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We show that for a random graph Lovász' ϑ function is of order $\sqrt{n}$.

In investigating certain properties of graphs, the behaviour of random graphs can serve as a guide. This motivates the study of the ϑ function for random graphs; one might hope to obtain a good estimate for the Shannon capacity. Unfortunately, this is not the case; the ϑ function is typically very large (of order $\sqrt{n}$), while the Shannon capacity is likely to be of order $\log n$ for a random graph.

Before stating our result we recall a theorem on the eigenvalues of random matrices.

Theorem 1. (Füredi, Komlós [1]). *Let $A=(a_{ij})$ be an $n\times n$ random symmetric matrix, in which the entries a_{ij} for $i>j$ are independent identically distributed bounded random variables with distribution function H, and $a_{ii}\equiv 0$. Denote the moments of H by*

$$\mu = \int x\,dH(x) \quad \text{and} \quad \sigma^2 = \int x^2\,dH(x)-\mu^2.$$

If $\lambda_1\geqq\lambda_2\geqq\ldots\geqq\lambda_n$ are the eigenvalues of A, then

(i) *in case $\mu>0$ $\lambda_1=\mu n+O(1)$ in measure, $\max\limits_{2\leqq i\leqq n}|\lambda_i|=2\sigma\sqrt{n}+O(n^{\frac{1}{3}}\log n)$ in probability,*

(ii) *in case $\mu=0$ $\max\limits_{1\leqq i\leqq n}|\lambda_i|=2\sigma\sqrt{n}+O(n^{\frac{1}{3}}\log n)$ in probability.* ∎

A similar theorem holds for non-symmetric matrices, see [2].

Now we state our theorem on the ϑ function.

Theorem 2. *Let $\{1, \ldots, n\}$ be the vertex-set of a random graph G, and denote the edge-*

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set by E. The measure is $P((i,j)\in E)=p,\ q=1-p$. *Then, with probability* $1-o(1)$,

$$\frac{1}{2}\sqrt{\frac{q}{p}}\sqrt{n}+O(n^{\frac{1}{3}}\log n)\leqq \vartheta(G)\leqq 2\sqrt{\frac{q}{p}}\sqrt{n}+O(n^{\frac{1}{3}}\log n).$$

Proof. Define the sets $\mathfrak{A}$ and $\mathfrak{B}$ of symmetric matrices as follows:

$$\mathfrak{A} = \{A=(a_{ij}):\ a_{ij}=1 \quad \text{if} \quad (i,j)\notin E, \text{ arbitrary real otherwise}\}$$

$$\mathfrak{B} = \{B=(b_{ij}):\ b_{ij}=0 \quad \text{if} \quad (i,j)\in E, \text{ arbitrary real otherwise}\}.$$

It is proved in [3] that

$$\vartheta(G)=\min_{A\in\mathfrak{A}}\lambda_1(A)=\max_{B\in\mathfrak{B}}\left(1-\frac{\lambda_1(B)}{\lambda_n(B)}\right)$$

where $\lambda_1(\cdot)$ and $\lambda_n(\cdot)$ stand for the largest and smallest eigenvalues.

Now for the upper bound in the theorem we define, for the random graph G, the matrix

$$\bar{A}=(\bar{a}_{ij})=\begin{cases}1 & \text{if} \quad (i,j)\notin E\\ -\dfrac{q}{p} & \text{if} \quad (i,j)\in E.\end{cases}$$

It is clear that $\bar{A}$ is a random matrix satisfying the conditions of Theorem 1, and $\mu=0$, $\sigma=\sqrt{\frac{q}{p}}$. Thus, with probability $1-o(1)$,

$$\vartheta(G)\leqq\lambda_1(\bar{A})\leqq 2\sigma\sqrt{n}+O(n^{\frac{1}{3}}\log n)=2\sqrt{\frac{q}{p}}\sqrt{n}+O(n^{\frac{1}{3}}\log n).$$

For proving the lower bound of our theorem, set

$$\bar{B}=(\bar{b}_{ij})=\begin{cases}0 & \text{if} \quad (i,j)\in E\\ 1 & \text{if} \quad (i,j)\notin E.\end{cases}$$

We get a random matrix with $\mu=q$, $\sigma=\sqrt{pq}$. Hence, with probability $1-o(1)$,

$$\vartheta(G)\geqq 1-\frac{\lambda_1(\bar{B})}{\lambda_n(\bar{B})}\geqq\frac{\mu n+O(1)}{2\sigma\sqrt{n}+O(n^{\frac{1}{3}}\log n)}=\frac{qn+O(1)}{2\sqrt{pq}\sqrt{n}+O(n^{\frac{1}{3}}\log n)}=$$

$$=\frac{1}{2}\sqrt{\frac{q}{p}}\sqrt{n}+O(n^{\frac{1}{3}}\log n).$$

Remark. One can easily extend Theorem 2 to the case $q=cn^{-1+\alpha}$. One obtains $\vartheta=O(n^{\frac{\alpha}{2}})$.

References

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