related to the union operation

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Abstract: There is a conjecture that if the union (also called sum) of graphic matroids is not graphic then it is nonbinary. Some special cases have been proved only, for example if several copies of the same graphic matroid are given. If there are two matroids and the first one can either be represented by a graph with two points, or is the direct sum of a circuit and some loops, then a necessary and sufficient condition is known for the other matroid to ensure the graphicity of the union and the above conjecture holds for these cases. We have proved the sufficiency of this condition for the graphicity of the union of two arbitrary graphic matroids. Then we present a weaker necessary condition which is of similar character. Finally we suggest a more general framework of the study of such questions by introducing matroid classes formed by those graphic (or arbitrary) matroids whose union with any graphic (or arbitrary) matroid is graphic (or either graphic or nonbinary).

Keywords: matroid theory, graphic matroid, union of matroids

1 Introduction

Matroids have been introduced as the generalizatons of graphs (and some other concepts) and many of the operations among matroids are straightforward generalizations of graph operations. Hence the class of graphic matroids is closed with respect to some basic operations like deletion, contraction, direct sum; or the class of matroids arising from planar graphs is closed with respect to duality. On the other hand, this is not the case for matroid union (also called sum). This operation is of great importance from the point of view of combinatorial optimization but only a few interesting subclasses of matroids are closed with respect to union. In particular, the union of two graphic matroids is rarely graphic and all the known nongraphic unions are nonbinary. In fact, the following conjecture was formulated long ago:

Conjecture 1 *[9] If the union of two graphic matroids is not graphic then it is non-binary.*

In what follows, some old and some new results in this area are presented, mostly without proofs. Throughout we use the notation of Oxley [7]. In particular, the definitions of direct sum and union are given as Definitions 16 and 17, respectively, in the Appendix, where some notational conventions are also given.

The fundamental contributions of Edmonds [4] and Nash-Williams [6] characterize those graphic matroids whose union is the free matroid (which is clearly graphic, namely the cycle matroid of a tree). If the union of several copies of the same graphic matroid is considered then one can decide if this union is graphic [5]. In fact, one can very easily characterize those graphs whose cycle matroids arise as the union

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of a given number of identical graphic matroids (or even as the union of a given number of identical matroids in general). These results settle Conjecture 1 as well for the case of identical addends. The general question (when is the union of two graphic matroids graphic) is still open for general addends. A possible approach is to fix a graph G_0 (or its cycle matroid $M_0 = M(G_0)$) and study those graphs G where the union of $M(G)$ and M_0 is graphic. If M_0 consists of loops only then the problem is trivial (every graphic matroid $M(G)$ will do). If M_0 contains bridges then these edges can clearly be disregarded both in $M(G)$ and M_0. Hence the first interesting question was if G_0 consists of a circuit of length two (two parallel edges) and any number of loops. This case has been solved in [8] where a Kuratowski-type characterization of G has been given. This result has recently been generalized for the case if G_0 consists of either several serial or several parallel edges in addition to the loops, see [1], [2] and [3].

Theorem 2 *[8] Let the matroid M_0 consist of loops and two parallel elements 1, 2. Then the union of M_0 and the cycle matroid $M(G)$ is graphic if and only if G does not contain as a subgraph either the graph H of Figure 1 or its subdivision (where the specific edges 1, 2 are in the specific position).*

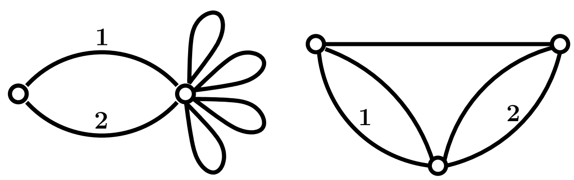

Figure 1: A graphic representation of M_0 (left) and the graph H (right)

Observe that if we delete the non-loop edges (namely 1 and 2) of M_0 from H then the remaining graph contains both a circuit and a spanning tree. This property turns out to be important in the more general cases as well, leading to some sufficient conditions for the graphicity of the union, see Section 2. A byproduct of the proof of Theorem 2 gives that if the union is not graphic then it is non-binary – a very special case of Conjecture 1. A generalization of this part of the proof leads to some necessary conditions for the graphicity, see Section 3. Finally in Section 4 we put Conjecture 1 in a more general framework, introducing some new matroid classes relating graphicity and union in several different ways.

