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Global stability and bifurcation analysis of a delayed predator–prey system with prey immigration

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Abstract. A delayed predator–prey system with a constant rate immigration is considered. Local and global stability of the equilibria are studied, a fixed point bifurcation appears near the boundary equilibrium and Hopf bifurcation occurs near the positive equilibrium when the time delay passes some critical values. We also show the existence of the global Hopf bifurcation, and the properties of the fixed point bifurcation and the stability and direction of the Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem.

Keywords: delay, stability, bifurcation, center manifold, normal form.

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1 Introduction

Since the Lotka–Volterra model was first prosed in 1920s, it has been studied in various models. Furthermore, many ecological concepts such as diffusion, functional responses and time delays have been added to the Lotka–Volterra equations to gain more accurate description and better understanding [1, 11, 15–18]. In [19] the author studied a Rosenzweig–MacArthur model first. In this model the prey has a logistic growth and the predator has a Holling II functional response. In [12–14, 20, 27] the global stability are discussed. There are also many researches on the limit cycle of Rosenweig–MacArthur model [6, 21, 26, 28]. Brauer et al. studied the stability of predator–prey systems with constant rate harvesting and stocking in [2–5]. Sugie et al. discussed the existence and uniqueness of limit cycles in predator–prey systems with a constant immigration in [23].

In this paper, we study the delayed Rosenweig–MacArthur model with a constant rate immigration, which has the following form:

$$\begin{aligned} \dot{x}(t) &= rx(t)\left(1-\frac{x(t)}{k}\right) - \frac{x(t)y(t)}{a+x(t)} + b, \\ \dot{y}(t) &= -dy(t) + \frac{\mu x(t-\tau)y(t-\tau)}{a+x(t-\tau)}, \end{aligned} \tag{1.1}$$

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where a, b, d, k, r, μ are all positive constants, and the meaning of them are the same as those in [23], and $\tau \geq 0$ is the constant delay due to the gestation of the predator. The initial conditions for system (1.1) take the form

$$\begin{array}{lll} x(\theta) = \phi_1(\theta), & y(\theta) = \phi_2(\theta), & \\ \phi_1(\theta) \geq 0, & \phi_2(\theta) \geq 0, & \theta \in [-\tau, 0), \\ \phi_1(0) > 0, & \phi_2(0) > 0, & \end{array} \tag{1.2}$$

where $(\phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], \mathbb{R}^2_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2_{+0}$, where $\mathbb{R}^2_{+0} = \{(x_1, x_2) : x_i \geq 0,\ i = 1, 2\}$.

Solving the algebraic equation

$$\begin{aligned} rx\left(1 - \frac{x}{k}\right) - \frac{xy}{a+x} + b = 0, \\ -dy + \frac{\mu xy}{a+x} = 0, \end{aligned}$$

we get that the system (1.1) has equilibria

$$E_0(x_0, 0) = \left(\frac{k}{2} \pm \frac{\sqrt{k^2r^2 + 4bkr}}{2r}, 0\right)$$

and

$$E_*(x_*, y_*) = \left(\frac{ad}{\mu - d}, \frac{\mu}{d}\left[rx_*\left(1 - \frac{x_*}{k}\right) + b\right]\right).$$

Combining the biological meaning, we take

$$x_0 = \frac{k}{2} + \frac{\sqrt{k^2r^2 + 4bkr}}{2r},$$

then the equilibrium $E_0(x_0, 0)$ always exists, which means that the predator will extinct.

It is easy to see that if $\mu < d$, then system (1.1) has no positive equilibrium and $E_0(x_0, 0)$ is the unique equilibrium of (1.1).

When $\mu > d$, we let

$$R_0 = \frac{bk(\mu - d)^2 + adkr(\mu - d)}{a^2d^2r},$$

then system (1.1) has no positive equilibrium and $E_0(x_0, 0)$ is the unique equilibrium of (1.1) when $R_0 \leq 1$; and system (1.1) has a positive equilibrium $E_*(x_*, y_*)$ besides $E_0(x_0, 0)$ when $R_0 > 1$, which means that R_0 is a critical value.

The rest of the present paper is organized as follows: in Section 2, we show the positiveness and boundedness of the solutions of (1.1). In Section 3, we analyze the local stability of E_0 and E_*, and the existence of Hopf bifurcation at E_*. In Section 4, we study the global stability of the equilibria E_0 and E_*. In Section 5, we determine the properties of the bifurcating periodic solution and discuss the existence of the global Hopf bifurcation. In Section 6, some numerical simulations are carried out to illustrate the analytic results.

2 Positiveness and boundedness of the solutions

In this section, we study the positiveness and boundedness of the solutions of system (1.1).

Theorem 2.1. *All the solutions of system* (1.1) *through initial conditions* (1.2) *are positive for* $t \geq 0$.

Proof. Solving the following ordinary differential equation

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{x(t)y(t)}{a + x(t)},$$

we can get the following solution

$$x(t) = \phi_1(0)\exp\left(\int_0^t \left[r(1 - \frac{x(s)}{k}) - \frac{y(s)}{a + x(s)}\right] \mathrm{d}s\right).$$

Obviously the solution is positive for all $t > 0$, furthermore, by the comparison theorem, we know that the solution $x(t)$ of system (1.1) is positive.

Based on the theory of Hale [9], we know that $y(t)$ is well defined on $[-\tau, +\infty)$ with the following form

$$y(t) = \phi_2(0)\mathrm{e}^{-td} + \int_0^t \mathrm{e}^{-d(t-s)} \frac{\mu x(s-\tau)y(s-\tau)}{a + x(s-\tau)} \mathrm{d}s,$$

since $\phi_2(0) > 0$, we have $y(t) > 0$ when $t \in [0, \tau]$, therefore $y(t) > 0$ for all $t \in [0, +\infty)$. □

Theorem 2.2. *All the solutions of system* (1.1) *through initial conditions* (1.2) *are uniformly ultimately bounded.*

Proof. Consider the following ordinary differential equation

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{k}\right) + b, \tag{2.1}$$

then all the solutions of equation (2.1) with positive initial conditions are positive, and

$$x_0 = \frac{k}{2} + \frac{\sqrt{k^2r^2 + 4bkr}}{2r}$$

is a positive equilibrium of equation (2.1).

Let

$$V(t) = \frac{1}{2}(x - x_0)^2,$$

then

$$\begin{aligned}\dot{V}(t) = (x - x_0)\dot{x} &= (x - x_0)\left[rx\left(1 - \frac{x}{k}\right) + b\right] \\ &= (x - x_0)\left[rx\left(1 - \frac{x}{k}\right) - rx_0\left(1 - \frac{x_0}{k}\right)\right] \\ &= r(x - x_0)^2\left(1 - \frac{x + x_0}{k}\right).\end{aligned}$$

Since $x_0 > k$ and all the solutions of (2.1) are positive, we have

$$\dot{V}(t) \leq 0$$

and $\dot{V}(t) = 0$ if and only if $x = x_0$, so the equilibrium x_0 of equation (2.1) is global asymptotically stable.

Basing on the comparison theorem, we know that the solution $x(t)$ of system (1.1) with initial conditions (1.2) satisfies

$$\limsup_{t\to\infty} x(t) \leq x_0,$$

then for $\epsilon > 0$, we have $x(t) \leq x_0 + \epsilon$ when t is sufficiently big.

