, the initial velocity is 0 m/s at the origin.

By using Richardson's extrapolation,

- find the acceleration of the parachutist at time $t = 10 \text{ s}$ with $h = 0.5$.
- find the absolute error if the exact acceleration is

$$a(t) = ge^{-\frac{c}{m}t}.$$

Solution

Given $g = 9.81 \text{ m/s}^2$, $m = 80 \text{ kg}$ and $c = 10 \text{ kg/s}$. Let $f(t)$ be the velocity function, $v(t)$, then

$$f(t) = v(t) = \frac{gm}{c}(1 - e^{-\frac{c}{m}t}) = 9.81(8) \left(1 - e^{-\frac{t}{8}}\right).$$

Thus acceleration function is given by $a(t) = f'(t) = v'(t)$ can be approximated by Richardson's extrapolation at $t = 10$ with $h = 0.5$.

The numerical solution of this example is shown in Figure 1 in Section 4, where $a(10)$ is approximately 2.810612056.

- (b) The absolute error is shown in cell G4 in Figure 1

THE IMPROVED RICHARDSON'S EXTRAPOLATION SPREADSHEET CALCULATOR

Figure 1 illustrates the improved Richardson's extrapolation spreadsheet calculator. The spreadsheet calculator is able to compute the Richardson's extrapolation value up to level 10. Users need to enter the function to be differentiated into cell C4, h into cell D4 and a chosen point of x into cell C3. Users can select the desired accuracy of one decimal place, two decimal places, up to nine decimal places calculation from the drop down menu in cell L3. To get the changes on the Richardson's extrapolation table up to the desired accuracy, users need to click on the **APPLY** button and the value in the Richardson's extrapolation table will be calculated automatically. The **APPLY** button is associated with the **MACRO** option in Excel spreadsheet recording the action of copying cell C4 to calculate $D(J, 0)$. The final answer is shown in cell E4. To calculate the absolute error, users need to enter the exact differentiated function into cell F4. The absolute error will then be calculated automatically in cell G4.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N		
1	Improved Richardson's Spreadsheet Calculator															
2																
3			x	10	h	Answer	Exact	Error			Accuracy	9D		APPLY		
4	Input:		f(x)	55.9551	0.5	2.810612056	2.810612057	1.2E-09								
5																
6																
7																
8			J	h		0	1	2	3	4	5	6	7	8	9	10
9			0	0.5		2.81244224										
10			1	0.25		2.811069536	2.810611968									
11			2	0.125		2.810726423	2.810612052	2.810612058								
12			3	0.0625		2.810640648	2.810612056	2.810612056	2.810612056							
13																
14																
15																
16																
17																
18																
19																

Figure 1: Solution of improved Richardson's extrapolation spreadsheet calculator

We typed in $=IF(D10<0,ROUND((4^E$7*D10-D9)/(4^E$7-1),SQ$4),0)$ in cell E10 to calculate $D(1,1)$, where cell Q4 corresponds to the number of decimal places chosen by user. The rest of the formulas in cells E11:N19 can be copied from cell E10.

The advantage of this improved Richardson's extrapolation spreadsheet calculator is that users only need to enter the function $f(x)$ to be differentiated instead of entering the formula of $D(0,0)$ using 3-point central difference as $\frac{f(x+h)-f(x-h)}{2h}$ in the previous Richardson's extrapolation spreadsheet calculator. Hence if the function $f(x)$ is complicated, many parentheses need to be put in the correct order in order to calculate $D(0,0)$, or it will lead to calculation errors. Besides, the improved spreadsheet calculator can calculate till level 10 if compared to the previous one, which can only calculate till level 4.

CONCLUSION

An improved Richardson's extrapolation spreadsheet calculator has been developed to approximate the derivative of the given function at a chosen point with a step size h . The Excel commands that calculate Richardson's extrapolation values are given in the previous section and the layout of the spreadsheet calculator is shown in Figure 1. The spreadsheet calculator is very user-friendly. It provides an alternative tool for approximating the numerical differentiation by Richardson's extrapolation. It can be used as a marking scheme for educators and students who need full solutions. Moreover, it reduces calculation time and is hoped to increase students' learning ability.

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