A maximum principle for some nonlinear cooperative elliptic PDE systems with mixed boundary conditions

János Karátson*

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Abstract

One of the classical maximum principles state that any nonnegative solution of a proper elliptic PDE attains its maximum on the boundary of a bounded domain. We suitably extend this principle to nonlinear cooperative elliptic systems with diagonally dominant coupling and with mixed boundary conditions. One of the consequences is a preservation of nonpositivity, i.e. if the coordinate functions or their fluxes are nonpositive on the Dirichlet or Neumann boundaries, respectively, then they are all nonpositive on the whole domain as well. Such a result essentially expresses that the studied PDE system is a qualitatively reliable model of the underlying real phenomena, such as proper reaction-diffusion systems in chemistry.

1 Introduction

The classical maximum principle, stating that the solution attains its maximum on the boundary under proper assumptions, is a widely studied property of elliptic boundary value problems together with its various generalizations, and has abundant literature, see e.g. [10, 18, 17, 19, 21, 22, 23, 24, 29] and the references therein. Maximum principles often essentially express that the studied equation is a qualitatively reliable model of the underlying real phenomenon, e.g. solutions that describe physical quantites like concentration etc. are indeed nonnegative. Our starting point is the principle stating that any nonnegative solution attains its maximum on the boundary of a bounded domain, i.e.

$$\max_{\overline{\Omega}} u \le \max\{0, \max_{\partial\Omega} u\} \tag{1.1}$$

whenever $Lu \le 0$ holds for a proper elliptic operator L that includes lower order terms (see e.g. [22]). This property produces in particular the preservation of nonpositivity, i.e. if u is nonpositive on the boundary then it is nonpositive on the whole domain as well. By reversing signs in the conditions, we analogously obtain a minimum principle and in particular the preservation of nonnegativity. Property (1.1) still concerns scalar equations.

*Department of Applied Analysis & MTA-ELTE NumNet Research Group, ELTE University; Department of Analysis, Technical University; Budapest, Hungary; karatson@cs.elte.hu. Supported by the Hungarian Scientific Research Fund OTKA, No. 112157.

The extension of the maximum principle from elliptic equations to systems has attracted much interest, and has been achieved in different forms (coordinatewise or for the modulus), but under strong restrictions only. The main class of problems where a maximum principle is characteristically valid is that of cooperative systems: as shown in [17], such systems essentially behave just as scalar equations. Important results of this type are found e.g. in [9, 17, 19, 21, 26] and some extensions to non-cooperative systems are also known, see [5] and references therein. Just as for scalar equations, the maximum principle is closely related to the preservation of nonnegativity, see e.g. [5, 19, 30]. In addition to cooperativity, one usually also assumes some kind of diagonal dominance of V. Moreover, as proved in [17], a maximum principle for a cooperative system holds if and only if the operator behaves like an M-matrix, i.e. preserves positivity of a suitable function, which is essentially a transformed variant of the diagonal dominance property in the cooperative case. This is why we also restrict our attention to diagonally dominant cooperative systems.

Whereas the above results cover general linear systems, the study of nonlinear systems has so far usually involved problems of more special form. Such results are often related to the p-Laplacian operator, to certain semilinear systems or involving the modulus of the solution vector, see e.g. [3, 8, 15, 16, 20, 25, 27, 29]. Alexandrov-Bakelman type estimates have been given in a similar context in [4, 6]. The most straightforward and very general result on nonlinear systems has been presented in [2], which allows fully nonlinear (i.e. non-divergence form of the) equations and viscosity solutions, moreover, they proved Alexandrov–Bakelman–Pucci and Harnack type estimates. However, these results only include Dirichlet boundary conditions. Thereby the maximum principle (1.1) is extended in a natural way as follows:

$$\max_{k=1,\dots,s} \max_{\overline{\Omega}} u_k \le \max_{k=1,\dots,s} \max\{0, \max_{\partial\Omega} u_k\}, \tag{1.2}$$

where $u = (u_1, \dots, u_s)$ is the solution vector. These bounds, i.e. either the scalar property (1.1) or the vector case (1.2), however, do not give a helpful information about the solution when mixed boundary conditions are imposed, because the solution is not known on the whole boundary $\partial\Omega$. To our knowledge, it was first clarified in the author's paper [13] what to expect instead of (1.1) to get a known bound on u. (In this paper a scalar maximum principle was also proved for a class of nonlinear equations, together with a discrete maximum principle. See also [7] for a later generalization in the case of linear problems.) Namely, for mixed boundary conditions one can simply replace $\partial\Omega$ with the Dirichlet boundary Γ_D in (1.1), provided that $Bu \le 0$ holds for the boundary operator B. Thus for systems one analogously expects

