Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **23**, 1–10; doi: 10.14232/ejqtde.2016.1.23 http://www.math.u-szeged.hu/ejqtde/

Global existence and stability for second order functional evolution equations with infinite delay

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Received 15 January 2016, appeared 2 May 2016
Communicated by Nickolai Kosmatov

Abstract. In this article, the authors give sufficient conditions for existence and attractivity of mild solutions for second order semi-linear functional evolution equation in Banach spaces using Schauder's fixed point theorem. An example is provided to illustrate the result.

Keywords: semilinear functional differential equations of second order, mild solution, attractivity, evolution system, fixed-point, infinite delay, infinite interval.

2010 Mathematics Subject Classification: 34G20, 34G25, 34K20, 34K30.

1 Introduction

In this paper, we consider the existence and attractivity of mild solutions of the second order evolution equation

$$y''(t) - A(t)y(t) = f(t, y_t), \qquad t \in J := [0, \infty), \tag{1.1}$$

$$y_0 = \phi, \qquad y'(0) = \tilde{y}, \tag{1.2}$$

where $(E, |\cdot|)$ a real Banach space, $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed operators from E into E that generate an evolution system of operators $\{\mathcal{U}(t,s)\}_{(t,s)\in J\times J}$ for $0 \le s \le t < +\infty$, $f : J \times \mathcal{B} \to E$ is a Carathéodory function, $\mathcal{B}$ is an abstract phase space to be specified later, $\tilde{y} \in E$, and $\phi \in \mathcal{B}$.

For any continuous function y and any $t \ge 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here, $y_t(\cdot)$ represents the history of the state up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

Functional differential equations arise in many areas of applications, and for basic results and background information, we refer the reader to the monographs of Hale and Verduyn Lunel [14] and Kolmanovskii and Myshkis [20]. There are many results concerning the second-order functional evolution equations; see, for example, Abbas and Benchohra [1], Balachandran *et al.* [5,6], Fattorini [12], Hernández [15], Hernández and McKibben [16], Henríquez and

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Vásquez [17], and Travis and Webb [23]. Fractional evolution equations and inclusions have been studied by Wang, Fečkan, and Zhou [24], Wang, Ibrahim, and Fečkan [25], Wang and Zhang [26], and Wang and Zhou [27].

Differential equations on infinite intervals frequently occur in mathematical modeling of various applied problems. For example, in the study of unsteady flow of a gas through a semi-infinite porous medium [3, 19], the analysis of the mass transfer on a rotating disk in a non-Newtonian fluid [4], heat transfer in the radial flow between parallel circular disks [22], investigation of the temperature distribution in the problem of phase change of solids with temperature dependent thermal conductivity [22], as well as numerous problems arising in the study of circular membranes [2, 9, 10], plasma physics [4], nonlinear mechanics, and non-Newtonian fluid flows [2].

This paper is organized as follows. In Section 2, we recall some definitions and facts about evolution systems. In Section 3, we prove the existence of mild solutions to the problem (1.1)–(1.2). In Section 4, we show the attractivity of mild solutions, and in the last section, an example is given to show the applicability of our results.

To our knowledge, no papers devoted to the global existence and the attractivity of mild solutions of problem (1.1)–(1.2) have appeared in the literature. The present work attempts to fill that gap.

2 Preliminaries

Let E be a Banach space with the norm $|\cdot|$ and let $BC(J,E)$ be the Banach space of all bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\| = \sup_{t\in J} |y(t)|.$$

Let $\mathcal{X}$ be the space defined by

$$\mathcal{X} = \left\{ y : \mathbb{R} \to E \mid y_{|_J} \in BC(J,E) \text{ and } y_0 \in \mathcal{B} \right\},$$

where by $y_{|_J}$ we mean the restriction of y to J.

In this paper, we will use an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [13] and follow the terminology used in [18]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms.

(A_1) If $y : (-\infty, b) \to E$, $b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for any $t \in [0, b)$ the following conditions hold:

- (i) $y_t \in \mathcal{B}$;
- (ii) there exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
- (iii) there exist functions $K, M : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\, |y(s)| : 0 \leq s \leq t \,\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Remark 2.1. In the sequel, we assume that K and M are bounded on J and

$$\gamma := \max\left\{\sup_{t\in\mathbb{R}_+}\{K(t)\}, \sup_{t\in\mathbb{R}_+}\{M(t)\}\right\}.$$

For additional details we refer the reader, for example, to the book by Hino *et al.* [18].

