Electronic Journal of Qualitative Theory of Differential Equations
2016, No. **27**, 1–13; doi: 10.14232/ejqtde.2016.1.27 http://www.math.u-szeged.hu/ejqtde/

Lyapunov-type inequalities for nonlinear impulsive systems with applications

Zeynep Kayar[✉ 1] and **Ağacık Zafer**[2, 3]

[1]Department of Mathematics, Faculty of Science, Yüzüncü Yıl University, 65080 Van, Turkey
[2]Department of Mathematics, Middle East Technical University, 06800 Ankara, Turkey
[3]Department of Mathematics and Statistics, American University of the Middle East, Kuwait

Received 21 November 2015, appeared 14 May 2016
Communicated by Leonid Berezansky

Abstract. We obtain new Lyapunov-type inequalities for systems of nonlinear impulsive differential equations, special cases of which include the impulsive Emden–Fowler equations and half-linear equations. By applying these inequalities, sufficient conditions are derived for the disconjugacy of solutions and the boundedness of weakly oscillatory solutions.

Keywords: differential equation, nonlinear, impulse, Lyapunov inequality, weakly oscillatory, disconjugate.

2010 Mathematics Subject Classification: 34A37, 34C10.

1 Introduction

Impulsive differential equations have played an important role in recent years because they provide better mathematical models in both physical and social sciences. Although there is an extensive literature on the Lyapunov-type inequalities for linear and nonlinear ordinary differential equations [1,2,7,9–12] and systems [3,8,14,16] as well as linear impulsive differential equations [5] and impulsive systems [4,6], there is not much done for nonlinear systems with or without impulse [15]. The present work stems from the corresponding ones in [11,15].

We consider the nonlinear impulsive system

$$\begin{aligned} &x' = \alpha_1(t)x + \beta_1(t)|u|^{\gamma-2}u, \quad u' = -\alpha_1(t)u - \beta_2(t)|x|^{\beta-2}x, \quad t \neq \tau_i \\ &x(\tau_i^+) = \xi_i x(\tau_i^-), \quad u(\tau_i^+) = \xi_i u(\tau_i^-) - \eta_i |x(\tau_i^-)|^{\beta-2}x(\tau_i^-), \\ &t \geq t_0, \quad i \in \mathbb{N} := \{1,2,\dots\}, \end{aligned} \tag{1.1}$$

where $\{\tau_i\}$ denotes the sequence of impulse moments of time such that

$$0 \leq t_0 < \tau_1 < \cdots < \tau_i < \cdots, \qquad \lim_{i\to\infty} \tau_i = +\infty.$$

We assume that the following conditions hold:

✉Corresponding author. Email: zeynepkayar@yyu.edu.tr, zykayar@gmail.com

(i) $\gamma > 1$, $\beta > 1$ are real constants,

(ii) $\alpha_1, \beta_1, \beta_2 \in \mathrm{PC}[t_0, \infty)$, $\beta_1(t) \geq 0$, where $\mathrm{PC}[t_0, \infty)$ denote the set of functions $f : [t_0, \infty) \to \mathbb{R}$ such that $f \in C([t_0, \infty) \setminus \{\tau_1, \tau_2, \ldots, \tau_i, \ldots\})$ and limits from the left $f(\tau_i^-)$ and from the right $f(\tau_i^+)$ exist (finite) at each τ_i for $i \in \mathbb{N}$,

(iii) ξ_i, η_i are real numbers such that $\xi_i \neq 0$ for $i = 1, 2, \ldots$

By a solution of system (1.1), we mean $x, u \in \mathrm{PC}[t_0, \infty)$ satisfying system (1.1) for $t \geq t_0$. Such a solution is said to be proper if

$$\sup\{|x(s)| + |u(s)| : t \leq s < \infty\} > 0$$

for any $t \geq t_0$. We tacitly assume that system (1.1) has proper solutions. For a classical theory of impulsive differential systems we refer the reader in particular to a seminal book [13].