$$\max_{k=1,\dots,s} \max_{\overline{\Omega}} u_k \le \max_{k=1,\dots,s} \max\{0, \max_{\Gamma_D} u_k\} \tag{1.3}$$

under proper conditions. Similarly to the scalar case, this property produces in particular the preservation of nonpositivity, i.e. if each u_k is nonpositive on the boundary then they are all nonpositive on the whole domain as well. Further, reversing signs in the conditions yields again a minimum principle and in particular the preservation of nonnegativity.

The goal of this paper is to verify (1.3) for some classes of nonlinear cooperative elliptic systems in divergence form and with diagonally dominant weak coupling. The novelty of our results, as shown by the above explanations, is the inclusion of the mixed boundary conditions, that is, for pure Dirichlet boundary conditions the results of [2] are more general than ours. In turn, our proofs are considerably shorter and elementary. The considered problems are formulated and the maximum principles are proved in Section 2, and some applications are shown in Section 3.

2 A maximum principle for nonlinear cooperative elliptic systems

Now we consider three types of systems, in which the lower order coupling terms are cooperative and form a weakly diagonally dominant system. First we study systems with nonlinear coefficients. Then systems with more general zeroth order terms are reduced to this one, allowing sublinear and superlinear growth of the zeroth order terms, respectively.

2.1 Systems with nonlinear coefficients

2.1.1 Formulation of the problem

Let us consider nonlinear elliptic systems of the form

$$\left.\begin{aligned} -\mathrm{div}\left(b_k(x,u,\nabla u)\,\nabla u_k\right) + \sum_{l=1}^{s} V_{kl}(x,u,\nabla u)\,u_l \;&=\; f_k(x) \quad \text{a.e. in } \Omega,\\ b_k(x,u,\nabla u)\tfrac{\partial u_k}{\partial\nu} \;&=\; \gamma_k(x) \quad \text{a.e. on } \Gamma_N,\\ u_k \;&=\; g_k(x) \quad \text{a.e. on } \Gamma_D \end{aligned}\right\} \qquad (k=1,\dots,s) \tag{2.1}$$

with unknown function $u = (u_1, \dots, u_s)^T$, under the following assumptions. Here ∇u denotes the $s \times d$ tensor with rows ∇u_k ($k = 1, \dots, s$), further, 'a.e.' means Lebesgue almost everywhere and inequalities for functions are understood a.e. pointwise for all possible arguments.

Assumptions 2.1.

(i) $\Omega \subset \mathbf{R}^d$ is a bounded piecewise C^1 domain; Γ_D, Γ_N are disjoint open measurable subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \neq \emptyset$.

(ii) (Smoothness and boundedness.) For all $k, l = 1, \dots, s$ we have $b_k \in L^\infty(\Omega \times \mathbf{R}^s \times \mathbf{R}^{s\times d})$ and $V_{kl} \in L^\infty(\Omega \times \mathbf{R}^s \times \mathbf{R}^{s\times d})$.

(iii) (Ellipticity.) There exists $m > 0$ such that $b_k \geq m$ holds for all $k = 1, \dots, s$.

(iv) (Cooperativity.) We have

$$V_{kl} \leq 0 \qquad (k, l = 1, \dots, s,\; k \neq l). \tag{2.2}$$

(v) (Weak diagonal dominance.) We have

$$\sum_{l=1}^{s} V_{kl} \geq 0 \quad \text{and} \quad \sum_{l=1}^{s} V_{lk} \geq 0 \qquad (k = 1, \ldots, s). \tag{2.3}$$

(vi) For all $k = 1, \ldots, s$ we have $f_k \in L^2(\Omega)$, $\gamma_k \in L^2(\Gamma_N)$, $g_k = g^*_{k|\Gamma_D}$ with $g^*_k \in H^1(\Omega)$.

Remark 2.1 Assumptions (2.2)–(2.3) imply $V_{kk} \geq 0 \quad (k = 1, \ldots, s)$.

