

# WHAT PROPERTIES OF NUMBERS ARE NEEDED TO MODEL ACCELERATED OBSERVERS IN RELATIVITY?

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**ABSTRACT.** We investigate the possible structures of numbers (as physical quantities) over which accelerated observers can be modeled in special relativity. We present a general axiomatic theory of accelerated observers which has a model over every real closed field. We also show that, if we would like to model certain accelerated observers, then not every real closed field is suitable, e.g., uniformly accelerated observers cannot be modeled over the field of real algebraic numbers. Consequently, the class of fields over which uniform acceleration can be investigated is not axiomatizable in the language of ordered fields.

## 1. INTRODUCTION

In this paper within an axiomatic framework, we investigate the possible structures of numbers (as physical quantities) over which accelerated observers can be modeled in special relativity.

There are several reasons for this kind of investigations. One of them is that we cannot experimentally verify whether the structure of quantities is isomorphic to  $\mathbb{R}$  (the field of real numbers). Thus we cannot have any direct empirical support to leave out of consideration the several other algebraic structures. Another reason is that these investigations lead to a deeper understanding of the relation of our mathematical and physical assumptions. For a more general perspective of this research direction, see [3].

In general we would like to investigate the question

*“What structure can numbers have in a certain physical theory?”*

To introduce our central concept, let  $\mathbf{Th}$  be a theory of physics. In this case, we can introduce notation  $Num(\mathbf{Th})$  for the class of the possible quantity structures of theory  $\mathbf{Th}$ :

$$Num(\mathbf{Th}) := \{\mathfrak{Q} : \mathfrak{Q} \text{ is a structure of quantities over which } \mathbf{Th} \text{ has a model.}\}$$

In this paper, our main question of interest is that what algebraic properties have to be satisfied by the numbers as physical quantities if we want to model accelerated observers in special relativity. So we will restrict our investigation to the case when  $\mathbf{Th}$  is a theory of special relativity extended with accelerated observers. However, this question can be investigated in any other physical theory the same way.

We introduce several theories and axioms of relativity theory. For example, our axiom system for  $d$ -dimensional special relativity ( $\mathbf{SpecRel}_d$ , see p.5) captures the

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kinematics of special relativity perfectly (if  $d \geq 3$ ) as it implies that the worldview transformations between inertial observers are Poincaré transformations, see [3]. Without any extra assumptions  $\text{SpecRel}_d$  has a model over every ordered field, i.e.,

$$\text{Num}(\text{SpecRel}_d) = \{\Omega : \Omega \text{ is an ordered field}\}.$$

Therefore,  $\text{SpecRel}_d$  has a model over the field of rational numbers  $\mathbb{Q}$ , too. However, if we also assume that inertial observers can move with arbitrary speed less than that of light, then every positive number has to have a square root if  $d \geq 3$ , see [3]. In particular, the number structure cannot be the field of rational numbers, but it can be the field of real algebraic numbers.

If we assume only that inertial observers can move only approximately with any speed slower than that of light, then we still can model special relativity over  $\mathbb{Q}$ , see [13].

Moving toward general relativity we will see that our theory of accelerated observers ( $\text{AccRel}_d$ , see p.6) requires the structure of quantities to be a real closed field, i.e., an ordered field in which every positive number has a square root and every odd degree polynomial has a root, see Theorem 1. Specially,  $\text{AccRel}_d$  does not have a model over  $\mathbb{Q}$ . However, any real closed field, e.g., the field of real algebraic numbers, can be the quantity structure of  $\text{AccRel}_d$ .

If we extend  $\text{AccRel}_d$  by an extra axiom stating that there are uniformly accelerated observers ( $\text{Ax}\exists\text{UnifOb}$ , see p.6), then the field of real algebraic numbers cannot be the structure of quantities anymore if  $d \geq 3$ , see Theorem 2. A surprising consequence of this result is that  $\text{Num}(\text{AccRel}_d + \text{Ax}\exists\text{UnifOb})$  is not a first-order logic axiomatizable class of fields, see Corollary 1. That is, in the language of ordered fields, it is impossible to axiomatize those fields over which uniformly accelerated observers can be modeled.