Note that many equations can be written as a special case of (1.1). For instance, impulsive Emden–Fowler type differential equation

$$\begin{aligned} &(p(t)|x'|^{\alpha-2}x')' + q(t)|x|^{\beta-2}x = 0, \quad t \neq \tau_i, \\ &x(\tau_i^+) = \xi_i \, x(\tau_i^-), \\ &p(\tau_i^+)|x'(\tau_i^+)|^{\alpha-2}x'(\tau_i^+) = \xi_i \, p(\tau_i^-)|x'(\tau_i^-)|^{\alpha-2}x'(\tau_i^-) - \eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\ &t \geq t_0, \quad i \in \mathbb{N} \end{aligned} \tag{1.2}$$

is equivalent to

$$\begin{aligned} &x' = \beta_1(t)|u|^{\gamma-2}u, \quad u' = -\beta_2(t)|x|^{\beta-2}x, \quad t \neq \tau_i \\ &x(\tau_i^+) = \xi_i x(\tau_i^-), \quad u(\tau_i^+) = \xi_i u(\tau_i^-) - \eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\ &t \geq t_0, \quad i \in \mathbb{N} \end{aligned}$$

with

$$\frac{1}{\gamma} + \frac{1}{\alpha} = 1, \qquad \beta_1(t) = p^{1-\gamma}(t), \qquad \beta_2(t) = q(t).$$

If we set $\alpha = \beta$, then we obtain the impulsive half-linear equation

$$\begin{aligned} &(p(t)|x'|^{\beta-2}x')' + q(t)|x|^{\beta-2}x = 0, \quad t \neq \tau_i, \\ &x(\tau_i^+) = \xi_i \, x(\tau_i^-), \\ &p(\tau_i^+)|x'(\tau_i^+)|^{\beta-2}x'(\tau_i^+) = \xi_i \, p(\tau_i^-)|x'(\tau_i^-)|^{\beta-2}x'(\tau_i^-) - \eta_i|x(\tau_i^-)|^{\beta-2}x(\tau_i^-) \\ &t \geq t_0, \quad i \in \mathbb{N}. \end{aligned} \tag{1.3}$$

Therefore, our results will be applied to important equations as special cases.

Let us recall the definition of a (generalized) zero.

Definition 1.1 ([4,5])**.** A real number c is called a generalized zero of a function f if $f(c^-) = 0$ or $f(c^+) = 0$. If f is continuous at c, then c becomes a real zero. If no such zero exists then we will write $f(t) \neq 0$.

2 Main results

Let k be a piece-wise constant function defined by

$$k(t) = \begin{cases} \xi_1\xi_2\cdots\xi_j, & t \in (\tau_j, \tau_{j+1}], \quad j \in \mathbb{N}, \\ 1, & t \in [t_0, \tau_1] \end{cases}$$

and $\{k_j\}$ be sequence given by

$$k_j = \begin{cases} \xi_1\xi_2\cdots\xi_j, & j \geq 1 \\ 1, & j = 0. \end{cases}$$

We will also use the usual notations

$$f^+(t) = \max\{f(t), 0\}, \qquad (f_i)^+ = \max\{f_i, 0\}.$$

Our first result is the following.

Theorem 2.1. *Let $(x(t), u(t))$ be a real solution of the impulsive system (1.1) and α be the conjugate number of γ, that is,*

$$\frac{1}{\alpha} + \frac{1}{\gamma} = 1.$$

If $x(t)$ has two consecutive generalized zeros at t_1 and t_2, then we have the following Lyapunov-type inequality:

$$\begin{aligned} M^{\beta-\alpha} \exp\left(\frac{\alpha}{2}\int_{t_1}^{t_2} |\alpha_1(u)| du\right) \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2} dt\right]^{\alpha-1} \\ \times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2} dt + \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+ |k_{i-1}|^{\beta-2}\right] \geq 2^\alpha, \end{aligned} \tag{2.1}$$

where

$$M = \sup_{t\in(t_1,t_2)} \left|\frac{x(t)}{k(t)}\right|. \tag{2.2}$$

Proof. Define

$$z(t) = \frac{x(t)}{k(t)}, \qquad v(t) = \frac{u(t)}{k(t)}, \qquad t \geq t_0.$$

It is easy to see that

$$\begin{aligned} & z' = \alpha_1(t)z + \beta_1(t)|k(t)|^{\gamma-2}|v|^{\gamma-2}v, \quad v' = -\alpha_1(t)v - \beta_2(t)|k(t)|^{\beta-2}|z|^{\beta-2}z, \quad t \neq \tau_i \\ & z(\tau_i^+) = z(\tau_i^-), \quad v(\tau_i^+) = v(\tau_i^-) - (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-), \\ & t \geq t_0, \quad i \in \mathbb{N}. \end{aligned} \tag{2.3}$$