Let us define the Sobolev space $H^1_D(\Omega) := \{z \in H^1(\Omega) : z_{|\Gamma_D} = 0\}$. The weak formulation of problem (2.1) then reads as follows: find $u \in H^1(\Omega)^s$ such that

$$\langle A(u), v\rangle = \langle \psi, v\rangle \qquad (\forall v \in H^1_D(\Omega)^s) \tag{2.4}$$

$$\text{and} \quad u - g^* \in H^1_D(\Omega)^s, \qquad \text{where} \tag{2.5}$$

$$\langle A(u), v\rangle = \int_\Omega \Big(\sum_{k=1}^{s} b_k(x, u, \nabla u)\, \nabla u_k \cdot \nabla v_k + \sum_{k,l=1}^{s} V_{kl}(x, u, \nabla u)\, u_l\, v_k \Big) \tag{2.6}$$

for given $u = (u_1, \ldots, u_s) \in H^1(\Omega)^s$ and $v = (v_1, \ldots, v_s) \in H^1_D(\Omega)^s$, further,

$$\langle \psi, v\rangle = \int_\Omega \sum_{k=1}^{s} f_k v_k + \int_{\Gamma_N} \sum_{k=1}^{s} \gamma_k v_k \tag{2.7}$$

for given $v = (v_1, \ldots, v_s) \in H^1_D(\Omega)^s$, and $g^* := (g^*_1, \ldots, g^*_s)$.

2.1.2 The maximum principle

Our goal is to establish property (1.3). Note that we can set $u_{k\,|\Gamma_D} = g_{k\,|\Gamma_D}$ on its r.h.s., in order to better express that we thus obtain a known bound for the solution.

Theorem 2.1 *Let Assumptions 2.1 hold and $u \in H^1(\Omega)^s$ be a weak solution of system* (2.1)*, such that $u \in C(\overline{\Omega})^s$. If*

$$f_k \leq 0 \quad \text{on } \Omega \quad \text{and} \quad \gamma_k \leq 0 \quad \text{on } \Gamma_N \qquad (k = 1, \ldots, s)\ ,$$

then

$$\max_{k=1,\ldots,s} \max_{\overline{\Omega}} u_k \leq \max_{k=1,\ldots,s} \max\{0, \max_{\Gamma_D} g_k\}. \tag{2.8}$$

Proof. Let $M := \max_{k=1,\ldots,s} \max\{0, \max_{\Gamma_D} g_k\}$, and introduce the functions

$$v_k^+ := \max\{u_k - M,\, 0\} \qquad (k = 1, \ldots, s).$$

Then $u_k \in H^1(\Omega)$ implies $v_k^+ \in H^1(\Omega)$ (see e.g. [11]), and the definition of M implies $v^+_{k\,|\Gamma_D} = 0$, hence $v^+ \in H^1_D(\Omega)^s$ and we can set $v := v^+$ into (2.4).

Consider first the left-hand side (2.6) of (2.4):

$$\langle A(u), v^+\rangle = \int_\Omega \sum_{k=1}^{s} b_k(x,u,\nabla u)\,\nabla u_k\cdot\nabla v_k^+ + \int_\Omega \sum_{k,l=1}^{s} V_{kl}(x,u,\nabla u)\,u_l\,v_k^+ \,.$$

Its first term is nonnegative, since all $b_k\geq 0$, and v_k^+ equals either 0 or $u_k - M$, hence $\nabla u_k\cdot\nabla v_k^+$ equals either 0 or $|\nabla u_k|^2\geq 0$.

The second term of (2.6) is also nonnegative. Namely, let us introduce the further notations

$$\widehat{V}_{kl}(x) := V_{kl}(x,u(x),\nabla u(x)), \qquad v_k^- := \max\{M-u_k,\, 0\}$$

($x\in\Omega,\ k,l=1,\dots,s$). Then, for all $l=1,\dots,s,$ we have $u_l = v_l^+ - v_l^- + M$ and hence the second integrand pointwise satisfies

$$\sum_{k,l=1}^{s} \widehat{V}_{kl}\,u_l\,v_k^+ = \sum_{k,l=1}^{s} \widehat{V}_{kl}\,v_l^+\,v_k^+ - \sum_{k=1}^{s} \widehat{V}_{kk}\,v_k^-\,v_k^+ + \sum_{k\neq l=1}^{s} (-\widehat{V}_{kl})\,v_l^-\,v_k^+ + M\sum_{k=1}^{s}\Big(\sum_{l=1}^{s}\widehat{V}_{kl}\Big)\,v_k^+ \,.$$