Let

$$W(t) = \mu x(t) + y(t+\tau),$$

then

$$\begin{aligned}\dot{W}(t) &= b\mu + \mu r x(t)\left(1 - \frac{x(t)}{k}\right) - dy(t+\tau)\\ &= b\mu + 2\mu r x(t) - \mu r x(t) - \frac{\mu r x^2(t)}{k} - dy(t+\tau)\\ &= b\mu + 2\mu r x(t) - \mu r x(t)\left(1 + \frac{x(t)}{k}\right) - dy(t+\tau)\\ &\leq b\mu + 2\mu r(x_0+\epsilon) - \min\{r,d\}[\mu x(t) + y(t+\tau)]\\ &= b\mu + 2\mu r(x_0+\epsilon) - \min\{r,d\}W(t),\end{aligned}$$

which implies

$$W(t) \leq \frac{b\mu + 2\mu r(x_0+\epsilon)}{\min\{r,d\}}.$$

This completes the proof. □

3 Local stability analysis

3.1 Local stability of the boundary equilibrium

Linearizing system (1.1) near the boundary equilibrium $E_0(x_0, 0)$, we get

$$\begin{aligned}\dot{x}(t) &= r\left(1 - \frac{2x_0}{k}\right)x(t) - \frac{x_0}{a+x_0}y(t),\\ \dot{y}(t) &= -dy(t) + \frac{\mu x_0}{a+x_0}y(t-\tau).\end{aligned} \tag{3.1}$$

and the characteristic equation

$$\left(\lambda - r\left(1 - \frac{2x_0}{k}\right)\right)\left(\lambda + d - \frac{\mu x_0}{a+x_0}e^{-\lambda\tau}\right) = 0. \tag{3.2}$$

Obviously,

$$\lambda = r(1 - \frac{2x_0}{k}) = -\frac{\sqrt{k^2r^2 + 4bkr}}{k}$$

is always a negative root of equation (3.2), so in the following we study the second factor of equation (3.2).

We denote

$$f(\lambda) = \lambda + d - \frac{\mu x_0}{a+x_0}e^{-\lambda\tau}.$$

If condition $\mu x_0/(a+x_0) > d$ holds, then it is easy to show that

$$f(0) = d - \frac{\mu x_0}{a+x_0} < 0, \qquad \lim_{\lambda\to+\infty} f(\lambda) = +\infty.$$

Hence $f(\lambda) = 0$ has at least one positive real root. Therefore, equilibrium $E_0(x_0, 0)$ is unstable.

If condition $\mu x_0/(a+x_0) < d$ holds, we need to consider the effect of the delay τ.

When $\tau = 0$, we get

$$\lambda = \frac{\mu x_0}{a + x_0} - d < 0,$$

which implies that the equilibrium $E_0(x_0, 0)$ is locally asymptotically stable when $\tau = 0$.

If $\lambda = i\omega$ $(\omega > 0)$ be a root of (3.2) when $\tau > 0$, substituting $\lambda = i\omega$ into (3.2) and separating the real and the imaginary parts, we have

$$d = \frac{\mu x_0}{a + x_0} \cos \omega\tau, \qquad -\omega = \frac{\mu x_0}{a + x_0} \sin \omega\tau,$$

and furthermore,

$$\omega^2 = \frac{\mu^2 x_0^2}{(a + x_0)^2} - d^2 < 0,$$

which implies that Eq. (3.2) has no purely imaginary root. Then by theorem in [22], we know that all roots of Eq. (3.2) have negative real part, and the equilibrium $E_0(x_0, 0)$ is locally asymptotically stable for all $\tau \geq 0$.

If condition $\mu x_0/(a + x_0) = d$ holds, then $\lambda = 0$ is a simple root of (3.2). So to determine the stability of E_0, we need to compute the restriction of system (1.1) on the center manifold. Here we use the center manifold theorem by [24], and the normal form method from [7,8]

Let $\Lambda = \{0\}$ and $B = 0$, clearly the non-resonance conditions relative to Λ are satisfied. Therefore there exists a 1-dimensional ODE which governs the dynamics of system (1.1) near E_0.

For convenience, we denote $d_0 = \mu x_0/(a + x_0)$. Firstly, we re-scale the time delay by $t \mapsto (t/\tau)$ to normalize the delay and let $d = d_0 + \varepsilon$, then $\varepsilon = 0$ is the critical value for the fixed point bifurcation, so system (1.1) can be written in the form:

$$\begin{aligned} \dot{x}(t) &= \tau\left(r - \frac{2rx_0}{k}\right)x(t) - \frac{\tau x_0}{a + x_0}y(t) - \frac{r\tau}{k}x^2(t) - \frac{a\tau}{(a + x_0)^2}x(t)y(t) + O(3), \\ \dot{y}(t) &= -\tau(d_0 + \varepsilon)y(t) + \tau d_0 y(t-1) + \frac{a\tau\mu}{(a + x_0)^2}x(t-1)y(t-1) + O(3). \end{aligned} \tag{3.3}$$

Clearly, the phase space for Eq. (3.3) is $C := C([-1, 0], \mathbb{R}^2)$. For $\varphi \in C$, define

$$L(\varepsilon)\varphi = \tau B_1 \varphi(0) + \tau B_2 \varphi(-1),$$

where

$$B_1 = \begin{pmatrix} r\left(1 - \frac{2x_0}{k}\right) & -\frac{x_0}{a + x_0} \\ 0 & -(d_0 + \varepsilon) \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & d_0 \end{pmatrix}.$$

Choosing

$$\eta(\theta, \varepsilon) = \begin{cases} \tau B_1, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\tau B_2, & \theta = -1, \end{cases}$$

then by the Riesz representation theorem, we obtain

$$L(\varepsilon)\varphi = \int_{-1}^{0} d\eta(\theta, \varepsilon)\varphi(\theta).$$

Furthermore, we choose

$$F(\varepsilon,\varphi)=\tau\begin{pmatrix}-\frac{r}{k}\varphi_1^2(0)-\frac{a}{(a+x_0)^2}\varphi_1(0)\varphi_2(0)+O(3)\\ \frac{a\mu}{(a+x_0)^2}\varphi_1(-1)\varphi_2(-1)+O(3)\end{pmatrix}.$$

Then Eq. (3.3) can be rewritten in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t)=L(\varepsilon)u_t+F(u_t,\varepsilon).$$

where $u=(u_1,u_2)$, and $u_t=u_t(\theta)=u(t+\theta),\ -1\le\theta\le 0$.

Let operator A_0, which satifies

$$A_0:D(A_0)\mapsto C,\qquad A_0\varphi=\dot{\varphi},$$

be the infinitesimal generator for the semigroup defined by the solutions of the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=L_0(x_t),\tag{3.4}$$

where $L_0=L(0)$, and $D(A_0)=\{\varphi\in C^1,\ \dot{\varphi}=L_0\varphi\}$

For $C^*=:C([0,1],\mathbb{R}^{2^*})$, here $\mathbb{R}^{2^*}$ denotes the space of row vectors, we consider the adjoint bilinear form on $C^*\times C$ defined by

$$(\psi,\varphi)=\psi(0)\varphi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\psi(\xi-\theta)\mathrm{d}\eta(\theta,0)\varphi(\xi)\mathrm{d}\xi.$$

For the eigenvalue of A_0, $\Lambda=0$, we use the formal adjoint theory for FDEs to decompose the phase space C by $\Lambda=\{0\}$. Let P be the center space of equation (3.4), the generalized eigenspace for A_0 associated with the eigenvalue zero, and P^* the center space of the adjoint equation of (3.4), then the phase space C can be decomposed by $\Lambda=\{0\}$ as $C=P\oplus Q$, where $Q=\{\varphi\in C:(\psi,\varphi)=0\quad \text{forall } \psi\in P^*\}$.