An interesting and related approach of Stannett introduces two structures one for the measurable numbers and one for the theoretical numbers and assumes that the set of measurable numbers is dense in the set of theoretical numbers, see [21].

We chose first-order predicate logic to formulate our axioms because experience (e.g., in geometry and set theory) shows that this logic is the best logic for providing an axiomatic foundation for a theory. A further reason for choosing first-order logic is that it is a well defined fragment of natural language with an unambiguous syntax and semantics, which do not depend on set theory. For further reasons, see, e.g., [1, §Why FOL?], [4], [22, §11], [24], [25].

## 2. THE LANGUAGE OF OUR THEORIES

To our investigation, we need an axiomatic theory of spacetimes.<sup>1</sup> The first important decision in writing up an axiom system is to choose the set of basic symbols of our logic language, i.e., what objects and relations between them we will use as basic concepts.

Here we will use the following two-sorted<sup>2</sup> language of first-order logic (FOL) parametrized by a natural number  $d \geq 2$  representing the dimension of spacetime:

$$\{B, Q; \text{Ob}, \text{IOb}, \text{Ph}, +, \cdot, \leq, W\}, \quad (1)$$

<sup>1</sup>In this paper, we will use the language and axiom systems of [2].

<sup>2</sup>That our theory is two-sorted means only that there are two types of basic objects (bodies and quantities) as opposed to, e.g., Zermelo–Fraenkel set theory where there is only one type of basic objects (sets).

where  $B$  (bodies) and  $Q$  (quantities) are the two sorts,  $\text{Ob}$  (observers),  $\text{IOb}$  (inertial observers) and  $\text{Ph}$  (light signals) are one-place relation symbols of sort  $B$ ,  $+$  and  $\cdot$  are two-place function symbols of sort  $Q$ ,  $\leq$  is a two-place relation symbol of sort  $Q$ , and  $W$  (the worldview relation) is a  $d + 2$ -place relation symbol the first two arguments of which are of sort  $B$  and the rest are of sort  $Q$ .

Relations  $\text{Ob}(o)$ ,  $\text{IOb}(m)$  and  $\text{Ph}(p)$  are translated as “ $o$  is an observer,” “ $m$  is an inertial observer,” and “ $p$  is a light signal,” respectively. To speak about coordinatization of observers, we translate relation  $W(k, b, x_1, x_2, \dots, x_d)$  as “body  $k$  coordinatizes body  $b$  at space-time location  $\langle x_1, x_2, \dots, x_d \rangle$ ,” (i.e., at space location  $\langle x_2, \dots, x_d \rangle$  and instant  $x_1$ ).

**Quantity terms** are the variables of sort  $Q$  and what can be built from them by using the two-place operations  $+$  and  $\cdot$ , **body terms** are only the variables of sort  $B$ .  $\text{IOb}(m)$ ,  $\text{Ph}(p)$ ,  $W(m, b, x_1, \dots, x_d)$ ,  $x = y$ , and  $x \leq y$  where  $m, p, b, x, y, x_1, \dots, x_d$  are arbitrary terms of the respective sorts are so-called **atomic formulas** of our first-order logic language. The **formulas** are built up from these atomic formulas by using the logical connectives *not* ( $\neg$ ), *and* ( $\wedge$ ), *or* ( $\vee$ ), *implies* ( $\rightarrow$ ), *if-and-only-if* ( $\leftrightarrow$ ) and the quantifiers *exists* ( $\exists$ ) and *for all* ( $\forall$ ).

To make them easier to read, we omit the outermost universal quantifiers from the formalizations of our axioms, i.e., all the free variables are universally quantified.

