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On Models of General Type-Theoretical Languages

Abstract. In the present paper we consider general type theoretical languages as the representations of the functor–argument decomposition and compositional semantics relying on it and find some theorems making explicit the theoretical presuppositions of general type theoretical languages and their total or partial semantics. After defining the notion of semantic categories in the spirit of Husserl, we characterize Tarskian and Husserlian models both in total and partial semantics and prove their characteristic theorems.

1 PERSONAL FOREWORD

I am greatly indebted to Professor Imre Ruzsa for the opportunity to work with him for almost two decades. After graduation I began to work as a research assistant at Kossuth University, Debrecen in 1979 and I wrote a letter to professor Imre Ruzsa. In spite of the fact that we had never met and did not know each other personally, he answered soon. The first personal meeting changed my scientific life profoundly. I have no opportunity to tell the whole story, but I should like to emphasize that I should be quoting his books and papers[1] in almost each sentence of the present paper, which is dedicated to the memory of Professor Imre Ruzsa.

2 INTRODUCTION

From the theoretical point of view, type theoretical languages (with a lambda operator) represent function abstraction and function application and rely on functor–argument decomposition, which goes back to Frege.

[1] I mention here only three of them: (Ruzsa 1989), (Ruzsa 1991), (Ruzsa 1997).

In Frege's view, one of the most important inventions of Begriffsschrift is the replacement of the subject–predicate decomposition by the functor–argument one. He wrote the following: "The very invention of this Begriffsschrift, it seems to me, has advanced logic. . . [L]ogic hitherto has always followed ordinary language and grammar too closely. In particular, I believe that the replacement of the concept *subject* and *predicate* by *argument* and *function* will prove itself in the long run. It is easy to see how taking a content as a function of an argument gives rise to concept formation. . . . The distinction between subject and predicate finds no place in my representation of a judgement."[2] (Frege 1879/1997, 51, 53.)

One of the most general theoretical representations of the functor–argument decomposition is the well–known type theory (or the different systems of type–theoretical language and/or logic[3]).

Generally, syntactic categories have to be distinguished from semantic ones. At the same time, our formal systems fulfill the following fundamental principle of formal type–theoretical semantics:

> [The mirror principle:] Associated with every syntactic category C is a counterpart semantic category C^*, whose *mathematical type* 'mirrors' the *grammatical type* of C. And, in particular, every expression of syntactic category C is interpreted by an object of semantic category C^*.
> (Dunn and Hardegree 2001, 142.)

On the basis of the mirror principle, in what follows, we are speaking about types, and using them to define and denote different syntactic categories and the corresponding sets of possible semantic values.

3 GENERAL FORMAL SYSTEM

At first, the system of types has to be defined. The system of types relies on primitive type(s). Generally we have only one requirement: the symbol o must be a primitive type. From the theoretical point of view, the main reason for this is that the symbol o is taken as the type of the most fundamental expressions of our formal language. Expressions of type o are called formulae. Formulae directly correspond to a special sort of conceptual content or information. It means that formulae are the structures of complete information or closed (and whole) conceptual content. In a given interpretation, formulae are intended to represent complete information called proposition in the literature.

There is another, mainly semantic reason for type o having been declared to be primitive. From the semantic point of view, Frege's context principle or as

[2] I use the expression 'functor' instead of 'function' in order to differentiate an incomplete expression of a language from its semantic value.

[3] It goes back to (Church 1940).

(Hodges 2001a) says, Frege's Dictum can be taken as a general leading idea. In *The Foundations of Arithmetic* Frege wrote the following, usually quoted as the context principle:

> never to ask for the meaning of a word in isolation, but only in the context of a proposition; (Frege 1884/1980, x.)
> It is enough if the proposition taken as a whole has sense; it is this that confers on its parts also their content. (Frege 1884/1980, 71.)

According to the context principle, an expression has sense (meaning) only in the sentence in which it occurs. Sometimes we need more than one primitive type (usually individual names constitute another primitive type). The main difference between primitive and non-primitive types is that the semantic domains of primitive types have to be given via definition, while the domains of non-primitive types are originated from them. Non-primitive types are usually called functor types.

Definition 1. *Let PT be an arbitrary set of symbols, the set of primitive types, such that $o \in PT$. Then the set $TYPE_{PT}$ is defined inductively as follows:*

(1) $PT \subseteq TYPE_{PT}$;
(2) $\alpha, \beta \in TYPE_{PT} \Rightarrow \langle \alpha, \beta \rangle \in TYPE_{PT}$.