We may define $z(\tau_i) = z(\tau_i^-)$ to make $z(t)$ continuous on $[t_1, t_2]$. Thus, $z(t_1) = z(t_2) = 0$ and $z(t) \neq 0$ for all $t \in (t_1, t_2)$. Without loss of generality, we can take $z(t) > 0$. Since $z(t)$ is continuous, we can choose $\tau \in (t_1, t_2)$ such that

$$z(\tau) = \max_{t\in(t_1,t_2)} \{z(t)\} = M.$$

From (2.3), we have

$$\begin{aligned}(vz)' &= \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma} - \beta_2(t)|k(t)|^{\beta-2}z^{\beta}(t), \quad t \neq \tau_i \\ (vz)(\tau_i^+) - (vz)(\tau_i^-) &= -(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i).\end{aligned} \tag{2.4}$$

Integrating the first equation in (2.4) from t_1 to t_2 and using $\beta_2^+(t) = \max\{\beta_2(t), 0\}$ and $(\eta_i/\xi_i)^+ = \max\{\eta_i/\xi_i, 0\}$ yields,

$$\begin{aligned}\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt \leq &\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}z^{\beta}(t)dt \\ &+ \sum_{t_1 \leq \tau_i < t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i).\end{aligned} \tag{2.5}$$

From the first equation in (2.3), we have

$$\left[z(t)\exp\left(-\int_{t_1}^{t} \alpha_1(u)du\right)\right]' = \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-2}v(t)\exp\left(-\int_{t_1}^{t} \alpha_1(u)du\right) \tag{2.6}$$

and

$$\left[z(t)\exp\left(\int_{t}^{t_2} \alpha_1(u)du\right)\right]' = \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-2}v(t)\exp\left(\int_{t}^{t_2} \alpha_1(u)du\right). \tag{2.7}$$

Integrating (2.6) from t_1 to τ, we get

$$z(\tau) = \int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-2}v(t)\exp\left(\int_{t}^{\tau} \alpha_1(u)du\right)dt,$$

which implies

$$z(\tau) \leq \exp\left(\int_{t_1}^{\tau} |\alpha_1(u)|du\right)\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt. \tag{2.8}$$

Similarly, by integrating (2.7) from τ to t_2, we have

$$z(\tau) \leq \exp\left(\int_{\tau}^{t_2} |\alpha_1(u)|du\right)\int_{\tau}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt. \tag{2.9}$$

Put

$$Q_1 = \frac{z(\tau)}{\exp\left(\int_{t_1}^{\tau} |\alpha_1(u)|du\right)} \quad \text{and} \quad Q_2 = \frac{z(\tau)}{\exp\left(\int_{\tau}^{t_2} |\alpha_1(u)|du\right)}.$$

Then we observe that

$$\begin{aligned}\frac{z(\tau)}{\exp\left(\frac{1}{2}\int_{t_1}^{t_2} |\alpha_1(u)|du\right)} &= \frac{z^{1/2}(\tau)z^{1/2}(\tau)}{\exp\left(\frac{1}{2}\int_{t_1}^{\tau} |\alpha_1(u)|du\right)\exp\left(\frac{1}{2}\int_{\tau}^{t_2} |\alpha_1(u)|du\right)} \\ &= \sqrt{Q_1Q_2} \\ &\leq \frac{Q_1+Q_2}{2} \\ &= \frac{z(\tau)}{2\exp\left(\int_{t_1}^{\tau} |\alpha_1(u)|du\right)} + \frac{z(\tau)}{2\exp\left(\int_{\tau}^{t_2} |\alpha_1(u)|du\right)}.\end{aligned} \tag{2.10}$$

Therefore, from (2.8), (2.9), and (2.10) we have

$$\frac{2z(\tau)}{\exp\left(\frac{1}{2}\int_{t_1}^{t_2}|\alpha_1(u)|du\right)} \leq \int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma-1}dt. \tag{2.11}$$

Applying Hölder's inequality to the right-hand side of (2.11) with indices α and γ and then using inequality (2.5) lead to

$$\begin{aligned}\frac{2z(\tau)}{\exp\left(\frac{1}{2}\int_{t_1}^{t_2}|\alpha_1(u)|du\right)} &\leq \left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}}\\ &\leq \left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\\ &\quad\times\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}z^{\beta}(t)dt+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i)\right]^{\frac{1}{\alpha}}.\end{aligned}$$

Since $z(\tau)\geq z(t)$ for all $t\in[t_1,t_2]$, we obtain

$$\begin{aligned}\frac{2z(\tau)}{\exp\left(\frac{1}{2}\int_{t_1}^{t_2}|\alpha_1(u)|du\right)} &\leq z^{\frac{\beta}{\alpha}}(\tau)\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\\ &\quad\times\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]^{\frac{1}{\alpha}}\end{aligned}$$

Finally, we use (2.2) in the last inequality to see that (2.1) holds. □

If we take $\beta=\alpha$, then the M dependence drops.

