The graphicity of the union of graphic matroids

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Abstract: There is a conjecture that if the union (also called sum) of graphic matroids is not graphic then it is nonbinary [7]. Some special cases have been proved only, for example if several copies of the same graphic matroid are given. If there are two matroids and the first one can either be represented by a graph with two points, or is the direct sum of a circuit and some loops, then a necessary and sufficient condition is given for the other matroid to ensure the graphicity of the union. These conditions can be checked in polynomial time. The proofs imply that the above conjecture holds for these cases.

Keywords: matroid theory, graphic matroids, union of matroids

1 Introduction

Graphic matroids form one of the most significant classes in matroid theory. When introducing matroids, Whitney concentrated on relations to graphs. The definition of some basic operations like deletion, contraction and direct sum were straightforward generalizations of the respective concepts in graph theory. Most matroid classes, for example those of binary, regular or graphic matroids, are closed with respect to these operations. This is not the case for the union. The union of two graphic matroids can be non-graphic.
The first paper studying the graphicity of the union of graphic matroids was probably that of Lovász and Recski [2], they examined the case if several copies of the same graphic matroid are given.
Another possible approach is to fix a graph G_0 and characterize those graphs G where the union of their cycle matroids $M(G_0) \vee M(G)$ is graphic. (Observe that we may clearly disregard the cases if G_0 consists of loops only, or if it contains coloops.) As a byproduct of some studies on the application of matroids in electric network analysis, this characterization has been performed for the case if G_0 consists of loops and a single circuit of length two only, see the first graph of Figure 1. (In view of the above observation this is the simplest nontrivial choice of G_0.)

Theorem 1 *[4] Let A and B be the cycle matroids of the graphs shown in Figure 1 on ground sets $E_A = \{1, 2, ..., n\}$ and $E_B = \{1, 2, i, j, k\}$, respectively. Let M be an arbitrary graphic matroid on E_A. Then the union $A \vee M$ is graphic if and only if B is not a minor of M with any triplet i, j, k.*

Recski [7] conjectured some thirty years ago that if the union of two graphic matroids is not graphic then it is nonbinary. This is known to be true if the two graphic matroids are identical or if one of them is A as given in Theorem 1 – these results follow in a straightforward way from [2] and from [4], respectively.

The main purpose of the present paper is to extend the result of Theorem 1 if G_0 either consists of loops and two points joined by n parallel edges ($n \geq 2$, see Section 4) or if it consists of loops and a single

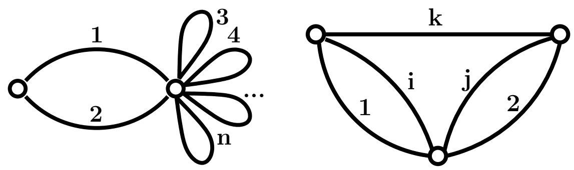

Figure 1: A graphic representation of A (left) and B (right)

circuit of length n ($n \geq 2$, see Section 3). We prove that deciding whether $M(G_0) \vee M(G)$ is graphic can be performed in polynomial time if G_0 is one of these two matroids (Theorems 20 and 12, respectively). Our results will then imply that the above conjecture is true if one of these two types of graphs play the role of G_0.

Observe that the first graph of Figure 1, representing A, has only two non-loop edges (1 and 2), while the second graph, representing B, has the property that the complement of the set $\{1, 2\}$ of non-loop edges of A contains both a circuit and a spanning tree. This property will turn out to be crucial if we consider a larger set of non-loop edges which are either all parallel or all serial, see Remark 24.
Then as a corollary, we can prove the conjecture in these two special cases: If the non-loop edges of a graph are either all parallel or all serial then the union of its cycle matroid with any graphic matroid is either graphic or contains a $U_{2,4}$ minor, hence it is nonbinary [8].
During our study of the union of the two graphic matroids $M_1 = M(G_0)$ and $M_2 = M(G)$ the former one will have a very special structure. Nevertheless, in Section 2 we formulate some reduction steps for arbitrary graphic matroids M_1 and M_2 on the same ground set (although we shall apply the results in the aforementioned two special cases only).

2 The reduction

Throughout M_1 and M_2 will be graphic matroids on the same ground set E. We shall refer to them as *addends*. It is well known that if a matroid is graphic then so are all of its submatroids and minors. Hence if a matroid has a non-graphic minor then the matroid is not graphic.