In fact, letting $L_0\Phi(\theta)=\dot{\Phi}(\theta)=(0,0)^T$, which implies that we can choose $\Phi(\theta)=(c_1,c_2)^T$ (here c_1,c_2 are constants), then we can get

$$\tau\begin{pmatrix}r\left(1-\frac{2x_0}{k}\right) & -\frac{x_0}{a+x_0}\\ 0 & -d_0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}+\tau\begin{pmatrix}0&0\\0&d_0\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

which equals

$$\begin{aligned}r\left(1-\frac{2x_0}{k}\right)c_1-\frac{x_0}{a+x_0}c_2&=0,\\ -d_0c_2+d_0c_2&=0.\end{aligned}$$

Thus, we can choose

$$\Phi(\theta)=(\phi_1,\phi_2)^T=\left(\frac{kd_0}{r\mu(k-2x_0)},1\right)^T.$$

Similarly, let us choose

$$\Psi(s)=(\psi_1,\psi_2)=\left(0,\frac{1}{1+\tau d_0}\right),$$

and we can verify that they are the bases of P and P^*, respectively, satisfying $(\Psi,\Phi)=1$. Thus the dual bases satisfy $\dot{\Phi}=\Phi B$ and $\dot{\Psi}=-B\Psi$, where $B=0$.

Taking the enlarged phase space

$$BC=\left\{\varphi:[-1,0)\mapsto\mathbb{R},\varphi \text{ is continuous on } [-1,0) \text{ and } \lim_{\theta\to 0}\varphi(\theta) \text{ exists}\right\},$$

we obtain the abstract ODE with the form:

$$\frac{\mathrm{d}}{\mathrm{dt}}u_t=Au_t+X_0G(u_t,\varepsilon), \tag{3.5}$$

where

$$\begin{aligned}G(\varphi(\theta),\varepsilon)&=[L(\varepsilon)-L_0]\varphi(\theta)+F(\varphi(\theta),\varepsilon)\\&=\tau\begin{pmatrix}-\frac{r}{k}\varphi_1^2(0)-\frac{a}{a+x_0}\varphi_1(0)\varphi_2(0)+O(3)\\-\varepsilon\varphi_2(0)+\frac{a\mu}{(a+x_0)^2}\varphi_1(-1)\varphi_2(-1)+O(3)\end{pmatrix}\end{aligned}$$

and A is an extension of the infinitesimal generator A_0, defined by $A:C^1\to BC$,

$$(A\varphi)(\theta)=\dot{\varphi}(\theta)+X_0(\theta)[L_0\varphi-\dot{\varphi}(0)]=\begin{cases}\dot{\varphi}(\theta), & -1\le\theta<0,\\ L_0\varphi, & \theta=0.\end{cases}$$

and $X_0(\theta)$ is given by

$$X_0(\theta)=\begin{cases}I, & \theta=0,\\ 0, & \theta\in[-1,0).\end{cases}$$

The definition of the continuous projection

$$\pi:BC\mapsto P,\qquad \pi(\varphi+X_0\alpha)=\Phi[(\Psi,\varphi)+\Psi(0)\alpha]$$

allows us to decompose the enlarged space by Λ as $BC=C\oplus\operatorname{Ker}\pi$. Since π commutes with A in C^1, and using the decomposition $u_t=\Phi x+y$, the abstract ODE (3.5) is therefore decomposed as the system

$$\begin{aligned}\dot{x}&=Bx+\Psi(0)G(\Phi x+y,\varepsilon),\\ \dot{y}&=A_{Q^1}y+(I-\pi)X_0G(\Phi x+y,\varepsilon).\end{aligned} \tag{3.6}$$

For $x\in\mathbb{R}$, $y\in Q^1=Q\cap C^1\subset\operatorname{Ker}\pi$, where A_{Q^1} is the restriction of A as an operator from Q^1 to the Banach space $\operatorname{Ker}\pi$. And by the expressions of $\Phi(\theta)$, we get

$$u_1(\theta)=\frac{kd_0}{r\mu(k-2x_0)}x+y_1(\theta),\qquad u_2(\theta)=x+y_2(\theta).$$

By Taylor's theorem, we expand the nonlinear terms in Eq. (3.6) at $(x,y,\varepsilon)=(0,0,0)$ as

$$\begin{aligned}\dot{x}(t)&=Bx+\frac{1}{2!}f_2^1(x,y,\varepsilon)+\text{h.o.t.},\\ \dot{y}(t)&=A_{Q^1}y+\frac{1}{2!}f_2^2(x,y,\varepsilon)+\text{h.o.t.},\end{aligned}$$

where

$$\frac{1}{2!}f_2^1 = \tau\psi_2[-\varepsilon(\phi_2 x + y_2(0)) + \frac{a\mu}{(a+x_0)^2}(\phi_1 x + y_1(-1))(\phi_2 x + y_2(-1))],$$

$$\frac{1}{2!}f_2^2 = (I-\pi)X_0\tau \begin{pmatrix} -\frac{r}{k}(\phi_1 x + y_1(0))^2 - \frac{a}{a+x_0}(\phi_1 x + y_1(0))(\phi_2 x + y_2(0)) \\ -\varepsilon(\phi_2 x + y_2(0)) + \frac{a\mu}{(a+x_0)^2}(\phi_1 x + y_1(-1))(\phi_2 x + y_2(-1)) \end{pmatrix}.$$

Then the ordinary differential equation for the flow of Eq. (3.6) on the center manifold which is given in normal form up to second order terms by letting $y = 0$ has the form

$$\dot{x}(t) = \frac{\tau}{1+\tau d_0}\left[-\varepsilon x + \frac{akd_0}{r(a+x_0)^2(k-2x_0)}x^2\right] =: f(x,\varepsilon),$$

and it is easy to check that

$$f(0,0) = 0, \qquad \frac{\partial f}{\partial x}(0,0) = 0, \qquad \frac{\partial f}{\partial \varepsilon}(0,0) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial \varepsilon}(0,0) = -\frac{\tau}{1+\tau d_0}, \qquad \frac{\partial^2 f}{\partial^2 x}(0,0) = \frac{2ak\tau d_0}{r(1+\tau d_0)(a+x_0)^2(k-2x_0)}.$$

Summarizing the analysis above and basing on the bifurcation theory [24], we have the following theorem.

Theorem 3.1. *For system* (1.1)

{i} *when* $(\mu - d)x_0 < ad$, $E_0(x_0, 0)$ *is asymptotically stable;*

{ii} *when* $(\mu - d)x_0 > ad$, $E_0(x_0, 0)$ *is unstable;*

{iii} *when* $(\mu - d)x_0 = ad$, $E_0(x_0, 0)$ *is unstable. Furthermore, system* (1.1) *undergoes a transcritical bifurcation at the critical value.*

3.2 Local stability of the positive equilibrium and the existence of Hopf bifurcation

Linearizing system (1.1) near the positive equilibrium $E_*(x_*, y_*)$, we get

$$\begin{aligned} \dot{x}(t) &= \left(r - \frac{2rx_*}{k} - \frac{ay_*}{(a+x_*)^2}\right)x(t) - \frac{x_*}{a+x_*}y(t), \\ \dot{y}(t) &= -dy(t) + \frac{a\mu y_*}{(a+x_*)^2}x(t-\tau) + \frac{\mu x_*}{a+x_*}y(t-\tau), \end{aligned} \tag{3.7}$$

and the characteristic equation

$$\lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)e^{-\lambda\tau} = 0, \tag{3.8}$$

where

$$p_0 = -d\left(r - \frac{2rx_*}{k} - \frac{ay_*}{(a+x_*)^2}\right), \qquad q_0 = \left(r - \frac{2rx_*}{k}\right)\frac{\mu x_*}{a+x_*},$$

$$p_1 = d - \left(r - \frac{2rx_*}{k} - \frac{ay_*}{(a+x_*)^2}\right), \qquad q_1 = -\frac{\mu x_*}{a+x_*}.$$

When $\tau = 0$, Eq. (3.8) becomes

$$\lambda^2 + (p_1 + q_1)\lambda + (p_0 + q_0) = 0, \tag{3.9}$$

notice that

$$\frac{\mu x_*}{a + x_*} = d \quad \text{and} \quad p_0 + q_0 = \frac{ady_*}{(a + x_*)^2} > 0,$$

so we know that both roots of Eq. (3.9) have negative real parts when $p_1 + q_1 > 0$.