Theorem 2.2. *Let $(x(t),u(t))$ be a real solution of the impulsive system (1.1) and β be the conjugate number of γ, that is,*

$$\frac{1}{\beta}+\frac{1}{\gamma}=1.$$

If $x(t)$ has two consecutive generalized zeros at t_1 and t_2, then we have the following Lyapunov-type inequality:

$$\begin{aligned}&\exp\left(\frac{\beta}{2}\int_{t_1}^{t_2}|\alpha_1(u)|du\right)\left[\int_{t_1}^{t_2}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\\ &\times\left[\int_{t_1}^{t_2}\beta_2^+(t)|k(t)|^{\beta-2}dt+\sum_{t_1\leq\tau_i<t_2}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]\geq 2^{\beta}.\end{aligned}$$

Corollaries below are immediate.

Corollary 2.3. *Let α be the conjugate number of γ.*

If the impulsive Emden–Fowler equation (1.2) has a real solution $x(t)$ having two consecutive generalized zeros at t_1 and t_2, then we have the following Lyapunov-type inequality:

$$M^{\beta-\alpha}\left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\ge 2^{\alpha},$$

where

$$M=\sup_{t\in(t_1,t_2)}\left|\frac{x(t)}{k(t)}\right|. \tag{2.12}$$

Corollary 2.4. *Let β be the conjugate number of γ.*

If the impulsive half-linear equation (1.3) has a real solution $x(t)$ having two consecutive generalized zeros at t_1 and t_2, then we have the following Lyapunov-type inequality:

$$\left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\ge 2^{\beta}.$$

Theorem 2.5. *Let α be the conjugate number of γ and M be given by (2.12).*

If the impulsive Emden–Fowler equation (1.2) has a real solution $x(t)$ having two consecutive generalized zeros at t_1 and t_2, then there exists $\tau\in(t_1,t_2)$ such that the following inequalities hold:

(i) If $\tau\in(\tau_{n-1},\tau_n)$ for some n, then

$$M^{\beta-\alpha}\left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\left[\int_{t_1}^{\tau} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<\tau}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\ge 1$$

and

$$M^{\beta-\alpha}\left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\left[\int_{\tau}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{\tau\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\ge 1.$$

(ii) If $\tau=\tau_n$, then

$$M^{\beta-\alpha}\left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\times\left[\int_{t_1}^{\tau} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<\tau}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}+\max_{i=1,2,\dots,m}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\ge 1$$

and

$$M^{\beta-\alpha}\left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\left[\int_{\tau}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{\tau\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]\ge 1.$$

Proof. (i) The proof is obtained by applying the proof of the Theorem 2.1 step by step for the intervals (t_1,τ) and (τ,t_2) separately and using $z'(\tau)=0$.

(ii) Let $\tau=\tau_n$ and $\tau_n<s<\tau_{n+1}$. Set

$$\beta_1=p^{1-\gamma}(t),\qquad \beta_2(t)=q(t).$$

If we repeat the procedure in the proof of Theorem 2.1, for the interval (t_1,s), we get

$$z(s) \le \left[\int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \left[\int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}}. \tag{2.13}$$

On the other hand, one can show that

$$\begin{aligned}\int_{t_1}^{s}(vz)'dt &= z(s)v(s^-) + \sum_{t_1\le\tau_i<s}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i)\\ &= \int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt - \int_{t_1}^{s} \beta_2(t)|k(t)|^{\beta-2}z^{\beta}(t).\end{aligned} \tag{2.14}$$

Substituting (2.14) into (2.13), we have

$$\begin{aligned} z(s) \le &\left[\int_{t_1}^{s} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\\ &\times\left[\int_{t_1}^{s} \beta_2(t)|k(t)|^{\beta-2}z^{\beta}(t)dt + \sum_{t_1\le\tau_i<s}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i) + z(s)v(s^-)\right]^{\frac{1}{\alpha}}.\end{aligned}$$