Definition 2 *We call some non-coloop edges of a matroid serial if they belong to exactly the same circuits.*

Definition 3 *Let $L(M)$ and $NL(M)$denote the set of loops and non-loops, respectively, in the matroid M.*

The following lemmata contain the main opportunities when we can simplify our addend matroids. Since they refer to graphic matroids only, we can use graph theoretical terminology. Throughout, $M \setminus X$ and M/X will denote deletion and contraction, respectively, of the set X in a matroid M, while $X - Y$ will denote the difference of the sets X and Y.

Lemma 4 *Let X and Y denote the set of coloops in M_1 and in M_2, respectively. The union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic.*

PROOF: If an element of a matroid M is a coloop then it will be a coloop in the union of M with any other matroid. Therefore if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is graphic then we can extend its representing graph with coloops for $X \cup Y$ and this way we get a graphic representation of $M_1 \vee M_2$.
On the other hand if $(M_1 \setminus (X \cup Y)) \vee (M_2 \setminus (X \cup Y))$ is non-graphic then $M_1 \vee M_2$ can't be graphic because it has a non-graphic submatroid. $\square$

Recall that a matroid is connected if it does not arise as the direct sum of two smaller matroids. If M is not connected and X is the underlying set of a connected component of M then $M/X = M \setminus X$

Lemma 5 *If a connected component X of the matroid M_1 is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is graphic.*

PROOF: It is easy to see that in this case the matroid which is the direct sum of $(M_1 \setminus X) \vee (M_2 \setminus X)$ and $M_1 \setminus (E - X)$ is isomorphic to $M_1 \vee M_2$. The direct sum of graphic matroids is also graphic, hence $M_1 \vee M_2$ is graphic.
On the other hand if $(M_1 \setminus X) \vee (M_2 \setminus X)$ is not graphic then $M_1 \vee M_2$ can't be graphic because it has a non-graphic submatroid. □

Recall that the cycle matroid of a graph is connected if and only if the graph is 2-vertex-connected.

Lemma 6 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ in which $X \subset E$ determines a connected subgraph and $E - X$ has exactly two common vertices with X (call them a and b).*
Let M_1' be the cycle matroid of $G' := G(V, (E - X) \cup \{(a, b)\})$ and $M_2' := (M_2 \setminus X) \cup loop(a, b)$ (Here $loop(a, b)$ denotes a loop corresponding to the edge (a, b) in G').
If X is a subset of $L(M_2)$ then the union $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.

PROOF: If $M_1' \vee M_2'$ is graphic then delete the edge (a, b) from the graph of the union and then put the original subgraph of X (from G) in the place of this deleted edge (put the original a and b to the endpoints of (a, b) in the union) and we get a graphic representation of $M_1 \vee M_2$.
On the other hand if $M_1' \vee M_2'$ is non-graphic then we show that this union arises as a minor of $M_1 \vee M_2$ hence this latter cannot be graphic either. There has to be a path between a and b in X in G; let α denote one of its edges. There is a subset C of X so that $\{\alpha\}$ will be a base in the contraction $[M_1 \setminus (E - X)]/C$.

$$((M_1 \vee M_2)/C) \setminus (X - (C \cup \{\alpha\})) = (M_1/C \setminus [X - (C \cup \{\alpha\})]) \vee (M_2 \setminus [X - \{\alpha\}]) = M_1' \vee M_2'$$

□

After these preliminaries we can define the reduction that will be the most important concept to reduce the infinite number of cases.

Definition 7 *We say that a pair M_1, M_2 is reduced if none of the lemmata above can help us to decrease the number of edges.*

Corollary 8 *Assume that the application of the previous lemmata to M_1 and M_2 leads to a reduced pair of matroids M_1', M_2'. Then $M_1 \vee M_2$ is graphic if and only if $M_1' \vee M_2'$ is graphic.*

Proposition 9 *Assume that M_1 and M_2 are given by their graphs G_1 and G_2, respectively. Then we can perform the reduction of these matroids in polynomial time.*