2 Sufficient conditions

The following two examples are minimal examples that the union of graphic matroids can be non-binary. They are minimal in the number of edges together with the number of non-loop edges.

Example 3 $(U_{1,3} \oplus U_{0,1}) \vee (U_{1,3} \oplus U_{0,1}) = U_{2,4}$ *if the loop of the first addend is not the same as the loop of the second.*

Example 4 *Let $M_1 = U_{1,2} \oplus U_{0,3}$ with parallel edges 1 and 2 and loops $3, 4$ and 5 and M_2 be the cycle matroid of the graph H of Figure 1. Then the contraction of edge 1 from $M_1 \vee M_2$ leads to $U_{2,4}$.*

As we have seen in the previous section the presence of loops and bridges can decrease the size of the matroids to be studied. There are some less obvious situations as well (like serial or parallel edges, low connectivity number, etc) which can also lead to the reduction of the size of the problem, still preserving graphicity or non-binarity of the union. This sequence of lemmata and the resulting concept of *reduced pair of matroids* were given in [3], we summarize them in the Appendix.

Definition 5 *Let $L(M)$ and $NL(M)$denote the set of loops and non-loops, respectively, in the matroid M.*

Theorem 6 *[3] If G_0 consists of loops and a single circuit of length n $(n \geq 2)$ and $M = M(G)$ is an arbitrary graphic matroid on the same ground set then their union is graphic if and only if for the reduced pair M_0', M' either $NL(M_0)$ contains a cut set in G' or $M' \setminus NL(M_0)$ is the free matroid.*

Theorem 7 *[3] If G_0 consists of loops and two points joined by n parallel edges and $M = M(G)$ is an arbitrary graphic matroid on the same ground set then their union is graphic if and only if for the reduced pair M_0', M' no 2-connected component of G' has two non-serial edges a and b from $NL(M_0)$ so that $M' \setminus \{a, b\}$ is not the free matroid.*

Now let G_0 be an arbitrary graph. The following theorems will show that these conditions can be formalized together to a sufficient but no longer necessary condition for the graphicity of the union.

Theorem 8 *Let M_1, M_2 be two matroids defined on the same ground set E. $M_1 \vee M_2$ is graphic if for every circuit C in M_1 either $r_2(E - C) < r_2(E)$ or $r_2(E - C) = |E - C|$ holds.*

PROOF: We shall apply the following observation:

Proposition 9 *If there exists an edge $\alpha \in E$ so that $E - \{\alpha\}$ is independent in a matroid M then M is graphic.*

PROOF: If E is independent as well then M is the free matroid which is the cycle matroid of a tree. Otherwise E contains a unique circuit C hence M is the cycle matroid of a graph composed of a circuit (formed by the edges of C) and some coloops (corresponding to the edges of $E - C$). $\square$

We consider the cases according to the circuits of M_1:

1. If there exists a circuit C of M_1 so that $r_2(E - C) = |E - C|$ then for every element α of C the set $C \setminus \{\alpha\}$ is independent in M_1 and $E \setminus C$ is independent in M_2. This means $E \setminus \{\alpha\}$ is independent in the union, hence $M_1 \vee M_2$ is graphic by Proposition 9.

2. Let $C_1, C_2, ..., C_k$ be the circuits of M_1. The only remaining case is that $r_2(E - C_i) < r_2(E)$ holds for every i. This means that every base of M_2 intersects every circuit C_i. Let $X \subseteq E$ be a base of M_2 then $E \setminus X$ must be independent in M_1 (since it cannot contain a circuit). This means that $X \cup (E \setminus X) = E$ is independent in the union $M_1 \vee M_2$ so the union is the free matroid.

In summary, the union contains at most one circuit. $\square$

$U_{0,2} \vee U_{0,2}$ is the simplest example to show that the condition of Theorem 8 is not necessary.

If the requirements of Theorem 8 are prescribed for circuits of length at least two only, then a slightly weaker condition will still suffice.

Theorem 10 *$M_1 \vee M_2$ is graphic if for every circuit C of length at least two in M_1 either $r_2(E - C) < r_2(E)$ or $r_2(E - C) = |E - C|$.*

Now $(U_{1,2} \oplus U_{0,1}) \vee U_{0,3}$ is the simplest example to show that this condition is still not necessary.