We use the notation  $Q^n$  for the set of all  $n$ -tuples of elements of  $Q$ . If  $\bar{x} \in Q^n$ , we assume that  $\bar{x} = \langle x_1, \dots, x_n \rangle$ , i.e.,  $x_i$  denotes the  $i$ -th component of the  $n$ -tuple  $\bar{x}$ . Specially, we write  $W(m, b, \bar{x})$  in place of  $W(m, b, x_1, \dots, x_d)$ , and we write  $\forall \bar{x}$  in place of  $\forall x_1 \dots \forall x_d$ , etc.

We use first-order logic set theory as a meta theory to speak about model theoretical terms, such as models, validity, etc. The **models** of this language are of the form

$$\mathfrak{M} = \langle B, Q; \text{Ob}_{\mathfrak{M}}, \text{IOb}_{\mathfrak{M}}, \text{Ph}_{\mathfrak{M}}, +_{\mathfrak{M}}, \cdot_{\mathfrak{M}}, \leq_{\mathfrak{M}}, W_{\mathfrak{M}} \rangle, \quad (2)$$

where  $B$  and  $Q$  are nonempty sets,  $\text{Ob}_{\mathfrak{M}}$ ,  $\text{IOb}_{\mathfrak{M}}$  and  $\text{Ph}_{\mathfrak{M}}$  are subsets of  $B$ ,  $+_{\mathfrak{M}}$  and  $\cdot_{\mathfrak{M}}$  are binary functions and  $\leq_{\mathfrak{M}}$  is a binary relation on  $Q$ , and  $W_{\mathfrak{M}}$  is a subset of  $B \times B \times Q^d$ . Formulas are interpreted in  $\mathfrak{M}$  in the usual way. For the precise definition of the syntax and semantics of first-order logic, see, e.g., [5, §1.3], [8, §2.1, §2.2].

### 3. NUMBERS REQUIRED BY SPECIAL RELATIVITY

First we formulate axioms for special relativity concerning inertial observers only in the logic language of Section 2.

The key axiom of special relativity states that the speed of light is the same in every direction for every inertial observers.

**AxPh:** For any inertial observer, the speed of light is the same everywhere and in every direction (and it is finite). Furthermore, it is possible to send out a light signal in any direction (existing according to the coordinate system) everywhere:

$$\begin{aligned} \text{IOb}(m) \rightarrow \exists c_m \left[ c_m > 0 \wedge \forall \bar{x} \bar{y} \left( \exists p \left[ \text{Ph}(p) \wedge W(m, p, \bar{x}) \wedge W(m, p, \bar{y}) \right] \right. \right. \\ \left. \left. \leftrightarrow (x_2 - y_2)^2 + \dots + (x_d - y_d)^2 = c_m^2 \cdot (x_1 - y_1)^2 \right) \right].^3 \quad (3) \end{aligned}$$

To get back the intended meaning of axiom **AxPh** (or even to be able to define subtraction from addition), we have to assume some properties of numbers.

In our next axiom, we state some basic properties of addition, multiplication and ordering true for real numbers.

**AxOField**: The quantity part  $\langle Q, +, \cdot, \leq \rangle$  is an ordered field, i.e.,

- $\langle Q, +, \cdot \rangle$  is a field in the sense of abstract algebra; and
- the relation  $\leq$  is a linear ordering on  $Q$  such that
  - i)  $x \leq y \rightarrow x + z \leq y + z$  and
  - ii)  $0 \leq x \wedge 0 \leq y \rightarrow 0 \leq xy$  holds.

Using axiom **AxOField** instead of assuming that the structure of quantities is the field of real numbers not just makes our theory more flexible, but also makes it possible to meaningfully investigate our main question. Another reason for using **AxOField** instead of  $\mathbb{R}$  is that we cannot experimentally verify whether the structure of physical quantities are isomorphic to  $\mathbb{R}$ . Hence the assumption that the structure of quantities is  $\mathbb{R}$  cannot have a direct empirical support. The two properties of real numbers which are the most difficult to defend from empirical point of view are the Archimedean property, see [17], [18, §3.1], [20], [19], and the supremum property,<sup>4</sup> see the remark after the introduction of **CONT** on p.5.