Here the first term on the r.h.s. equals the quadratic form $\widehat{V}v^+\cdot v^+$. The cooperativity and the weak diagonal dominance of V w.r.t. both rows and columns imply that $\widehat{V}$ is positive semidefinite, hence $\widehat{V}v^+\cdot v^+\geq 0$. The second term equals zero, since either v_k^- or v_k^+ vanishes for all k. The third term is nonnegative, since $\widehat{V}_{kl}\leq 0$ from (2.2) and v_l^-, $v_k^+\geq 0$ by definition. The last term is also nonnegative, since $\sum_{l=1}^{s}\widehat{V}_{kl}\geq 0$ from (2.3).

Altogether, we obtain $\langle A(u), v^+\rangle\geq 0$. On the other hand, the assumptions $f_k\leq 0$ and $\gamma_k\leq 0$ imply that the right-hand side (2.7) of (2.4) satisfies

$$\langle \psi, v^+\rangle = \int_\Omega \sum_{k=1}^{s} f_k v_k^+ + \int_{\Gamma_N} \sum_{k=1}^{s} \gamma_k v_k^+ \leq 0.$$

This implies that $\langle A(u), v^+\rangle = \langle \psi, v^+\rangle = 0$. Moreover, both integrands in $\langle A(u), v^+\rangle$ vanish. Introducing the notation $\Omega_k^+ := \{x\in\Omega:\ u_k(x)\geq M\}$, the first integrand in $\langle A(u), v^+\rangle$ satisfies

$$0 = \int_\Omega \sum_{k=1}^{s} b_k(x,u,\nabla u)\,\nabla u_k\cdot\nabla v_k^+ = \sum_{k=1}^{s}\int_{\Omega_k^+} b_k(x,u,\nabla u)\,|\nabla v_k^+|^2.$$

Using condition $b_k\geq m>0$, we obtain that the integrals on each Ω_k^+ vanish, moreover, if Ω_k^+ has a positive measure then $\nabla v_k^+\equiv 0$, i.e. v_k^+ is constant, and (using $v_{k|\Gamma_D}^+ = 0$ and $\Gamma_D\neq\emptyset$) we obtain $v_k^+\equiv 0$, which means that $u_k\leq M$ on Ω. On the other hand, if Ω_k^+ has zero measure then $u_k\leq M$ on Ω again, now by the definition of v_k^+.

Altogether, we obtain $u_k\leq M$ on Ω for all k, which is equivalent to (2.8). ∎

Remark 2.2 (i) If $u\in C(\overline{\Omega})^s$ is not assumed then the same proof can be repeated, provided that g_k are bounded on Γ_D: then $\max u_k$ and $\max g_k$ in (2.8) are replaced by $\operatorname{ess\,sup} u_k$ and $\operatorname{ess\,sup} g_k$, respectively.

(ii) The result holds as well if there are additional terms $\sum_l \omega_{kl}(x,u,\nabla u)\,u_l$ on the Neumann boundary Γ_N, where the functions ω_{kl} satisfy similar properties as assumed for V_{kl} in (2.2)–(2.3).

Remark 2.3 By reversing signs in Theorem 2.1, one obtains the minimum principle: if $f_k \geq 0$ on Ω and $\gamma_k \geq 0$ on $\Gamma_N \quad (k = 1, \ldots, s)$, then

$$\min_{k=1,\ldots,s} \min_{\overline{\Omega}} u_k \geq \min_{k=1,\ldots,s} \min\{0, \min_{\Gamma_D} g_k\}.$$

In particular, this implies the nonnegativity principle: if

$$f_k \geq 0, \quad \gamma_k \geq 0 \text{ and } \quad g_k \geq 0,$$

then

$$u_k \geq 0 \quad \text{on } \Omega \qquad (k = 1, \ldots, s). \tag{2.9}$$

Remark 2.4 As proved in [14], proper finite element discretizations of system (2.1) satsify the discrete counterpart of (2.8).

