A SIMPLE PROOF OF THE LEBESGUE DECOMPOSITION THEOREM

TAMÁS TITKOS

The aim of this short note is to present an elementary, self-contained, and direct proof for the classical Lebesgue decomposition theorem. In fact, I will show that the absolutely continuous part just measures the squared semidistance of the characteristic functions from a suitable subspace.

This approach also gives a decomposition in the finitely additive case, but it differs from the Lebesgue-Darst decomposition [1], because the involved absolute continuity concepts are different.

Notations. Let $\mathcal{A}$ be a σ-algebra over $X \neq \emptyset$, and consider the finite measures $\mu, \nu : \mathcal{A} \to \mathbb{R}_+$ on it. The measure μ is ν-absolutely continuous ($\mu \ll \nu$, in symbols) if $\nu(A) = 0$ implies $\mu(A) = 0$ for all $A \in \mathcal{A}$. Singularity of μ and ν (denoted by $\mu \perp \nu$) means that the only measure dominated by both μ and ν is the zero measure. As it is known, this is equivalent with the existence of a measurable set $P \in \mathcal{A}$ such that $\mu(P) = \nu(X \setminus P) = 0$.

Theorem. *Let μ and ν be finite measures on $\mathcal{A}$. Then μ splits uniquely into $\mu_{\mathrm{ac}} \ll \nu$ and $\mu_{\mathrm{s}} \perp \nu$.*

Proof. Consider the real vector space $\mathscr{E}$ of real valued $\mathcal{A}$-measurable step-functions and let $\mathscr{N}$ be the linear subspace generated by the characteristic functions of those measurable sets A such that $\nu(A) = 0$. Define the set function μ_{ac} by

$$\mu_{\mathrm{ac}}(A) := \inf_{\psi \in \mathscr{N}} \int_X |\mathbb{1}_A - \psi|^2 \, \mathrm{d}\mu \qquad (A \in \mathcal{A}).$$

It is clear that $\mu_{\mathrm{ac}} \leq \mu$ ($\psi := \mathbb{1}_\emptyset$), and that $\nu(A) = 0$ implies $\mu_{\mathrm{ac}}(A) = 0$ ($\psi := \mathbb{1}_A$). Furthermore, trivial verification shows that if A and B are disjoint elements of $\mathcal{A}$, then

$$\inf_{\psi \in \mathscr{N}} \int_X |\mathbb{1}_{A \cup B} - \psi|^2 \, \mathrm{d}\mu = \inf_{\psi \in \mathscr{N}} \int_X |\mathbb{1}_A - \psi|^2 \, \mathrm{d}\mu + \inf_{\psi \in \mathscr{N}} \int_X |\mathbb{1}_B - \psi|^2 \, \mathrm{d}\mu.$$

Since μ_{ac} is nonnegative, additive, and dominated by the measure μ, we infer that μ_{ac} is a measure itself.

What is left is to show that $\mu_{\mathrm{s}} := \mu - \mu_{\mathrm{ac}}$ and ν are singular, and that the decomposition is unique. Both follow immediately from the fact that μ_{ac} is maximal among those measures ϑ such that $\vartheta \leq \mu$ and $\vartheta \ll \nu$. Indeed, let ϑ be such a measure, $\psi \in \mathscr{N}$, and observe that

$$\vartheta(A) = \int_X |\mathbb{1}_A|^2 \, \mathrm{d}\vartheta = \int_X |\mathbb{1}_A - \psi|^2 \, \mathrm{d}\vartheta \leq \int_X |\mathbb{1}_A - \psi|^2 \, \mathrm{d}\mu.$$

Taking the infimum over $\mathscr{N}$ we obtain that $\vartheta \leq \mu_{\mathrm{ac}}$.

Now, let η be a measure, such that $\eta \leq \nu$ and $\eta \leq \mu - \mu_{\mathrm{ac}}$. In this case, $\mu_{\mathrm{ac}} + \eta \leq \mu$ and $\mu_{\mathrm{ac}} + \eta \ll \nu$, thus $\eta = 0$. If $\mu = \mu_1 + \mu_2$, where $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$, then $\mu_{\mathrm{ac}} - \mu_1$ is a measure, which is simultaneously ν-absolutely continuous and ν-singular. This yields that $\mu_1 = \mu_{\mathrm{ac}}$. $\square$

References

[1] Tarcsay, Zs., *A functional analytic proof of the Lebesgue-Darst decomposition theorem*, Real Analysis Exchange, Vol. 39(1), 2013/2014, 241–248.

Tamás Titkos, Department of Applied Analysis, Eötvös Loránd University, Pázmány Péter sétány 1/c, H-1117, Budapest, Hungary;

E-mail address: `titkos@cs.elte.hu`

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