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A new regularity criterion for the Navier–Stokes equations in terms of the two components of the velocity

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Abstract. This paper establishes a new regularity criterion for the Navier–Stokes equations in terms of two velocity components. We show that if the two velocity components $\widetilde{u} = (u_1, u_2, 0)$ satisfy

$$\int_0^T \|\tilde{u}(s)\|^2_{\dot{B}^0_{\infty,\infty}} ds < \infty,$$

then the solution can be smoothly extended after $t = T$. This gives an answer to an open problem in [B. Q. Dong, Z. Zhang, *Nonlinear Anal. Real World Appl.* **11**(2010), 2415–2421].

Keywords: Navier–Stokes equations, regularity criterion, Besov space.

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1 Introduction

Consider the Navier–Stokes equations for the viscous incompressible fluid flow in the whole space $\mathbb{R}^3$:

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p = 0, \qquad & (x,t) \in \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0, \qquad & (x,t) \in \mathbb{R}^3 \times (0,T), \\ u(x,0) = u_0(x), \qquad & x \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $u = u(x,t)$ is the velocity field, $p = p(x,t)$ is the scalar pressure and $u_0(x)$ with $\operatorname{div} u_0 = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included in the pressure gradient.

It is well-known that Navier–Stokes equations are an important mathematical model in fluid dynamics (see [12, 22]). The question of global regularity for smooth solutions in the 3D case remains generally open. Therefore, it is interesting to consider the regularity criterion for solutions under some additional growth conditions. The research on this direction started

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from Serrin in the 1960's and attracted more and more attention over the last few decades (see e.g. [1,6,10,26,31,32] and the references therein).

Introducing the class $L^{\alpha}(0,T;L^{q}(\mathbb{R}^3))$, it is shown that if we have a Leray–Hopf weak solution u belonging to $L^{\alpha}((0,T);L^{q}(\mathbb{R}^3))$ with the exponents α and q satisfying $\frac{2}{\alpha}+\frac{3}{q}\leq 1$, $2\leq\alpha<\infty$, $3<q\leq\infty$, then the solution $u(x,t)\in C^{\infty}(\mathbb{R}^3\times(0,T])$ [9,11,23–25,29,30], while the limit case $\alpha=\infty, q=3$ was covered much later by Iskauriaza, Serëgin and Shverak in [8]. One may also refer to the interesting results devoted to finding sufficient conditions to ensure the smoothness of the solutions; see [13–18] and the references therein.

Another approach is to consider the regularity condition in terms of the two velocity components $\widetilde{u}=(u_1,u_2,0)$. In [4] (see also [2]), Bae and Choe proved that if

$$\widetilde{u}\in L^{\alpha}(0,T;L^{q}(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{\alpha}+\frac{3}{q}\leq 1 \quad \text{and} \quad 3<q\leq\infty, \tag{1.2}$$

then the solution is smooth on $(0,T)$. Later on, Zhang et al. [33] extended the condition (1.2) into BMO space in the marginal case when $q=\infty$, i.e.

$$\widetilde{u}\in L^{2}(0,T;BMO(\mathbb{R}^3)). \tag{1.3}$$

Here BMO denotes the space of the bounded mean oscillation defined by

$$f\in L^{1}_{\text{loc}}(\mathbb{R}^3), \qquad \sup_{x,R}\frac{1}{|B(x,R)|}\int_{B(x,R)}\left|f(y)-\overline{f}_{B(x,R)}\right|dy<\infty,$$

with $\overline{f}_{B(x,R)}$ is the average of f over $B(x,R)=\{y\in\mathbb{R}^3:|x-y|<R\}$ (cf. Stein [27]).

Based on the Littlewood–Paley decomposition of equations (1.1), Dong and Zhang [7] extended the regularity criterion by means of horizontal derivatives of the two velocity components $\nabla_h\tilde{u}=(\partial_1\tilde{u},\partial_2\tilde{u},0)$ in the homogeneous Besov space $\dot{B}^{0}_{\infty,\infty}$:

$$\int_0^T\|\nabla_h\tilde{u}(s)\|^2_{\dot{B}^0_{\infty,\infty}}ds<\infty. \tag{1.4}$$

It is still an open question, asked by Dong and Zhang [7, p. 2417], whether (1.4) can be replaced by the following condition

$$\int_0^T\|\tilde{u}(s)\|^2_{\dot{B}^0_{\infty,\infty}}ds<\infty. \tag{1.5}$$

The purpose of this paper is to give an answer to this problem posed in [7] and we claim that we obtain the regularity for weak solutions if the two components are in the sharp critical space

$$\int_0^T\|\tilde{u}(s)\|^2_{\dot{B}^0_{\infty,\infty}}ds<\infty.$$

Before giving the main result, we recall the following definition of Leray–Hopf weak solution.