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ that is dense in E and independent of t. The existence of solutions to the problem (1.1)–(1.2) is related to the existence of an evolution operator $\mathcal{U}(t,s)$ for the homogeneous problem

$$y''(t) = A(t)y(t), \qquad t \in J. \tag{2.1}$$

This concept of evolution operator has been developed by Kozak [21].

Definition 2.2. A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in J \times J : s \leq t\}$, is called an evolution operator of the equation (2.1) if the following conditions hold.

(D_1) For any $x \in E$ the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is continuously differentiable and:

  (a) for any $t \in J$, $\mathcal{U}(t,t) = 0$;

  (b) for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)x\big|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x\big|_{t=s} = -x$.

(D_2) For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of class C^2, and:

  (a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x = A(t)\mathcal{U}(t,s)x$;

  (b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x = \mathcal{U}(t,s)A(s)x$;

  (c) $\frac{\partial^2}{\partial s\partial t}\mathcal{U}(t,s)x\big|_{t=s} = 0$.

(D_3) For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, $\frac{\partial^3}{\partial t^2\partial s}\mathcal{U}(t,s)x$ and $\frac{\partial^3}{\partial s^2\partial t}\mathcal{U}(t,s)x$ exist, and:

  (a) $\frac{\partial^3}{\partial t^2\partial s}\mathcal{U}(t,s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$;

  (b) $\frac{\partial^3}{\partial s^2\partial t}\mathcal{U}(t,s)x = \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$ is continuous.

The following compactness criterion in $C(\mathbb{R}_+, E)$ is particularly useful.

Lemma 2.3 (Corduneanu [7]). *Let $C \subset BC(\mathbb{R}_+, E)$ be a set satisfying the following conditions:*

(i) C is bounded in $BC(\mathbb{R}_+, E)$;

(ii) the functions belonging to C are equicontinuous on any compact interval of $\mathbb{R}_+$;

(iii) the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of $\mathbb{R}_+$;

Then C is relatively compact in $BC(\mathbb{R}_+, E)$.

Our final lemma is the well known Schauder fixed point theorem [11].

Lemma 2.4. *Let C be a nonempty closed convex bounded subset of a Banach space E. Then any continuous compact mapping $T : C \to C$ has a fixed point.*

3 Main result

We begin with the definition of a mild solution to our problem.

Definition 3.1. A function $y \in \mathcal{X}$ is called a mild solution to the problem (1.1)–(1.2), if y is continuous and

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0, \\ -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y} + \int_0^t \mathcal{U}(t,s) f(s, y_s) ds, & \text{if } t \in J. \end{cases} \tag{3.1}$$

To prove our results we introduce the following conditions.

(H_1) There exists a constant $\widehat{M} \geq 1$ and $\omega > 0$ such that

$$\|\mathcal{U}(t,s)\|_{B(E)} \leq \widehat{M} e^{-\omega(t-s)} \quad \text{for any } (t,s) \in \Delta.$$

(H_2) There exists a constant $\tilde{M} \geq 0$ such that

$$\left\| \frac{\partial}{\partial s}\mathcal{U}(t,s) \right\|_{B(E)} \leq \tilde{M}.$$

(H_3) There exists a function $p \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t)(\|u\|_{\mathcal{B}} + 1) \quad \text{for a.e. } t \in J \text{ and any } u \in \mathcal{B}.$$

(H_4) For any $(t,s) \in \Delta$, we have

$$\lim_{t \to +\infty} \int_0^t e^{-w(t-s)} p(s) ds = 0.$$

Theorem 3.2. *If conditions (H_1)–(H_4) hold, then the problem (1.1)–(1.2) admits at least one mild solution.*

Proof. It is clear that the fixed points of the operator $T : \mathcal{X} \to \mathcal{X}$ defined by

$$T(y)(t) = \begin{cases} \phi(t), & \text{if } t \leq 0, \\ -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y} + \int_0^t \mathcal{U}(t,s) f(s, y_s) ds, & \text{if } t \in J, \end{cases} \tag{3.2}$$

are mild solutions of problem (1.1)–(1.2).