PROOF: The number of edges decreases with every step of reduction so we have to see that each step can be performed in polynomial time and that we can check in polynomial time whether we can apply a reduction step.
If we found a coloop or a component which is a subset of $L(M_{3-i})$ in order to apply Lemma 4 or Lemma 5, respectively, then we can delete them quickly and it is also easy to replace a subset by an edge as in Lemma 6 (once we have found the subset).
We can find the 2-vertex-connected components of a graph in polynomial time. This way we can easily identify all the coloops. Moreover we can determine the number of the edges from $NL(M_i)$ in any set and delete those components in M_{3-i} which have none.
In order to apply Lemma 6 we have to recognize these sets X effectively in spite of the fact that the same matroid may have many graphic representations. For this purpose we define a relation on the edge set of a coloopless graph G so that e and f are in relation if and only if either $e = f$ or $\{e, f\}$ is a cut set

in G. This is an equivalence relation and using the operation "twisting" (see [3], Section 5.3) one can change G to G' so that edges in each equivalence class form paths in G' and $M(G) = M(G')$. This can be performed in polynomial time (for each equivalence class contract all but one of the edges and replace the remaining edge by a path formed by all the edges in this class). Finally pick all pairs of points in both graphs and decide whether they separate their component into two parts so that one of them is a subset of the set of loops in the other graph. $\square$

Now we are ready to give this polynomial algorithm which reduces a pair of given graphic matroids. We formulate the algorithm for coloopless matroids only, in order to keep its later application simpler.

Algorithm 10 *INPUT: Two coloopless graphic matroids M_1 and M_2 on the same ground set given by their graphs G_1 and G_2, respectively.*
OUTPUT: The reduced pair M_1', M_2'.

1. *If G_i has a 2-vertex-connected component which does not have edges from $NL(M_{3-i})$ then delete this component from G_i and the corresponding loops from G_{3-i}.*
2. *Change G_i if necessary, to a new one where the equivalence classes (as described in the proof of Proposition 9) are paths.*
3. *If $X \subseteq E$ determines a connected subgraph of G_i which does not have edges from $NL(M_{3-i})$ and the subgraph has exactly two common vertices a and b with $E - X$ in G_i, then delete from G_{3-i} all the loops of X except a single (arbitrary) one denoted by x and replace G_i by $(G_i \setminus X) \cup edge(a, b)$ (where $edge(a, b)$ will play the role of x).*
4. *If during the last step the matroids are changed then go to Step 2 otherwise let M_1', M_2' denote the reduced pair.*

We close this section with one more reduction related statement which will be needed in Theorem 25 only.

Lemma 11 *Assume that M_1 is the cycle matroid of a graph $G(V, E)$ where E_0 is the edge set of a 2-connected component X of G which has only one edge x from $NL(M_2)$. Then the union $M_1 \vee M_2$ is graphic if and only if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic.*

Proof: If $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is graphic then we can obtain the graph of $M_1 \vee M_2$ by replacing edge x with the subgraph X in the following way:
Let a and b denote the end vertices of x in X. Cut vertex a into two vertices a_1 and a_2 in X. Among the edges incident to a in X, join x to a_1 (b remains the other endpoint of x) and all the others to a_2, let X' denote the resulting graph. Now replace x in the graph of $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ with the graph X' along the vertices a_1 and a_2.
On the other hand if $((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$ is non-graphic then since this union arises as a minor of $M_1 \vee M_2$, this latter cannot be graphic either.

$$(M_1 \vee M_2)/\,(E_0 - \{x\}) = ((M_1 \setminus E_0) \cup loop(x)) \vee (M_2 \setminus (E_0 - \{x\}))$$

$\square$

3 The case when all the non-loop edges of G_0 are serial

From now on we study the union $M_1 \vee M_2$ where $M_1 = M(G_0)$ is the matroid which consists of a circuit of length n and k loops and $M_2 = M(G)$ is an arbitrary graphic matroid. We shall write $[n]$ for the set of the edges of the circuit in G_0.

Theorem 12 *If G_0 consists of loops and a single circuit of length n ($n \geq 2$) and $M(G)$ is an arbitrary graphic matroid on the same ground set then the graphicity of their union can be decided in polynomial time.*

Algorithm 13 *INPUT: Two matroids M_1 and M_2 on the same ground set given by the graphs G_0 and G, respectively, where G_0 consists of two parts: a circuit (with edge set $[n]$) and loops.*
OUTPUT: Decision whether the union $M_1 \vee M_2$ is graphic.

1. *If $[n]$ has an element which is a coloop in G then the union is graphic. If the complement of $[n]$ has elements which are coloops in G then delete these elements from both G and G_0.*
2. *Run Algorithm 10 to the pair M_1, M_2, that gives us the reduced pair M_1', M_2'.*
3. *If $[n]$ contains a cut set in M_2' or if $M_2' \setminus [n]$ is the free matroid then the union is graphic otherwise it is not (neither binary).*

Step 1 uses the statement of Lemma 4 and the special structure of G_0. Algorithm 10 preserves the graphicity or non-graphicity of the union according to Lemma 5 and Lemma 6. The correctness of the algorithm will then follow from Proposition 14 below.
In view of Proposition 9 this algorithm is polynomial − in Step 3 we only have to check whether the deletion of the edges of $[n]$ disconnects G' or leads to a circuit-free subgraph.