We also have to support **AxPh** with the assumption that all observers coordinatize the same “external” reality (the same set of events). By the **event** occurring for observer  $m$  at point  $\bar{x}$ , we mean the set of bodies  $m$  coordinatizes at  $\bar{x}$ :

$$\text{ev}_m(\bar{x}) := \{b : W(m, b, \bar{x})\}. \quad (4)$$

**AxEv** All inertial observers coordinatize the same set of events:

$$\text{IOb}(m) \wedge \text{IOb}(k) \rightarrow \exists \bar{y} \forall b [W(m, b, \bar{x}) \leftrightarrow W(k, b, \bar{y})].$$

From now on, we will use  $\text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y})$  to abbreviate the subformula  $\forall b [W(m, b, \bar{x}) \leftrightarrow W(k, b, \bar{y})]$  of **AxEv**.

The three axioms above are enough to capture the essence of special relativity. However, let us assume two more simplifying axioms.

**AxSelf** Any inertial observer is stationary relative to himself:

$$\text{IOb}(m) \rightarrow \forall \bar{x} [W(m, m, \bar{x}) \leftrightarrow x_2 = \dots = x_d = 0].$$

Our last axiom on inertial observers is a symmetry axiom saying that they use the same units of measurement.

**AxSymD** Any two inertial observers agree as to the spatial distance between two events if these two events are simultaneous for both of them; furthermore, the speed of light is 1 for all observers:

$$\begin{aligned} \text{IOb}(m) \wedge \text{IOb}(k) \wedge x_1 = y_1 \wedge x'_1 = y'_1 \wedge \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{x}') \wedge \text{ev}_m(\bar{y}) = \text{ev}_k(\bar{y}') \rightarrow \\ (x_2 - y_2)^2 + \dots + (x_d - y_d)^2 = (x'_2 - y'_2)^2 + \dots + (x'_d - y'_d)^2 \text{ and} \\ \text{IOb}(m) \rightarrow \exists p [Ph(p) \wedge W(m, p, 0, \dots, 0) \wedge W(m, p, 1, 1, 0, \dots, 0)]. \end{aligned}$$

<sup>3</sup>That is, if  $m$  is an inertial observer, there is a positive quantity  $c_m$  such that for all coordinate points  $\bar{x}$  and  $\bar{y}$  there is a light signal  $p$  coordinatized at  $\bar{x}$  and  $\bar{y}$  by observer  $m$  if and only if equation  $(x_2 - y_2)^2 + \dots + (x_d - y_d)^2 = c_m^2 \cdot (x_1 - y_1)^2$  holds.

<sup>4</sup>The supremum property (i.e., that every nonempty and bounded subset of the numbers has a least upper bound) implies the Archimedean property. So if we want to get ourselves free from the Archimedean property, we have to leave this one, too.

Let us introduce an axiom system for special relativity as the collection of the five axioms above:

$$\text{SpecRel}_d := \text{AxPh} + \text{AxOField} + \text{AxEv} + \text{AxSelf} + \text{AxSymD}.$$

Streamlined axiom system  $\text{SpecRel}_d$  perfectly captures the kinematics of special relativity since it implies that the worldview transformations between inertial observers are Poincaré transformations, see [3].

#### 4. NUMBERS IMPLIED BY ACCELERATED OBSERVERS

Now we are going to investigate what happens with the possible structures of quantities if we extend our theory  $\text{SpecRel}_d$  with accelerated observers. To do so, let us recall our first-order logic axiom system of accelerated observers  $\text{AccRel}_d$ . The key axiom of  $\text{AccRel}_d$  is the following:

AxCmv At each moment of its worldline, each observer sees the nearby world for a short while as an inertial observer does.

For formalization of  $\text{AxCmv}$  in the first-order language of Section 2, see [22]. In  $\text{AccRel}_d$  we will also use the following localized version of axioms  $\text{AxEv}$  and  $\text{AxSelf}$  of  $\text{SpecRel}_d$ .

