Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **21**, 1–15; doi: 10.14232/ejqtde.2016.1.21 http://www.math.u-szeged.hu/ejqtde/

# On pseudomonotone elliptic operators with functional dependence on unbounded domains

**Mihály Csirik**✉

Department of Applied Analysis, Eötvös Loránd University,
Hungary, H-1117 Budapest, Pázmány Péter sétány 1/C.

Received 17 February 2016, appeared 1 May 2016
Communicated by László Simon

**Abstract.** We generalize F. E. Browder's results concerning pseudomonotone elliptic partial differential operators defined on unbounded domains. Browder treated equations for quasilinear operators of divergence form

$$\sum_{|\alpha|\le k} D_\alpha a_\alpha(x, u(x), \dots, D^\beta u(x)) = f(x),$$

on an arbitrary unbounded domain $\Omega$, where $|\beta| \le k$ for some $k \ge 1$. We show that under suitable assumptions, Browder's result holds true if the functions $a_\alpha$ are functionals of $u$.

**Keywords:** elliptic operators, nonlocal, nonlinear, pseudomonotone operators.

**2010 Mathematics Subject Classification:** 35J88, 35J87.

## 1 Introduction

The theory of pseudomonotone operators proved to be highly useful for establishing existence and uniqueness theorems for divergence-form elliptic problems with standard growth conditions (see [10] too, for general growth conditions). The concept of pseudomonotonicity was introduced by H. Brezis [2] in 1968.

**Definition 1.1.** Let $X$ be a Banach space. A bounded operator $A\colon X \to X^*$ is said to be *pseudomonotone* if for any sequence $\{u_j\} \subset X$, such that

$$u_j \rightharpoonup u \quad (\text{in } X) \quad \text{and} \quad \limsup_{j\to\infty} \langle A(u_j), u_j - u\rangle \le 0,$$

then

**(PM1)** $\langle A(u_j), u_j - u\rangle \to 0$ as $j \to \infty$ and

**(PM2)** $A(u_j) \rightharpoonup A(u)$ in $X^*$ as $j \to \infty$.

✉ Email: csirik@gmail.com

As usual, the symbol $\rightharpoonup$ denotes weak convergence.

The following abstract surjectivity result [9, Theorem 2.12] is widely used in the literature for proving the existence of a weak solution to a nonlinear elliptic partial differential equation.

**Theorem 1.2.** *Let $X$ be a reflexive separable Banach space and $A\colon X \to X^*$ a bounded, coercive and pseudomonotone operator. Then for arbitrary $F \in X^*$, there exists $u \in X$, such that $A(u) = F$ in $X^*$.*

In this context, coercivity is defined as follows:

**Definition 1.3.** An operator $A\colon X \to X^*$ is called *coercive* if

$$\frac{\langle A(u), u\rangle}{\|u\|} \to +\infty \quad (\text{as } \|u\| \to \infty).$$

Guaranteeing boundedness and coercivity is usually a trivial matter. The proof of pseudomonotonicity usually involves the Rellich–Kondrachov compactness theorem as a crucial step. On unbounded domains however, a compact embedding result seems to require more complicated conditions on the domain, see e.g. [1, Theorem 6.52]. F. E. Browder managed to avoid the use of such compactness results in [3]. To establish pseudomonotonicity, it turns out that the main task is to prove the a.e. convergence of the sequences $\{D^\alpha u_j\}_{j=1}^\infty$. Browder's idea is a natural one: let the unbounded domain $\Omega$ be exhausted by an increasing sequence $\{\Omega_i\}$ of bounded domains with smooth boundary – such that on each $\Omega_i$ the Rellich–Kondrachov theorem holds. Combining this with a diagonal argument, we extract a subsequence of the lower-order derivatives $\{D^\alpha u_j\}$ converging a.e. to $D^\alpha u$ ($|\alpha| \le k-1$). Proving a.e. convergence of the highest-order derivatives $D^\alpha u_j \to D^\alpha u$ ($|\alpha| = k$) is more involved.

The results of F. E. Browder on nonlinear elliptic equations on unbounded domains have been extended in [10], [4] and [6] to strongly nonlinear elliptic equations, i.e. equations containing a term which is arbitrarily quickly increasing with respect to the values of unknown function $u$. Further, there are some results in [7] and [8] on elliptic problems where the lower order terms or the boundary condition contains nonlocal (e.g. integral type) dependence on $u$.

The aim of this paper is to extend Browder's theorem to elliptic operators with nonlocal dependence in the main (highest order) terms, too: we shall modify the assumptions and the proof of the original theorem for $2k$-order divergence-type nonlinear functional elliptic equations. After formulating sufficient conditions for such a nonlocal operator to be bounded, coercive and pseudomonotone, we prove our main result. Finally, we give concrete examples that satisfy our assumptions.

## 2 Problem formulation

Let $\Omega \subset \mathbb{R}^n$ be a possibly unbounded domain with sufficiently smooth boundary, and let $W_0^{k,p}(\Omega) \subset V \subset W^{k,p}(\Omega)$ be a closed linear subspace with $1 < p < \infty$ and $k \ge 1$. Let $A\colon V \to V^*$ be defined by

$$\langle A(u), v\rangle = \sum_{|\alpha|\le k} \int_\Omega a_\alpha(x, u(x), \dots, D^\beta u(x), \dots; u) D^\alpha v(x)\, dx \tag{2.1}$$

for all $u, v \in V$, where $|\beta| \le k$ is a multiindex. The function $a_\alpha$ may depend on the pointwise values of any of the partial derivatives of $u$. Furthermore, "$;u$" notation signifies that $a_\alpha$ may be a *functional* of $u$. In other words, $a_\alpha$ may depend on the *whole* solution $u$.

The arguments of the functions $a_\alpha$ are denoted as $a_\alpha(x,\eta;u)$, and we sometimes split $\eta$ as $\eta = (\zeta,\xi)$ where $\zeta \in \mathbb{R}^{N_1}$ and $\xi \in \mathbb{R}^{N_2}$, so that $\eta \in \mathbb{R}^N$ with $N = N_1 + N_2$ and write $a_\alpha(x,\zeta,\xi;u)$, where the numbers $N_1$ and $N_2$ denote number of multiindexes $\beta$ such that $|\beta| \le k-1$ and $|\beta| = k$, respectively. Furthermore, the notation

$$\eta^{(\ell)} = \{\eta_\beta \,:\, |\beta| = \ell\}$$

is used, where $\ell = 0,1,\ldots,k$. Note that

$$\zeta = \left\{\eta^{(\ell)} \,:\, \ell = 0,1,\ldots,k-1\right\} \quad \text{and} \quad \xi = \left\{\eta^{(\ell)} \,:\, \ell = k\right\}.$$

We impose the following assumptions on the structure of $A$ and $\Omega$.

**(A0)** Suppose that there exist a sequence $\{\Omega_i\} \subset \mathbb{R}^n$ of bounded domains such that $\Omega_i \subset \Omega_{i+1}$ ($i = 1,2,\ldots$) and $\Omega = \bigcup_{i=1}^\infty \Omega_i$. Furthermore, assume that each $\partial\Omega_i$ is sufficiently smooth so that the Rellich–Kondrachov theorem holds: $W^{k,p}(\Omega_i) \subset\subset W^{k-1,p}(\Omega_i)$ ($i = 1,2,\ldots$).