Remark 1. *Here o is the type of formulae from the syntactic point of view, and the type of their possible semantic values from the semantic point of view. $\langle \alpha, \beta \rangle$ is the type of functors which, when they are filled in by an argument of type α, yield an expression of type β in syntax (in the formal language), and it stands for the type of function from objects of type α to objects of type β in semantics.*

The type–theoretical language is the most general one concerning the functor–argument decomposition. It has only two syntactic operations: filling a functor with an argument (function application from the semantic point of view) and lambda abstraction. The latter produces a way to create a functor from an expression.

Definition 2. *A type–theoretical language is an ordered quadruple*

$$L = \langle LC, Var, Con, Cat \rangle$$

satisfying the following conditions:

(1) *LC is the set of theoretical constants.*[4] $LC = \{\lambda, (,)\}$
(2) $Var = \cup_{\alpha \in TYPE_{PT}} Var(\alpha)$ *and $Var(\alpha)$ is a denumerably infinite set of symbols*[5].

[4]A theoretical constant has the same semantic value (or sense) in every interpretation as a logical constant does in a logical system.

[5]$Var(\alpha)$ is the set of variables of the type α.

(3) $Con = \cup_{\alpha \in TYPE_{PT}} Con(\alpha)$, *where* $Con(\alpha)$ *is a denumerably set of symbols.*[6]
(4) *All mentioned sets of symbols are assumed to be pairwise disjoint ones.*
(5) $Cat = \cup_{\alpha \in TYPE_{PT}} Cat(\alpha)$, *where the sets* $Cat(\alpha)$ *are defined by the inductive rules (a)... (c) as follows:*[7]
   (a) $Var(\alpha) \cup Con(\alpha) \subseteq Cat(\alpha)$;
   (b) $C \in Cat(\langle\alpha, \beta\rangle)$, $B \in Cat(\alpha) \Rightarrow$ '$C(B)$' $\in Cat(\beta)$;
   (c) $A \in Cat(\beta)$, $\tau \in Var(\alpha) \Rightarrow$ '$(\lambda\tau A)$' $\in Cat(\langle\alpha, \beta\rangle)$;

The (total or partial) functor–argument frame is the compositional mirror of a type–theoretical language. It can be said that the functor–argument frame gives *possible* semantic values.

Definition 3. *A total functor–argument frame* F *is the system of sets* $\langle Dom_F(\gamma)\rangle_{\gamma \in TYPE_{PT}}$ *such that*
(1) *If* $\gamma \in PT$, *then* $Dom_F(\gamma)$ *is an arbitrary nonempty set.*
(2) $Dom_F(\langle\alpha, \beta\rangle) = Dom_F(\beta)^{Dom_F(\alpha)}$ *for all* $\langle\alpha, \beta\rangle \in TYPE_{PT}$

Definition 4. *A partial functor–argument frame* PF *is the system of sets* $\langle Dom_{PF}(\gamma)\rangle_{\gamma \in TYPE_{PT}}$ *such that*
(1) *if* $\gamma \in PT$, *then* $Dom_{PF}(\gamma)$ *is an arbitrary set with a distinguished member* Θ_γ, *which is called the null entity of type* γ, *such that* $Dom_{PF}(\gamma) \setminus \{\Theta_\gamma\} \neq \emptyset$;
(2) $Dom_{PF}(\langle\alpha, \beta\rangle) = Dom_{PF}(\beta)^{Dom_{PF}(\alpha)}$ *for all* $\langle\alpha, \beta\rangle \in TYPE_{PT}$ *and* $\Theta_{\langle\alpha,\beta\rangle} = g$ *where* $g \in Dom_{PF}(\langle\alpha, \beta\rangle)$ *and* $g(u) = \Theta_\beta$ *for all* $u \in Dom_{PF}(\alpha)$.

Interpretive function and assignment associate the constants and the variables of the type–theoretical language with their semantic values. In a model, which consists of a frame, an interpretive function and an assignment, semantic rules can be defined to determine the semantic values of compound expressions with respect to the given model.

Definition 5. *A (total or partial) model* M *on* G *is an ordered triple* $\langle G, \varrho, v\rangle$ *where*
(1) G *is a (total or partial) functor–argument frame;*
(2) ϱ, v *are functions with domains* Con *and* Var *respectively*[8] *such that*
   (a) *if* $a \in Con(\alpha)$, *then* $\varrho(a) \in Dom_G(\alpha)$;
   (b) *if* $\tau \in Var(\alpha)$, *then* $v(\tau) \in Dom_G(\alpha)$.