Proposition 14 *Let $M_1' = M(G_0')$ and $M_2' = M(G')$ be the matroids after all the possible reductions using Step 1 and Algorithm 10. $M_1' \vee M_2'$ is graphic if and only if either $[n]$ contains a cut set in G' or $M_2' \setminus [n]$ is the free matroid.*

Proof: For the if part of the proof the following two propositions solve the two possible cases separately. It is easy to see that if we have edges from $NL(M_1')$ which are serial in M_2' then the union will be graphic (because these edges can destroy the circuit of G_0', so $[n]$ will consists of coloops in the union). In fact, the slightly more general statement of the following proposition is also true.

Proposition 15 *If $[n]$ contains a cut set $[c]$ in M_2' then the union $M_1' \vee M_2'$ will be the cycle matroid of the graph obtained from G' by replacing the edges of $[n]$ with coloops.*

Proof: For every set $X \subset L(M_1')$ we have to prove that $[n] \cup X$ is independent in the union if and only if X is independent in M_2'.
If X is not independent in M_2' then it will not be independent in the union either because every edge of X is in $L(M_1')$. It means $[n] \cup X$ is not independent either in the union.
On the other hand if X is independent in M_2' then it is a part of a base. A cut set intersects all the bases so $[c]$ has an element a such that $X \cup \{a\}$ is independent in M_2'. $[n]$ is a circuit in M_1', so every proper subset of it is independent thus $[n] - \{a\}$ is independent. Now $[n] \cup X$ has a partition so that $[n] - \{a\}$ is independent in M_1' and $X \cup \{a\}$ is independent in M_2' so it is independent in the union $M_1' \vee M_2'$. □

Notice that this proposition is more general than the if part of Proposition 14, since the matroids need not to be reduced.
If a set contains no cut set in a matroid that means it is spanned by its complement. This means $[n]$ is spanned in M_2' by edges from $L(M_1')$.

Proposition 16 *If $M_2' \setminus [n]$ is the free matroid and $[n]$ does not contain a cut set in M_2' then the union $M_1' \vee M_2'$ is graphic, namely it is a circuit formed by all the edges.*

Proof: Now $L(M_1')$, that is $E - [n]$, is a spanning forest of G' according to the assumption.
$L(M_1') \cup [n]$ is not independent in the union because we can pick only $n-1$ independent elements in M_1' and for every element i of $[n]$ the set $L(M_1') \cup \{i\}$ is not independent in M_2' (because $L(M_1')$ spans every edge of $[n]$). On the other hand every proper subset of $L(M_1') \cup [n]$ is independent in the union $M_1' \vee M_2'$. In order to prove this we give a suitable partition for all cases when we delete only one edge α:

- If $\alpha \in [n]$ then $[n] - \{\alpha\}$ is independent in M_1' and $L(M_1')$ is independent in M_2'.
- If $\alpha \in L(M_1')$ then let F' denote the spanning tree of G' in $L(M_1')$ which contains α and let F_1' and F_2' be the two parts of F' (in the two sides of α). There exists a subset of $NL(M_1')$ which is a path in G' between F_1' and F_2', since otherwise α would be a coloop in G' which contradicts Step 1. There exists such a path with exactly one edge e, because if the lengths of all these paths are at least two then there will be a point which is not covered by F' in G' so there will be a cut set in $[n]$. In that case $L(M_1') \cup \{e\} - \{\alpha\}$ is independent in M_2' and $[n] - \{e\}$ is independent in M_1'.

□

This means that we proved the if part of Proposition 14 because if G' contains more edges than a spanning forest and $[n]$ then there must be a circuit in it which is a subset of $L(M_1')$.

For the only if part suppose that $M_2' \setminus [n]$ is not the free matroid and $[n]$ does not contain a cut set in M_2'. M_2' contains a circuit I in $E - [n]$ by the assumption, and its length is at least three due to the reduction (length 1 or 2 would contradict to Lemmata 4 and 6, respectively).
In the proof we shall give a non-graphic minor of the union, to do this the following "non-equivalent reduction" will be our tool.