Definition 1.1. Let $u_0\in L^2(\mathbb{R}^3)$ and $\nabla\cdot u_0=0$. A measurable vector field $u(x,t)$ is called a Leary–Hopf weak solution to the Navier–Stokes equations (1.1) on $(0,T)$, if u satisfies the following properties:

(i) $u\in L^{\infty}(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$;

(ii) $\partial_t u+(u\cdot\nabla)u+\nabla\pi=\Delta u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$;

(iii) $\nabla \cdot u = 0$ in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$;

(iv) u satisfy the energy inequality

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(x,s)|^2\, dxds \le \|u_0\|_{L^2}^2, \quad \text{for } 0 \le t \le T. \tag{1.6}$$

Next, we recall the definition of the space that we are going to use (see e.g. [3,28]).

Definition 1.2. Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity that satisfies $\widehat{\varphi} \in C_0^\infty(B_2 \backslash B_{\frac{1}{2}})$, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$ and

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}_j(\xi) = 1 \quad \text{for any} \quad \xi \ne 0,$$

where B_R is the ball in $\mathbb{R}^3$ centered at the origin with radius $R > 0$. The homogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^3)$ are defined to be

$$\dot{B}^s_{p,q}(\mathbb{R}^3) = \left\{ f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} \left\| 2^{js} \varphi_j * f \right\|_{L^p}^q \right)^{\frac{1}{q}} & \text{if } 1 < q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js} \left\| \varphi_j * f \right\|_{L^p} & \text{if } q = \infty, \end{cases}$$

for $s \in \mathbb{R}$, $1 \le p, q \le \infty$, where $\mathcal{S}'$ is the space of tempered distributions and $\mathcal{P}$ is the space of polynomials.

This definition ensures the following homogeneous property

$$\|f(\lambda .)\|_{\dot{B}^s_{p,q}} = \lambda^{s-\frac{3}{p}} \|f(.)\|_{\dot{B}^s_{p,q}}.$$

Throughout this paper, C will denote a generic positive constant which can vary from line to line. Our main result is the following.

Theorem 1.3. *Let $u_0 \in H^3(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in $\mathbb{R}^3$. Assume that $u(x,t)$ is a Leray–Hopf weak solution of (1.1) on $(0,T)$. If u satisfies*

$$\int_0^T \|\tilde{u}(s)\|_{\dot{B}^0_{\infty,\infty}}^2 ds < \infty,$$

then the solution can be extended after $t = T$. In other words, if the solution blows up at $t = T$, then

$$\int_0^T \|\tilde{u}(s)\|_{\dot{B}^0_{\infty,\infty}}^2 ds = \infty.$$

Duality relation $(\mathcal{H}^1)^* = BMO$ and the fact that $\omega.\nabla\omega \in \mathcal{H}^1$ due to [5] play an important role of our apriori estimate, where $\mathcal{H}^1$ denotes the Hardy space. For the proof of our result, we recall the following logarithmic Sobolev inequality in Besov spaces due to Kozono–Ogawa–Taniuchi [20, Theorem 2.1].

Lemma 1.4. *Let $s > \frac{3}{2}$. There exists a constant C such that the estimate*

$$\|f\|_{\dot{B}^0_{\infty,2}} \leq C\left(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \ln^{\frac{1}{2}}\left(1 + \|f\|_{H^s}\right)\right) \tag{1.7}$$

holds for all $f \in H^s\left(\mathbb{R}^3\right)$.

In order to prove our main result, we need the following lemma.

Lemma 1.5. *Assume that $u = (u_1, u_2, u_3)$ is a smooth and divergence-free $(\nabla.u = 0)$ vector field, let $\widetilde{u} = (u_1, u_2, 0)$. Then we have, for the generic constant C*

$$\left|\int_{\mathbb{R}^3} (u \cdot \nabla u).\Delta u dx\right| \leq C \int_{\mathbb{R}^3} |\widetilde{u}| \, |\nabla u| \, |\Delta u| \, dx.$$

Proof. Due to the divergence-free condition $\nabla.u = \sum_{i=1}^3 \partial_i u_i = 0$ and integration by parts, observe first that

$$\begin{aligned}
\int_{\mathbb{R}^3} (u \cdot \nabla u).\Delta u dx &= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx = -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k (u_i \partial_i u_j) \partial_k u_j dx \\
&= -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx - \frac{1}{2} \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i (\partial_k u_j \partial_k u_j) dx \\
&= -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx = RHS,
\end{aligned} \tag{1.8}$$

where we have used the fact

$$\begin{aligned}
-\frac{1}{2} \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i (\partial_k u_j \partial_k u_j) dx &= \frac{1}{2} \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_i u_i (\partial_k u_j \partial_k u_j) dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^3} \left(\sum_{i=1}^3 \partial_i u_i\right) \sum_{j,k=1}^3 (\partial_k u_j)^2 dx = 0.
\end{aligned}$$