When $\tau > 0$, let $\mathrm{i}\omega$ $(\omega > 0)$ be a root of Eq. (3.8), substituting $\mathrm{i}\omega$ into (3.8) and separating the real and the imaginary part, we get

$$\begin{aligned} \omega^2 - p_0 &= q_0 \cos \omega\tau + q_1 \omega \sin \omega\tau, \\ p_1 \omega &= q_0 \sin \omega\tau - q_1 \omega \cos \omega\tau, \end{aligned}$$

which implies that

$$\sin \omega\tau = \frac{q_1\omega^3 + (p_1 q_0 - p_0 q_1)\omega}{q_0^2 + q_1^2 \omega}, \qquad \cos \omega\tau = \frac{(q_0 - p_1 q_1)\omega^2 - p_0 q_0}{q_0^2 + q_1^2\omega},$$

and

$$\omega^4 + (p_1^2 - q_1^2 - 2p_0)\omega^2 + (p_0^2 - q_0^2) = 0. \tag{3.10}$$

Moreover, it is easy to get that

$$p_1^2 - q_1^2 - 2p_0 = \left(r - \frac{2rx_*}{k} - \frac{ay_*}{(a + x_*)^2} \right)^2 > 0.$$

Thus, as $p_0 + q_0 > 0$, we know that (3.10) has no positive real root when $p_0 > q_0$, and has one real root when $p_0 < q_0$.

Lemma 3.2. *Suppose $p_0 < q_0$, then Eq. (3.8) has a pair of conjugate purely imaginary root $\pm \mathrm{i}\omega_0$, where*

$$\omega_0 = \frac{1}{2} \left[(q_1^2 - p_1^2 + 2p_0) + \sqrt{(q_1^2 - p_1^2 + 2p_0)^2 - 4(p_0^2 - q_0^2)} \right]^{\frac{1}{2}},$$

when $\tau = \tau_j$, $j = 0, 1, 2, \ldots$,

$$\tau_j = \begin{cases} \dfrac{1}{\omega_0}\left(\arcsin \dfrac{a^*}{c^*} + 2j\pi\right), & a^* > 0, \quad b^* > 0, \\ \dfrac{1}{\omega_0}\left(\pi - \arcsin \dfrac{a^*}{c^*} + 2j\pi\right), & a^* > 0, \quad b^* < 0, \\ \dfrac{1}{\omega_0}\left(\pi + \arcsin \dfrac{a^*}{c^*} + 2j\pi\right), & a^* < 0, \quad b^* < 0, \\ \dfrac{1}{\omega_0}\left(2\pi - \arcsin \dfrac{a^*}{c^*} + 2j\pi\right), & a^* < 0, \quad b^* > 0, \end{cases} \qquad j = 0, 1, 2, \ldots, \tag{3.11}$$

where $a^* = q_1\omega_0^3 + (p_1 q_0 - p_0 q_1)\omega_0$, $b^* = (q_0 - p_1 q_1)\omega_0^2 - p_0 q_0$, $c^* = q_0^2 + q_1^2\omega_0^2$.

Furthermore, since

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{2\lambda + p_1 + q_1 e^{-\lambda\tau} - \tau(q_1\lambda + q_0)e^{-\lambda\tau}}{\lambda(q_1\lambda + q_0)e^{-\lambda\tau}}$$
$$= \frac{2\lambda + p_1}{-\lambda(\lambda^2 + p_1\lambda + p_0)} + \frac{q_1}{\lambda(q_1\lambda + q_0)} - \frac{\tau}{\lambda},$$

we can get that

$$\mathrm{Sgn}\left\{\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\right\}_{\lambda=\mathrm{i}\omega_0} = \mathrm{Sgn}\left\{\mathrm{Re}\left(\frac{p_1 + 2\mathrm{i}\omega_0}{p_1\omega_0^2 - \mathrm{i}\omega_0(p_0 - \omega_0^2)} + \frac{q_1}{-q_1\omega_0^2 + \mathrm{i}q_0\omega_0} - \frac{\tau}{\mathrm{i}\omega_0}\right)\right\}$$
$$= \mathrm{Sgn}\left\{\frac{p_1^2 - 2(p_0 - \omega_0^2)}{p_1^2\omega_0^2 + (p_0 - \omega_0^2)^2} - \frac{q_1^2}{q_1^2\omega_0^2 + q_0^2}\right\}$$
$$= \mathrm{Sgn}\left\{\frac{2\omega_0^2 + p_1^2 - q_1^2 - 2p_0}{q_1^2\omega_0^2 + q_0^2}\right\} > 0,$$

which implies that the transversal condition holds.

In conclusion, we have the following results.

Theorem 3.3. *For system (1.1), let condition $p_1 + q_1 > 0$ hold, then we have the following:*

{i} *if $p_0 > q_0$, then the coexistence equilibrium $E_*(x_*, y_*)$ is locally asymptotically stable for all $\tau \geq 0$;*

{ii} *if $p_0 < q_0$, then $E_*(x_*, y_*)$ is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Furthermore, system (1.1) undergoes a Hopf bifurcation near E_* when $\tau = \tau_j$, $j = 0, 1, 2, \dots$.*

4 Global stability analysis

In this section, we investigate the global stability of equilibria E_0 and E_*.

Theorem 4.1. *If $R_0 < 1$, then $E_0(x_0, 0)$ is globally asymptotically stable.*

Proof. From Section 1, we know that system (1.1) has no positive equilibrium when $R_0 < 1$, and $E_0(x_0, 0)$ is the unique equilibrium.