2.2 Systems with general reaction terms of sublinear growth

It is somewhat restrictive in (2.1) that both the principal and lower-order parts of the equations are given as containing products of coefficients with ∇u_k and u_l, respectively. Whereas this is widespread in real models for the principal part (and often the coefficient of ∇u_k depends only on x, or x and $|\nabla u|$), on the contrary, the lower order terms are usually not given in such a coefficient form. Now we consider problems where the dependence on the lower order terms is given as general functions of x and u. In this section these functions are allowed to grow at most linearly, in which case one can reduce the problem to the previous one (2.1) directly. (Superlinear growth of q_k will be dealt with in the next section.) Accordingly, let us now consider the system

$$\left.\begin{aligned} -\text{div}\left(b_k(x,u,\nabla u)\,\nabla u_k\right) + \; q_k(x,u_1,\ldots,u_s) &= f_k(x) \quad \text{a.e. in } \Omega, \\ b_k(x,u,\nabla u)\tfrac{\partial u_k}{\partial \nu} &= \gamma_k(x) \quad \text{a.e. on } \Gamma_N, \\ u_k &= g_k(x) \quad \text{a.e. on } \Gamma_D \end{aligned}\right\} \quad (k = 1, \ldots, s) \tag{2.10}$$

with the following properties:

Assumptions 2.2.

(i) $\Omega \subset \mathbf{R}^d$ is a bounded piecewise C^1 domain; Γ_D, Γ_N are disjoint open measurable subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$.

(ii) (Smoothness and boundedness.) For all $k, l = 1, \ldots, s$ we have $b_k \in L^\infty(\Omega \times \mathbf{R}^s \times \mathbf{R}^{s\times d})$ and $q_k \in W^{1,\infty}(\Omega \times \mathbf{R}^s)$.

(iii) (Ellipticity.) There exists $m > 0$ such that $b_k \geq m$ holds for all $k = 1, \ldots, s$.

(iv) (Cooperativity.) We have

$$\frac{\partial q_k}{\partial \xi_l}(x,\xi) \leq 0 \qquad (k, l = 1, \ldots, s, \; k \neq l; \; x \in \Omega, \; \xi \in \mathbf{R}^s). \tag{2.11}$$

(v) (Weak diagonal dominance for the Jacobians.) We have

$$\sum_{l=1}^{s} \frac{\partial q_k}{\partial \xi_l}(x,\xi) \ge 0 \quad \text{and} \quad \sum_{l=1}^{s} \frac{\partial q_l}{\partial \xi_k}(x,\xi) \ge 0 \qquad (k=1,\dots,s;\ x\in\Omega,\ \xi\in\mathbf{R}^s). \tag{2.12}$$

(vi) For all $k=1,\dots,s$ we have $f_k \in L^2(\Omega)$, $\gamma_k \in L^2(\Gamma_N)$, $g_k = g^*_{k|\Gamma_D}$ with $g^* \in H^1(\Omega)$.

The basic idea to deal with problem (2.10) is to reduce it to (2.1) via suitably defined functions $V_{kl} : \Omega \times \mathbf{R}^s \to \mathbf{R}$. Namely, let

$$V_{kl}(x,\xi) := \int_0^1 \frac{\partial q_k}{\partial \xi_l}(x,t\xi)\,dt \qquad (k,l=1,\dots,s;\ x\in\Omega,\ \xi\in\mathbf{R}^s). \tag{2.13}$$

Here the assumption $q_k \in W^{1,\infty}(\Omega\times\mathbf{R}^s)$ implies that $V_{kl} \in L^\infty(\Omega\times\mathbf{R}^s)$ $(k,l=1,\dots,s)$. Then the Newton-Leibniz formula yields

$$q_k(x,\xi) = q_k(x,0) + \sum_{l=1}^{s} V_{kl}(x,\xi)\,\xi_l \qquad (k=1,\dots,s;\ x\in\Omega,\ \xi\in\mathbf{R}^s). \tag{2.14}$$