We now estimate *RHS*. When $i = 1, 2$ or $j = 1, 2$, by integrating by parts,

$$\begin{aligned}
RHS &= -\sum_{i,j=1}^2 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_k u_j \partial_i u_j dx \\
&= -\sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_k u_j \partial_i u_j dx - \sum_{j=1}^2 \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u_3 \partial_k u_j \partial_3 u_j dx \\
&= \sum_{i=1}^2 \sum_{j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_k (\partial_k u_j \partial_i u_j) dx + \sum_{j=1}^2 \sum_{k=1}^3 \int_{\mathbb{R}^3} u_j \partial_3 (\partial_k u_3 \partial_k u_j) dx \\
&\leq C \int_{\mathbb{R}^3} (|u_1| + |u_2|) \, |\nabla u| \, |\Delta u| \, dx \\
&\leq C \int_{\mathbb{R}^3} |\widetilde{u}| \, |\nabla u| \, |\Delta u| \, dx.
\end{aligned}$$

In the case $i = j = 3$, by using the fact $-\partial_3 u_3 = \partial_1 u_1 + \partial_2 u_2$ and integrating by parts, we have

$$\begin{aligned}
\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx &= - \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k (u_i \partial_i u_j) \partial_k u_j dx \\
&= - \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k (u_3 \partial_3 u_3) \partial_k u_3 dx \\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_3 (-\partial_3 u_3) \partial_k u_3 dx \\
&= \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_3 (\partial_1 u_1 + \partial_2 u_2) \partial_k u_3 dx \\
&= - \sum_{k=1}^{3} \int_{\mathbb{R}^3} u_1 \partial_1 \partial_k u_3 \partial_k u_3 dx - \sum_{k=1}^{3} \int_{\mathbb{R}^3} u_2 \partial_k u_3 \partial_2 \partial_k u_3 dx \\
&\le C \int_{\mathbb{R}^3} (|u_1| + |u_2|) |\nabla u| |\Delta u| dx \\
&\le C \int_{\mathbb{R}^3} |\widetilde{u}| |\nabla u| |\Delta u| dx.
\end{aligned}$$

Combining the two inequalities with (1.8) yields

$$\left| \int_{\mathbb{R}^3} (u \cdot \nabla u).\Delta u dx \right| \le C \int_{\mathbb{R}^3} |\widetilde{u}| |\nabla u| |\Delta u| dx.$$

The proof of Lemma 1.5 is complete. □

2 Proof of Theorem 1.3

Now we are in a position to prove our main result.

Proof. We take L^2-inner products on (1.1) with u, integrate by parts and use incompressibility to obtain

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 = 0.$$

This identity allows us to get

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2 .$$

Next, multiplying the first equation in (1.1) by Δu, after integration by parts and taking the divergence free property into account, we have by Lemma 1.5

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 = - \int_{\mathbb{R}^3} (u \cdot \nabla u).\Delta u dx \le C \int_{\mathbb{R}^3} |\widetilde{u}| |\nabla u| |\Delta u| dx. \tag{2.1}$$

We recall the following property of Hardy spaces and BMOs

$$\int_{\mathbb{R}^3} fgh dx \le \|fg\|_{\mathcal{H}^1} \|h\|_{BMO} \le \|f\|_{L^2} \|g\|_{L^2} \|h\|_{BMO} , \tag{2.2}$$

for any $\nabla.f = 0$ and $\nabla \times g = 0$. According to above inequality (2.2) and Young's inequality, we estimate

$$\begin{aligned}\int_{\mathbb{R}^3} |\tilde{u}|\,|\nabla u|\,|\Delta u|\,dx &\le \|\nabla u.\Delta u\|_{\mathcal{H}^1}\,\|\tilde{u}\|_{BMO}\\ &\le C\,\|\nabla u\|_{L^2}\,\|\Delta u\|_{L^2}\,\|\tilde{u}\|_{BMO}\\ &\le \frac{1}{2}\,\|\Delta u\|_{L^2}^2 + C\,\|\nabla u\|_{L^2}^2\,\|\tilde{u}\|_{BMO}^2 . \end{aligned} \tag{2.3}$$

Due to a fact that $\dot{B}^0_{\infty,2} \subset BMO$, inserting the above estimates into (2.1), we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,\|\nabla u(t)\|_{L^2}^2 + \|\Delta u(t)\|_{L^2}^2 &\le C\,\|\tilde{u}(t)\|_{BMO}^2\,\|\nabla u(t)\|_{L^2}^2\\ &\le C(1+\|\tilde{u}(t)\|^2_{\dot{B}^0_{\infty,\infty}}\ln(e+\|\tilde{u}(t)\|_{H^3}))\,\|\nabla u(t)\|_{L^2}^2\\ &\le C(1+\|\tilde{u}(t)\|^2_{\dot{B}^0_{\infty,\infty}}\ln(e+\|u(t)\|_{H^3}))\,\|\nabla u(t)\|_{L^2}^2 . \end{aligned}\tag{2.4}$$