Let

$$V_{11}(t) = x(t) - x_0 - x_0 \ln\frac{x(t)}{x_0} + c_1 y(t),$$

where $c_1 = x_0/(ad)$, then we get the derivative of $V_{11}(t)$ along solutions of system (1.1)

$$\dot{V}_{11}(t) = \left(1 - \frac{x_0}{x(t)}\right)\left[rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{x(t)y(t)}{a + x(t)} + b\right] + c_1\left[-dy(t) + \frac{\mu x(t-\tau)y(t-\tau)}{a + x(t-\tau)}\right]$$
$$= \left(1 - \frac{x_0}{x(t)}\right)\left[rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{x(t)y(t)}{a + x(t)} - rx_0\left(1 - \frac{x_0}{k}\right)\right]$$
$$+ c_1\left[-dy(t) + \frac{\mu x(t-\tau)y(t-\tau)}{a + x(t-\tau)}\right]$$
$$= \frac{r}{x(t)}(x(t) - x_0)^2\left[1 - \frac{x(t) + x_0}{k}\right] - \left(1 + \frac{x_0}{a}\right)\frac{x(t)y(t)}{a + x(t)} + \left(\frac{x_0}{a} - c_1 d\right)y(t)$$
$$+ \frac{c_1\mu x(t-\tau)y(t-\tau)}{a + x(t-\tau)}.$$

Defining

$$V_1(t) = V_{11}(t) + c_1\mu \int_{t-\tau}^{t} \frac{x(s)y(s)}{a+x(s)}\mathrm{d}s,$$

then

$$\dot{V}_1(t) = \frac{r}{x(t)}(x(t)-x_0)^2\left[1-\frac{x(t)+x_0}{k}\right] + \frac{1}{ad}[\mu x_0 - (a+x_0)d]\frac{x(t)y(t)}{a+x(t)}.$$

From the positiveness of $x(t)$ and $x_0 > k$, combined with the condition

$$R_0 < 1 \Leftrightarrow \frac{\mu x_0}{a+x_0} < d,$$

we have $\dot{V}_1(t) \le 0$, and $\dot{V}_1(t) = 0 \Leftrightarrow (x(t), y(t)) = (x_0, 0)$. By LaSalle's invariant set principle we know that E_0 is global asymptotically stable. □

Theorem 4.2. *When $\mu > d,\ R_0 > 1$, if condition*

$$x_* > k \Leftrightarrow \frac{ad}{\mu - d} > k$$

holds, then $E_*(x_*, y_*)$ is globally asymptotically stable.

Proof. System (1.1) has a positive equilibrium $E_*(x_*, y_*)$ when $\mu > d,\ R_0 > 1$.

Let

$$V_{21}(t) = x(t) - x_* - x_* \ln\frac{x(t)}{x_*} + c_2\left(y(t) - y_* - y_*\ln\frac{y(t)}{y_*}\right),$$

where $c_2 = x_*/(ad)$, then we get the derivative of $V_{21}(t)$ along solutions of system (1.1)

$$\begin{aligned}
\dot{V}_{21}(t) =& \left(1-\frac{x_*}{x(t)}\right)\left[rx(t)\left(1-\frac{x(t)}{k}\right) - \frac{x(t)y(t)}{a+x(t)} + b\right] \\
&+ c_2\left(1-\frac{y_*}{y(t)}\right)\left[-dy(t) + \frac{\mu x(t-\tau)y(t-\tau)}{a+x(t-\tau)}\right] \\
=& \frac{r}{x(t)}[x(t)-x_*]^2\left[1-\frac{x(t)+x_*}{k}\right] - \left(1-\frac{x_*}{x(t)}\right)\frac{x(t)y(t)}{a+x(t)} + \left(1-\frac{x_*}{x(t)}\right)\frac{x_*y_*}{a+x_*} \\
&+ c_2\left[-dy(t) + \frac{\mu x(t-\tau)y(t-\tau)}{a+x(t-\tau)} + dy_* - \frac{y_*\mu x(t-\tau)y(t-\tau)}{y(t)[a+x(t-\tau)]}\right] \\
=& \frac{r}{x(t)}[x(t)-x_*]^2\left[1-\frac{x(t)+x_*}{k}\right] - \frac{(a+x_*)x(t)y(t)}{a[a+x(t)]} + \frac{x_*}{a}y(t) - c_2dy(t) \\
&+ \left(1-\frac{x_*}{x(t)}\right)\frac{x_*y_*}{a+x_*} + c_2dy_* + \frac{c_2\mu x(t-\tau)y(t-\tau)}{a+x(t-\tau)} - \frac{c_2y_*\mu x(t-\tau)y(t-\tau)}{y(t)[a+x(t-\tau)]}.
\end{aligned}$$

Defining

$$V_2(t) = V_{21}(t) + c_2\mu\int_{t-\tau}^{t}\left[\frac{x(s)y(s)}{a+x(s)} - \frac{x_*y_*}{a+x_*} - \frac{x_*y_*}{a+x_*}\ln\frac{(a+x_*)x(s)y(s)}{x_*y_*[a+x(s)]}\right]\mathrm{d}s,$$

then

$$\begin{aligned}
\dot{V}_2(t) =& \frac{r}{x(t)}[x(t)-x_*]^2\left[1-\frac{x(t)+x_*}{k}\right] - \frac{(a+x_*)x(t)y(t)}{a[a+x(t)]} + \left(1-\frac{x_*}{x(t)}\right)\frac{x_*y_*}{a+x_*} \\
&+ \frac{c_2\mu x_*y_*}{a+x_*} + \frac{c_2\mu x(t-\tau)y(t-\tau)}{a+x(t-\tau)} - \frac{c_2y_*\mu x(t-\tau)y(t-\tau)}{y(t)[a+x(t-\tau)]} \\
&+ c_2\mu\left[\frac{x(t)y(t)}{a+x(t)} - \frac{x_*y_*}{a+x_*}\ln\frac{(a+x_*)x(t)y(t)}{x_*y_*[a+x(t)]}\right] \\
&- c_2\mu\left[\frac{x(t-\tau)y(t-\tau)}{a+x(t-\tau)} - \frac{x_*y_*}{a+x_*}\ln\frac{(a+x_*)x(t-\tau)y(t-\tau)}{x_*y_*[a+x(t-\tau)]}\right].
\end{aligned}$$

Since

$$c_2\mu = \frac{a+x_*}{a}, \qquad \left(1-\frac{x_*}{x(t)}\right)\frac{x_*y_*}{a+x_*} = \frac{c_2\mu x_*y_*}{a+x_*}\left[1-\frac{x_*[a+x(t)]}{x(t)(a+x_*)}\right],$$

we have

$$\begin{aligned}\dot{V}_2(t) &= \frac{r}{x(t)}[x(t)-x_*]^2\left[1-\frac{x(t)+x_*}{k}\right] + \frac{c_2\mu x_*y_*}{a+x_*}\left[1-\frac{x_*[a+x(t)]}{x(t)(a+x_*)}\right]\\ &\quad + \frac{c_2\mu x_*y_*}{a+x_*} - \frac{c_2\mu x_*y_*(a+x_*)x(t-\tau)y(t-\tau)}{(a+x_*)x_*y(t)[a+x(t-\tau)]}\\ &\quad + \frac{c_2\mu x_*y_*}{a+x_*}\ln\frac{x_*[a+x(t)](a+x_*)x(t-\tau)y(t-\tau)}{x(t)(a+x_*)x_*y(t)[a+x(t-\tau)]}\\ &= \frac{r}{x(t)}[x(t)-x_*]^2\left[1-\frac{x(t)+x_*}{k}\right] - \frac{c_2\mu x_*y_*}{a+x_*}\left[\frac{x_*[a+x(t)]}{x(t)(a+x_*)} - 1 - \ln\frac{x_*[a+x(t)]}{x(t)(a+x_*)}\right]\\ &\quad - \frac{c_2\mu x_*y_*}{a+x_*}\left[\frac{(a+x_*)x(t-\tau)y(t-\tau)}{x_*y(t)[a+x(t-\tau)]} - 1 - \ln\frac{(a+x_*)x(t-\tau)y(t-\tau)}{x_*y(t)[a+x(t-\tau)]}\right].\end{aligned}$$

From the positiveness of $x(t)$, combining the condition, we have $\dot{V}_2(t) \le 0$, and $\dot{V}_2(t) = 0$ if and only if $x(t) = x_*$, $y(t) = y_*$. By LaSalle's invariant set principle we know that $E_*(x_*, y_*)$ is global asymptotically stable. □

5 Hopf bifurcation analysis

In Section 3, we found that under some conditions the system undergoes a Hopf bifurcation when τ passes through some critical value. In this section we study some properties of the Hopf bifurcation and the global existence of the periodic solutions. For convenience, we assume that the condition for Hopf bifurcation

$$(\mathrm{H}_1) \qquad p_1+q_1>0, \qquad p_0-q_0<0$$

is always satisfied in this section.