Letting $s \to \tau^+$ gives

$$\begin{aligned} z(\tau) \le &\left[\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\\ &\times\left[\int_{t_1}^{\tau} \beta_2(t)|k(t)|^{\beta-2}z^{\beta}(t)dt + \sum_{t_1\le\tau_i<\tau^+}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i) + z(\tau)v(\tau^+)\right]^{\frac{1}{\alpha}}.\end{aligned}$$

In view of $z(\tau)v(\tau^+) \le 0$ and $z(\tau)v(\tau^-) \ge 0$, we get

$$\begin{aligned} z(\tau) \le z(\tau)^{\frac{\beta}{\alpha}} &\left[\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\\ &\times\left[\int_{t_1}^{\tau} \beta_2(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<\tau}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2} + \frac{\eta_n}{\xi_n}|k_{n-1}|^{\beta-2}\right]^{\frac{1}{\alpha}}.\end{aligned}$$

Since $z(t) \le z(\tau)$ for all $t \in [t_1, t_2]$, we obtain the desired inequality

$$\begin{aligned} 1 \le M^{\beta-\alpha} &\left[\int_{t_1}^{\tau} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\\ &\times\left[\int_{t_1}^{\tau} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<\tau}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2} + \max_{i=1,2,\dots,m}(\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right].\end{aligned}$$

Now, let $\tau = \tau_n$ and $s < \tau_n < \tau_{n+1}$. By the same procedure worked on (s, t_2), we get

$$z(s) \le \left[\int_{s}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \left[\int_{s}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}|v(t)|^{\gamma}dt\right]^{\frac{1}{\alpha}},$$

which in a similar manner above leads to

$$\begin{aligned} z(s) \le &\left[\int_{s}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}}\\ &\times\left[\int_{s}^{t_2} \beta_2(t)|k(t)|^{\beta-2}z^{\beta}(t)dt + \sum_{s\le\tau_i<t_2}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i) - z(s)v(s^+)\right]^{\frac{1}{\alpha}},\end{aligned}$$

and so as $s \to \tau^-$, we obtain

$$z(\tau) \le \left[\int_{\tau}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\frac{1}{\gamma}} \times \left[\int_{\tau}^{t_2} \beta_2(t)|k(t)|^{\beta-2}z^{\beta}(t)dt + \sum_{\tau\le\tau_i<t_2} (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}z^{\beta}(\tau_i) - z(\tau)v(\tau^-)\right]^{\frac{1}{\alpha}},$$

which yields

$$1 \le M^{\beta-\alpha}\left[\int_{\tau}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}\left[\int_{\tau}^{t_2} \beta_2^{+}(t)|k(t)|^{\beta-2}dt + \sum_{\tau\le\tau_i<t_2} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right].$$

□

Theorem 2.6. *Let β be the conjugate number of γ.*

If the impulsive half-linear equation (1.3) has a real solution $x(t)$ having two consecutive zeros at t_1 and t_2, then there exists $\tau \in (t_1, t_2)$ such that the following inequalities hold:

(i) If $\tau \in (\tau_{n-1}, \tau_n)$, for some $n = 1, 2, \ldots, m$, then

$$\left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\left[\int_{t_1}^{\tau} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<\tau} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right] \ge 1$$

and

$$\left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\left[\int_{\tau}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{\tau\le\tau_i<t_2} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right] \ge 1.$$

(ii) If $\tau = \tau_n$, then

$$\left[\int_{t_1}^{\tau} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \times \left[\int_{t_1}^{\tau} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<\tau} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2} + \max_{i=1,2,\ldots,m} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right] \ge 1$$

and

$$\left[\int_{\tau}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1}\left[\int_{\tau}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt + \sum_{\tau\le\tau_i<t_2} (\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right] \ge 1.$$

Remark 2.7. It may not be plausible at first to have a constant M in the inequalities, however they are still very useful in several applications, see the next section. Moreover, if β and γ are conjugate, then M disappears.

Remark 2.8. If there is no impulse, i.e. $\xi_i = 1$ and $\eta_i = 0$ for all $i \in \mathbb{N}$, then inequality (2.1) improves and generalizes inequality (10) in [15, Theorem 1]. Theorem 2.5 may be considered as an extension of [15, Theorem 3] to the impulsive equations.

Remark 2.9. If $\alpha = \beta = \gamma = 2$, system (1.1) is reduced to a system of 2 first-order linear impulsive differential equations studied in [4, 6]. We see that Theorem 2.1 is more general than both [4, Theorem 5.1] and [6, Theorem 3.1], and extends [14, Theorem 2.4] with $n = 1$ to the impulsive equations. In the absence of impulse effect, inequality (2.1) in Theorem 2.1 is sharper than the corresponding ones in [8] and [3].

