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Abstract: There is a conjecture that if the sum of graphic matroids is not graphic then it is nonbinary [5]. Some special cases have been proved only, for example if several copies of the same graphic matroid are given. If there are two matroids and the first one can be represented by a graph with two points, then a necessary and sufficient condition is given for the other matroid to ensure the graphicity of the sum. Hence the conjecture holds for this special case.

Keywords: matroid theory, graphic matroids, sum of matroids

1 Introduction

Graphic matroids form one of the most significant classes in matroid theory. When introducing matroids, Whitney concentrated on relations to graphs. The definition of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the sum. The sum of two graphic matroids can be nongraphic.

The purpose of our work is to study the graphicity of the sum of graphic matroids. The first paper in this area was that of Lovász and Recski: they examined the case if several copies of the same graphic matroid are given [1]. Then Recski conjectured thirty years ago that if the sum of graphic matroids is not graphic then it is nonbinary [5]. He also studied the case if we fix one simple graphic matroid and take its sum with every possible graphic matroid. His main result is the following theorem.

Theorem 1 *[2] Let A and B be the cycle matroids of the graphs shown in Figure 1 on ground sets $E_A = \{1, 2, ..., n\}$ and $E_B = \{1, 2, i, j, k\}$, respectively.*
Then the sum $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.

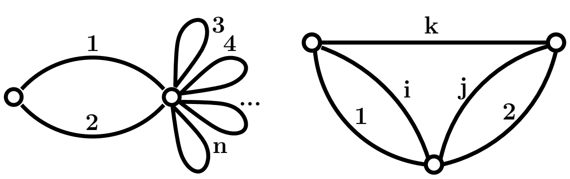

Figure 1: A graphic representation of A (left) and B (right)

We shall use Tutte's theorem which is fundamental in matroid theory:

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Theorem 2 *[6]*

- *A matroid is binary if and only if it does not contain $U_{4,2}$ as a minor.*
- *A matroid is regular if and only if it does not contain $U_{4,2}$, F_7 and F_7^* as minors.*
- *A matroid is graphic if and only if it does not contain $U_{4,2}$, F_7, F_7^*, $M^*(K_5)$ and $M^*(K_{3,3})$ as minors.*
- *A matroid is the circuit matroid of a plane graph if and only if it does not contain $U_{4,2}$, F_7, F_7^*, $M^*(K_5)$, $M^*(K_{3,3})$, $M(K_5)$ and $M(K_{3,3})$ as minors.*

Forbidden minors will be of importance in our forthcoming results as well, although they will not appear in our final statement.

2 Main result

In most of the cases we speak about graphic matroids so I will call the elements of the matroids edges. Observe that all but two of the edges of the graph representing the matroid A of Theorem 1 are loops (see Figure 1 as well). Also observe that bridges in a matroid remain bridges of its sum with any other matroid. Hence, in order to generalize Theorem 1 I started to analyze the case when we have only three edges which are neither bridges nor loops. There are two types of matroids with this property, the one with a circuit of length three, and the other with three parallel edges. I found in both cases that there are some forbidden minors so that if any of them appears in the other matroid then the sum is not graphic, while if the matroid doesn't contain any of them then the sum is graphic.
After these results [7] I started to work with the cases with n parallel edges or with circuits of length n (of course there may be many loops and bridges). After some disappointing results (that the n long circuit's cases surely can't lead to the same type of conditions that I wanted to prove in the other case) the case with n parallel edges lead to a very useful result.
For a transparent presentation of the theorems that will follow, we have to formulate some definitions and prove some lemmata, which help us to reduce the infinite number of cases. We study the sum of two graphic matroids M_1 and M_2 (of course they have identical ground sets). Throughout we shall refer to M_1 and M_2 as addends.
It is well known that if a matroid is graphic then so are all of its submatroids and minors. Hence if a matroid has a non graphic minor then the matroid can't be graphic.

Definition 3 *We call some edges of the matroid serial if they belong to exactly the same circuits.*

Definition 4 *We call an edge of M_1 essential if it is not a loop in M_2 and we call it irrelevant otherwise.*

Definition 5 *We call a submatroid of an addend devoid if it contains irrelevant edges only.*

The following lemmata contain the main opportunities when we want to simplify our addend matroids. It is important that these are valid for graphic matroids only, so I can use graph theoretical concepts.

Lemma 6 *Let X and Y denote the set of bridges in M_1 and in M_2 respectively.*
The sum $M_1 \vee M_2$ is graphic if and only if $M_1 \backslash (X \cup Y) \vee M_2 \backslash (X \cup Y)$ is graphic.