In order to obtain further, gradually weaker conditions which will still suffice, first we may form a symmetric version of Theorem 10, that is, the union is graphic if the circuits of one of the matroids satisfy the rank requirements in the other matroid. However, $(U_{1,2} \oplus U_{0,2}) \vee (U_{0,2} \oplus U_{1,2})$ is the simplest example to show that this condition is still not necessary (the loops of the first matroid are the parallel edges in the second matroid).

Next it is enough to require this property to a reduced pair of matroids only. However $(U_{1,3} \oplus U_{0,3}) \vee (U_{1,2} \oplus U_{1,2} \oplus U_{1,2})$, where every component of the second matroid has exactly one loop from the first matroid shows that even this condition is not necessary.
It is easy to see that Lemma 27 eliminates this case because there exist serial edges in M_2 so that one is a loop in M_1. The following example shows that even with all these extensions, and with the help of Lemmata 25, 26 and 27 the property is not necessary for the graphicity of the union.

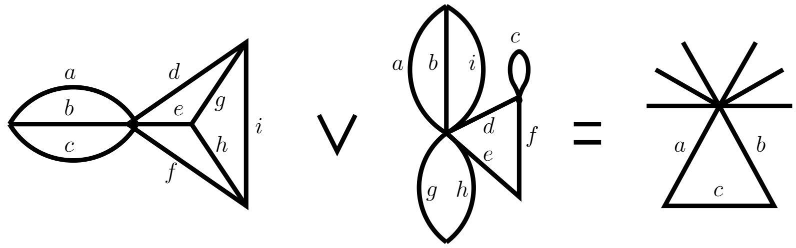

Figure 2: The graphic representation of two matroids and their union

Example 11 *Let M_1 and M_2 be two graphic matroids represented by the first two graphs of Figure 2. The union will have a circuit a, b, c and coloops hence graphic see the third graph of Figure 2. However a, b is a length two circuit in M_1 so that $M_2 \setminus \{a, b\}$ contains a spanning tree and a circuit too. Nevertheless M_1, M_2 is a reduced pair. This means this is a counterexample for the necessity of the property.*

In fact in Theorem 7 where one of the matroids consists of parallel edges and loops, we stated this property in a slightly different way: C_1 is a circuit in M_1, the elements of C_1 are in the same component of M_2 and $M_2 \setminus C_1$ contains a spanning tree and a circuit too. Observe that a and b are in the same component of M_2 in Example 11 so that remains a counterexample for the necessity even if we add this condition.

However if we use Lemma 28 and Lemma 29 then this example can be reduced too. In fact either of the two will do, because a and b are parallel in both matroids (Lemma 28), on the other hand c is a loop in M_2 and c is parallel to a in M_1 (Lemma 29). Recall that these lemmata are not about equivalent reductions like Lemmata 18-20 or 24-27 (just in the case where the union is binary), so we can no longer speak about necessity of the extended version of the conditions.

3 A necessary condition

In this section we present a necessary condition for the binarity of the union of two graphic matroids. This condition is formalized in a way similar to the sufficient condition in the previous section. The symmetrized version of Theorem 10 implies that if $M_1 \vee M_2$ is not graphic then there exist two subsets C_1, C_2 of size at least two so that C_i is a circuit in M_i for both $i = 1$ and 2 and $r_i(E) = r_i(E - C_{3-i}) < |E - C_{3-i}|$. This is still not necessary; however, requiring some further relations between these two sets lead to a necessary condition.

Theorem 12 *Let M_1 and M_2 be graphic matroids. If all the following conditions hold then the union $M_1 \vee M_2$ is not binary.*

1. *There exist dependent sets X_i in M_i for both $i = 1$ and 2*
2. *$X_1 \cap X_2 = \emptyset$*
3. *There exists a circuit C_i of M_i in X_i so that $|C_i| \geq 2$ for both $i = 1$ and 2*
4. *$r_i(X_i) = r_i(X_1 \cup X_2)$ for both $i = 1$ and 2*
5. *There exist two distinct elements $a, b \in C_1 \cup C_2$ so that one of the following holds*
   - *if $a \in C_i$ and $b \in C_{3-i}$ then a and b are in the same component in both matroids*
   - *if $a, b \in C_i$ then there exists $X'_{3-i} \subset X_{3-i}$ so that if we contract X'_{3-i} in M_{3-i} then a and b are distinct diagonals of C_{3-i}*

Unfortunately there remains a gap which consists of those cases where there might exist a counterexample for Conjecture 1 (a pair of graphic matroids which have a nongraphic but binary union). Such a counterexample, if exists, must be unreducable (as described by Lemmata 18 through 27). The study of the above gap was the motivation of Lemmata 28 and 29. These lemmata imply that a possible minimal counterexample must have some special properties.
Observe that the two minimal examples of graphic pairs which have non binary union (see Examples 3 and 4) motivate the last condition.