AxEv<sup>-</sup> Observers coordinatize all the events in which they participate:

$$\text{Ob}(k) \wedge \text{W}(m, k, \bar{x}) \rightarrow \exists \bar{y} \text{ ev}_m(\bar{x}) = \text{ev}_k(\bar{y}).$$

AxSelf<sup>-</sup> In his own worldview, the worldline of any observer is an interval of the time-axis containing all the coordinate points of the time-axis where the observer sees something:

$$\begin{aligned} & [\text{W}(m, m, \bar{x}) \rightarrow x_2 = \dots = x_d = 0] \wedge \\ & [\text{W}(m, m, \bar{y}) \wedge \text{W}(m, m, \bar{z}) \wedge x_1 < t < y_1 \rightarrow \text{W}(m, m, t, 0, \dots, 0)] \wedge \\ & \exists b [\text{W}(m, b, t, 0, \dots, 0) \rightarrow \text{W}(m, m, t, 0, \dots, 0)]. \end{aligned}$$

Let us now introduce a promising theory of accelerated observers as  $\text{SpecRel}_d$  extended with the three axioms above.

$$\text{AccRel}_d^0 := \text{SpecRel}_d + \text{AxCmv} + \text{AxEv}^- + \text{AxSelf}^-$$

Axiom  $\text{AxCmv}$  ties the behavior of accelerated observers to the inertial ones and  $\text{SpecRel}_d$  captures the kinematics of special relativity perfectly (it implies that the worldview transformations between inertial observers are Poincaré transformations, see [3]). Therefore, it is quite natural to hope that  $\text{AccRel}_d^0$  is a strong enough theory of accelerated observers to prove the most fundamental results about accelerated observers. However,  $\text{AccRel}_d^0$  does not imply even the most basic predictions about accelerated observers such as the twin paradox or that stationary observers measure the same time between two events [12], [22, §7]. Moreover, it can be proved that even if we add the whole first-order logic theory of real numbers to  $\text{AccRel}_d^0$  is not enough to get a theory that implies the twin paradox, see [12], [22, §7].

In the models of  $\text{AccRel}_d^0$  in which  $\text{TwP}$  is not true, there are some definable gaps in  $Q$ . Our axiom scheme  $\text{CONT}$  excludes these gaps.

CONT Every parametrically definable, bounded and nonempty subset of  $Q$  has a supremum (i.e., least upper bound) with respect to  $\leq$ .

In **CONT** “definable” means “definable in the language of  $\text{AccRel}_d$ , parametrically.” For a precise formulation of **CONT** in the first-order language of Section 2, see [12, p.692] or [22, §10.1].

Axiom scheme **CONT** makes the supremum postulate of real numbers closer to the physical/empirical level because **CONT** speaks only about “physically meaningful” sets of the quantities which can be defined in the language of our (physical) theory and not “any fancy subset.”

Our axiom scheme of continuity (**CONT**) is Tarski’s first-order logic version of Hilbert’s continuity axiom in his axiomatization of geometry, see [10, pp.161-162], fitted to the language of  $\text{AccRel}_d$ .

When  $Q$  is the ordered field of real numbers, **CONT** is automatically true. Let us introduce our axioms system  $\text{AccRel}_d$  as the extension of  $\text{AccRel}_d^0$  by axiom scheme **CONT**.

$$\text{AccRel}_d := \text{AccRel}_d^0 + \text{CONT}$$

It can be proved that axiom system  $\text{AccRel}_d$  implies the twin paradox, see [12], [22, §7.2].

An ordered field is called **real closed field** if a first-order logic sentence of the language of ordered fields is true in it exactly when it is true in the field of real numbers, or equivalently if it is an ordered field in which every positive number has a square root and every polynomial of odd degree has a root in it, see, e.g., [23].

Axiom scheme **CONT** is so powerful that it implies that the possible structures of quantities have to be real closed fields:

**Theorem 1.** *For all  $d \geq 2$ ,*

$$\text{Num}(\text{AccRel}_d) = \{\Omega : \Omega \text{ is real closed fields}\}.$$

For the proof of Theorem 1, see [3].