**(A1)** Let $a_\alpha$ be Carathéodory functions for fixed $u \in V$ and all multiindex $|\alpha| \le k$, i.e. let $a_\alpha(\,\cdot\,,\eta;u)$ be measurable for every fixed $\eta \in \mathbb{R}^N$, and let $a_\alpha(x,\,\cdot\,;u)$ be continuous for almost every fixed $x \in \Omega$.

**(A2)** Suppose that there exist a bounded functional $g_1 \colon V \to \mathbb{R}_+$ and a compact map

$$k_1^\alpha \colon V \to L^{r'_\ell}(\Omega)$$

with $k_1^\alpha(u) \ge 0$, where $p' = p/(p-1)$, $r'_\ell = r_\ell/(r_\ell - 1)$ and

$$p \le r_\ell < p_\ell^*, \qquad p_\ell^* = \begin{cases} \dfrac{np}{n-(k-\ell)p}, & \text{if } n > (k-\ell)p \\ > 0, & \text{otherwise.} \end{cases}$$

such that

$$|a_\alpha(x,\eta;u)| \le g_1(u)\left[|\eta^{(\ell)}|^{p-1} + |\eta^{(\ell)}|^{r_\ell - 1}\right] + [k_1^\alpha(u)](x)$$

for each multiindex $\ell = |\alpha| \le k$, almost all $x \in \Omega$, all $\eta \in \mathbb{R}^N$ and all $u \in V$. Note that for $|\alpha| = \ell = k$, we must have $r_k = p$. Here, we introduce the notation

$$[\mathcal{K}_1^{(\ell)}(u)](x) = \max_{|\alpha|=\ell} [k_1^\alpha(u)](x)$$

for all $\ell = 1,\ldots,k$.

**(A3)** Suppose that

$$\sum_{|\alpha|=k} \big(a_\alpha(x,\zeta,\xi;u) - a_\alpha(x,\zeta,\xi';u)\big)(\xi_\alpha - \xi'_\alpha) > 0$$

for almost all $x \in \Omega$, all $\zeta \in \mathbb{R}^{N_1}$, $\xi \ne \xi' \in \mathbb{R}^{N_2}$ and all $u \in V$.

**(A4)** Suppose that there exist a bounded and lower semicontinuous functional $g_2 \colon V \to \mathbb{R}_+$ and a compact map $k_2 \colon V \to L^1(\Omega)$ such that

$$\sum_{|\alpha|\le k} a_\alpha(x,\eta;u)\eta_\alpha \ge g_2(u)|\xi|^p - [k_2(u)](x)$$

for almost all $x \in \Omega$, every $u \in V$, and all $\eta = (\zeta,\xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

Note that the preceding coercivity-like assumption requires the inequality to hold for *all* $u \in V$ and $\eta$ – contrary to usual asymptotic version, which is prescribed only for *large* $\|u\|_V$

and $|\eta|$. The reason for this is that the proof of pseudomonotonicity employs a certain inequality which is needed for all $u$ and $\eta$ and is derived from this coercivity estimate. We now state a significant strengthening of **(A4)** that ensures coercivity in the sense of Definition 1.3.

**(A4′)** Suppose that there exist a bounded functional $g_2\colon V \to \mathbb{R}_+$ and a compact map $k_2\colon V \to L^1(\Omega)$ such that

$$\sum_{|\alpha|\le k} a_\alpha(x,\eta;u)\eta_\alpha \ge \begin{cases} g_2(u)|\xi|^p - [k_2(u)](x), & \text{for every } u \in V \\ g_2(u)\left[|\xi|^p + \sum_{\ell=0}^{k-1}(|\eta^{(\ell)}|^p + |\eta^{(\ell)}|^{r_\ell})\right] - [k_2(u)](x), & \text{for large } \|u\|_V \end{cases}$$

for almost all $x \in \Omega$ and all $\eta = (\zeta,\xi) \in \mathbb{R}^N$. Here, the functional $g_2$ satisfies the estimate

$$g_2(u) \ge c^*\|u\|_V^{-\sigma^*}$$

for all $u \in V$ with sufficiently large $\|u\|_V$, with some $c^* > 0$ and $0 \le \sigma^* < p-1$. Also, the map $k_2$ satisfies

$$\|k_2(u)\|_{L^1(\Omega)} \le c^*\|u\|_V^{\sigma}$$

for all $u \in V$ with sufficiently large $\|u\|_V$ and some $0 \le \sigma < p - \sigma^*$.

**(A5)** Whenever $u_j \rightharpoonup u$ in $V$ and $\{\eta_j\} \subset \mathbb{R}^N$ with $\eta_j \to \eta$, then $a_\alpha(x,\eta_j;u_j) \to a_\alpha(x,\eta;u)$ for a.e. $x \in \Omega$ up to a subsequence.

## 3 Pseudomonotonicity

**Theorem 3.1.** *Assume* **(A0)**, **(A1)**, **(A2)**, **(A3)** *and* **(A4)**. *Then the operator $A\colon V \to V^*$ defined in* (2.1) *is pseudomonotone.*

*Proof.* Let $\{u_j\} \subset V$ be a sequence that satisfies $u_j \rightharpoonup u$ in $V$ and

$$\limsup_{j\to\infty} \langle A(u_j), u_j - u\rangle \le 0. \tag{3.1}$$

Assumption **(A0)** implies that there exists a sequence $\{\Omega_i\} \subset \mathbb{R}^n$ of bounded domains such that $\Omega_i \subset \Omega_{i+1}$, $\Omega = \bigcup_{i=1}^\infty \Omega_i$ and the Rellich–Kondrachov theorem holds on each $\Omega_i$: $W^{k,p}(\Omega_i) \subset\subset W^{k-1,p}(\Omega_i)$. For every $i \in \mathbb{N}$ there is a subsequence $\{u_j^{(i)}\}_{j=1}^\infty \subset \{u_j\}_{j=1}^\infty$ (indexed by the same $j$ for simplicity) such that $\{u_j^{(i)}\}_{j=1}^\infty \supset \{u_j^{(i+1)}\}_{j=1}^\infty$ and $u_j^{(i)} \to u$ in $W^{k-1,p}(\Omega_i)$ as $j \to \infty$. The diagonal sequence $\{u_j\}_{j=1}^\infty = \{u_j^{(j)}\}_{j=1}^\infty$ satisfies $u_j \to u$ in $W^{k-1,p}(\Omega_i)$ for any $i \in \mathbb{N}$. Then

$$D^\gamma u_j \to D^\gamma u \quad \text{a.e. in } \Omega \text{ for all } |\gamma| \le k-1 \tag{3.2}$$

up to a subsequence. Further, by **(A2)** and **(A4)** we may assume that the sequences $\{\mathcal{K}_1^{(\ell)}(u_j)\} \subset L^{r'_\ell}(\Omega)$ (for every $\ell = 1,\dots,k$) and $\{k_2(u_j)\} \subset L^1(\Omega)$ are convergent. Note however, that we *do not* have $u_j \to u$ in $W^{k-1,p}(\Omega)$.