Defining

$$\hat f_k(x) := f_k(x) - q_k(x,0) \qquad (k=1,\dots,s), \tag{2.15}$$

problem (2.10) then becomes

$$\left.\begin{aligned} -\mathrm{div}\Big(b_k(x,u,\nabla u)\,\nabla u_k\Big) + \sum_{l=1}^{s} V_{kl}(x,u)\,u_l &= \hat f_k(x) \quad \text{a.e. in } \Omega,\\ b_k(x,u,\nabla u)\tfrac{\partial u_k}{\partial\nu} &= \gamma_k(x) \quad \text{a.e. on } \Gamma_N,\\ u_k &= g_k(x) \quad \text{a.e. on } \Gamma_D \end{aligned}\right\} \qquad (k=1,\dots,s), \tag{2.16}$$

which is a special case of (2.1). First, as seen above, $V_{kl} \in L^\infty(\Omega\times\mathbf{R}^s)$ $(k,l=1,\dots,s)$. Further, assumptions (2.11) and (2.12) imply that the functions V_{kl} satisfy (2.2) and (2.3), respectively. The remaining items of Assumptions 2.1 and 2.2 coincide, therefore system (2.16) satisfies Assumptions 2.2.

Consequently, our results obtained for (2.1) can be applied to (2.16) too. For the original system (2.10), we thus obtain

Corollary 2.1 *Let Assumptions 2.2 hold and $u \in H^1(\Omega)^s$ be a weak solution of system* (2.10)*, such that $u \in C(\overline{\Omega})^s$. If*

$$f_k \le q_k(x,0) \quad \textit{and} \quad \gamma_k \le 0 \qquad (k=1,\dots,s), \tag{2.17}$$

then

$$\max_{k=1,\dots,s} \max_{\overline{\Omega}} u_k \le \max_{k=1,\dots,s} \max\{0, \max_{\Gamma_D} g_k\}. \tag{2.18}$$

Finally, just as mentioned in Remark 2.2, if $u \in C(\overline{\Omega})^s$ is not assumed but g_k are bounded on Γ_D, then $\max u_k$ and $\max g_k$ in (2.18) are replaced by $\operatorname{ess\,sup} u_k$ and $\operatorname{ess\,sup} g_k$, respectively.

2.3 Systems with general reaction terms of superlinear growth

In the previous section we have required the functions q_k to grow at most linearly via the condition $q_k \in W^{1,\infty}(\Omega \times \mathbf{R}^s)$. However, this is a strong restriction and is not satisfied even by nonlinear polynomials of u_k that often arise in reaction-diffusion problems. In this section we extend the previous results to problems where the functions q_k may grow polynomially.

Accordingly, let us now consider the system

$$\left.\begin{aligned} -\mathrm{div}\Big(b_k(x,u,\nabla u_k)\,\nabla u_k\Big) +\ q_k(x,u_1,\dots,u_s) &= f_k(x) \quad \text{a.e. in } \Omega, \\ b_k(x,\nabla u_k)\tfrac{\partial u_k}{\partial \nu} &= \gamma_k(x) \quad \text{a.e. on } \Gamma_N, \\ u_k &= g_k(x) \quad \text{a.e. on } \Gamma_D \end{aligned}\right\} \qquad (k=1,\dots,s) \tag{2.19}$$

and modify Assumptions 2.2 as follows:

Assumptions 2.3. Let the items (i) and (iii)–(vi) of Assumptions 2.2 hold, and let its item (ii) be replaced by

(ii)' (Smoothness and growth.) For all $k,l=1,\dots,s$ we have $b_k \in L^\infty(\Omega \times \mathbf{R}^s \times \mathbf{R}^{s\times d})$ and $q_k \in C^1(\Omega \times \mathbf{R}^s)$. Further, let

$$2 \le p < p^*, \quad \text{where } p^* := \tfrac{2d}{d-2} \ \text{ if } d \ge 3 \ \text{ and } \ p^* := +\infty \text{ if } d=2; \tag{2.20}$$

then there exist constants $\beta_1, \beta_2 \ge 0$ such that

$$\left|\frac{\partial q_k}{\partial \xi_l}(x,\xi)\right| \le \beta_1 + \beta_2|\xi|^{p-2} \qquad (k,l=1,\dots,s;\ x\in\Omega,\ \xi \in \mathbf{R}^s). \tag{2.21}$$

To handle system (2.19), we start as in the previous subsection by reducing it to a system with nonlinear coefficients: if the functions V_{kl} and $\hat{f}_k$ $(k,l=1,\dots,s)$ are defined as in (2.13) and (2.15), respectively, then problem (2.19) takes a form similar to (2.16). However, in contrast to system (2.10) before, the superlinear growth allowed in (2.21) does not let us now apply the results of subsection 2.1 directly. On the other hand, it is still easy to verify that the allowed superlinear growth does not affect the main result.