For $\phi \in \mathcal{B}$, let $x : (-\infty, +\infty) \to E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ -\frac{\partial}{\partial s}\mathcal{U}(t,0)\phi(0) + \mathcal{U}(t,0)\tilde{y} & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For any function $z \in \mathcal{X}$, we set

$$y(t) = x(t) + z(t).$$

It is clear that y satisfies (3.2) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$z(t) = \int_0^t \mathcal{U}(t,s) f(t, x_s + z_s) ds. \tag{3.3}$$

In the sequel, we always take $\mathcal{X}_0$ to be the Banach space

$$\mathcal{X}_0 = \{z \in \mathcal{X} : \; z_0 = 0\}$$

endowed with the norm

$$\|z\|_{\mathcal{X}_0} = \sup_{t\in J} |z(t)| + \|z_0\|_{\mathcal{B}} = \sup_{t\in J} |z(t)|.$$

Now, we can consider the operator $L : \mathcal{X}_0 \to \mathcal{X}_0$ given by

$$Lz(t) = \int_0^t \mathcal{U}(t,s) f(s, z_s + x_s) ds, \quad \text{for } t \in J.$$

The problem (1.1) having a solution is equivalent to L having a fixed point. To prove that problem (1.1) does in fact have a solution, we begin with the following estimation.

For any $z \in \mathcal{X}_0$ and $t \in J$, we have

$$\begin{aligned}
\|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\
&\leq K(t)|z(t)| + K(t)\|\frac{\partial}{\partial s}\mathcal{U}(t,0)\|_{B(E)}\|\phi\|_{\mathcal{B}} \\
&\quad + K(t)\|\mathcal{U}(t,0)\|_{B(E)}|\tilde{y}| + M(t)\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{X}_0} + \gamma\tilde{M}\|\phi\|_{\mathcal{B}} + \gamma\hat{M}e^{-\omega t}|\tilde{y}| + \gamma\|\phi\|_{\mathcal{B}} \\
&\leq \gamma\|z\|_{\mathcal{X}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma\hat{M}|\tilde{y}|. \qquad (3.4)
\end{aligned}$$

Now, we will show that the operator L satisfied the conditions of Schauder's fixed point theorem.

Step 1. L is continuous.

Let $(z^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{X}_0$ such that $z^k \to z$ in $\mathcal{X}_0$; then for any $t \in J$, we obtain

$$\begin{aligned}
|L(z^k)(t) - L(z)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |f(t, x_s + z_s^k) - f(t, x_s + z_s)| \, ds \\
&\leq \hat{M}\int_0^t e^{-\omega(t-s)} |f(s, z_s^k + x_s) - f(s, z_s + x_s)| \, ds.
\end{aligned}$$

Hence, from the continuity of the function f and the Lebesgue dominated convergence theorem, we obtain

$$\|Lz_k - Lz\|_{\mathcal{X}_0} \to 0 \quad \text{as } k \to +\infty.$$

So L is continuous.

Step 2. L maps bounded sets in $\mathcal{X}_0$ into bounded sets.

Let $\eta > 0$ satisfy

$$\eta \geq \frac{\hat{M}\big(\gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma\hat{M}|\tilde{y}| + 1\big)\|p\|_{L^1}}{1 - \hat{M}\gamma\|p\|_{L^1}},$$

and consider the set $D_\eta = \{z \in \mathcal{X}_0 : \|z\|_{\mathcal{X}_0} \leq \eta\}$. If $z \in D_\eta$, then from (H_3) and (3.4),

$$\begin{aligned}
|L(z)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |f(s, x_s + z_s)| \, ds \\
&\leq \hat{M}\int_0^t e^{-\omega(t-s)} p(s)(\|z_s + x_s\|_{\mathcal{B}} + 1) ds. \\
&\leq \hat{M}\Big(\gamma\|z\|_{\mathcal{X}_0} + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma\hat{M}|\tilde{y}| + 1\Big)\int_0^t e^{-\omega(t-s)} p(s) ds \\
&\leq \hat{M}\xi\|p\|_{L^1} \leq \eta,
\end{aligned}$$

where

$$\xi := \gamma\eta + \gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1) + \gamma\hat{M}|\tilde{y}| + 1.$$

Thus, the operator L maps D_η into itself.

Step 3. $L(D_\eta)$ relatively compact.

Let D_η be a bounded subset of $\mathcal{X}_0$. To show that $L(D_\eta)$ is relatively compact we will use Lemma 2.3.

♦ $L(D_\eta)$ is equicontinuous.