The following notations are used throughout the proof:

$$\left.\begin{aligned} \zeta(x) &= \{D^\beta u(x) \ : \ |\beta| \le k-1\}, \\ \zeta_j(x) &= \{D^\beta u_j(x) \ : \ |\beta| \le k-1\}, \\ \xi(x) &= \{D^\beta u(x) \ : \ |\beta| = k\}, \\ \xi_j(x) &= \{D^\beta u_j(x) \ : \ |\beta| = k\} \\ \eta^{(\ell)}(x) &= \{D^\beta u(x) \ : \ |\beta| = \ell\}, \\ \eta_j^{(\ell)}(x) &= \{D^\beta u_j(x) \ : \ |\beta| = \ell\}, \\ \eta(x) &= \{\eta^{(\ell)}(x) \ : \ \ell = 1, \dots, k\}, \\ \eta_j(x) &= \{\eta_j^{(\ell)}(x) \ : \ \ell = 1, \dots, k\}. \end{aligned}\right\} \tag{3.3}$$

Using these, we may write

$$\langle A(u_j) - A(u), u_j - u\rangle = \int_\Omega p_j,$$

where

$$p_j(x) = \sum_{|\alpha| \le k} \left[a_\alpha(x, \zeta_j(x), \xi_j(x); u_j) - a_\alpha(x, \zeta(x), \xi(x); u)\right](D^\alpha u_j - D^\alpha u),$$

Also, (3.2) may be written as $\zeta_j \to \zeta$ a.e. or $\eta_j^{(\ell)} \to \eta^{(\ell)}$ a.e. for all $\ell = 0, 1, \dots, k-1$.

First we derive conclusion **(PM1)** of pseudomonotonicity. The following trivial lemma is well-known.

**Lemma 3.2.** *Relation* (3.1) *implies*

$$\limsup_{j\to\infty} \langle A(u_j) - A(u), u_j - u\rangle \le 0.$$

*Proof.* We have

$$\limsup_{j\to\infty} \langle A(u_j) - A(u), u_j - u\rangle \le \limsup_{j\to\infty} \langle A(u_j), u_j - u\rangle - \liminf_{j\to\infty} \langle A(u), u - u_j\rangle.$$

By (3.1), the first term is nonpositive. For the second term, note that the functional $v \mapsto \langle A(u), u - v\rangle$ is weakly lower semicontinuous, so $\liminf \langle A(u), u - u_j\rangle \ge 0$. ☐

The conclusion of Lemma 3.2 can be written briefly as

$$\limsup_{j\to\infty} \int_\Omega p_j \le 0. \tag{3.4}$$

Using the positive-negative decomposition $p_j(x) = p_j^+(x) - p_j^-(x)$, we have $0 \le p_j^+(x) = p_j(x) + p_j^-(x)$ hence (3.4) immediately implies

$$\int_\Omega p_j^+ \to 0 \tag{3.5}$$

as $j \to \infty$. Hence, the convergence $\int_\Omega p_j^- \to 0$ $(j \to \infty)$ needs to be established, so that $\int_\Omega p_j \to 0$ $(j \to \infty)$ holds, which implies **(PM1)**. This will be done via Vitali's convergence theorem (see Theorem A.6) applied to the sequence $\{p_j^-\}$.

**Lemma 3.3.** *The sequence $\{p_j^-\}$ is equiintegrable and tight over $\Omega$. Furthermore, there exist $C_1 > 0$ and an a.e. bounded function $\beta\colon \Omega \to \mathbb{R}_+$ such that for a.a. $x \in \Omega$,*

$$p_j(x) \geq C_1|\xi_j(x)|^p - \beta(x) \tag{3.6}$$

*Proof.* Expand $p_j(x)$ as

$$p_j(x) = \sum_{|\alpha|=k} a_\alpha(x, \zeta_j, \xi_j; u_j)D^\alpha u_j + \sum_{|\alpha|\leq k-1} a_\alpha(x, \zeta_j, \xi_j; u_j)D^\alpha u_j - w_j(x),$$

where

$$\begin{aligned} w_j(x) &= \sum_{|\alpha|\leq k} \Big[a_\alpha(x, \zeta, \xi; u)\big(D^\alpha u_j - D^\alpha u\big) + a_\alpha(x, \zeta_j, \xi_j; u_j)D^\alpha u\Big] \\ &=: \sum_{\ell=0}^{k} w_j^{(\ell)}(x). \end{aligned}$$

We prove that $\{w_j\}$ is equiintegrable and tight. Assumption **(A2)** implies that

$$\begin{aligned} |w_j^{(\ell)}(x)| &\leq C_2\Big(g_1(u)\Big[|\eta^{(\ell)}|^{p-1} + |\eta^{(\ell)}|^{r_\ell-1}\Big] + [\mathcal{K}_1^{(\ell)}(u)](x)\Big)\big(|\eta_j^{(\ell)}| + |\eta^{(\ell)}|\big) \\ &\quad + C_2\Big(g_1(u_j)\Big[|\eta_j^{(\ell)}|^{p-1} + |\eta_j^{(\ell)}|^{r_\ell-1}\Big] + [\mathcal{K}_1^{(\ell)}(u_j)](x)\Big)|\eta^{(\ell)}| \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\leq C_3\Big(|\eta^{(\ell)}|^{p-1}|\eta_j^{(\ell)}| + |\eta^{(\ell)}|^p + |\eta^{(\ell)}|^{r_\ell-1}|\eta_j^{(\ell)}| + |\eta^{(\ell)}|^{r_\ell} \\ &\quad + |\eta_j^{(\ell)}|^{p-1}|\eta^{(\ell)}| + |\eta_j^{(\ell)}|^{r_\ell-1}|\eta^{(\ell)}| \\ &\quad + [\mathcal{K}_1^\ell(u)](x)\big(|\eta_j^{(\ell)}| + |\eta^{(\ell)}|\big) + [\mathcal{K}_1^{(\ell)}(u_j)](x)|\eta^{(\ell)}|\Big) \end{aligned} \tag{3.8}$$

where $C_2, C_3 > 0$ are constants. We shall apply Proposition A.3 to prove that the function dominating $w_j^{(\ell)}(x)$ is equiintegrable and tight. The weak convergence $u_j \rightharpoonup u$ in $V \subset W^{k,p}(\Omega)$ implies that the sequence $\{\eta_j^{(\ell)}\} \subset W^{k-\ell,p}(\Omega)$ is bounded, hence by the Sobolev embedding $W^{k-\ell,p}(\Omega) \subset L^q(\Omega)$ (where $p \leq q \leq p_\ell^*$) we have that $\{\eta_j^{(\ell)}\} \subset L^q(\Omega)$ is bounded. In particular, $\{|\eta_j^{(\ell)}|^{r_\ell}\}, \{|\eta_j^{(\ell)}|^p\} \subset L^1(\Omega)$ are bounded.