Remark 2. *A model* M *on* G *is total or partial if* G *is a total or partial functor–argument frame respectively.*

[6] Con is the set of non–theoretical symbols of L. The semantic value of an expression belonging to the set Con is given by an interpretation.

[7] Cat is the set of all well–formed expressions of L. The set $Cat(\alpha)$ is the α–category of L ($\alpha \in TYPE_{PT}$).

[8] ϱ is an interpretive function, v is an assignment.

If $M = \langle F, \varrho, v \rangle$ *is a total model on* F, *then*

$$Dom_M(\alpha) = Dom_F(\alpha).$$

If $PM = \langle PF, \varrho, v \rangle$ *is a partial model on* PF, *then*

$$Dom_{PM}(\alpha) = Dom_{PF}(\alpha) \setminus \{\Theta_\alpha\}.$$

If M $(= \langle G, \varrho, v \rangle)$ *is a total or partial model,* $\xi \in Var(\gamma)$ *and* $u \in Dom_G(\gamma)$, *then the model* M_ξ^u $(= \langle G, \varrho, v[\xi : u] \rangle)$ *is like* M *except that* $v[\xi : u](\xi) = u$.

Definition 6. *A total or partial model* M $(= \langle G, \varrho, v \rangle)$ *assigns each expression* A *of type* α *a semantic value* $[\![A]\!]_M$ *according to the following semantic rules:*

(1) *if* $a \in Con(\gamma)$, *then* $[\![a]\!]_M = \varrho(a)$;
(2) *if* $\xi \in Var(\gamma)$, *then* $[\![\xi]\!]_M = v(\xi)$;
(3) *if* $A \in Cat(\langle \alpha, \beta \rangle)$ *and* $B \in Cat(\alpha)$, *then* $[\![A(B)]\!]_M = [\![A]\!]_M([\![B]\!]_M)$;
(4) *if* A *is an expression of type* β *and* $\xi \in Var(\alpha)$, *then* $[\![\lambda\xi A]\!]_M = g$, *where* g *is a function from* $Dom_G(\alpha)$ *to* $Dom_G(\beta)$ *such that* $g(u) = [\![A]\!]_{M_\tau^u}$ *for all* $u \in Dom_G(\alpha)$.

Proposition 1. *If* M *is a total model and* $A \in Cat(\alpha)$, *then* $[\![A]\!]_M \in Dom_M(\alpha)$. *If* M *is a partial model, then* $[\![A]\!]_M \in Dom_M(\alpha) \cup \{\Theta_\alpha\}$.

Definition 7. *If* M *is a total or partial model, then* A *is meaningful with respect to* M, *in symbols* $A \in Cat_{mf}^M$ *if* $A \in Cat(\alpha)$ *for some type* α *and* $[\![A]\!]_M \in Dom_M(\alpha)$.

Remark 3. *If* M *is a total model, then all* $A \in Cat$ *are meaningful, i.e. there is no difference at all between the notions of well–formedness and meaningfulness. We can only make a real differentiation between them in the case of partial models.*

Theorem 1. *If* $A \in Cat$, $M_1 = \langle G, \varrho, v_1 \rangle$ *and* $M_2 = \langle G, \varrho, v_2 \rangle$ *are two (total or partial) models of* L *with the same frame* G *and interpretive function* ϱ *such that* $v_1(\tau) = v_2(\tau)$ *for all* $\tau \in V(A)$[9], *then* $[\![A]\!]_{M_1} = [\![A]\!]_{M_2}$.

Proposition 2. *If* $A \in Cat$ *is a closed expression, then* $[\![A]\!]_M$ *is independent from* v *i.e.* $[\![A]\!]_M = [\![A]\!]_{M_\tau^u}$ *for all* $\tau \in Var(\gamma)$ *and* $u \in Dom_F(\gamma)$.[10]

To prove lambda–conversion law, we need the Law of replacement 2 and Lemma 1. The first one says that in semantics, we only take into consideration semantic values and don't pay any attention to the expression itself—except its type—whose semantic value is given. It doesn't matter how we get a

[9] The definitions of subterms, free variables, open and close expressions and the substitutability are usual ones. The set $V(A)$, is the set of free variables of the expression A.