4 New questions

In order to put Conjecture 1 into a more general framework, we formally define eight matroid classes as follows.
Let A be the set of those graphic matroids which give a graphic or non-binary union with any graphic matroid.
Let B be the set of those graphic matroids which give a graphic union with any graphic matroid.
Let C be the set of those graphic matroids which give a graphic or non-binary union with any matroid.
Let D be the set of those graphic matroids which give a graphic union with any matroid.
Let E be the set of those matroids which give a graphic or non-binary union with any graphic matroid.
Let F be the set of those matroids which give a graphic union with any graphic matroid.
Let G be the set of those matroids which give a graphic or non-binary union with any matroid.
Let H be the set of those matroids which give a graphic union with any matroid.

Observe that Conjecture 1 states that A is the set of all graphic matroids.

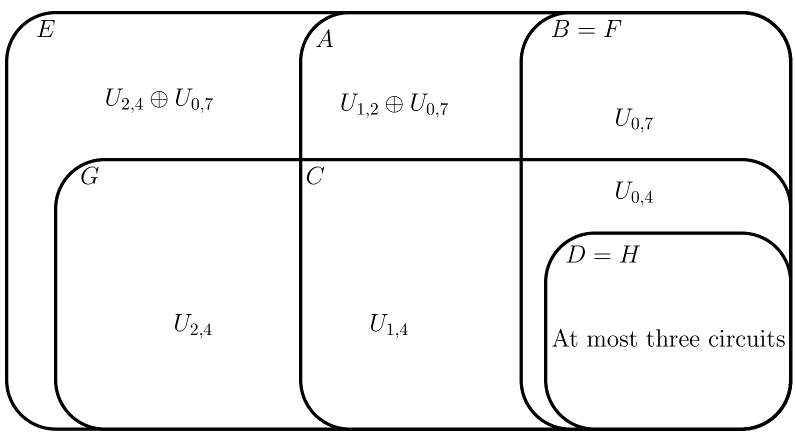

Figure 3: Examples for all nonempty subsets

Most of the relationships between the sets are trivial ($D \subseteq C \subseteq A$, $D \subseteq B \subseteq A$, $H \subseteq G \subseteq E$, $H \subseteq F \subseteq E$, $A \subseteq E$, $B \subseteq F$, $C \subseteq G$, $D \subseteq H$). For $D = H$ recall that the union of M and $U_{0,k}$ is M so if the union is graphic then M is also graphic. Since $U_{0,k}$ is graphic $F = B$ follows similarly. $(A \cap G) \setminus C$ is empty because if a matroid is in G but not in C then it is not graphic.
These remarks are summarized in the diagram of Figure 3. All the containments as indicated in Figure 3 are proper, as shown by the examples. The position of these examples are straightforward for all but one case, see the following result.

Theorem 13 *The set $E - (G \cup A)$ is not empty, it contains the matroid $K = U_{0,7} \oplus U_{2,4}$.*

To our best knowledge, only one of these classes can easily be characterized.

Theorem 14 *A matroid is in D if and only if it contains at most three circuits.*

The following lemma is the key for this theorem.

Lemma 15 *If a graphic matroid contains at least four circuits then it contains at least one of the following three minors: $U_{1,4}$, $U_{0,4}$, $U_{1,3} \oplus U_{0,1}$.*

PROOF: It is easy to see that a matroid M containing at least three circuits either contains three pairwise disjoint circuits or a θ-graph – that is a graph consisting of three internally disjoint paths (each of length at least one) between two points. In the former case the extension of three disjoint circuits with a fourth one either leads to a minor $U_{0,4}$ (if the fourth circuit is disjoint to the previous ones) or to $U_{1,3} \oplus U_{0,1}$ (if the fourth circuit intersects at least one of the old ones).
On the other hand, if M contains a θ-graph then the fourth circuit may be disjoint to it, leading to $U_{1,3} \oplus U_{0,1}$ or contributes to the θ-graph and we obtain a $U_{1,4}$ as a minor. □

5 Appendix

Throughout, $M \setminus X$ and M/X will denote deletion and contraction, respectively, of the set X in a matroid M, while $X - Y$ will denote the difference of the sets X and Y. We shall write $Y \cup x$, $Y - x$, $M \setminus x$ and M/x instead of $Y \cup \{x\}$, $Y - \{x\}$, $M \setminus \{x\}$ and $M/\{x\}$, respectively.