Lemma 17 *If we contract an edge $e \in L(M_i)$ in an addend M_{3-i} and delete the corresponding loop in the other one (M_i) then the union of the new matroids will be a minor of that of the originals.*

Proof: Contract e in the union. Then the independent sets will be those which can be partitioned so that the first part is independent in M_i and the second is independent with e in M_{3-i} because e is a loop in M_i. This description is exactly the union of the above described matroids. □

Thus we can reduce our study to the 2-connected cases only, as follows. We can contract (like in the previous lemma) a spanning tree of edges from $L(M_1')$ in all 2-connected components but the one containing I. We can apply now the steps of the reduction, so we get a reduced connected matroid from it which contains the circuit $I \subset L(M_1')$ and the new M_1' will still consist of a circuit and loops.
We can clearly assume that there is only one circuit from $L(M_1')$ because if there were more we could delete one edge from both matroids so that there remain at least one circuit from $L(M_1')$ in M_2' (but less than before) and $[n]$ still does not contain a cut set. After that we can reduce the matroids and we still have all the necessary conditions.
An edge from $NL(M_1')$ will be called an *essential diagonal* of I if it connects two distinct vertices of I.

Proposition 18 *If we contract all edges from $L(M_1')$ except the edges of I in M_2' then there will be at least two different essential diagonals of I.*

Proof: Indirectly suppose that I has at most two vertices incident to edges from $NL(M_1')$ after we contracted all other edges of $L(M_1')$ in M_2'. It means that all paths between the vertices of I and the endpoints of the edges from $NL(M_1')$ go through these points. This is a contradicton because $I \subseteq L(M_1')$ and I is a connected component which has at most two common vertices with $E - I$ so I either disappears or must be a simple edge in the reduced matroid (see Lemma 5 and Lemma 6). □

Now use Lemma 17 to contract M_2' in two steps. First contract the subset as in Proposition 18. We get a circuit which is a subset of $L(M_1')$, with at least two different essential diagonals. Then contract all but three suitable edges of I such that we get a circuit of length three with at least two different essential diagonals. (See the second graph of Figure 2 below)
Now we have a minor M_2'' of M_2' and the corresponding submatroid M_1'' of M_1' (which is M_1' without the loops which we contracted in M_2') so that if the union $M_1'' \vee M_2''$ is non-graphic then the original union is not graphic either.
We have to apply Lemma 17 again to the loops from $NL(M_1'')$ in M_2'' to contract the corresponding edges in M_1' and after that we can delete all the remaining edges which are loops in both matroids. Let M_2''' and M_1''' denote the new matroids.

Proposition 19 *$M_1''' \vee M_2'''$ is neither graphic nor binary.*

Proof: For now M_1''' consists of a circuit of length h (let $[h]$ denote the set of its edges) and three loops m, n and o while M_2''' has a circuit formed by m, n and o and $[h]$ can be partitioned to sets of edges M, N and O such that all elements of the sets are parallel to the corresponding edge from $L(M_1''')$ (see Figure 2).

According to Proposition 18 at least two of the three sets M, N and O are non-empty. Then suppose

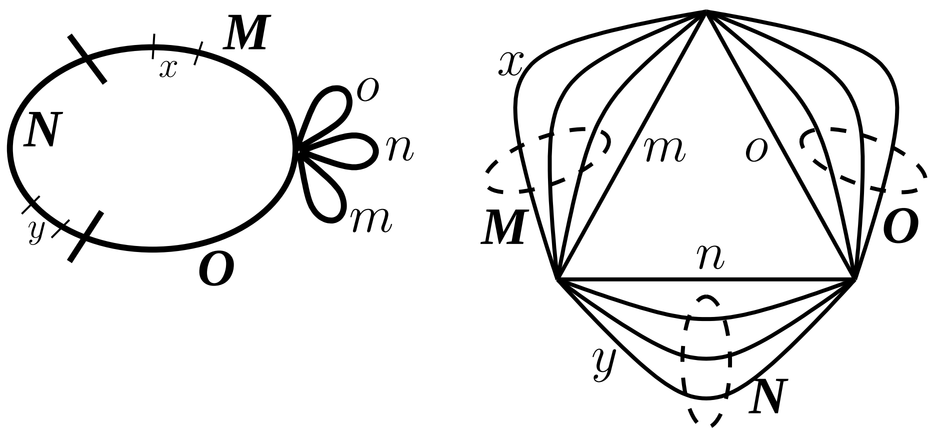

Figure 2: Graphic representation of M_1''' and M_2'''