[10] In the case of closed expressions we can speak about models as ordered pairs of frames and interpretive functions.

semantic value, what form of the compound expression gets the semantic value. We may formulate the property in the law of replacement by means of universal replacement of expressions belonging to the same type with the same semantic value. >From the logical–philosophical point of view, the law of replacement is a special type–theoretical formulation of a version of the principle of compositionality called the substitutivity principle, which goes back to Leibniz.

> [The Substitutivity Principle:] If two expressions have the same meaning, then substitution of one for the other in a third expression does not change the meaning of the third expression. (Szabó 2000, 490.)

I must emphasize that the law of replacement can only be considered as a restricted version of the substitutivity principle, the unrestricted form of the substitutivity principle holds only in Husserlian models dealt with in Section 6. The next definition introduces the notion of 1–compositionality. 1–compositional systems fulfill a restricted version of the substitutivity principle, and Corollary 1 of Law of replacement 2 says that our general system is compositional in the sense of 1–compositionality.

Definition 8. *Let M be a model of L. We say that M is 1–compositional if for all well–formed expressions A, B, C $(A, B, C \in Cat)$ and variable τ $(\tau \in Var)$ such that* $(\lambda\tau C)(A), (\lambda\tau C)(B) \in Cat^M_{mf}$

$$[\![A]\!]_M = [\![B]\!]_M \Rightarrow [\![(\lambda\tau C)(A)]\!]_M = [\![(\lambda\tau C)(B)]\!]_M$$

Theorem 2 (Law of replacement). [11]
If $A \in Cat$, $B, C \in Cat(\gamma)$, and M is a (total or partial) model of L, then

$$[\![B]\!]_M = [\![C]\!]_M \Rightarrow [\![A]\!]_M = [\![A[C \downarrow B]]\!]_M.$$

Corollary 1. *If M is a (total or partial) model of L, then M is 1–compositional.*

Lemma 1. *If B is substitutable for variable τ in A, M is a (total or partial) model, and $[\![B]\!]_M = u$, then $[\![A^B_\tau]\!]_M = [\![A]\!]_{M^u_\tau}$.*

Theorem 3 (Lambda–conversion law). *If $A \in Cat$, $\tau \in Var(\beta)$, $B \in Cat(\beta)$ and B is substitutable for τ in A, then $[\![(\lambda\tau A)(B)]\!]_M = [\![A^B_\tau]\!]_M$ for all (total or partial) models M.*

4 PROPERTIES OF TOTAL AND PARTIAL MODELS

Let us turn our attention to different, total or partial models.[12] We need some notions to compare and combine different models. In the following

[11] If $A \in Cat$ and $B, C \in Cat(\gamma)$, then $A[C \downarrow B]$ $(\in Cat)$ is obtained by replacing a subterm occurrence (i.e. not preceded immediately by λ) of B by C.

[12] The proofs of theorems in Section 4.5 can be found in (Mihálydeák 2010, 127–131.).

definitions let $L\ (= \langle LC, Var, Con, Cat\rangle)$ be a type–theoretical language and $M\ (= \langle G, \varrho, v\rangle)$ be its total or partial model.

Definition 9.

(1) *If $\approx$ is an equivalence relation on the set Cat' $(\subseteq Cat)$, then $\approx$ is a synonymy for L. The set Cat' is the field of synonymy $\approx$.*
(2) *Syntactic synonymy for L is the synonymy $\cong_L$ generated by the syntax of L, i.e. $A \cong_L B$ if and only if there is a type γ such that $A, B \in Cat(\gamma)$.*
(3) *Synonymy generated by the model M is a synonymy $\approx_M$ for L with the field Cat^M_{mf} such that $A \approx_M B \Leftrightarrow [\![A]\!]_M = [\![B]\!]_M$.*
(4) *Closed synonymy (or c–synonymy) generated by the model M is a synonymy $\approx_{Mc}$ for L with the field $\{A : A \in Cat, A \text{ is closed}\} \cap Cat^M_{mf}$ such that $A \approx_{Mc} B \Leftrightarrow [\![A]\!]_M = [\![B]\!]_M$.*
(5) *A synonymy $\approx$ for L is semantic if there is a model M of L such that $\approx_M$ equals $\approx$.*

The next proposition shows that in a general type–theoretical compositional framework, syntactic synonymy can be treated as a degenerate semantic one.

Proposition 3. *The syntactic synonymy for L is semantic (in a degenerate sense).*

Remark 4. *In what follows, a model of L generating the synonymy $\cong_L$ is denoted by M_L and called 'syntactic' model.*

Definition 10.