Let $s, t \in [0, b]$ with $t > s$ and $z \in D_\eta$. Then, we have

$$\begin{aligned}
|(Lz)(t) - (Lz)(s)| &= \left| \int_0^s (\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)) f(\tau, z_\tau + x_\tau) d\tau + \int_s^t \mathcal{U}(t,\tau) f(\tau, z_\tau + x_\tau) d\tau \right| \\
&\leq \int_0^s \|\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\|_{B(E)}\ p(\tau) \Big(\|z_\tau + x_\tau\|_{\mathcal{B}} + 1 \Big) d\tau \\
&\quad + \hat{M} \int_s^t e^{-\omega(t-\tau)}\ p(\tau) \Big(\|z_\tau + x_\tau\|_{\mathcal{B}} + 1 \Big) d\tau.
\end{aligned}$$

From inequality (3.4), we obtain

$$|(Lz)(t) - (Lz)(s)| \leq \xi \int_0^s \|\mathcal{U}(t,\tau) - \mathcal{U}(s,\tau)\|_{B(E)}\ p(\tau)\ d\tau + \hat{M}\xi \int_s^t p(\tau) d\tau.$$

The right-hand side of the above inequality tends to zero as $t - s \to 0$, which implies that $L(D_\eta)$ is equicontinuous.

♦ $\Lambda := \{(Lz)(t) : z \in D_\eta\}$ is relatively compact in E.

Let $t \in J$ be a fixed and let $0 < \varepsilon < t \leq b$. For $z \in D_\eta$, we define

$$L_\varepsilon(z)(t) = \mathcal{U}(t, t-\varepsilon) \int_0^{t-\varepsilon} \mathcal{U}(t-\varepsilon, s) f(s, z_s + x_s) ds.$$

Since $\mathcal{U}(t,s)$ is a compact operator, and the set $\Lambda_\varepsilon := \{(L_\varepsilon z)(t) : z \in D_\eta\}$ is the image of bounded set in E by $\mathcal{U}(t,s)$, we see that Λ_ε is precompact in E. Furthermore, for $z \in D_\eta$, we have

$$\begin{aligned}
|L(z)(t) - L_\varepsilon(z)(t)| &\leq \int_{t-\varepsilon}^t \|\mathcal{U}(t,s)\|_{B(E)} \Big| f(s, z_s + x_s) \Big| ds \\
&\leq \int_{t-\varepsilon}^t \|\mathcal{U}(t,s)\|_{B(E)} p(s) \Big(\|z_s + x_s\|_{\mathcal{B}} + 1 \Big) ds \\
&\leq \xi\hat{M} \int_{t-\varepsilon}^t e^{-\omega(t-s)} p(s) ds.
\end{aligned}$$

The right-hand side tends to zero as $\varepsilon \to 0$, so $L_\varepsilon(z)$ converge uniformly to $L(z)$, which implies that $D_\eta(t)$ is precompact in E.

♦ L is equiconvergent.

Let $z \in D$; then from conditions (H_1)–(H_3) and (3.4), we have

$$|(Lz)(t)| \leq \hat{M}\xi \int_0^t e^{-\omega(t-s)} p(s) ds,$$

and it follows immediately from (H_4) that $|(Lz)(t)| \to 0$ as $t \to +\infty$. Hence,

$$\lim_{t \to +\infty} |(Lz)(t) - (Lz)(+\infty)| = 0,$$

which implies that L is equiconvergent.

Therefore, by Lemma 2.3, $L(D_\eta)$ is relatively compact. Hence, by Lemma 2.4, the operator L has at least one fixed point which in turn is a mild solution of problem (1.1)–(1.2). □

4 Attractivity of solutions

In this section we study the local attractivity of solutions the problem (1.1)–(1.2).

Definition 4.1 ([8])**.** Solutions of (1.1) are locally attractive if there exists a closed ball $\bar{B}(z^*,\sigma)$ in the space $\mathcal{X}_0$ for some $z^* \in \mathcal{X}$ such that, for any solutions z and $\tilde{z}$ of (1.1)–(1.2) belonging to $\bar{B}(z^*,\sigma)$, we have

$$\lim_{t\to+\infty}(z(t)-\tilde{z}(t)) = 0.$$

Under the assumptions of Section 3, let z^* be a solution to (1.1)–(1.2) and $\bar{B}(z^*,\sigma)$ the closed ball in $\mathcal{X}_0$ where σ satisfies

$$\sigma \geq \frac{2\hat{M}\Big(\gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1)+\gamma\hat{M}|\tilde{y}|+1\Big)\|p\|_{L^1}}{1-2\hat{M}\gamma\|p\|_{L^1}}.$$