that $x \in M$ and $y \in N$ are two edges from $NL(M_1''')$. We show that $(M_1''' \vee M_2''') / ([h] - \{x\})$ is $U_{2,4}$. The rank of M_1''' is $h-1$ and the rank of M_2''' is 2 so in order to obtain a base of the union we can choose $h-1$ elements of the first matroid and 2 elements of the second one. It is easy to check that $\{x, m\}$, $\{x, n\}$, $\{x, o\}$, $\{m, n\}$, $\{m, o\}$ and $\{n, o\}$ are independent in $(M_1''' \vee M_2''') / ([h] - \{x\})$: for the first pick $[h] - \{y\}$ from M_1''' and y and m from M_2''' and for the last five simply pick $[h] - \{x\}$ from M_1''' and the others from M_2'''. This means it is really a $U_{2,4}$. □

This proposition completes the proof of the only if part of Proposition 14 because we gave a non-graphic minor of the union. □

4 The case when all the non-loop edges of G_0 are parallel

From now on we study the union $M_1 \vee M_2$ where M_1 is the cycle matroid of G_0 which consists of loops and two points joined by n parallel edges and $M_2 = M(G)$ is an arbitrary graphic matroid. We shall write $[n]$ for the set of the parallel edges in M_1.

In this section we shall prove a neccessary and sufficient condition for M_2, like in the previous section, for the graphicity of the union $M_1 \vee M_2$.

Theorem 20 *If G_0 consists of loops and two points joined by n parallel edges and $M(G_1)$ is an arbitrary graphic matroid on the same ground set then the graphicity of their union can be decided in polynomial time.*

Algorithm 21 *INPUT: Two matroid M_1 and M_2 on the same ground set given by the graphs G_0 and G, respectively, where M_1 consists of two parts: a set $[n]$ of parallel edges and loops.*
OUTPUT: Decision whether the union $M_1 \vee M_2$ is graphic.

1. *Delete all the coloops of G (if any) from both G_0 and G.*
2. *Run Algorithm 10 to the pair M_1, M_2, that gives us the reduced pair M'_1, M'_2.*
3. *If there exist two elements a and b of $[n]$ so that $M'_2 \setminus \{a, b\}$ is not the free matroid and a and b are not serial in M'_2 then the union is not graphic (neither binary) otherwise it is graphic.*

Step 1 uses the statement of Lemma 4 and the special structure of G_0. Algorithm 10 preserves the graphicity or non-graphicity of the union according to Lemma 5 and Lemma 6. The correctness of the algorithm will then follow from Proposition 22 below.
In view of Proposition 9 this algorithm is polynomial – in Step 3 there are $\binom{n}{2}$ possible choices of a and b, and in each case a spanning forest has to be constructed only.

Proposition 22 *Let $M'_1 = M(G'_0)$ and $M'_2 = M(G')$ be the matroids after all the possible reductions using Step 1 and Algorithm 10. Then $M'_1 \vee M'_2$ is graphic if and only if no 2-connected component of G' has two non-serial edges a and b from $[n]$ so that $M'_2 \setminus \{a, b\}$ is not the free matroid.*

PROOF: For the **only if part** assume that there exist two such edges a and b. The assumption implies that:

- There exists a circuit C_1 in G' containing a and b (since they are in the same component)
- There exists a circuit C_2 in G' containing a but not b (since they are not serial)
- There exists a circuit C_3 in G' containing none of them (since the remaining is not the free matroid)

Observe that since a and b are not coloops in G', the condition that a and b are not serial is equivalent to that $\{a, b\}$ does not contain a cutset (which more strongly resembles Proposition 14).
The existence of C_1 and C_2 must generate a subgraph of G' as shown in the first graph of Figure 3. Here

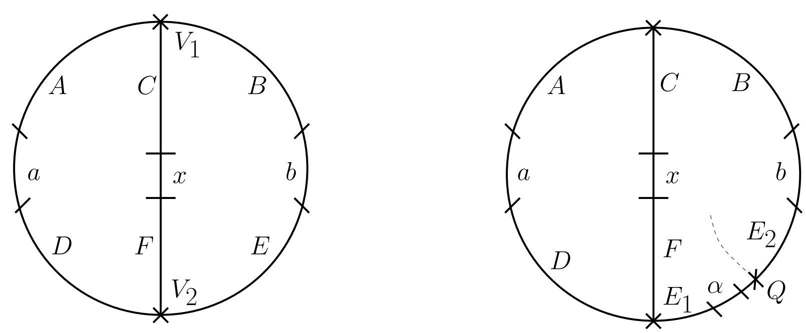

Figure 3: The main structure

there are three internally disjoint paths between the vertices V_1 and V_2 (let Θ denote this structure). Each path must have length at least one, hence there exists at least one edge x in the "central" path. However any of the paths indicated by A, B, C, D, E and F may be of length zero.
We have to consider two different cases.
Case 1. Let C_3 be in the same component of G' as a and b are. That is, Θ is a proper subgraph of the component of G' under consideration. The following lemma handles this case. We speak about paths in a matroid in the sense of equivalence classes, see the proof of Proposition 9.