(1) *Two models M_1, M_2 of a language L are said to be equivalent (closed equivalent, c–equivalent) if $\approx_{M_1}$ equals $\approx_{M_2}$ ($\approx_{M_1c}$ equals $\approx_{M_2c}$), i.e. their generated synonymies (c–synonymies) are equivalent.*
(2) *Given two synonymies $\approx$ and $\approx'$ for L, we say that $\approx$ is compatible with $\approx'$ if for all expressions A, B $(\in Cat)$ in the field of both synonymies, $A \approx B \Leftrightarrow A \approx' B$*
(3) *Given two synonymies $\approx$ and $\approx'$ for L, we say that $\approx$ is closed compatible with (or c–compatible with) $\approx'$ if for all closed expressions $A, B(\in Cat)$ in the field of both synonymies $A \approx B \Leftrightarrow A \approx' B$*
(4) *We say that two models M_1, M_2 of L are compatible (closed compatible) if their generated synonymies $\approx_{M_1}, \approx_{M_2}$ are compatible (c–compatible).*

Proposition 4. *If the models M_1, M_2 of L are equivalent, then M_1 and M_2 are compatible and c–compatible.*

Proposition 5. *If the models M_1, M_2 of L are equivalent, then M_1 and M_2 are c–equivalent.*

Proposition 6. *If $M\ (= \langle G, \varrho, v\rangle)$ is a model of L, $\tau \in Var(\gamma)$ and $u \in Dom_G$, then the models M and M^u_τ are c–equivalent.*

Proposition 7. *If the models M_1, M_2 of L are compatible, then M_1, M_2 are c–compatible.*

Proposition 8. *Let the models $M_1,\ M_2$ of L be total. $M_1,\ M_2$ are compatible if and only if $M_1,\ M_2$ are equivalent.*

In order to investigate the connection between total and partial semantic systems, we need a 'total' or 'pseudo partial' part of a partial frame PF, which is denoted by PF^t.

Definition 11. *Let PF be a partial frame. The total part PF^t of the partial frame PF is the system of sets $\langle Dom^t_{PF}(\gamma)\rangle_{\gamma\in TYPE_{PT}}$ such that*

(1) *if $\gamma \in PT$, then $Dom^t_{PF}(\gamma) = Dom_{PF}(\gamma) \setminus \{\Theta_\gamma\}$;*
(2) *if $\gamma = \langle\alpha,\beta\rangle$ then $Dom^t_{PF}(\gamma) \subseteq Dom_{PF}(\gamma)$ such that for all $f \in Dom^t_{PF}(\langle\alpha,\beta\rangle)$ $f(u) \in Dom^t_{PF}(\beta)$ if $u \in Dom^t_{PF}(\alpha)$ and $f(u) = \Theta_\beta$ otherwise.*

Remark 5.

(1) *For the sake of brevity, we use the notation 'Dom^t_F' in the case of a total frame F. Of course, in this case $Dom^t_F(\gamma) = Dom_F(\gamma)$ for all $\gamma \in TYPE_{PT}$.*
(2) *If M $(= \langle G, \varrho, v\rangle)$ is a total or partial model, then $Dom^t_M(\gamma) = Dom^t_G(\gamma)$ for all $\gamma \in TYPE_{PT}$.*

Definition 12. *An expression A of type γ is total with respect to M if $[\![A]\!]_M \in Dom^t_M(\gamma)$.*

Proposition 9.

(1) *If a non–logical constant A of a primitive type is meaningful with respect to a model M of L, then A is total, i.e. if $A \in Con(\gamma)$ where $\gamma \in PT$, and $A \in Cat^M_{mf}$, then $[\![A]\!]_M \in Dom^t_M(\gamma)$.*
(2) *If $A \in Cat(\langle\alpha,\beta\rangle)$ and $B \in Cat(\alpha)$ are total with respect to M, then $A(B)$ is total with respect to M.*

Definition 13.