Then, for $z\in\bar{B}(z^*,\sigma)$, from (H_1)–(H_3) and (3.4), we have

$$\begin{aligned}
|(Lz)(t)-z^*(t)| &= |(Lz)(t)-(Lz^*)(t)|\\
&\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}\Big|f(s,z_s+x_s)-f(s,z_s^*+x_s)\Big|ds\\
&\leq \hat{M}\int_0^t e^{-\omega(t-s)}p(t)\Big(\|z_s+x_s\|_{\mathcal{B}}+\|z_s^*+x_s\|_{\mathcal{B}}+2\Big)ds\\
&\leq 2\hat{M}(\gamma\sigma+\gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1)+\gamma\hat{M}|\tilde{y}|+1)\|p\|_{L^1}\\
&\leq \sigma.
\end{aligned}$$

Therefore, $L(\bar{B}(z^*,\sigma))\subset\bar{B}(z^*,\sigma)$. So, for any solution $z\in\bar{B}(z^*,\rho)$ to problem (1.1) and $t\in J$, we have

$$\begin{aligned}
|z(t)-z^*(t)| &= |(Lz)(t)-(Lz^*)(t)|\\
&\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}\Big|f(s,z_s+x_s)-f(s,z_s^*+x_s)\Big|ds\\
&\leq \hat{M}\int_0^t e^{-\omega(t-s)}p(t)\Big(\|z_s+x_s\|_{\mathcal{B}}+\|z_s^*+x_s\|_{\mathcal{B}}+2\Big)ds\\
&\leq 2\hat{M}(\gamma\sigma+\gamma\|\phi\|_{\mathcal{B}}(\tilde{M}+1)+\gamma\hat{M}|\tilde{y}|+1)\int_0^t e^{-\omega(t-s)}p(t)ds.
\end{aligned}$$

Hence, from (H_4), we conclude that

$$\lim_{t\to\infty}|z(t)-\tilde{z}(t)|=0.$$

Consequently, the solutions of problem (1.1)–(1.2) are locally attractive.

5 An example

Consider the second order Cauchy problem

$$\begin{cases}
\dfrac{\partial^2}{\partial t^2}y(t,\tau) = \dfrac{\partial^2}{\partial\tau^2}y(t,\tau)+a(t)\dfrac{\partial}{\partial t}y(t,\tau)\\
\qquad\qquad +\displaystyle\int_{-\infty}^t b(t-s)y(s,\tau)ds, & t\in J:=[0,\infty),\ \tau\in[0,2\pi],\\
y(t,0)=y(t,2\pi)=0, & t\in J,\\
y(\theta,\tau)=\phi(\theta,\tau),\quad \dfrac{\partial}{\partial t}y(0,\tau)=\psi(\tau), & \theta\in(-\infty,0],\tau\in[0,2\pi],
\end{cases}\tag{5.1}$$

where $a, b : J \to \mathbb{R}$ are continuous functions and $\phi(\theta, \cdot) \in \mathcal{B}$.

Let $X = L^2(\mathbb{R}, \mathbb{C})$ the space of 2π-periodic square-integrable functions from $\mathbb{R}$ into $\mathbb{C}$, and let $H^2(\mathbb{R}, \mathbb{C})$ denote the Sobolev space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{R}, \mathbb{C})$.

We consider the operator $A_1 y(\tau) = y''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R}, \mathbb{C})$. In addition, we take $A_2(t)y(s) = a(t)y'(s)$ defined on $H^1(\mathbb{R}, \mathbb{C})$, and consider the closed linear operator $A(t) = A_1 + A_2(t)$ which generates an evolution operator $\mathcal{U}$ defined by

$$\mathcal{U}(t,s) = \sum_{n \in \mathbb{Z}} z_n(t,s) \langle x, w_n \rangle w_n,$$

where z_n is a solution to the scalar initial value problem

$$\begin{cases} z''(t) = -n^2 z(t) + ina(t)z(t), \\ z(s) = 0, \quad z'(s) = z_1. \end{cases} \tag{5.2}$$

Define the operator $f : J \times \mathcal{B} \to X$ by

$$\begin{aligned} f(t,\varphi)(\tau) &= \int_{-\infty}^{t} b(t-s)\varphi(s)(\tau)ds, && \tau \in [0, 2\pi], \\ w(t)(\tau) &= y(t,\tau), && t \geq 0,\ \tau \in [0, 2\pi], \\ \phi(s)(\tau) &= y(s,\tau), && -\infty < s \leq 0,\ \tau \in [0, 2\pi], \end{aligned}$$

and

$$\frac{d}{dt} w(0)(\tau) = \frac{\partial}{\partial t} y(0,\tau), \qquad \tau \in [0, 2\pi].$$

Then, (5.1) can be written in the abstract form (1.1)–(1.2) with A and f defined above. Now, the existence and attractivity of a mild solution can be concluded from an application of Theorem 3.2.

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