**Question 1.** *Can **CONT** be replaced in  $\text{AccRel}_d$  with some natural assumptions such that they (together with  $\text{AccRel}_d^0$ ) imply all (or certain) important predictions of relativity theory about accelerated observers (e.g., the twin paradox) yet they do not require that the structure of quantities is a real closed field.*

## 5. NUMBERS IMPLIED BY UNIFORMLY ACCELERATED OBSERVERS

In paper [3], we have seen that assuming existence of observers can ensure the existence of numbers. So let us investigate another axiom of this kind, which postulates the existence of uniformly accelerated observers. To introduce this axiom, let us define the **life-curve**  $\text{lc}_{km}$  of observer  $k$  according to observer  $m$  as the worldline of  $k$  according to  $m$  parametrized by the time measured by  $k$ , formally:

$$\text{lc}_{km} := \{ \langle t, \bar{x} \rangle : \exists \bar{y} \ k \in \text{ev}_k(\bar{y}) = \text{ev}_m(\bar{x}) \wedge y_1 = t \}. \quad (5)$$

Now we can introduce our axiom ensuring the existence of uniformly accelerated observers.

**Ax∃UnifOb** It is possible to accelerate an observer uniformly:<sup>5</sup>

$$\text{IOb}(m) \rightarrow \exists k \left[ \text{Ob}(k) \wedge \text{Dom } \text{lc}_{km} = Q \wedge \text{lc}_{km}(0) = \bar{y} \wedge \text{lc}_{km}(1)_1 > y_1 \wedge \forall \bar{x} \left[ \bar{x} \in \text{Ran } \text{lc}_{km} \leftrightarrow (x_2 - y_2)^2 - (x_1 - y_1)^2 = a^2 \wedge x_3 - y_3 = \dots = x_d - y_d = 0 \right] \right].$$

We use the notation  $\mathfrak{Q} \in \text{Num}(\text{Th})$  for algebraic structure  $\mathfrak{Q}$  the same way as the model theoretic notation  $\mathfrak{Q} \in \text{Mod}(\text{AxField})$ , e.g.,  $\mathbb{Q} \in \text{Num}(\text{Th})$  means that  $\mathbb{Q}$ , the field of rational numbers, can be the structure of quantities (numbers) in  $\text{Th}$ .

Let  $\mathbb{A} \cap \mathbb{R}$  denote the ordered field of real algebraic numbers. Theorem 2 states that the ordered field of algebraic real numbers cannot be the structure of quantities of theory  $\text{AccRel}_d + \text{Ax}\exists\text{UnifOb}$  if  $d \geq 3$ :

**Theorem 2.** *Let  $d \geq 3$ . Then*

$$\mathbb{A} \cap \mathbb{R} \notin \text{Num}(\text{AccRel}_d + \text{Ax}\exists\text{UnifOb}).$$

The proof of Theorem 2 is in Section 6 on p.12.

Since the ordered fields of real numbers and real algebraic numbers are elementarily equivalent, Theorem 2 implies that the quantity part of the models of  $\text{AccRel}_d + \text{Ax}\exists\text{UnifOb}$  is not an elementary class.<sup>6</sup>

**Corollary 1.** *Let  $d \geq 3$ . Then  $\text{Num}(\text{AccRel}_d + \text{Ax}\exists\text{UnifOb})$  is not axiomatizable in the language of ordered fields.*

By Theorem 2, we know that not every real closed field can be the quantity structure of  $\text{AccRel}_d + \text{Ax}\exists\text{UnifOb}$ , e.g., it cannot be the field of real algebraic numbers. However, the problem that exactly which ordered fields can be the quantity structures of  $\text{AccRel}_d + \text{Ax}\exists\text{UnifOb}$  is still open:

**Question 2.** *Exactly which ordered fields are the elements of class  $\text{Num}(\text{AccRel}_d + \text{Ax}\exists\text{UnifOb})$ ?*

We can also ask what properties of numbers do axiom  $\text{Ax}\exists\text{UnifOb}$  requires without  $\text{CONT}$ :

**Question 3.** *Exactly which ordered fields are the elements of class  $\text{Num}(\text{AccRel}_d^0 + \text{Ax}\exists\text{UnifOb})$ ?*

Theorem 5 below suggests that the answer to Questions 2 and 3 may have something to do with ordered exponential fields, see [6, §4], [11].