Definition 16 *Let $M_1 \oplus M_2$ denote the direct sum of the matroids $M_1(E_1, I_1)$ and $M_2(E_2, I_2)$ (where $E_1 \cap E_2 = \emptyset$). Then $E_1 \cup E_2$ is the groundset of the direct sum. A subset S is independent in the direct sum if $S \cap E_i$ is independent in M_i for both $i = 1$ and 2.*

Definition 17 *Let $M_1 \vee M_2$ denote the union of the matroids $M_1(E, I_1)$ and $M_2(E, I_2)$. Then E is the groundset of the union. A subset S is independent in the union if it has a partition $S = S_1 \cup S_2$ so that S_i is independent in M_i for both $i = 1$ and 2.*

Lemma 18 *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

Lemma 19 *If a connected component X of the matroid M_1 is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

Lemma 20 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset E$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b).
Let M_1' be the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').
If X is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

Definition 21 *We say that a pair M_1, M_2 is reduced if none of the Lemmata 18-20 can help us to decrease the number of edges.*

Corollary 22 *Assume that the application of the Lemmata 18-20 to M_1 and M_2 leads to a reduced pair of matroids M_1', M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

Proposition 23 *Assume that M_1 and M_2 are given by their graphs G_1 and G_2, respectively. Then we can perform the reduction of these matroids in polynomial time.*

Lemma 24 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ and E_0 is the edge set of a 2-connected component X of G which has only one edge x from $NL(M_2)$. Then the union $M_1 \vee M_2$ is graphic if and only if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - x))$ is graphic.*

Lemma 25 *If two parallel edges x and y of M_1 are serial in M_2 then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus x) \vee (M_2/x)$ is graphic.*

Lemma 26 *If two serial edges x and y of M_1 are serial in M_2 as well then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus \{x, y\}) \vee (M_2 \setminus \{x, y\})$ is graphic.*

Lemma 27 *If two serial edges x and y of M_1 are not contained in any common circuit in M_2 then the union $M_1 \vee M_2$ is graphic if and only if $(M_1/x) \vee M_2'$ is graphic, where M_2' denotes the serial connection of the two components of M_2 along y such as in Figure 4.*

This case includes the subcase where x (or y) is a loop in M_2. Then M_2' will be $M_2 \setminus x$. Recall that if both x and y are loops in M_2 then we can apply Lemma 20.

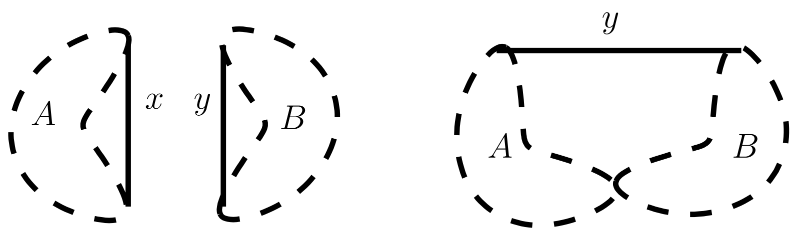

Figure 4: The structure of M_2 and M_2'

Lemma 28 *Let two parallel edges x and y of M_1 be parallel in M_2 too. Then if x and y are coloops or serial in the union then the union is graphic if and only if the right choice of $(M_1/x) \vee (M_2 \setminus x)$ and $(M_1 \setminus x) \vee (M_2/x)$ is graphic, while if they are neither serial nor coloops then the union is not binary.*

Lemma 29 *Let x and y be two parallel edges of M_1 and let x be a loop in M_2. Then if x and y are coloops or serial in the union then the union is graphic if and only if $(M_1/x) \vee (M_2 \setminus x)$ is graphic, while if they are neither serial nor coloops then the union is not binary.*

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