(1) *If $\approx, \approx'$ are synonymies for L, we say that $\approx'$ extends $\approx$ (or it is an extension of $\approx$) if the field of $\approx'$ includes that of $\approx$ and the two synonymies are compatible.*
(2) *If M_1, M_2 are models of L, we say that M_2 extends M_1 (or that it is an extension of M_1) if $[\![A]\!]_{M_2} = [\![A]\!]_{M_1}$ for all $A \in Cat^{M_1}_{mf}$.*
(3) *If M_1, M_2 are models of L, we write $M_2 \geq M_1$ to mean that $\approx_{M_2} \supseteq \approx_{M_1}$.*

Remark 6. *If $M_2 \geq M_1$, then the domain of M_2 includes that of M_1, but within that domain, M_1 may make more distinctions than M_2 does.*

Proposition 10. *The models M_1 and M_2 of L are equivalent if and only if both $M_2 \geq M_1$ and $M_1 \geq M_2$.*

Proposition 11. *If M_2 extends M_1, then $M_2 \geq M_1$. (In this case M_2 makes exactly the same distinctions in the field of M_1 as M_1 does.)*

Proposition 12. *A total model is maximal in the sense that all of its extensions are equivalent to it.*

Proposition 13. *A total model M of L is minimal in the sense that there is no total model M' such that M extends M' and M and M' are not equivalent.*

Corollary 2. *If a total model M extends M' such that M and M' are not equivalent, then M' is a partial model of L.*

5 TARSKIAN MODELS

In Section 4 we investigated the properties of models by means of their generated synonymies. In his well–known paper (Tarski 1936/1983) Tarski introduces a new classification. The classification and therefore the associated synonymy is—at least in some cases—between syntactic synonymy and synonymies generated by non–degenerate models of our language.

Definition 14. *If L is a type–theoretical language, M is a model of L and A, B are well–formed expressions (or grammatical terms, i.e. $A, B \in Cat$), then we say that A, B belong to the same semantic category with respect to M (they have the same M–category), in symbols $A \sim_M B$, if for every expression C $(\in Cat)$ and a variable τ $(\in Var)$*

$$(\lambda\tau C)(A) \in Cat^M_{mf} \Leftrightarrow (\lambda\tau C)(B) \in Cat^M_{mf}.$$

In a very general sense, the next proposition has been mentioned by Tarski. In our case it sounds as follows:

Proposition 14. *If M is a (total or partial) model of L, then $\sim_M$ is a synonymy with the field of Cat.*

Theorem 4. *$A \sim_M B \Rightarrow A \cong_L B$ (and so $\cong_L \supseteq \sim_M$), where M is a (total or partial) model of L.*

Corollary 3. *If A, B are well–formed but not meaningful expressions with respect to a partial model M, i.e. $A, B \in Cat \setminus Cat^M_{mf}$, then*

$$A \sim_M B \Leftrightarrow A \cong_L B$$

By means of the notion of semantic category, Tarski lays down a very important principle called the first principle of the theory of semantic categories,[13] which is, as he says, very natural "from the standpoint of ordinary usage of language" (Tarski 1936/1983, 216.). In our terminology the principle sounds informally as follows:

[13]Its original version can be found in (Tarski 1936/1983, 216.).

> [The first principle of the theory of semantic categories:] Two expressions of our language have the same semantic category if there is an expression of our language such that it produces meaningful expressions when combined with them.[14]

The following definition formulates the first principle of the theory of semantic categories formally, and gives the notion of a Tarskian model:

Definition 15. *We say that the model M of L is Tarskian if it is the case that if there is a meaningful expression C and a variable τ such that $(\lambda\tau C)(A)$ and $(\lambda\tau C)(B)$ are both meaningful, then A and B have the same M–category.*

Remark 7. *A model M of L is Tarskian if and only if it fulfills Tarski's first principle of the theory of semantic categories.*

Theorem 5 (Characteristic theorem of Tarskian models)**.** *The model M of L is Tarskian, if and only if the synonymies $\sim_M$ and $\cong_L$ are equivalent, i.e. $\sim_M$ equals $\cong_L$.*

Remark 8. *According to the Characteristic theorem of Tarskian models 5, all Tarskian models of L have the same system of semantic categories and this system is equivalent to the system of syntactic categories.*

Proposition 15. *If the model M of L is total, then the synonymies $\sim_M$ and $\cong_L$ are equivalent, i.e. $\sim_M$ equals $\cong_L$.*

Theorem 6. *If M is a total model of L, then M is Tarskian.*

Corollary 4. *Non–Tarskian models are partial.*

6 HUSSERLIAN MODELS

In Section 5 we dealt with the connection between syntactic and semantic categories. The next step we have to take is the investigation of the bridge between the system of semantic categories and the classification generated by the equivalence relation $\approx_M$.

Definition 16.