## 6. PROOF OF THEOREM 2

In this section, we prove Theorem 2. To do so, let us introduce some concepts. The **space component** of  $\bar{x} \in Q^d$  is defined as

$$\bar{x}_s := \langle x_2, \dots, x_d \rangle. \tag{6}$$

The (signed) **Minkowski length** of  $\bar{x} \in Q^d$  is

$$\mu(\bar{x}) := \begin{cases} \sqrt{x_1^2 - |\bar{x}_s|^2} & \text{if } x_1^2 \geq |\bar{x}_s|^2, \\ -\sqrt{|\bar{x}_s|^2 - x_1^2} & \text{in other cases,} \end{cases} \tag{7}$$

<sup>5</sup>In relativity theory, uniformly accelerated observers are moving along hyperbolas, see, e.g., [7, §3.8, pp.37-38], [15, §6], [16, §12.4, pp.267-272].

<sup>6</sup>Of course, it is a pseudo elementary class, i.e., it is a class of the reducts of an elementary class.

and the **Minkowski distance** between  $\bar{x}$  and  $\bar{y}$  is  $\mu(\bar{x}, \bar{y}) := \mu(\bar{x} - \bar{y})$ . We use the signed version of the Minkowski length because it contains two kinds of information: (i) the length of  $\bar{x}$ , and (ii) whether it is spacelike, lightlike or timelike. Let  $H \subseteq Q$ . We say that  $H$  is an **interval** iff  $z \in H$  when there are  $x, y \in H$  such that  $x < z < y$ . We say that a function  $\gamma : H \rightarrow Q^d$  is a **curve** if  $H$  is an interval and has at least two distinct elements.

The usual (first-order logic) formula can be used to define the differentiability function over any ordered field  $\mathfrak{Q}$ . The **derivative of** function  $f : Q \rightarrow Q^n$  is  $A \in Q^n$  at  $x_0 \in \text{Dom } f$ :

$$\text{Diff}(f, x_0, A) \stackrel{\text{def}}{\iff} \forall \varepsilon > 0 \exists \delta > 0 \forall x \ 0 < |x - x_0| < \delta \\ \wedge x \in \text{Dom } f \rightarrow |f(x) - f(x_0) - A(x - x_0)| < \varepsilon |x - x_0|. \quad (8)$$

In the case when there is one and only one  $A$  such that  $\text{Diff}(f, x_0, A)$  holds, we write  $f'(x_0) = A$ . It can be proved that there is at most one  $A$  such that  $\text{Diff}(f, x_0, A)$  holds if  $\text{Dom } f$  is open, see [22, Thm.10.3.9]

A curve  $\gamma$  is called **timelike curve** iff it is differentiable, and  $\gamma'(t)$  is timelike, i.e.,  $\mu(\gamma'(t)) > 0$ , for all  $t \in \text{Dom } \gamma$ . We call a timelike curve  $\alpha$  **well-parametrized** if  $\mu(\alpha'(t)) = 1$  for all  $t \in \text{Dom } \alpha$ .

**Theorem 3.** *Let  $d \geq 3$ . Assume  $\text{AccRel}_d$ . Let  $k$  be an observer and  $m$  be an inertial observer. Then  $\text{lc}_{km}$  is a well-parametrized timelike curve.*

For the proof of Theorem 3, see [22, Thm.6.1.11].