Lemma 23 *If a 2-connected component of the reduced matroid M'_2 contains Θ as a proper subgraph then the union $M'_1 \vee M'_2$ has a $U_{2,4}$ minor.*

Proof: First delete all the other 2-connected components of M_2' from both matroids. If there are more than a Θ structure and an additional path remaining, then we can delete at least one path such that the result reduces to a matroid which still has at least four paths (but less than before). That way we can suppose that there is exactly one path in addition to Θ (throughout we refer to this path as the *additional path*) in the reduced matroid M_2'. The three paths of the Θ structure are shown in the first graph of Figure 3, let P and Q denote the endpoints of the additional path.
The following case study examines all the possible cases. Recall that if there are two paths (equivalence classes) between two points then at least one of the paths must contain an edge from $NL(M_1')$, since we consider a pair of reduced matroids.

1. If both P and Q are in $C \cup F$ (the points V_1, V_2 are also permitted) then there must be at least one edge c from $NL(M_1')$ in the additional path or in the one which forms a circuit with it, let y denote an arbitrary edge from the other one. If we contract all edges but a, b, c and y in the union we get a $U_{2,4}$ (since any pair of edges will be independent and any triple will be dependent).

2. Otherwise without loss of generality we may suppose that Q is in E and it is separated from V_2 by at least one edge α, see the second graph of Figure 3. The other endpoint P can be in A, B, C, D, E_1, E_2 and F (see the second graph of Figure 3). Let p denote an edge in the additional path.

   (a) If P is in A, B or C: if we contract all edges but b, x, p and α in the union then we get a $U_{2,4}$.

   (b) If $P = V_2$ or $P \in E_1$: there must be an edge c from $NL(M_1')$ in one of the two paths between P and Q (according to the reduction). If we contract all edges except b, c, x and the one of α or p which is not serial with c in M_2' in the union then we get a $U_{2,4}$.

   (c) If $P \in E_2$: as before there must be an edge c from $NL(M_1')$ in one of the two paths between P and Q. If p is in the same path as c then let the role of p be played by an arbitrary edge from the part of the other path which forms a circuit with the additional path. If we contract all edges but b, c, x and p in the union then we get a $U_{2,4}$.

   (d) If $P \in D$ (but not V_2): there must be an edge α_2 in D between V_2 and P. If we contract all edges but b, α, α_2 and p in the union then we get a $U_{2,4}$.

   (e) If $P \in F$ (but not V_2): there must be an edge β in F between V_2 and P. If we contract all edges but b, α, β and p in the union then we get a $U_{2,4}$.

□

This case study is mainly the same as in [5: Figure 18.11], where Lemma 6 has implicitly been used as well.

Case 2 (in the proof of the only if part of Proposition 22). C_3 is in another component of G, but according to Lemma 4, that component must contain at least one edge c from $NL(M_1')$. Let C_3' be a circuit of G' containing c.
We shall find a $U_{2,4}$ minor in $M_1' \vee M_2'$. If we contract all edges of $C_1 \cup C_2 \cup C_3'$ but $\{a, b, c, x\}$ and delete all the other edges in the union then we get a $U_{2,4}$ in $M_1' \vee M_2'$.

For the **if part** of Proposition 22 suppose that there are no such edges a and b. Then we distinguish two cases again.
Either there are two non-serial edges a, b from $NL(M_1')$ in G' but $G' \setminus \{a, b\}$ is the free matroid. Then we have the situation as in the first graph of Figure 3 and so the union is graphic (a large circuit formed by all the edges which were not coloops in G').
In the other case all components of G' are circuits, consisting of all but one edges from $NL(M_1')$ (see Figure 4). Let $X \subseteq E$ be an arbitrary subset and let $k(X)$ denote the number of circuits of G' which are fully contained in X. If $k(X) \geq 2$ then X is dependent in the union, because we can only choose one edge from M_1' and $j-1$ edges of a circuit of length j from M_2'. If $k(X) = 1$ then it is independent in

the union, because we can choose one element from M_1' (one from the circuit of G' which is in X) and all the others from M_2'. If $k(X) = 0$ then X is independent in M_2' so it is independent in $M_1' \vee M_2'$ too. This means that a set is independent in the union if and only if it does not contain all the edges of more than one circuit of G'. Then the union is graphic, the circuits of G' will become parallel paths (see an example in Figure 4). $\square$