The second and fourth terms in (3.8) are equiintegrable and tight by part (1) of Proposition A.3. Further, the first term is equiintegrable and tight by part (3) of Proposition A.3 applied to the constant sequence $|\eta^{(\ell)}|^{p-1} \in L^{p'}(\Omega)$ (with $|\eta^{(\ell)}|^p \in L^1(\Omega)$ being equiintegrable and tight by part (1) of the said Proposition) and to the bounded sequence $\{|\eta_j^{(\ell)}|\} \subset L^p(\Omega)$. The third term is similar. The fifth term is also equiintegrable and tight by part (3) of Proposition A.3 applied to the bounded $\{|\eta_j^{(\ell)}|^{p-1}\} \subset L^{p'}(\Omega)$ and the constant $|\eta^{(\ell)}| \in L^p(\Omega)$ sequences. The sixth term is handled in a similar way. Finally, $\{\mathcal{K}_1^{(\ell)}(u_j)^{r_\ell'}\} \subset L^1(\Omega)$ is convergent by construction. Therefore the last two terms are equiintegrable and tight, too.

Moreover, assumption **(A4)** implies that

$$p_j(x) \geq g_2(u_j)|\xi_j|^p - k_2(u_j)(x) - |w_j(x)| \geq -k_2(u_j)(x) - |w_j(x)|. \tag{3.9}$$

It follows that

$$0 \leq p_j^-(x) \leq [k_2(u_j)](x) + |w_j(x)|,$$

hence $\{p_j^-\}$ is equiintegrable and tight, where we have used the fact that $\{k_2(u_j)\}$ is equiintegrable and tight, since it is convergent in $L^1(\Omega)$.

Finally, we turn to the proof of inequality (3.6). Young's inequality applied to the products on the right side of (3.7) implies that

$$|w_j^{(\ell)}(x)| \le K_3(\varepsilon)\Big(|\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta^{(\ell)}|^{r_\ell} + [\mathcal{K}_1^{(\ell)}(u)](x)^{r'_\ell}\Big) + C_3\varepsilon\Big(|\eta_j^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{r_\ell}\Big)$$
$$+ C_4\varepsilon\Big(|\eta_j^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + [\mathcal{K}_1^{(\ell)}(u_j)](x)^{r'_\ell}\Big) + K_4(\varepsilon)|\eta^{(\ell)}|^{r_\ell}.$$

By summing over $j = 0, 1, \ldots, k$, and noting that $r_k = p$, we get

$$\begin{aligned}
|w_j(x)| &\le C_5\varepsilon|\xi_j|^p + K_5(\varepsilon)\Big(2|\xi|^p + [\mathcal{K}_1^{(k)}(u)](x)^{p'} + [\mathcal{K}_1^{(k)}(u_j)](x)^{p'}\Big) + \sum_{\ell=0}^{k-1}|w_j^{(\ell)}(x)| \\
&\le C_5\varepsilon|\xi_j|^p + K(\varepsilon)\bigg(2|\xi|^p + [\mathcal{K}_1^{(k)}(u)](x)^{p'} + [\mathcal{K}_1^{(k)}(u_j)](x)^{p'} \\
&\qquad + \sum_{\ell=0}^{k-1}\Big[|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + |\eta_j^{(\ell)}|^{(p-1)r'_\ell} \\
&\qquad\qquad + [\mathcal{K}_1^{(\ell)}(u)](x)^{r'_\ell} + [\mathcal{K}_1^{(\ell)}(u_j)](x)^{r'_\ell}\Big]\bigg) \\
&=: C_5\varepsilon|\xi_j|^p \\
&\quad + K(\varepsilon)\bigg(2|\xi|^p + \sum_{\ell=0}^{k-1}\Big[|\eta^{(\ell)}|^{r_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{(p-1)r'_\ell}\Big] + [\mathcal{K}_3(u,u_j)](x)\bigg),
\end{aligned}$$

where $\{\mathcal{K}_3(u,u_j)\} \subset L^1(\Omega)$ is convergent, hence it is convergent a.e. up to a subsequence, thus it is a.e. bounded. Therefore, using the a.e. convergence $\eta^{(\ell)} \to \eta$ ($\ell = 0, \ldots, k-1$) we have that the function

$$\beta_1(x) = 2|\xi|^p + \sum_{\ell=0}^{k-1}\Big[|\eta^{(\ell)}|^{r_\ell} + |\eta_j^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^{(p-1)r'_\ell} + |\eta_j^{(\ell)}|^{(p-1)r'_\ell}\Big] + [\mathcal{K}_3(u,u_j)](x)$$

is bounded a.e.

The first inequality of (3.9) combined with the preceding estimate and assumption **(A4)** leads to

$$\begin{aligned}
p_j(x) &\ge g_2(u_j)|\xi_j|^p - [k_2(u_j)](x) - |w_j(x)| \\
&\ge g_2(u_j)|\xi_j|^p - C_5\varepsilon|\xi_j|^p - [k_2(u_j)](x) - K(\varepsilon)\beta_1(x) \\
&\ge |\xi_j|^p(A - C_5\varepsilon) - \beta(x)
\end{aligned}$$

where $g_2(u_j) \ge A > 0$ (due to the weak lower semicontinuity of $g_2\colon V \to \mathbb{R}_+$ and the weak convergence $u_j \rightharpoonup u$) and $\beta(x) = K(\varepsilon)\beta_1(x) + [k_2(u_j)](x)$ is still bounded a.e., because $\{k_2(u_j)\} \subset L^1(\Omega)$ is bounded and therefore convergent a.e. up to a subsequence. The desired inequality follows by choosing $\varepsilon = A/(2C_5)$. □

**Claim 3.4.** *The convergence $p_j^- \to 0$ a.e. holds.*

*Proof.* Split $p_j(x)$ as

$$\begin{aligned} p_j(x) &= \sum_{|\alpha|=k} \big[a_\alpha(x,\zeta_j,\xi_j;u_j) - a_\alpha(x,\zeta_j,\xi;u_j)\big](D^\alpha u_j - D^\alpha u) \\ &\quad + \sum_{|\alpha|=k} \big[a_\alpha(x,\zeta_j,\xi;u_j) - a_\alpha(x,\zeta,\xi;u)\big](D^\alpha u_j - D^\alpha u) \\ &\quad + \sum_{|\alpha|\le k-1} \big[a_\alpha(x,\zeta_j,\xi_j;u_j) - a_\alpha(x,\zeta,\xi;u)\big](D^\alpha u_j - D^\alpha u) \\ &=: q_j(x) + r_j(x) + s_j(x) \end{aligned} \tag{3.10}$$