5.1 Properties of bifurcating periodic solutions

In this part, we will study the properties of the bifurcating periodic solutions such as the orbital stability and the direction of Hopf bifurcation. The method we used is based on the normal form method and the center manifold theory introduced by Hassard et al. [10].

Re-scale the time by $t \to (t/\tau)$ to normalize the delay, and let $\tau = \tau_0 + \varepsilon$, $\varepsilon \in \mathbb{R}$, then we can rewrite system (1.1) in the following form

$$\begin{aligned}\dot{x}(t) &= (\tau_0+\varepsilon)[l_1x(t)+l_2y(t)+l_3x^2(t)+l_4x(t)y(t)+O(3)],\\ \dot{y}(t) &= (\tau_0+\varepsilon)[m_1y(t)+m_2x(t-1)+m_3y(t-1)\\ &\qquad +m_4x^2(t-1)+m_5x(t-1)y(t-1)+O(3)],\end{aligned} \tag{5.1}$$

where

$$\begin{aligned} &l_1 = r-\frac{2rx_*}{k}-\frac{ay_*}{(a+x_*)^2}, \quad l_2 = \frac{-x_*}{a+x_*}, \quad l_3 = -\frac{r}{k}+\frac{ay_*}{(a+x_*)^3}, \quad l_4 = \frac{-a}{(a+x_*)^2},\\ &m_1=-d, \quad m_2 = \frac{a\mu y_*}{(a+x_*)^2}, \quad m_3 = \frac{\mu x_*}{a+x_*}, \quad m_4 = -\frac{a\mu y_*}{(a+x_*)^3}, \quad m_5 = \frac{a\mu}{(a+x_*)^2}.\end{aligned}$$

Clearly, the phase space is $\mathcal{C} = C([-1,0], \mathbb{R}^2)$. From the analysis above we know that $\varepsilon = 0$ is the Hopf bifurcation value for system (5.1).

For $\phi = (\phi_1, \phi_2) \in \mathcal{C}$, let

$$L_\varepsilon(\phi) = (\tau_0 + \varepsilon)B\phi(0) + (\tau_0 + \varepsilon)C\phi(-1),$$

where

$$B = \begin{pmatrix} l_1 & l_2 \\ 0 & m_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ m_2 & m_3 \end{pmatrix},$$

and

$$f(\varepsilon, \phi) = (\tau_0 + \varepsilon) \begin{pmatrix} l_3\phi_1^2(0) + l_4\phi_1(0)\phi_2(0) + O(3) \\ m_4\phi_1^2(-1) + m_5\phi_1(-1)\phi_2(-1) + O(3) \end{pmatrix}.$$

By the Riesz representation theorem, there exists a 2×2 matrix, $\eta(\theta, \varepsilon)$ $(-1 \le \theta \le 0)$, whose elements are of bounded variation functions such that

$$L_\varepsilon(\phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta, \varepsilon)\phi(\theta), \qquad \phi \in \mathcal{C}.$$

In fact, we can choose

$$\eta(\theta, \varepsilon) = \begin{cases} (\tau_0 + \varepsilon)B, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -(\tau_0 + \varepsilon)C, & \theta = -1. \end{cases}$$

Then Eq. (5.1) is satisfied.

For $\phi \in \mathcal{C} \cap C^1$, define the operator $A(\varepsilon)$ as

$$A(\varepsilon)\phi(\theta) = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\eta(\theta, \varepsilon)\phi(\theta), & \theta = 0, \end{cases}$$

and $R(\varepsilon)\phi$ as

$$R(\varepsilon)\phi(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\varepsilon, \phi), & \theta = 0, \end{cases}$$

then system (5.1) is equivalent to the following operator equation

$$\dot{u}_t = A(\varepsilon)u_t + R(\varepsilon)u_t, \tag{5.2}$$

where $u(t) = (x(t), y(t))^T, u_t = u(t + \theta)$, for $\theta \in [-1, 0]$.

For $\psi \in \mathcal{C} \cap C^1$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0, 1], \\ \displaystyle\int_{-1}^{0} \psi(-\xi)\mathrm{d}\eta(\xi, 0), & s = 0. \end{cases}$$

For $\phi \in C([-1, 0], \mathbb{C}^2)$ and $\psi \in C([-1, 0], \mathbb{C}^{2^*})$, define the bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{0}^{\theta} \bar{\psi}(\xi - \theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$. Then $A(0)$ and A^* are adjoint operators.

Let $q(\theta), q^*(s)$ are eigenvectors of $A(0)$ and A^* associated to $\mathrm{i}\omega_0\tau_0$ and $-\mathrm{i}\omega_0\tau_0$ respectively, it's not difficult to verify that

$$q(\theta) = (1, \alpha)^T e^{i\omega_0\tau_0\theta}, \qquad q^*(s) = \frac{1}{\bar{D}}(1, \beta) e^{i\omega_0\tau_0 s},$$

where

$$\alpha = \frac{\mathrm{i}\omega_0 - l_1}{l_2}, \qquad \beta = \frac{e^{\mathrm{i}\omega_0\tau_0}}{m_2}(\mathrm{i}\omega_0 - l_1),$$
$$D = (1 + \alpha\bar{\beta}) + \tau_0 e^{-\mathrm{i}\omega_0\tau_0}\bar{\beta}(m_2 + \alpha m_3),$$

then $\langle q^*(s), q(\theta)\rangle = 1, \langle q^*(s), \bar{q}(\theta)\rangle = 0$.

Following the algorithms given by Hassard et al. [10], we can obtain the coefficients which will be used in determining the important quantities:

$$\begin{aligned}
g_{20} &= \frac{2\tau_0}{D}[l_3 + \alpha l_4 + \bar{\beta}(m_4 + \alpha m_5)e^{-2\mathrm{i}\omega_0\tau_0}],\\
g_{11} &= \frac{\tau_0}{D}[2l_3 + l_4(\alpha + \bar{\alpha}) + 2\bar{\beta}m_4 + \bar{\beta}m_5(\alpha + \bar{\alpha})],\\
g_{02} &= \frac{2\tau_0}{D}[l_3 + \bar{\alpha} l_4 + \bar{\beta}(m_4 + \bar{\alpha} m_5)e^{2\mathrm{i}\omega_0\tau_0}],\\
g_{21} &= \frac{\tau_0}{D}\Big[2l_3(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)) + l_4(2W_{11}^{(2)}(0) + 2\alpha W_{11}^{(1)}(0) + W_{20}^{(2)}(0) + \bar{\alpha}W_{20}^{(1)}(0))\\
&\quad + 2\bar{\beta}m_4(2e^{-\mathrm{i}\omega_0\tau_0}W_{11}^{(1)}(-1) + e^{\mathrm{i}\omega_0\tau_0}W_{20}^{(1)}(-1)) + 2\bar{\beta}m_5 e^{-\mathrm{i}\omega_0\tau_0}(W_{11}^{(2)}(-1) + \alpha W_{11}^{(1)}(-1))\\
&\quad + \bar{\beta}m_5 e^{\mathrm{i}\omega_0\tau_0}(W_{20}^{(2)}(-1) + \bar{\alpha}W_{20}^{(1)}(-1))\Big],
\end{aligned}$$