Namely, the weak form of our problem now involves an operator similar to (2.6):

$$\langle A(u), v\rangle = \int_\Omega \Big(\sum_{k=1}^s b_k(x,u,\nabla u)\,\nabla u_k \cdot \nabla v_k + \sum_{k,l=1}^s V_{kl}(x,u)\,u_l\,v_k\Big) \tag{2.22}$$

where the functions V_{kl} are defined as in (2.13). Here one must see that the second integral makes sense, since the V_{kl} are now not necessarily bounded. The growth condition (2.21) implies the Sobolev embeddings

$$H^1_D(\Omega) \subset L^p(\Omega), \tag{2.23}$$

see [1], hence u_l and v_k are in $L^p(\Omega)$ and assumption (2.20) yields that $V_{kl}(x,u) \in L^{\frac{p}{p-2}}$. Altogether, by Hölder's inequality, the functions $V_{kl}(x,u)\,u_l\,v_k \in L^1(\Omega)$ and thus the integral

$$\int_\Omega \sum_{k,l=1}^{s} V_{kl}(x,u)\,u_l\,v_k$$

exists. That is, the weak form of system (2.19) can be defined just as in (2.4).

From here the proof of Theorem 2.1 can be repeated in the same form, since it only uses the weak formulation but does not exploit the boundedness V_{kl}. The final result is also reduced to Theorem 2.1 in the same way as in subsection 2.2.

Corollary 2.2 *Let Assumptions 2.3 hold and $u \in H^1(\Omega)^s$ be a weak solution of system* (2.19) *such that $u \in C(\overline{\Omega})^s$. Then under the sign conditions* (2.17) *the maximum principle* (2.18) *holds. Further, if $u \in C(\overline{\Omega})^s$ is not assumed but g_k are bounded on Γ_D, then* $\max u_k$ *and* $\max g_k$ *in* (2.18) *are replaced by* $\operatorname{ess\,sup} u_k$ *and* $\operatorname{ess\,sup} g_k$, *respectively.*

Similarly to Remark 2.3, one can derive the corresponding minimum principle and, as a special case, the nonnegativity principle:

Corollary 2.3 *Let Assumptions 2.3 hold and $u \in H^1(\Omega)^s$ be a weak solution of system* (2.19) *such that $u \in C(\overline{\Omega})^s$. If $f_k \ge q_k(x,0)$, $\gamma_k \ge 0$ and $g_k \ge 0$, then*

$$u_k \ge 0 \quad \text{on } \Omega \qquad (k = 1,\dots,s). \tag{2.24}$$

3 Some applications

3.1 Reaction-diffusion systems in chemistry

The steady states of certain reaction-diffusion processes in chemistry are described by systems of the following form:

$$\left.\begin{aligned} -b_k\Delta u_k + P_k(x,u_1,\dots,u_s) &= f_k(x) \quad \text{in } \Omega, \\ b_k\frac{\partial u_k}{\partial \nu} &= \gamma_k(x) \quad \text{on } \Gamma_N, \\ u_k &= g_k(x) \quad \text{on } \Gamma_D \end{aligned}\right\} \qquad (k = 1,\dots,s). \tag{3.1}$$

Here, for all k, the quantity u_k describes the concentration of the kth species, and P_k is a polynomial which characterizes the rate of the reactions involving the k-th species. A common way to describe such reactions is the so-called mass action type kinetics [12], which implies that P_k has no constant term for any k, in other words, $P_k(x,0) \equiv 0$ on Ω for all k. The reaction between different species is often proportional to the product of their concentration. The function $f_k \ge 0$ describes a source independent of concentrations. The boundary data similarly satisfy $\gamma_k, g_k \ge 0$.

We consider system (3.1) under the following conditions, such that it becomes a special case of system (2.19).

Assumptions 3.1.

(i) $\Omega \subset \mathbf{R}^d$ is a bounded piecewise C^1 domain, where $d = 2$ or 3, and Γ_D, Γ_N are disjoint open measurable subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$.

(ii) (Smoothness and growth.) For all $k, l = 1, \ldots, s$, the functions P_k are polynomials of arbitrary degree if $d = 2$ and of degree at most 4 if $d = 3$, further, $P_k(x, 0) \equiv 0$ on Ω.

(iii) (Ellipticity.) $b_k > 0$ ($k = 1, \ldots, s$) are given numbers.