Let $\chi_j$ be the characteristic function of the level set $\{x \in \Omega \ : \ p_j^-(x) > 0\}$ and write

$$-p_j^- = \chi_j q_j + \chi_j r_j + \chi_j s_j.$$

First, note that $\chi_j q_j \ge 0$ a.e. due to the monotonicity assumption **(A3)**, so it is enough to prove $\chi_j r_j \to 0$ a.e. and $\chi_j s_j \to 0$ a.e. Lemma 3.3 ensures that there exists $\beta\colon \Omega \to \mathbb{R}$ a.e. bounded such that

$$|\xi_j(x)|^p \le \beta(x),$$

for all $x \in \Omega$ such that $p_j(x) < 0$. Therefore $\{\chi_j(x)\xi_j(x)\}$ is bounded for a.e. $x \in \Omega$. By **(A2)**, **(A5)** and $\zeta_j \to \zeta$ a.e. (from (3.2)), we find that $\chi_j r_j \to 0$ a.e. and $\chi_j s_j \to 0$ a.e. for a subsequence, from which $p_j^- \to 0$ a.e. follows. □

In summary, we have that $\{p_j^-\}$ is equiintegrable and tight, and $p_j^- \to 0$ a.e. A corollary of the Vitali convergence theorem (Theorem A.6 below) yields that these conditions are actually *necessary and sufficient* to ensure the convergence

$$\int_\Omega p_j^- \to 0,$$

as $j \to \infty$. Recalling (3.5), we have in summary

$$\int_\Omega p_j \to 0 \tag{3.11}$$

as $j \to \infty$. Then conclusion **(PM1)** of pseudomonotonicity is established:

$$\begin{aligned} \langle A(u_j), u_j - u\rangle &= \langle A(u_j) - A(u), u_j - u\rangle + \langle A(u), u_j - u\rangle \\ &= \int_\Omega p_j + \langle A(u), u_j - u\rangle \to 0. \end{aligned}$$

Turning to the proof of **(PM2)**, first note that (3.11) implies that $p_j \to 0$ a.e. up to a subsequence.

**Claim 3.5.** *The convergence $\xi_j \to \xi$ a.e. holds.*

*Proof.* It follows from estimate (3.6) that $\{\xi_j\}$ is bounded a.e. Fix an $x_0 \in \Omega$ such that $\{\xi_j(x_0)\}$ is bounded and $p_j(x_0) \to 0$. Assume for contradiction that $\xi_j(x_0) \to \xi'$ for a subsequence and some $\xi'$ such that $\xi' \ne \xi(x_0)$. Since we have $\zeta_j \to \zeta$ a.e., by using decomposition (3.10) and **(A1)**, it follows that $r_j \to 0$ and $s_j \to 0$ a.e. But then the continuity assumption **(A5)** implies

$$p_j(x_0) \to 0 = \sum_{|\alpha|=k} \big[a_\alpha(x_0,\zeta,\xi';u) - a_\alpha(x_0,\zeta,\xi;u)\big](\xi'_\alpha - D^\alpha u(x_0))$$

Thus **(A3)** yields $\xi'_\alpha = D^\alpha u(x_0)$, which is a contradiction. □

Finally, we prove $A(u_j) \rightharpoonup A(u)$ in $V^*$. By the Vitali convergence theorem

$$\begin{aligned}\langle A(u_j), v\rangle &= \sum_{|\alpha|\le k}\int_\Omega a_\alpha(x,\eta_j;u_j)D^\alpha v(x)\,dx \\ &\to \sum_{|\alpha|\le k}\int_\Omega a_\alpha(x,\eta;u)D^\alpha v(x)\,dx,\end{aligned}$$

because the integrand is equiintegrable and tight by Proposition A.3 (3) and the a.e. convergence $a_\alpha(x,\eta_j;u_j) \to a_\alpha(x,\eta;u)$ follows from **(A5)**. □

**Proposition 3.6.** *If* ***(A4′)*** *holds then* $A\colon V \to V^*$ *is coercive.*

*Proof.* We have for $u \in V$ with sufficiently large $\|u\|_V$,

$$\begin{aligned}\langle A(u), u\rangle &\ge g_2(u)\int_\Omega |\xi|^p + \sum_{\ell=0}^{k-1}(|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^p)\,dx - \int_\Omega [k_2(u)](x)\,dx \\ &\ge C\|u\|_V^{-\sigma^*}\|u\|_V^p - c^*\|u\|_V^\sigma \\ &\ge C'\|u\|_V^{p-\sigma^*}\end{aligned}$$

for some $C, C' > 0$. Therefore $\langle A(u), u\rangle/\|u\|_V \to +\infty$ if $\|u\|_V \to \infty$, because $p - \sigma^* > 1$. □

# 4 Examples

Here we formulate examples satisfying **(A1)–(A5)** and **(A4′)**. For all $|\alpha| = \ell$, with $\ell = 0, 1, \ldots, k$ consider

$$a_\alpha(x,\eta;u) = \Psi_\ell(H_\ell(u))\Big[a_\ell(x)\chi_\ell(G_\ell(u))\big(|\eta^{(\ell)}|^{r_\ell-2} + |\eta^{(\ell)}|^{p-2}\big)\eta_\alpha + b_\alpha(x)M_\alpha(u)\Big],$$

where $p \le r_\ell \le p_\ell^*$ and $m \le a_\ell(x) \le M$ for some constants $m, M > 0$. (We remind the reader that $\eta^{(k)} = \xi$ and $p_k^* = p$, so that the highest order $a_\alpha$ reads

$$a_\alpha(x,\eta;u) = \Psi_k(H_k(u))\big[a_k(x)\chi_k(G_k(u))|\xi|^{p-2}\xi_\alpha + b_\alpha(x)M_\alpha(u)\big],$$

where $|\alpha| = k$, which is reminiscent of the $p$-Laplacian.) We propose the following two possibilities for the choice of $\Psi_\ell$ and $H_\ell$.

1. Let $H_\ell\colon W^{k-1,p}(\Omega') \to L^\infty(\Omega)$ be a bounded linear map (with $\Omega' \subset \Omega$ a bounded domain) and let $\Psi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $\Psi_\ell(\nu) \ge C_\Psi/(1+|\nu|)^{-\sigma^*}$ for some $C_\Psi > 0$ and large $|\nu|$.

2. Let $H_\ell\colon V \to \mathbb{R}$ be a bounded linear functional and let $\Psi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $\Psi_\ell(\nu) \ge C_\Psi/(1+|\nu|^{\sigma^*})$ for some $C_\Psi > 0$.

Again, we may choose $\chi_\ell$ and $G_\ell$ as follows.

1. Let $G_\ell\colon W^{k-1,p}(\Omega') \to L^{p'}(\Omega)$ be a bounded linear map and let $\chi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $m \le \chi_\ell(\nu) \le M$ for some constants $m, M > 0$.

2. Let $G_\ell\colon V \to \mathbb{R}$ be a bounded linear functional and let $\chi_\ell\colon \mathbb{R} \to \mathbb{R}_+$ be continuous with $m \le \chi_\ell(\nu) \le M$ for some constants $m, M > 0$.