Remark 2.10. When $\alpha_1(t) = 0$ and $\alpha = \beta = \gamma = 2$, we recover [5, Theorem 4.5] from Corollary 2.3 and/or Corollary 2.4.

3 Applications

3.1 Disconjugacy

In this section, by using the inequalities obtained in Section 2, we establish some disconjugacy results.

We first recall the disconjugacy definition.

Definition 3.1 ([4, 6]). The system (1.1) is called disconjugate (relatively disconjugate with respect to x) on an interval $[a, b]$ if and only if there is no real solution $(x(t), u(t))$ of (1.1) with a nontrivial x having two or more zeros on $[a, b]$.

Theorem 3.2. *Let α be the conjugate number of γ, and M be as in (2.2). If*

$$\begin{aligned} M^{\beta-\alpha} \exp\left(\frac{\alpha}{2}\int_{t_1}^{t_2} |\alpha_1(u)|du\right) \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ \times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right] < 2^\alpha, \end{aligned} \tag{3.1}$$

then system (1.1) is disconjugate on $[t_1, t_2]$.

Proof. Suppose on the contrary that there is a real solution $y(t) = (x(t), u(t))$ with nontrivial $x(t)$ having two zeros $s_1, s_2 \in [t_1, t_2]$ $(s_1 < s_2)$ such that $x(t) \neq 0$ for all $t \in (s_1, s_2)$. Applying Theorem 2.1 we see that

$$\begin{aligned} 2^\alpha &\le M^{\beta-\alpha} \exp\left(\frac{\alpha}{2}\int_{s_1}^{s_2} |\alpha_1(u)|du\right) \left[\int_{s_1}^{s_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ &\quad\times \left[\int_{s_1}^{s_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{s_1\le\tau_i<s_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right] \\ &\le M^{\beta-\alpha} \exp\left(\frac{\alpha}{2}\int_{t_1}^{t_2} |\alpha_1(u)|du\right) \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \\ &\quad\times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right]. \end{aligned}$$

Clearly, the last inequality contradicts (3.1). The proof is complete. □

Theorem 3.3. *Let β be the conjugate number of γ. If*

$$\begin{aligned} \exp\left(\frac{\beta}{2}\int_{t_1}^{t_2} |\alpha_1(u)|du\right) \left[\int_{t_1}^{t_2} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \\ \times \left[\int_{t_1}^{t_2} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1\le\tau_i<t_2} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2}\right] < 2^\beta, \end{aligned}$$

then system (1.1) *is disconjugate on* $[t_1, t_2]$.

We have the corresponding corollaries.

Corollary 3.4. *Let* α *be the conjugate number of* γ *and* M *be given by* (2.12). *If*

$$M^{\beta-\alpha}\left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} \times\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]<2^{\alpha},$$

then equation (1.2) *is disconjugate on* $[t_1, t_2]$.

Corollary 3.5. *Let* β *be the conjugate number of* γ. *If*

$$\left[\int_{t_1}^{t_2} p^{1-\gamma}(t)|k(t)|^{\gamma-2}dt\right]^{\beta-1} \times\left[\int_{t_1}^{t_2} q^{+}(t)|k(t)|^{\beta-2}dt+\sum_{t_1\le\tau_i<t_2}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}\right]<2^{\beta},$$

then equation (1.3) *is disconjugate on* $[t_1, t_2]$.

3.2 Weakly oscillatory solutions

We shall make use of the following definitions.

Definition 3.6. A proper solution $(x(t), u(t))$ of system (1.1) is said to be weakly oscillatory if $x(t)$ has arbitrarily large (generalized) zeros.

Definition 3.7. A proper solution $(x(t), u(t))$ of system (1.1) is said to be weakly bounded if $x(t)/k(t)$ is bounded on $[t_0, \infty)$. The solution is said to be bounded if both $x(t)/k(t)$ and $u(t)/k(t)$ are bounded on $[t_0, \infty)$.

Theorem 3.8. *Suppose that*

$$\begin{aligned} \exp\left(\frac{\alpha}{2}\int^{\infty}|\alpha_1(t)|dt\right)\left[\int^{\infty}\beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1}&<\infty,\\ \int^{\infty}\beta_2^{+}(t)|k(t)|^{\beta-2}dt&<\infty,\\ \sum_{\tau_i<\infty}(\eta_i/\xi_i)^{+}|k_{i-1}|^{\beta-2}&<\infty. \end{aligned} \tag{3.2}$$

Then the following hold.