where

$$W_{20}(\theta) = \frac{g_{20}q(0)}{-\mathrm{i}\omega_0\tau_0}e^{\mathrm{i}\omega_0\tau_0\theta} + \frac{\bar{g}_{20}\bar{q}(0)}{-3\mathrm{i}\omega_0\tau_0}e^{-\mathrm{i}\omega_0\tau_0\theta} + Ee^{2\mathrm{i}\omega_0\tau_0\theta},$$
$$W_{11}(\theta) = \frac{g_{11}q(0)}{\mathrm{i}\omega_0\tau_0}e^{\mathrm{i}\omega_0\tau_0\theta} + \frac{\bar{g}_{11}\bar{q}(0)}{-\mathrm{i}\omega_0\tau_0}e^{-\mathrm{i}\omega_0\tau_0\theta} + F,$$

and

$$E = \begin{pmatrix} 2\mathrm{i}\omega_0 - l_1 & -l_2 \\ -m_2 e^{-2\mathrm{i}\omega_0\tau_0} & 2\mathrm{i}\omega_0 - m_1 - m_3 e^{-2\mathrm{i}\omega_0\tau_0} \end{pmatrix}^{-1} \begin{pmatrix} 2l_3 + 2\alpha l_4 \\ 2(m_4 + \alpha m_5)e^{-2\mathrm{i}\omega_0\tau_0} \end{pmatrix},$$
$$F = \begin{pmatrix} l_1 & l_2 \\ m_2 & m_1 + m_3 \end{pmatrix}^{-1} \begin{pmatrix} -2l_3 - l_4(\alpha + \bar{\alpha}) \\ -2m_4 - m_5(\alpha + \bar{\alpha}) \end{pmatrix}.$$

Consequently, the above g_{21} can be expressed by the parameters in system (1.1). Thus, we can compute the following quantities:

$$\begin{aligned}
c_1(0) &= \frac{\mathrm{i}}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}\\
\mu_2 &= -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re}\lambda'(\tau_0)},\\
\beta_2 &= 2\operatorname{Re} c_1(0),\\
T_2 &= -\frac{\operatorname{Im} c_1(0) + \mu_2 \operatorname{Im}\lambda'(\tau_0)}{\omega_0\tau_0},
\end{aligned}$$

which determine the properties of bifurcating periodic solutions at the critical value τ_0. The direction and stability of Hopf bifurcation in the center manifold can be determined by μ_2 and β_2 respectively. In fact, if $\mu_2 > 0$ ($\mu_2 < 0$), then the bifurcating periodic solutions are forward (backward); the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

From the discussion in Section 2, we know that the transversal condition is positive, therefore we have the following result.

Theorem 5.1. *For system* (1.1)*, if conditions $p_1 + q_1 > 0$ and $p_0 - q_0 < 0$ hold, then the Hopf bifurcation at E_* when $\tau = \tau_j$ is forward (backward) and the bifurcating periodic solutions are orbitally asymptotically stable (unstable) when* $\mathrm{Re}(c_1(0)) < 0$ (> 0).

5.2 Global existence of periodic solutions

In this subsection, we shall study the global existence of periodic solutions bifurcating from the point (E_*, τ_j), $j = 0, 1, 2, \ldots$ for system (1.1) by a global Hopf bifurcation theorem by Wu [25].

For simplification of notations, setting $z_t = (x_t, y_t)$, we may rewrite systems (1.1) as the following functional differential equation:

$$\dot{z}(t) = F(z_t, \tau, p),$$

where $z_t(\theta) = z(t+\theta) \in C([-\tau, 0], \mathbb{R}^2)$, and p is the period of the solution of the above equation.

Following the work of Wu [25], we introduce some notations:

$$\begin{aligned} &X = \mathcal{C}([-\tau, 0], R), \\ &\Sigma = \mathcal{C}l\{(z, \tau, p) : z \text{ is a } p-\text{periodic solution of (1.1)}\} \subset X \times \mathbb{R}_+ \times \mathbb{R}_+, \\ &N = \{(\bar{z}, \bar{\tau}, \bar{p}) : F(\bar{z}, \bar{\tau}, \bar{p}) = 0\}. \end{aligned}$$

Let $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ denote the connected component of $(z_*, \tau_j, 2\pi/\omega_0)$ in Σ, where τ_j and ω_0 are defined in Lemma 3.2.

Lemma 5.2. *If condition*

$$(\mathrm{H}_2) \qquad \mu > d, \qquad k > x_*, \qquad (\mu + d) + \frac{bk(\mu - d)}{rx_*} \geq dk$$

is satisfied, then system (1.1) *has no nontrivial τ-periodic solution.*

Proof. To the contrary, suppose that system (1.1) has a τ-periodic solution, then the following system of ordinary differential equations also has a periodic solution

$$\begin{aligned} \dot{x}(t) &= rx(t)\left(1 - \frac{x(t)}{k}\right) - \frac{x(t)y(t)}{a + x(t)} + b, \\ \dot{y}(t) &= -dy(t) + \frac{\mu x(t)y(t)}{a + x(t)}. \end{aligned} \tag{5.3}$$

As to the existence of the limit cycle of this ODE system, Sugie and his coworkers have obtained some results in [23]. Based on Theorem 2.3 in [23], we know that the ODE system has no limit cycles when condition (H_2) holds, which completes the proof. □

Theorem 5.3. *Suppose that the conditions* (H_1) *and* (H_2) *are satisfied, then for each* $\tau > \tau_j$, $j = 0, 1, 2, \ldots$, *system* (1.1) *has at least one periodic solution.*

Proof. It is sufficient to prove that the projection of $\mathcal{C}\{(z_*, \tau_j, p)\}$ onto the $\tau-$space is $[\bar{\tau}, \infty)$ for each $j \geq 0$, where $\bar{\tau} \leq \tau_j$.

Firstly, we note that $F(z_t, \tau, p)$ satisfies the hypotheses (A_1), (A_2) and (A_3) in Wu [25], and

$$\Delta_{(z_*,\tau,p)} = \lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)e^{-\lambda\tau} = 0.$$

It can also be verified that $(z_*, \tau_j, 2\pi/\omega_0)$ are isolated centers, then by Lemma 3.2, there exist $\varepsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to \mathbb{C}$ such that

$$\Delta(\lambda(\tau)) = 0, \qquad |\lambda(\tau) - \mathrm{i}\omega_0| < \varepsilon$$

for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$, and

$$\lambda(\tau_j) = \mathrm{i}\omega_0, \qquad \left.\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_j} > 0.$$

Denote $p_k = 2\pi/\omega_0$, and let

$$\Omega_\varepsilon = \{(0, p) : 0 < u < \varepsilon,\ |p - p_k| < \varepsilon\}.$$

Clearly, if $|\tau - \tau_k| \leq \delta$ and $(u, p) \in \Omega_\varepsilon$, such that $\Delta_{(z_*,\tau,p)}(u + 2\mathrm{i}\pi/p) = 0$, then $\tau = \tau_j$, $u = 0$ and $p = p_j$. This verifies the assumption (A_4) in Wu [25] for $m = 1$. Moreover, putting

$$H^{\pm}(z_*, \tau_j, 2\pi/\omega_0)(u, p) = \Delta_{(z_*,\tau_j\pm\delta,p)}(u + 2\mathrm{i}\pi/\omega_0),$$

we have the crossing number

$$\gamma_1(z_*, \tau_j, 2\pi/\omega_0) = \deg_B(H^-(z_*, \tau_j, 2\pi/\omega_0), \Omega_\varepsilon) - \deg_B(H^+(z_*, \tau_j, 2\pi/\omega_0), \Omega_\varepsilon) = -1.$$

By Theorem 3.2 given by Wu [25], we conclude that the connected component $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ through $(z_*, \tau_j, 2\pi/\omega_0)$ in Σ is nonempty. Meanwhile, we have

$$\sum_{(z,\tau,p)\in\mathcal{C}(z_*,\tau_j,2\pi/\omega_0)} \gamma_1(z, \tau, p) < 0,$$

then by Theorem 3.3 given by Wu [25], $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ is unbounded.