Finally, for fixed any $|\alpha| = \ell$, let $2 \le p_1 \le p$, $m = 1, \ldots, k$ and let

$$M_\alpha \colon V \to W^{m,p_1}(\Omega) \quad (\text{or } \mathbb{R})$$

be a bounded map such that

$$\|M_\alpha(u)\|_{W^{m,p_1}(\Omega)} \le \text{const}\|u\|_V^{\gamma_\alpha}, \tag{4.1}$$
$$\|M_\alpha(u) - M_\alpha(v)\|_{W^{m,p_1}(\Omega)} \le \text{const}\|u - v\|_V^{\gamma_\alpha}, \tag{4.2}$$

where $0 < \gamma_\alpha < \frac{p}{r'_\ell}$; also, let $\lambda_\alpha = q_\alpha / r'_\ell$ and $b_\alpha \in L^{r'_\ell \lambda'_\alpha}(\Omega)$ where

$$\begin{cases} p_1 < q_\alpha < \frac{np_1}{n - mp_1} & \text{if} \quad m < \frac{n}{p_1} \\ q_\alpha > 0 & \text{otherwise} \end{cases}$$

(or, if $M_\alpha \colon V \to \mathbb{R}$, then

$$|M_\alpha(u)| \le \text{const}\|u\|_V^{\gamma_\alpha}, \tag{4.3}$$
$$|M_\alpha(u) - M_\alpha(v)| \le \text{const}\|u - v\|_V^{\gamma_\alpha}, \tag{4.4}$$

with $\gamma_\alpha = \sigma / r'_\ell$, $b_\alpha \in L^{r'_\ell}(\Omega)$).

Under these hypotheses, **(A1)** and **(A3)** are satisfied. Note that the continuous embeddings $W^{m,p_1}(\Omega) \subset L^{q_\alpha}(\Omega)$ hold, so

$$\|M_\alpha(u)\|_{L^{q_\alpha}(\Omega)} \le \text{const}\|M_\alpha(u)\|_{W^{m,p_1}(\Omega)} \le \text{const}\|u\|_V^{\gamma_\alpha}.$$

Therefore, by Hölder's inequality and (4.1)

$$\begin{aligned} \int_\Omega |b_\alpha(x)|^{r'_\ell} |M_\alpha(u)|^{r'_\ell}\, dx &\le \|b_\alpha\|^{r'_\ell}_{L^{r'_\ell \lambda'_\alpha}(\Omega)} \Big[ \int_\Omega |M_\alpha(u)|^{r'_\ell \lambda_\alpha} \Big]^{1/\lambda_\alpha} \\ &= \|b_\alpha\|^{r'_\ell}_{L^{r'_\ell \lambda'_\alpha}(\Omega)} \|M_\alpha(u)\|^{q_\alpha/\lambda_\alpha}_{L^{q_\alpha}(\Omega)} \\ &\le \text{const}\|b_\alpha\|^{r'_\ell}_{L^{r'_\ell \lambda'_\alpha}(\Omega)} \|u\|_V^{q_\alpha \gamma_\alpha / \lambda_\alpha} \\ &\le c^* \|u\|_V^\sigma, \end{aligned} \tag{4.5}$$

where $\sigma = q_\alpha \gamma_\alpha / \lambda_\alpha = r'_\ell \gamma_\alpha < p$ for the case $M_\alpha \colon V \to W^{m,p_1}(\Omega)$. The case $M_\alpha \colon V \to \mathbb{R}$ is treated similarly.

**Claim 4.1.** *Assumption **(A2)** holds.*

*Proof.* The growth condition reads

$$\begin{aligned} |a_\alpha(x, \eta, \xi; u)| \le{}& \Psi_\ell(H_\ell(u)) |a_\ell(x)| \chi_\ell(G_\ell(u)) \big( |\eta^{(\ell)}|^{r_\ell - 1} + |\eta^{(\ell)}|^{p-1} \big) \\ &+ \Psi_\ell(H_\ell(u)) |b_\alpha(x) M_\alpha(u)|. \end{aligned}$$

Then $g_1(u) = \Psi_\ell(H_\ell(u)) M^2$ is a bounded functional by assumption. Letting

$$[k_1^\alpha(u)](x) = \Psi_\ell(H_\ell(u)) |b_\alpha(x) M_\alpha(u)|,$$

we find by (4.5) that $k_1^\alpha \colon V \to L^{r'_\ell}(\Omega)$ is bounded.

Proving the compactness of $k_1^\alpha$ requires more effort (except when $M_\alpha \colon V \to \mathbb{R}$). To this end, suppose that $\{u_j\} \subset V$ is a bounded sequence. Let $\{\Omega_i\}$ be the sequence guaranteed to exist by assumption **(A0)**. Then $\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega\setminus\Omega_i)} \to 0$. Using the compact embedding $W^{m,p_1}(\Omega_i) \subset\subset L^{q_\alpha}(\Omega_i)$ we can choose subsequences of $\{u_j\}$ as follows. Let $\{u_{1j}\} \subset \{u_j\}$ be a subsequence such that

$$\|M_\alpha(u_{1j}) - M_\alpha(u_{1m})\|_{L^{q_\alpha}(\Omega_1)} < 1 \quad \text{for } j,m = 1,2,3,\dots$$

Let $\{u_{2j}\} \subset \{u_{1j}\}$ be a subsequence such that

$$\|M_\alpha(u_{2j}) - M_\alpha(u_{2m})\|_{L^{q_\alpha}(\Omega_2)} < \frac{1}{2} \quad \text{for } j,m = 2,3,\dots$$

Continuing this way, for fixed $i$ let $\{u_{ij}\} \subset \{u_{i-1,j}\}$ be a subsequence such that

$$\|M_\alpha(u_{ij}) - M_\alpha(u_{im})\|_{L^{q_\alpha}(\Omega_i)} < \frac{1}{i} \quad \text{for } j,m = i,i+1,\dots$$

It follows that the diagonal sequence $\{u_{jj}\}$ satisfies

$$\|M_\alpha(u_{jj}) - M_\alpha(u_{mm})\|_{L^{q_\alpha}(\Omega_i)} < \frac{1}{i} \quad \text{for } j,m = i,i+1,\dots$$

Using Hölder's inequality, we find for $j,m \geq i$

$$\begin{aligned}
&\int_\Omega |b_\alpha(x)|^{r'_\ell} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{r'_\ell}\, dx \\
&\quad = \left( \int_{\Omega\setminus\Omega_i} + \int_{\Omega_i} \right) |b_\alpha(x)|^{r'_\ell} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{r'_\ell}\, dx \\
&\quad \leq \text{const} \|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega\setminus\Omega_i)} \left[ \int_{\Omega\setminus\Omega_i} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{q_\alpha}\, dx \right]^{1/\lambda_\alpha} \\
&\qquad + \text{const} \|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega_i)} \left[ \int_{\Omega_i} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{q_\alpha}\, dx \right]^{1/\lambda_\alpha}.
\end{aligned}$$