(a) *Every weakly oscillatory proper solution* $(x(t), u(t))$ *of* (1.1) *is weakly bounded.*

(b) *For each weakly oscillatory proper solution* $(x(t), u(t))$ *of* (1.1), *we have*

$$\lim_{t\to\infty}\frac{x(t)}{k(t)}=0.$$

Proof. (a) Let $(x(t), u(t))$ be a weakly oscillatory proper solution of (1.1). Let $z(t) = x(t)/k(t)$. Suppose on the contrary that $z(t)$ is unbounded. Then there is a positive number T sufficiently large such that $|z(t)| > 1$ for some $t > T$. Since z is also oscillatory, there exist an interval (t_1, t_2) with $t_1 \geq T$ such that $z(t_1) = z(t_2) = 0$, and $\tau \in (t_1, t_2)$ such that

$$M = |z(\tau)| = \max\{|z(t)| : t_1 < t < t_2\} > 1.$$

Because of (3.2), increasing the size of t_1 if necessary, we may write that

$$\exp\left(\frac{\alpha}{2}\int_{t_1}^{\infty} |\alpha_1(t)|dt\right)\left[\int_{t_1}^{\infty} \beta_1(t)|k(t)|^{\gamma-2}dt\right]^{\alpha-1} < M^{\alpha-\beta} \tag{3.3}$$

and

$$\int_{t_1}^{\infty} \beta_2^+(t)|k(t)|^{\beta-2}dt + \sum_{t_1 \leq \tau_i < \infty} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2} < 1. \tag{3.4}$$

In view of (3.3) and (3.4), we see from (2.1) that

$$2 \leq M^{\frac{\beta-\alpha}{\alpha}} M^{\frac{\alpha-\beta}{\alpha}} = 1$$

This is a contradiction.

(b) From (a) we know that every weakly oscillatory solution is weakly bounded. Suppose on the contrary that $z(t)$ does not approach zero as $t \to \infty$. Then

$$\limsup_{t\to\infty} |z(t)| = L > 0.$$

Since z has arbitrarily large zeros, there exists an interval (t_1, t_2) with $t_1 \geq T$, where T is sufficiently large, such that $z(t_1) = z(t_2) = 0$. Choose τ in (t_1, t_2) such that

$$M = |z(\tau)| = \max\{|z(t)| : t \in (t_1, t_2)\} > L/2.$$

The remainder of the proof is similar to that of part (a), hence it is omitted. □

Corollary 3.9. *Let α be the conjugate number of γ. Suppose that*

$$\int^{\infty} p^{1-\gamma}(t)|k(t)|^{\gamma-2} < \infty, \qquad \int^{\infty} q^+(t)|k(t)|^{\beta-2}dt < \infty, \qquad \sum_{\tau_i<\infty} (\eta_i/\xi_i)^+|k_{i-1}|^{\beta-2} < \infty.$$

Then every oscillatory solution $x(t)$ of impulsive Emden–Fowler equation (1.2) satisfies

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = 0.$$

Remark 3.10. Corollary 3.9 is valid for solutions of impulsive half-linear equation (1.3) by taking $\alpha = \beta$.

We conclude by a theorem on boundedness of the weakly bounded solutions of (1.1).

Theorem 3.11. *Suppose that*

$$\int^{\infty} \alpha_1(u)du > -\infty,$$
$$\int^{\infty} |\beta_2(u)||k(u)|^{\beta-2} \exp\left(-\int_u^{\infty} \alpha_1(s)ds\right) du < \infty,$$
$$\sum_{\tau_i<\infty} |\eta_i/\xi_i||k_{i-1}|^{\beta-2} \exp\left(-\int_{\tau_i}^{\infty} \alpha_1(s)ds\right) < \infty.$$

Then every weakly bounded solution of (1.1) is bounded.