(iv) (Cooperativity.) We have $\dfrac{\partial P_k}{\partial \xi_l}(x, \xi) \leq 0 \quad (k, l = 1, \ldots, s,\ k \neq l;\ x \in \Omega,\ \xi \in \mathbf{R}^s)$.

(v) (Weak diagonal dominance for the Jacobians.) We have

$$\sum_{l=1}^{s} \frac{\partial P_k}{\partial \xi_l}(x, \xi) \geq 0, \qquad \sum_{l=1}^{s} \frac{\partial P_l}{\partial \xi_k}(x, \xi) \geq 0 \qquad (k = 1, \ldots, s;\ x \in \Omega,\ \xi \in \mathbf{R}^s). \tag{3.2}$$

(vi) For all $k = 1, \ldots, s$ we have $f_k \in L^2(\Omega)$, $\gamma_k \in L^2(\Gamma_N)$, $g_k = g^*_{k|\Gamma_D}$ with $g^* \in H^1(\Omega)$.

Similarly to Remark 2.1, assumptions (iv)–(v) now imply

$$\frac{\partial P_k}{\partial \xi_k}(x, \xi) \geq 0 \qquad (k = 1, \ldots, s;\ x \in \Omega,\ \xi \in \mathbf{R}^s). \tag{3.3}$$

From the point of view of the chemical model described by system (3.1), the meaning of the cooperativity is cross-catalysis, whereas (3.3) means autoinhibiton. Cross-catalysis arises e.g. in gradient systems [28]. Condition (3.2) means that autoinhibition is strong enough to ensure both weak diagonal dominances.

By definition, the concentrations u_k are nonnegative, therefore a proper model must produce such solutions. We can use Corollary 2.3 to obtain the required property: since $f_k \geq P_k(x, 0) \equiv 0$, $\gamma_k \geq 0$ and $g_k \geq 0$, we have

Corollary 3.1 *Let system (3.1) satisfy Assumptions 3.1, and $u \in H^1(\Omega)^s$ be a weak solution such that $u \in C(\overline{\Omega})^s$. Then*

$$u_k \geq 0 \quad \text{on } \Omega \qquad (k = 1, \ldots, s). \tag{3.4}$$

3.2 Linear elliptic systems

As mentioned in the introduction, even for linear systems maximum principles are still of interest in the case of mixed boundary conditions, hence it is worthwile to derive the corresponding result from the previous sections. Let us therefore consider linear elliptic systems of the form

$$\left.\begin{aligned} -\mathrm{div}\,(b_k(x)\,\nabla u_k) + \sum_{l=1}^{s} V_{kl}(x)\,u_l &= f_k(x) \quad \text{a.e. in } \Omega, \\ b_k(x)\tfrac{\partial u_k}{\partial \nu} &= \gamma_k(x) \quad \text{a.e. on } \Gamma_N, \\ u_k &= g_k(x) \quad \text{a.e. on } \Gamma_D \end{aligned}\right\} \qquad (k = 1, \ldots, s) \tag{3.5}$$

where for all $k, l = 1, \dots, s$ we have $b_k \in W^{1,\infty}(\Omega)$ and $V_{kl} \in L^\infty(\Omega)$.

Let the functions b_k and V_{kl} satisfy Assumptions 2.1, i.e. the sign conditions, cooperativity and diagonal dominance. Then (3.5) is a special case of (2.1), hence Theorem 2.1 holds, as well as the analogous results mentioned in Remark 2.3. Here we formulate the maximum and nonnegativity principles:

Corollary 3.2 *Let system (3.5) satisfy Assumptions 2.1, and $u \in H^1(\Omega)^s$ be a weak solution such that $u \in C(\overline{\Omega})^s$.*

(1) If $f_k \le 0$ on Ω and $\gamma_k \le 0$ on Γ_N $(k = 1, \dots, s)$, then

$$\max_{k=1,\dots,s} \max_{\overline{\Omega}} u_k \le \max_{k=1,\dots,s} \max\{0, \max_{\Gamma_D} g_k\}. \tag{3.6}$$

(2) If $f_k \ge 0$, $\gamma_k \ge 0$ and $g_k \ge 0$ $(k = 1, \dots, s)$, then

$$u_k \ge 0 \quad \text{on } \Omega \qquad (k = 1, \dots, s). \tag{3.7}$$

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