PROOF: If an element of a matroid M is a bridge then it will be a bridge in the sum of M with any other matroid. Therefore if $M_1 \backslash (X \cup Y) \vee M_2 \backslash (X \cup Y)$ is graphic then we can extend this graph that represents the sum with bridges for $X \cup Y$ and this way we get a graph of $M_1 \vee M_2$.
On the other hand if $(M_1 \backslash (X \cup Y)) \vee (M_2 \backslash (X \cup Y))$ is not graphic then $M_1 \vee M_2$ can't be graphic because it has a non graphic submatroid. $\square$

Lemma 7 *If a connected component X of the matroid M_1 is a devoid submatroid then the sum $M_1 \vee M_2$ is graphic if and only if $(M_1 \backslash X) \vee (M_2 \backslash X)$ is graphic.*

Proof: It is easy to see that the matroid which is the direct sum of $(M_1 \backslash X) \vee (M_2 \backslash X)$ and X is isomorphic to $M_1 \vee M_2$. The direct sum of graphic matroids is also graphic, hence $M_1 \vee M_2$ is graphic. On the other hand if $(M_1 \backslash X) \vee (M_2 \backslash X)$ is not graphic then $M_1 \vee M_2$ can't be graphic because it has a non graphic submatroid. □

Lemma 8 *Assume that M_1 is the circuit matroid of a graph $G(V, E)$ in which X is a connected set of edges and $E \backslash X$ has exactly two common vertices with X (call them a and b).*
Let M_1' be the circuit matroid of $G' := G(V, E \cup \{(a, b)\} \backslash X)$ and $M_2' := M_2 \backslash X \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').
If X is devoid then the sum $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.

Proof: If $M_1' \vee M_2'$ is graphic then we can replace the edge (a, b) in its graph with the subgraph of X (where $a, b \in X$ will be the two common vertices of X and its complement) and we get a graph of $M_1 \vee M_2$.
On the other hand if $M_1' \vee M_2'$ is not graphic then since this sum arises as a minor of $M_1 \vee M_2$ this latter cannot be graphic either. □

Lemma 9 *Assume that M_1 is the circuit matroid of a graph $G(V, E)$ where X is a connected component of G. If X has only one essential edge x then the sum $M_1 \vee M_2$ is graphic if and only if $(M_1 \backslash X \cup loop(x)) \vee (M_2 \backslash (X \backslash x))$ is graphic.*

Proof: If $(M_1 \backslash X \cup loop(x)) \vee (M_2 \backslash (X \backslash x))$ is graphic then we can obtain the graph of $M_1 \vee M_2$ by replacing edge x with the graph X in the following way:
Let a and b denote the end vertices of x in X. Cut vertex a into two vertices a_1 and a_2 in X. Among the edges incident to a, join x to a_1 and all the others to a_2. Replace x with the result in the graph of $(M_1 \backslash X \cup loop(x)) \vee (M_2 \backslash (X \backslash x))$ along the vertices a_1 and a_2.
On the other hand if $(M_1 \backslash X \cup loop(x)) \vee (M_2 \backslash (X \backslash x))$ is not graphic then since this sum arises as a minor of $M_1 \vee M_2$, this latter cannot be graphic either. □

The next lemma will be less general, it is only for the cases with parallel edges and loops.

Lemma 10 *M_1 consist of n parallel edges and k loops. If two essential edges x and y in M_2 are serial then the sum $M_1 \vee M_2$ is graphic if and only if $(M_1 \backslash x) \vee M_2'$ is graphic where M_2' is defined as follows: Simply replace the edges x and y in M_2 by a single edge xy so that a set S containing xy is independent if and only if $S \backslash xy \cup \{x, y\}$ was independent. Then xy will play the role of y in $M_1 \backslash x$.*

Proof: If $(M_1 \backslash x) \vee M_2'$ is graphic then we can separate edge y of that graph into two serial edges x and y. Then the circuit matroid of the resulting graph will be exactly $M_1 \vee M_2$.
On the other hand if $(M_1 \backslash x) \vee M_2'$ is not graphic then since this sum arises as a minor of $M_1 \vee M_2$ this latter cannot be graphic either. □

After these preliminaries we can define the reduction, that will be the most important concept to reduce the infinite number of cases.