Here, $\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega\setminus\Omega_i)} \to 0$ and $\|b_\alpha\|_{L^{r'_\ell \lambda'_\alpha}(\Omega_i)}$ is bounded. By assumption (4.2), the first integral is bounded and for the second integral we have

$$\int_{\Omega_i} |M_\alpha(u_{jj}) - M_\alpha(u_{mm})|^{q_\alpha}\, dx \leq \frac{1}{i^{q_\alpha}} \to 0$$

if $j,m \geq i$ and $i \to \infty$. □

We now show that **(A4')** holds. It is enough to estimate the terms of

$$\begin{aligned}
&\sum_{|\alpha|=\ell} a_\alpha(x,\eta;u)\eta_\alpha \\
&\quad = \Psi_\ell(H_\ell(u)) a_\ell(x) \chi_\ell(G_\ell(u)) (|\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^p) + \sum_{|\alpha|=\ell} \Psi_\ell(H_\ell(u)) b_\alpha(x) M_\alpha(u) \eta_\alpha
\end{aligned}$$

for all $\ell = 0,1,\dots,k$. The first term may be estimated from below by

$$C\Psi_\ell(H_\ell(u)) \left( |\eta^{(\ell)}|^{r_\ell} + |\eta^{(\ell)}|^p \right)$$

for some constant $C > 0$. Here, the quantity $\Psi_\ell(H_\ell(u))$ satisfies

$$\Psi_\ell(H_\ell(u)) \geq \frac{C_\Psi}{|H_\ell(u)|^{\sigma^*}+1} \geq \frac{C_\Psi}{\|H_\ell(u)\|^{\sigma^*}_{L^\infty(\Omega)}+1} \geq \frac{C'_\Psi}{\|u\|^{\sigma^*}_{W^{k-1,p}(\Omega')}+1} \geq \frac{C'_\Psi}{\|u\|^{\sigma^*}_V+1}.$$

The terms of the sum may be bounded from above by Young's inequality,

$$\begin{aligned}\Psi_\ell(H_\ell(u))|b_\alpha(x)M_\alpha(u)\eta_\alpha| &\leq \varepsilon\Psi_\ell(H_\ell(u))|\eta_\alpha|^{r_\ell} + C^*(\varepsilon)|b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell}\\ &\leq \varepsilon\Psi_\ell(H_\ell(u))|\eta^{(\ell)}|^{r_\ell} + C^*(\varepsilon)|b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell}.\end{aligned}$$

Choosing a sufficiently small $\varepsilon > 0$, it turns out that it is enough to estimate the $L^1(\Omega)$-norm of the expression

$$[k_2^\alpha(u)](x) = |b_\alpha(x)|^{r'_\ell}|M_\alpha(u)|^{r'_\ell},$$

which, using (4.5), satisfies

$$\|k_2^\alpha\|_{L^1(\Omega)} \leq c^*\|u\|^\sigma_V.$$

The proof of compactness of $k_2^\alpha$ is analogous to that of $k_1^\alpha$. The required $k_2$ in Assumption **(A4')** is given by the pointwise maximum of $k_2^\alpha$ over all $|\alpha| \leq k$.

To finish the argument, note that assumption **(A5)** is satisfied since the functions $\Phi_\ell$, $\chi_\ell$ and $\Psi_\alpha$ are continuous and the operators $H_\ell$, $G_\ell$ and $M_\alpha$ are continuous in the respective Sobolev and Lebesgue spaces. Thus if $u_j \rightharpoonup u$ in $V$, then for a subsequence $H_\ell(u_j)$, $G_\ell(u_j)$, $M_\alpha(u_j)$ are convergent a.e. in $\Omega$.

**Example 4.2.** For a more concrete example to $M_\alpha$, consider the following. In the case $M_\alpha\colon V \to W^{m,p_1}(\Omega)$, let $M_\alpha(u) = \widetilde{H}_\alpha(u)$ where $\widetilde{H}_\alpha\colon V \to W^{m,p_1}(\Omega)$ is a continuous linear operator. For a more concrete example, consider

$$[\widetilde{H}_\alpha(u)](x) = \sum_{|\alpha|\leq k}\int_\Omega G_\alpha(x,y)D^\alpha u(y)\,dy,$$

where the functions $G_\alpha\colon \Omega\times\Omega \to \mathbb{R}$ satisfy

$$x \mapsto \left[\int_\Omega |D^\beta G_\alpha(x,y)|^{p'}\,dy\right]^{1/p'} \in L^{p_1}(\Omega) \quad \text{for } |\beta| \leq m.$$

In the case $M_\alpha\colon V \to \mathbb{R}$, let $M_\alpha(u) = \Phi_\alpha(\widetilde{H}_\alpha(u))$, where $\widetilde{H}_\alpha\colon V \to \mathbb{R}_+$ is a bounded linear functional and $\Phi_\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $|\Phi_\alpha(\nu_1) - \Phi_\alpha(\nu_2)| \leq C_\Phi|\nu_1 - \nu_2|^{\sigma/r'_\ell}$. Note that $\Phi_\alpha(\nu) \leq C_\Phi|\nu|^{\sigma/r'_\ell}$ follows automatically.

The operators $H_\ell\colon W^{k-1,p}(\Omega') \to L^\infty(\Omega)$ and $G_\ell\colon W^{k-1,p}(\Omega') \to L^{p'}(\Omega)$ can be defined by the formula

$$(Bu)(x) = \sum_{|\alpha|\leq k-1}\int_{\Omega'} G_\alpha(x,y)D^\alpha u(u)\,dy,$$

where the measurable functions $G_\alpha\colon \Omega\times\Omega' \to \mathbb{R}$ satisfy

$$x \mapsto \left[\int_{\Omega'} |G_\alpha(x,y)|^{p'}\,dy\right]^{1/p'} \in L^\infty(\Omega) \ \text{ and } \ L^{p'}(\Omega),$$

respectively.

**Example 4.3.** Now suppose that $\Omega' \subset \Omega$ is a bounded domain with sufficiently smooth boundary. Let $V = H_0^1(\Omega)$, $V_1 = H_0^1(\Omega') \subset W^{1,2}(\Omega')$, $m = 1$, $p_1 = 2$ and let $B_\alpha \colon V_1 \to V_1^*$ be an elliptic operator given by

$$\langle B_\alpha(v), w\rangle = \int_\Omega \left[ \sum_{j,k=1}^{n} a_{jk}^\alpha(x) D_j v D_k w + c_\alpha(x) v w \right] dx,$$

where $v, w \in V_1$ and $a_{jk}^\alpha \in L^\infty(\Omega)$ form a uniformly elliptic coefficient matrix and $c_\alpha(x) \geq c_0 > 0$. The strong form of this operator is "$-\operatorname{div} \mathbf{A}^\alpha Dv + c_\alpha v$", where $\mathbf{A}^\alpha = (a_{jk}^\alpha)$. Then we may take $M_\alpha(u) = \widetilde{H}_\alpha(u) = v$, where $v \in V_1$ is a unique solution to $B_\alpha(v) = u\big|_{\Omega'} \in V_1^*$. Then $\widetilde{H}_\alpha = B_\alpha^{-1} \colon V \to V_1 \subset W^{1,2}(\Omega)$ is a continuous linear operator. (A function $v \in H_0^1(\Omega')$ belongs to $W^{1,2}(\Omega)$ if it is extended by 0 in $\Omega \setminus \Omega'$.)