(1) *Let M_1, M_2 be models. We say that M_1 and its generated synonymy $\approx_{M_1}$ are M_2–Husserlian if $A \approx_{M_1} B \Rightarrow A \sim_{M_2} B$ for all $A, B \in Cat$.*

(2) *We say that a model M of L is Husserlian if it is M–Husserlian. (That is $A \approx_M B \Rightarrow A \sim_M B$ for all $A, B \in Cat$.)*

(3) *We say that a model M $(= \langle G, \varrho, v\rangle)$ of L is strictly Husserlian if M' $(= \langle G, \varrho, v'\rangle)$ is Husserlian for all assignments v'.*

(4) *We say that the generated synonymy $\approx_M$ of a model M is Husserlian (strictly Husserlian) if the model M is Husserlian (strictly Husserlian).*

[14] A version of the principle is quoted by (Hodges 2001b, 11.).

The notion of a Husserlian model creates a connection between generated synonymy and M–category. It requires that two expressions with the same semantic value with respect to M have to belong to the same M–category, and so according to Theorem 4 they have to have the same type. More precisely:

Proposition 16. *If a model M of L is Husserlian and $A \approx_M B$ for some $(A, B \in Cat)$, then $A \cong_L B$ i.e. there is a $\gamma \in TYPE_{PT}$ such that $A, B \in Cat(\gamma)$.*

Corollary 5. *If a model M of L is Husserlian, then $\cong_L \supseteq \approx_M$, i.e. $M_L \geq M$.*

Corollary 6. *Let M_1, M_2 be models of L. If M_1 is M_2–Husserlian, then it is M_L–Husserlian.*

Theorem 7. *Let M be a Tarskian model of L. The model M is Husserlian if and only if $\cong_L \supseteq \approx_M$, i.e. $M_L \geq M$.*

Corollary 7. *Let M be a total model of L. The model M is Husserlian, if and only if $\cong_L \supseteq \approx_M$, i.e. $M_L \geq M$.*

Law of replacement 2 says that an expression can substitute for another one without changing the semantic value of the compound expression, if the semantic value of the first expressions equals that of the second one. In the law, there is a special condition usually regarded as not too important. The condition requires that the two expressions have to belong to the same syntactic category. Without supposing it, the law of replacement holds only in Husserlian models. That is why I said that Law of replacement 2 is only a restricted version of the substitutivity principle (see in Section 3), a version of the principle of compositionality. Its unrestricted type–theoretical formulation is the following Husserlian law of replacement.

Theorem 8 (Husserlian law of replacement)**.** *If $A, B, C \in Cat$ and M is a Husserlian model of L, then*

$$[\![B]\!]_M = [\![C]\!]_M \Rightarrow [\![A]\!]_M = [\![A[C \downarrow B]]\!]_M.$$

Theorem 9 (Conversion of Husserlian law of replacement)**.** *If for all $A, B, C \in Cat$*

$$[\![B]\!]_M = [\![C]\!]_M \Rightarrow [\![A]\!]_M = [\![A[C \downarrow B]]\!]_M,$$

then M is a Husserlian model of L.

Proof. The proof is indirect. Suppose that the model M is not Husserlian. Then there are $B, C \in Cat$ such that $B \approx_M C$ ($[\![B]\!]_M = [\![C]\!]_M$) and $B \not\sim_M C$. Therefore there is some $D \in Cat, \tau \in Var$, such that $(\lambda\tau D)(B) \in Cat^M_{mf}$ and $(\lambda\tau D)(C) \notin Cat^M_{mf}$. According to Law of replacement 2, it is impossible that $B,\ C \in Cat(\gamma)$ for some $\gamma \in TYPE_{PT}$ because in contrary $[\![(\lambda\tau D)(B)]\!]_M = [\![(\lambda\tau D)(C)]\!]_M$. Therefore there are $\alpha, \beta \in TYPE_{PT}$ such that $\alpha \neq \beta$ and $B \in Cat(\alpha),\ C \in Cat(\beta)$. Let $A =$ '$(\lambda\xi\xi)(B)$' where $\xi \in Var(\alpha)$. $A \in Cat$ and $A[C \downarrow B] \notin Cat$ and so $[\![A]\!]_M \neq [\![A[C \downarrow B]]\!]_M$. □

Definition 17. *A model M of L fulfills the substitutivity principle if for all $A, B, C \in Cat$*

$$[\![B]\!]_M = [\![C]\!]_M \Rightarrow [\![A]\!]_M = [\![A[C \downarrow B]]\!]_M.$$

The next theorem shows that the substitutivity principle is a strong version of the principle of compositionality. In our theoretical framework all models are compositional, but a model fulfills the substitutivity principle if and only if it is Husserlian.