Definition 11 *We want to know if the sum $M_1 \vee M_2$ is graphic. We call M_2 reduced if none of the lemmata above can help us to decrease the number of edges. (Recall that Lemma 8 can be applied for a special case only while the other four types of simplifications can be applied in any case.)*

Corollary 12 *Assume that M_1 and M_2 are graphic matroids. Application of the previous lemmata to M_2 leads to a reduced matroid M_2' while M_1 changes to M_1' that we shall call the pair of M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

We are only one step away from our theorem, but to write it in a pretty way we have to define the following:

Definition 13 *Suppose that a matroid has at least three circuits C_1, C_2, C_3 so that each circuit C_i has at least one element a_i satisfying $a_i \notin \bigcup_{j \neq i} C_j$. (These edges will be called proper edges.) Such a matroid fulfils the three circuits property if the circuits C_1, C_2 and C_3 are not pairwise disjoint.*

Theorem 14 *Let M_1 be a matroid which consists of n parallel edges and k loops and let M_2' be the matroid reduced from M_2. Then $M_1 \vee M_2$ is non graphic if and only if M_2' fulfils the three-circuit property.*

PROOF: First we prove that if M_2' does not fulfil the three-circuit property then $M_1' \vee M_2'$ is graphic, which means $M_1 \vee M_2$ is graphic due to the previous corollary.
There can be two different cases, one with any number of essential loops and nothing else in M_2' and the other is like A_1 (see Figure 2). Both cases lead us to a graphic sum, the first to M_1' and the second to a circuit with a, b and c.
Now we have to prove that if M_2' fulfils the three-circuit property then $M_1' \vee M_2'$ is not graphic, which

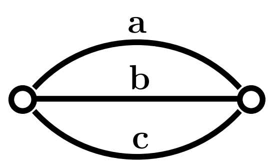

Figure 2: A graph which belongs to matroid A_1

means $M_1 \vee M_2$ is not graphic because it has a non graphic minor.
Without loss of generality, we may suppose that M_2 has exactly three circuits (delete one proper edge from an appropriately chosen extra circuit otherwise, and then perform reduction for the resulting matroid). Notice that if we have more than three circuits with proper edges then we can delete one proper edge of a circuit and the three-circuit property still holds. In this case we can delete as many proper edges as we can, but we have to reduce the matroid after each deletion. So we can assume that M_2' has exactly three circuits with proper edges.
It is easy to see that if M_2' has at least two components (it means exactly two by the previous assumption) then it has F_1 as a minor (see Figure 3), and then the sum can't be graphic, moreover the sum contains $U_{4,2}$ as a minor.
Then we only have to analyze the cases when M_2' is connected therefore has exactly three circuits with

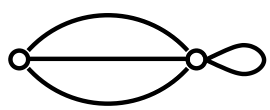

Figure 3: A graph which belongs to matroid F_1

proper edges in one component.
All the cases with this assumption are minors of the following matroid F_2. Let F_2 be the circuit matroid of a graph obtained from K_4 by replacing each edge with a path of length 2 (see Figure 4). There are several cases, but all we have to do is to verify that the sum has $U_{4,2}$ as a minor.
I show the way of the test and then leave the cases to the reader. So let M_1 be the matroid with four parallel edges (a, b, c, d) and two loops (e, f) and M_2 be the matroid which is a minor of F_2 such that the irrelevant edges are neighbours as in Figure 5.
The bases of $M_1 \vee M_2$ will be: $\{a, b, c, d\}; \{a, b, c, e\}; \{a, b, c, f\}; \{a, b, d, e\}; \{a, b, d, f\};$

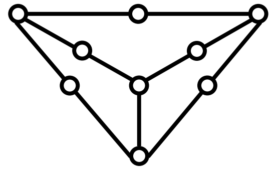

Figure 4: A graph which belongs to matroid F_2. Any essential edge may be serial with an irrelevant one after the reductions.

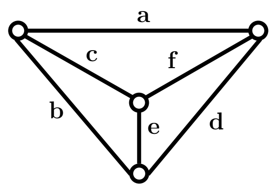

Figure 5: A graph which belongs to matroid M_2

$\{a, b, e, f\}; \{a, c, d, e\}; \{a, c, d, f\}; \{a, c, e, f\}; \{a, d, e, f\}; \{b, c, d, e\}; \{b, c, d, f\};$
$\{b, c, e, f\}; \{b, d, e, f\}; \{c, d, e, f\}$ So the sum is $U_{6,2}$ then it has $U_{4,2}$ as minor, so it is not binary, let alone graphic. $\square$

3 Summary

In spite of these results the main problem is still open. The conjecture holds for all the examined cases. We expect that the reductions described by Lemmata 4 through 7 will also be useful in the future study of more general cases.

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