Proof. Given $z(t) = x(t)/k(t)$ is bounded, we only need to show that $v(t) = u(t)/k(t)$ is bounded as well. We know that

$$v' + \alpha_1(t)v = -\beta_2(t)|k(t)|^{\beta-2}|z|^{\beta-2}z, \quad t \neq \tau_i,$$
$$v(\tau_i^+) = v(\tau_i^-) - (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-).$$

Hence

$$\left[v(t)\exp\left(\int_\tau^t \alpha_1(s)ds\right)\right]' = -\exp\left(\int_\tau^t \alpha_1(s)ds\right)\beta_2(t)|k(t)|^{\beta-2}|z|^{\beta-2}z, \qquad t \neq \tau_i,$$
$$v(\tau_i^+) = v(\tau_i^-) - (\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-).$$

Integrating from τ to $t, \tau \leq t \leq t_2$, we get

$$v(t) = v(\tau)\exp\left(-\int_\tau^t \alpha_1(s)ds\right) - \int_\tau^t \exp\left(-\int_u^t \alpha_1(s)ds\right)\beta_2(u)|k(u)|^{\beta-2}|z|^{\beta-2}z du$$
$$- \sum_{\tau\leq\tau_i\leq t}(\eta_i/\xi_i)|k_{i-1}|^{\beta-2}|z(\tau_i^-)|^{\beta-2}z(\tau_i^-)\exp\left(\int_{\tau_i}^t -\alpha_1(s)ds\right),$$

which implies that $v(t)$ is bounded. □

References

[1] G. Borg, On a Liapounoff criterion of stability, *Amer. J. Math.* **71**(1949), No. 1, 67–70. MR0028500

[2] O. Došlý, P. Řehák, *Half-linear differential equations*, Mathematics Studies, Vol. 202, North-Holland 2005. MR2158903

[3] G. Sh. Guseinov, B. Kaymakçalan, Lyapunov inequalities for discrete linear Hamiltonian systems, *Comput. Math. Appl.* **45**(2003), 1399–1416. MR2000605; url

[4] G. Sh. Guseinov, A. Zafer, Stability criteria for linear periodic impulsive Hamiltonian systems, *J. Math. Anal. Appl.* **335**(2007), 1195–1206. MR2346900; url

[5] G. Sh. Guseinov, A. Zafer, Stability criterion for second order linear impulsive differential equations with periodic coefficients, *Math. Nachr.* **281**(2008), 1273–1282. MR2442705; url

[6] Z. Kayar, A. Zafer, Stability criteria for linear Hamiltonian systems under impulsive perturbations, *Appl. Math. Comput.* **230**(2014), 680–686. url

[7] M. G. Kreĭn, On certain problems on the maximum and minimum of characteristic values and on the Lyapunov zones of stability, *Prikl. Math. Mekh.* **15**(1951), 323–348; *Amer. Math. Soc. Transl. Ser. 2* **1**(1955), 139–148. MR0043980

[8] M. G. Kreĭn, The basic propositions of the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients (in Russian), in: *In memory of Aleksandr Aleksandrovič Andronov*, Izdat. Acad. Nauk SSSR (Moscow), 1955, pp. 413–498; translation in: *Topics in differential and integral equations and operator theory*, Operator Theory: Advances and Applications, Vol. 7, pp. 1–105 MR0075382; url

[9] C. Lee, C. Yeh, C. Hong, R. P. Agarwal, Lyapunov and Wirtinger inequalities, *Appl. Math. Lett.* **17**(2004), No. 7, 847–853. MR2072845; url.

[10] M. A. Liapounoff, Problème général de la stabilité du mouvement (in French), *Ann. Fac. Sci. Toulouse Math.* **9**(1907), 203–474; *Annals of Mathematics Studies*, No. 17, Princeton University Press, Princeton, N. J.; Oxford University Press, London, 1947, pp. 203–474. MR0021186

[11] B. G. Pachpatte, Lyapunov type integral inequalities for certain differential equations, *Georgian Math. J.* **4**(1997) 139–148. MR1439592; url

[12] J. P. Pinasco, Lower bounds for the eigenvalues of the one dimensional p-Laplacian, *Abstr. Appl. Anal.* **2004**, No. 2, 147–153. MR2058270; url

[13] A. M. Samoilenko, N. A. Perestyuk, *Impulsive differential equations*, World Scientific, Singapore, 1995. MR1355787; url

[14] X. H. Tang, M. Zhang, Lyapunov inequalities and stability for linear Hamiltonian systems, *J. Differential Equations* **252**(2012), 358–381. MR2852210; url

[15] A. Tiryaki, M. Ünal, D. Çakmak, Lyapunov-type inequalities for nonlinear systems, *J. Math. Anal. Appl.* **332**(2007), 497–511. MR2319679; url

[16] X. Wang, Stability criteria for linear periodic Hamiltonian systems, *J. Math. Anal. Appl.* **367**(2010), 329–336. MR2600402; url