Theorem 10 (Characteristic theorem of Husserlian models)**.** *A model M of L is Husserlian if and only if it fulfills the substitutivity principle.*

Definition 18. *A model M of L is strongly compositional if it fulfills the substitutivity principle.*

Remark 9. *Characteristic theorem of Husserlian models 10 says the property of being strongly compositional is equivalent to being Husserlian.*

Corollary 8. *If M is a Tarskian model of L and $\cong_L \supseteq \approx_M$, then it fulfills the substitutivity principle.*

Theorem 11. *A model M of L is strictly Husserlian if and only if the sets $Dom_M(\gamma)$ $(\gamma \in PT)$ are pairwise disjoint ones.*

Proof. I have to note that the sets $Dom_M(\gamma)$ $(\gamma \in PT)$ are pairwise disjoint ones if and only if the sets $Dom_M(\gamma)$ $(\gamma \in TYPE_{PT})$ are pairwise disjoint ones.

At first we prove that if M $(= \langle G, \varrho, v\rangle)$ is strictly Husserlian, then the sets $Dom_M(\gamma)$ $(\gamma \in TYPE_{PT})$ are pairwise disjoint ones. The proof is indirect. Suppose that M is strictly Husserlian and there is a semantic value u such that $u \in Dom_M(\alpha) \cap Dom_M(\beta)$ where $\alpha \neq \beta$. Let $\tau_1 \in Var(\alpha)$ $\tau_2 \in Var(\beta)$ and v' be an assignment such that $v'(\tau_1) = u = v'(\tau_2)$. If $M' = \langle G, \varrho, v'\rangle$, then $[\![\tau_1]\!]_{M'} = [\![\tau_2]\!]_{M'}$ but $\tau_1 \not\cong_L \tau_2$ and according to Proposition 16 M' is not Husserlian. So M is not strictly Husserlian.

Secondly it is enough to prove that if M is a model of L and the sets $Dom_M(\gamma)$ $(\gamma \in TYPE_{PT})$ are pairwise disjoint ones, then M is Husserlian. The proof is indirect. Suppose that $A \approx_M B$ and $A \not\sim_M B$ where $A, B \in Cat^M_{mf}$. Then $A \not\cong_L B$, and so there are $\alpha, \beta \in TYPE_{PT}$ such that $A \in Cat(\alpha)$, $B \in Cat(\beta)$ and $\alpha \neq \beta$. According to Proposition 1 $[\![A]\!]_M \in Dom_M(\alpha)$, $[\![B]\!]_M \in Dom_M(\beta)$. Since $[\![A]\!]_M = [\![B]\!]_M$, $Dom_M(\alpha) \cap Dom_M(\beta) \neq \emptyset$. □

Definition 19. *A (total or partial) frame G is strictly Husserlian if the sets $Dom_G(\gamma)$ $(\gamma \in PT)$ are pairwise disjoint ones.*

Corollary 9. *If M is a model on a strictly Husserlian frame then the model M of L is strictly Husserlian.*

Corollary 10. *The degenerate model M_L, which generates the syntactic synonymy $\cong_L$, is strictly Husserlian, and so the synonymy $\cong_L$ is strictly Husserlian.*

Theorem 12. *A model M is Husserlian if and only if there is a strictly Husserlian model M' such that $M' \geq M$.*

Proof. According to Corollary 10, the model M_L is strictly Husserlian. If M is Husserlian, then according to Corollary 5, $M_L \geq M$.

Let M' be a strictly Husserlian model such that $M' \geq M$. Then according to Corollary 5 $M_L \geq M'$ and so $M_L \geq M$. It means that if $A \approx_M B$, then $A \cong_L B$ i.e. there is $\gamma \in TYPE_{PT}$ such that $A, B \in Cat(\gamma)$. Therefore $(\lambda\tau C)(A) \in Cat$ if and only if $(\lambda\tau C)(B) \in Cat$ for any $C \in Cat$ and $\tau \in Var$. According to Law of replacement 2 $[\![(\lambda\tau C)(A)]\!]_M = [\![(\lambda\tau C)(B)]\!]_M$ and so $(\lambda\tau C)(A) \in Cat^M_{mf} \Leftrightarrow (\lambda\tau C)(B) \in Cat^M_{mf}$, i.e. $A \sim_M B$. □

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