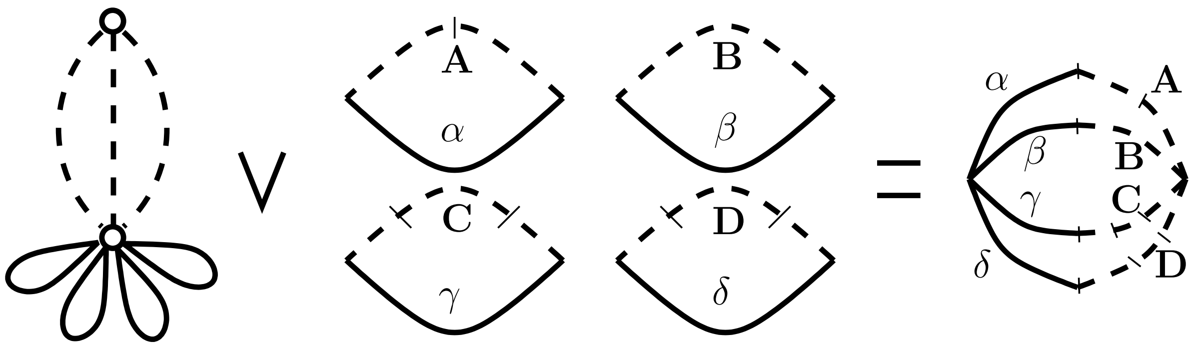

Figure 4: A schematic representation of the second case of the proof of the if part

Remark 24 *Observe that the final conditions in Algorithms 13 and 21 can be formulated in a uniform way as well:* $M_1 \vee M_2$ *is graphic if and only if for every circuit* C *of length* ≥ 2 *in* M_1' *either* $M_2' \setminus C$ *is the free matroid or* C *contains a cut set in* M_2'*.*

One can see that the lemmata in Section 2 can be applied to any pair of graphic matroids. In addition we only use the structure of the reduced matroids in Propositions 14 and 22. Hence we have a slightly more general result:

Theorem 25 *We can decide in polynomial time whether the union of* M_1 *and* M_2 *is graphic if using all the possible reductions described by Lemmata 4-5-6-11 leads to a case where* M_1' *or* M_2' *consists of loops and either a single circuit, or some parallel edges.*

PROOF: We only have to verify that we can reduce the matroids in polynomial time until one of them consists of loops and a single circuit, or loops and parallel edges (as in Algorithms 13 or 21). This is true since every step of the reduction (including the steps described by Lemma 11) can be performed in polynomial time and decreases the number of edges. The last part of the proof follows from Theorems 12 and 20. $\square$

The general problem (when is the union of two graphic matroids graphic) and the conjecture (if it is not graphic then it is non-binary) are still open. We have given necessary and sufficient conditions for the problem in two special cases. These conditions can be checked in polynomial time. Our results also prove the conjecture in these two cases.
We expect that these results will also be useful in the future study of more general cases. However, the straightforward generalization of the statement in Remark 24 is not true.

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References

[1] Cs.Gy. Csehi, *Master's Thesis*, Budapest University of Technology and Economics (2012)

[2] L. Lovász and A. Recski, On the sum of matroids, *Acta Math. Acad. Sci. Hungar.* (1973) **24**, 329-333.

[3] J. Oxley, *Matroid Theory,* Second Edition, Oxford University Press (2011)

[4] A. Recski, On the sum of matroids II, *Proc. 5th British Combinatorial Conf. Aberdeen* (1975) 515-520.

[5] A. Recski, Matroids – the Engineers' Revenge, W J Cook, L Lovász, J Vygen (eds.) *Research trends in combinatorial optimization,* Springer (2008) 387-398.

[6] A. Recski, *Matroid Theory and its Applications in Electric Network Theory and in Statics,* Springer, Berlin (1989)

[7] A. Recski, Some open problems of matroid theory, suggested by its applications, *Coll. Math. Soc. J. Bolyai* (1982) **40**, 311-325.

[8] W.T. Tutte, *Introduction to the Theory of Matroids,* American Elsevier, New York (1971)