By (3.11), we know that when $j > 0$,

$$\frac{2\pi}{\omega_0} < \tau_j.$$

Now we prove that the projection of $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ onto τ-space is $[\bar{\tau}, \infty)$, where $\bar{\tau} \leq \tau_j$. Similar to Lemma 5.2, we know that system (1.1) with $\tau = 0$ has no nonconstant periodic solutions. Therefore, the projection of $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ onto the τ-space is bounded below.

For a contradiction, we assume that the projection of $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ onto the τ-space is bounded, which implies that the projection of $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ onto the $\tau-$ space is included in a interval $(0, \tau^*)$. Since $2\pi/\omega_0 < \tau_j$ and applying Lemma 5.2, we have $0 < p < \tau^*$ for $(z(t), \tau, p) \in \mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$, which implies that the projection of $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ onto the p-space is bounded. Then combining this with Theorem 2.2 we get that the connected component $\mathcal{C}(z_*, \tau_j, 2\pi/\omega_0)$ is bounded. This completes the proof. □

6 Numerical simulation

We choose a set of data as follows:

$$a = 1, \quad b = 0.4, \quad d = 0.4, \quad k = 4, \quad r = 6, \quad \mu = 2, \tag{a}$$

which are the same as those in [23], then $E_0 = (4.06559, 0)$, $E_* = (0.25, 9.03125)$. We can get $\mu x_0/(a + x_0) = 1.60518 > 0.4 = d$, which implies that E_0 is unstable.

Basing on the analysis in Section 4, we can get

$$p_0 = 0.212, \quad p_1 = 0.93, \quad q_0 = 2.1, \quad q_1 = -0.4, \quad p_1 + q_1 = 0.53 > 0, \quad p_0 - q_0 = -1.888,$$

which implies that the conditions for the Hopf bifurcation are satisfied, and by the previous algorithm we can get

$$\omega_0 \doteq 1.3973, \quad \tau_0 \doteq 0.2727, \quad \operatorname{Re} c_1(0) \doteq -122.1746, \quad \mu_2 \doteq 162.6826, \quad \beta_2 \doteq -244.3492.$$

Therefore, we know that the equilibrium E_* is asymptotically stable when $\tau \in [0, 0.2727)$, which is shown in Figure 6.1, here we choose $\tau = 0.25$ and the initial value is taken as $(x_0, y_0) = (0.1, 10.5)$.

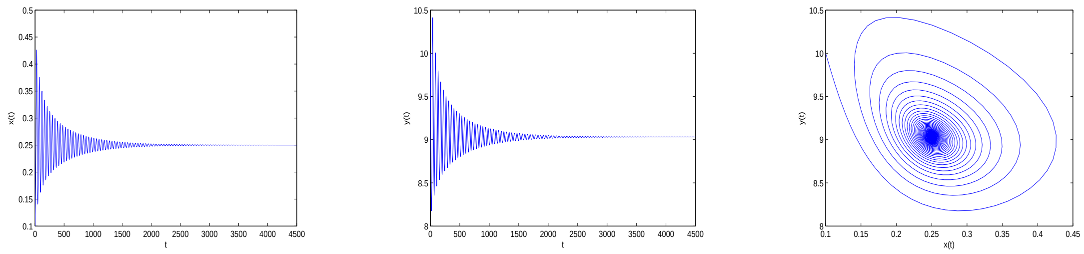

Figure 6.1: When $\tau = 0.25 < \tau_0$, the positive equilibrium of system (1.1) is asymptotically stable with parameters given in (a), and the initial value is $(x_0, y_0) = (0.1, 10.5)$.

Furthermore, we know that the equilibrium E_* is unstable when $\tau > 0.2727$ and the Hopf bifurcation is forward and the bifurcating periodic solutions are orbitally asymptotically stable, which is shown in Figure 6.2, here we choose $\tau = 0.3$ and the initial value is taken as $(x_0, y_0) = (0.1, 10.5)$.

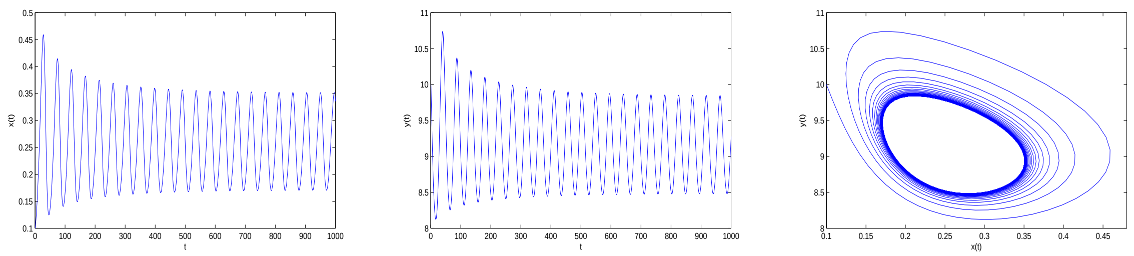

Figure 6.2: When $\tau = 0.3 > \tau_0$, periodic solutions bifurcating from the positive equilibrium of system (1.1) with parameters given in (a), and the initial value is $(x_0, y_0) = (0.1, 10.5)$.

Finally, by the analysis in Subsection 5.2, we know that the bifurcating periodic solutions exist for all $\tau \in (0.2727, +\infty)$, which is shown in Figure 6.3.

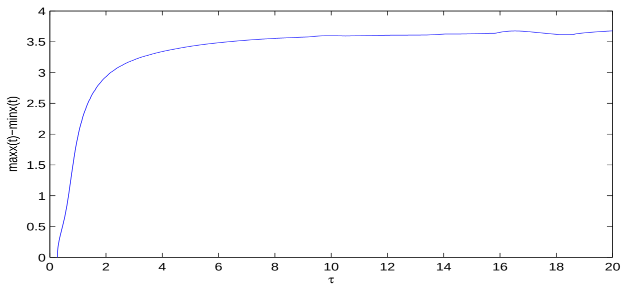

Figure 6.3: Global Hopf branch from $\tau_0 = 0.2727$ with parameters given in (a).

7 Conclusion

Sugie et al. studied the existence of the limit cycles of system (5.3) in [23]. They showed that when condition (H_2) holds, the coexistence equilibrium $E_*(x_*, y_*)$ is globally asymptotically stable and system (5.3) has no limit cycle. In this paper, we consider the same system with a constant delay τ due to the gestation of predator. We find that even if the condition (H_2) is satisfied, combining with the condition (H_1), the equilibrium $E_*(x_*, y_*)$ loses its stability and an orbitally asymptotically stable periodic solution arises from the Hopf bifurcation when the delay τ passes through some critical value τ_0, and the bifurcating periodic solution always exists for all $\tau \in (\tau_0, +\infty)$. This shows the important influence of the time delay τ on the system.

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