**Example 4.4.** More generally, let $V_1 \subset W^{m,p_1}(\Omega)$ be a closed subspace (which may depend on $\alpha$) and let $N_\alpha \colon V_1 \to V_1^*$ be a bounded, strictly monotone and coercive operator that satisfies

$$\langle N_\alpha(v_1) - N_\alpha(v_2), v_1 - v_2\rangle \geq c_2 \|v_1 - v_2\|_{V_1}^{p_1},$$

and

$$\langle N_\alpha(v), v\rangle \geq c_3 \|v\|_{V_1}^{p_1}.$$

Then for every $w \in V_1^*$ there exists a unique element $v \in V_1$ such that $N_\alpha(v) = w$ and the mapping $N_\alpha^{-1} \colon V_1^* \to V_1$ is Hölder continuous:

$$\|N_\alpha^{-1}(w_1) - N_\alpha^{-1}(w_2)\|_{V_1}^{1/(p_1-1)} \leq \text{const}\|w_1 - w_2\|_{V_1^*}.$$

Now let

$$M_\alpha(u) := N_\alpha^{-1}(h_\alpha u),$$

for all $u \in V$, where $h_\alpha \in L^{p_1' r}(\Omega)$ is some fixed function that makes $h_\alpha u \in L^{p_1'}(\Omega) \subset V_1^*$ if $p > 2$, and we may take $h_\alpha \equiv 1$ if $p = 2$. We have that $M_\alpha(u) \in V_1$ and $M_\alpha \colon V \to W^{m,p_1}(\Omega)$ is bounded map:

$$\begin{aligned}
\|M_\alpha(u)\|_{W^{m,p_1}(\Omega)} = \|N_\alpha^{-1}(h_\alpha u)\|_{W^{m,p_1}(\Omega)} &\leq \|h_\alpha u\|_{V_1^*}^{1/(p_1-1)} \\
&\leq \text{const}\|h_\alpha u\|_{L^{p_1'}(\Omega)}^{1/(p_1-1)} = \text{const}\left[\int_\Omega |h_\alpha u|^{p_1'}\right]^{1/p_1} \\
&\leq \text{const}\left[\left[\int_\Omega |h_\alpha|^{p_1' r}\right]^{1/r} \left[\int_\Omega |u|^p\right]^{p_1'/p}\right]^{1/p_1} \\
&\leq \text{const}\|h_\alpha\|_{L^{p_1' r}(\Omega)}^{1/(p_1-1)} \|u\|_V^{p_1'-1},
\end{aligned}$$

where $r = p/(p - p_1')$. The exponent $\gamma_\alpha' = p_1' - 1$ satisfies $\gamma_\alpha < p/r_\ell'$ if $2 \leq p_1 < p$.

## A Equiintegrability and tightness

This appendix collects some results used in the paper; see e.g. [5] for proofs.

**Definition A.1.** A sequence $\{f_j\}$ of measurable functions $f_j \colon \Omega \to \mathbb{R}$ is said to be *equiintegrable* over $\Omega$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f_j| < \varepsilon \quad \text{for all } j \in \mathbb{N} \text{ and all } E \subset \Omega \text{ measurable with } |E| < \delta.$$

**Definition A.2.** A sequence $\{f_j\}$ is said to be *tight* over $\Omega$ if for all $\varepsilon > 0$ there exists $E_0 \subset \Omega$ measurable with $|E_0| < \infty$ such that

$$\int_{\Omega \setminus E_0} |f_j| < \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

Clearly, a dominated sequence inherits equiintegrability (tightness). More precisely, if $|g_j| \leq |f_j|$ and $\{f_j\}$ is equiintegrable (tight), then $\{g_j\}$ is equiintegrable (tight). Similarly, equiintegrability (tightness) is inherited to a smaller domain $\Omega' \subset \Omega$. The following useful properties are easily established.

**Proposition A.3.** *The following statements hold.*

1. *If $\{f_j\} \subset L^1(\Omega)$, $f \in L^1(\Omega)$ and $f_j \to f$ in $L^1(\Omega)$, then $\{f_j\}$ is equiintegrable and tight.*
2. *If $\{f_j\}$ and $\{g_j\}$ are equiintegrable and tight, then $\{\alpha f_j + \beta g_j\}$ is equiintegrable and tight for all $\alpha, \beta \in \mathbb{R}$.*
3. *If $\{f_j\} \subset L^q(\Omega)$ is bounded and $\{g_j\} \subset L^{q'}(\Omega)$ (where $q' = q/(q-1)$ and $1 < q < \infty$) with $\{|g_j|^{q'}\}$ equiintegrable and tight, then $\{f_j g_j\}$ is equiintegrable and tight.*

**Theorem A.4** (Vitali convergence theorem)**.** *Suppose that $|\Omega| < \infty$ and let $\{f_j\}$ be equiintegrable over $\Omega$. If $f_j \to f$ a.e. on $\Omega$, then $f \in L^1(\Omega)$ and*

$$\int f_j \to \int f \quad \text{as } j \to \infty.$$

**Theorem A.5** (Vitali convergence theorem)**.** *Let $\{f_j\}$ be equiintegrable and tight over $\Omega$. If $f_j \to f$ a.e. on $\Omega$, then $f \in L^1(\Omega)$ and*

$$\int f_j \to \int f \quad \text{as } j \to \infty.$$

**Theorem A.6** (Sharp Vitali convergence theorem)**.** *Suppose that $h_j \geq 0$ a.e. on $\Omega$. Then*

$$\int h_j \to 0 \quad \text{as } j \to \infty$$

*if and only if $h_j \to 0$ a.e. on $\Omega$ and $\{h_j\}$ is equiintegrable and tight over $\Omega$.*

## References


[1] R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Pure and Applied Mathematics, Vol. 140, Elsevier Science, 2003. MR2424078

[2] H. Brezis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité (in French), *Ann. Inst. Fourier (Grenoble)* **18**(1968), 115–175. MR0270222

[3] F. E. Browder, Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc. Nat. Acad. Sci. U.S.A.* **74**(1977), 2659–2661. MR0445124

[4] R. Landes, V. Mustonen, Boundary value problems for strongly nonlinear second order elliptic equations, *Boll. Un. Mat. Ital. B (6)* **4**(1985), 15–32. MR783329

[5] H. L. Royden, *Real analysis*, Vol. 198, Macmillan, New York, 1988. MR1013117

[6] L. Simon, On strongly nonlinear elliptic equations in unbounded domains, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **28**(1985), 241–252. MR856997

[7] L. Simon, Strongly nonlinear elliptic variational inequalities with nonlocal boundary conditions, in: *Qualitative theory of differential equations (Szeged, 1988)*, *Coll. Math. Soc. J. Bolyai*, Vol. 53, 1988, pp. 605–620. MR1062683

[8] L. Simon, Existence results for strongly nonlinear functional elliptic problems *Coll. Math. Soc. J. Bolyai*, **62**(1991), 271–287. MR1468761

[9] L. Simon, Application of monotone type operators to nonlinear PDE's, 2013. url

[10] J. R. L. Webb, Boundary value problems for strongly nonlinear elliptic equations, *J. London Math. Soc. (2)* **21**(1980), 123–132. MR576188; url