For any $T_0 < t \le T$, we set

$$z(t) := \sup_{T_0\le s\le t} \left\|\Lambda^3 u(t)\right\|_{L^2} . \tag{2.5}$$

By the Gronwall inequality on (2.4) for the interval $[T_0, t]$, one has

$$\|\nabla u(t)\|_{L^2}^2 \le \|\nabla u(T_0)\|_{L^2}^2 \exp\left(C\int_{T_0}^t (1+\|\tilde{u}(s)\|^2_{\dot{B}^0_{\infty,\infty}}\ln(e+z(s)))ds\right).$$

Hence, we obtain from the above estimate

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 &\le C_0\exp(C\epsilon(1+\ln(e+z(t))))\\ &\le C_0\exp(2C\epsilon\ln(e+z(t)))\\ &\le C_0(e+z(t))^{2C\epsilon},\end{aligned}$$

provided that for any small constant $\epsilon > 0$, there exists $T_0 < T$ such that

$$\int_{T_0}^T \|\tilde{u}(s)\|^2_{\dot{B}^0_{\infty,\infty}} ds < \epsilon \ll 1, \tag{2.6}$$

here C_0 means a constant depending on T_0.

Now we do the estimate for $z(t)$ defined by (2.5). Taking the operation $\Lambda^3 = (-\Delta)^{\frac{3}{2}}$ on both sides of (1.1), then multiplying them by $\Lambda^3 u$, after integrating over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 u(t)\right\|_{L^2}^2 + \left\|\Lambda^3\nabla u(t)\right\|_{L^2}^2 = -\int_{\mathbb{R}^3}\Lambda^3\,(u\cdot\nabla u)\,\Lambda^3 u dx. \tag{2.7}$$

Noting that $\nabla\cdot u = 0$ and integrating by parts, we write (2.7) as

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^3 u(t)\right\|_{L^2}^2 + \left\|\Lambda^3\nabla u(t)\right\|_{L^2}^2 = -\int_{\mathbb{R}^3}\left[\Lambda^3(u\cdot\nabla u) - u\cdot\Lambda^3\nabla u\right]\Lambda^3 u dx = \Pi. \tag{2.8}$$

In what follows, we will use the following inequality due to Kenig, Ponce and Vega [19]:

$$\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \le C\left(\|\Lambda^{\alpha-1}g\|_{L^{q_1}}\,\|\nabla f\|_{L^{p_1}} + \|\Lambda^\alpha f\|_{L^{p_2}}\,\|g\|_{L^{q_2}}\right), \tag{2.9}$$

for $\alpha > 1$, and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Hence Π can be estimated as

$$\begin{aligned}\Pi &\le C\|\nabla u\|_{L^3}\|\Lambda^3 u\|_{L^3}^2 \le C\|\nabla u\|_{L^2}^{\frac{13}{12}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{4}}\|\Lambda^4 u\|_{L^2}^{\frac{5}{3}} \\ &\le \frac{1}{6}\|\Lambda^4 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{\frac{13}{2}}\|\Lambda^3 u\|_{L^2}^{\frac{3}{2}},\end{aligned} \tag{2.10}$$

where we used (2.9) with $\alpha = 3, p = \frac{3}{2}, p_1 = q_1 = p_2 = q_2 = 3$, and the following Gagliardo–Nirenberg inequalities

$$\|\nabla u\|_{L^3} \le C\|\nabla u\|_{L^2}^{\frac{3}{4}}\|\Lambda^3 u\|_{L^2}^{\frac{1}{4}},$$

and

$$\|\Lambda^3 u\|_{L^3} \le C\|\nabla u\|_{L^2}^{\frac{1}{6}}\|\Lambda^4 u\|_{L^2}^{\frac{5}{6}}.$$

If we use the existing estimate (2.6) for $T_0 < t < T$, (2.10) reduces to

$$\Pi \le \frac{1}{6}\|\Lambda^4 u\|_{L^2}^2 + C_0 C(e + z(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon}.$$

Combining (2.10) and (2.7), we easily get

$$\frac{d}{dt}\left\|\Lambda^3 u(t)\right\|_{L^2}^2 \le C_0 C(e + z(t))^{\frac{3}{2}+\frac{13}{2}C\epsilon}.$$

Gronwall's inequality implies the boundedness of H^3-norm of u provided that $\epsilon < \frac{1}{13C}$, which can be achieved by the absolute continuous property of integral (1.5). This completes the proof of Theorem 1.3. □

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