A part of real analysis can be generalized for arbitrary ordered fields without any real difficulty, see [22, §10]. However, a certain fragment of real analysis can only be generalized within first-order logic for *definable* functions and their proofs need axiom schema **CONT**. We refer to these generalizations by marking them “**CONT-**.” The first-order logic generalizations of some theorems, such as Chain Rule can be proved without **CONT**, so they are naturally referred to without the “**CONT-**” mark.

**Lemma 1.** *Assume  $\text{AxOField}$  and **CONT**. Let  $f : Q \rightarrow Q$  be an injective definable continuous function. Then  $f$  is also monotonous.*

For the proof of Lemma 1, see [22, Thm.10.2.4].

**Proposition 1.** *Assume **CONT** and  $\text{AxOField}$ . Let  $\gamma, \delta : Q \rightarrow Q^d$  be definable and differentiable well-parametrized timelike curves such that  $\text{Ran } \gamma = \text{Ran } \delta$ . Then there are  $\varepsilon \in \{-1, +1\}$  and  $c \in Q$  such that  $\delta(t) = \gamma(\varepsilon t + c)$  for all  $t \in Q$ .*

*Proof.* By [22, Lem.10.5.4], we have that there is a (definable) differentiable function  $h : Q \rightarrow Q$  such that  $|h'| = 1$  and  $\delta(t) = \gamma(h(t))$  for all  $t \in Q$ . By **CONT**-Darboux’s Theorem [22, p.110],  $h'(t) = 1$  for all  $t \in Q$  or  $h'(t) = -1$  for all  $t \in Q$ . By the **CONT** version of the fundamental theorem of integration [22, Prop.10.3.19],  $h(t) = t + c$  or  $h(t) = -t + c$  for some  $c \in Q$ .  $\square$

**Lemma 2.** *Let  $d \geq 3$ . Assume  $\text{AccRel}_d$ . Let  $m_1, m_2$  be inertial observers and let  $k_1$  and  $k_2$  be observers such that*

- (1)  $\text{Dom } \text{lc}_{k_1 m_1} = \text{Dom } \text{lc}_{k_2 m_2} = Q$ ,
- (2)  $\text{Ran } \text{lc}_{m_1 k_1} = \text{Ran } \text{lc}_{m_2 k_2}$ ,
- (3)  $\text{lc}_{k_1 m_1}(0) = \text{lc}_{k_2 m_2}(0)$ ,



$$(4) \text{lc}_{k_1 m_1}(1)_1 > \text{lc}_{k_1 m_1}(0)_1 \text{ and } \text{w}_{k_2 m_2}(1)_1 > \text{w}_{k_1 m_1}(0)_1.$$

Then  $\text{lc}_{k_1 m_1} = \text{lc}_{k_2 m_2}$ .

*Proof.* We are going to prove our statement by applying Proposition 1 to  $\text{lc}_{k_1 m_1}$  and  $\text{lc}_{k_2 m_2}$ . So let  $\gamma := \text{lc}_{k_1 m_1}$  and  $\delta := \text{lc}_{k_2 m_2}$ . By Theorem 3,  $\gamma$  and  $\delta$  are well-parametrized timelike curves. By assumptions (1) and (2)  $\text{Dom } \gamma = \text{Dom } \delta = Q$  and  $\text{Ran } \gamma = \text{Ran } \delta$ . Therefore, by Proposition 1, there is a  $c \in Q$  and  $\varepsilon \in \{-1, +1\}$  such that  $\delta(t) = \gamma(\varepsilon t + c)$  for all  $t \in Q$ . By assumption (3),  $\gamma(0) = \delta(0)$ . Therefore,  $c = 0$ . Since  $\gamma$  and  $\delta$  are timelike curves  $\gamma_1$  and  $\delta_1$  are either strictly increasing or strictly decreasing functions. By assumption (4),  $\gamma(1)_1 > \gamma(0)_1$  and  $\delta(1)_1 > \delta(0)_1$ . Thus both  $\gamma_1$  and  $\delta_1$  are strictly increasing. Consequently,  $\gamma'(0)_1 > 0$  and  $\delta'(0)_1 > 0$ . Therefore,  $\varepsilon$  cannot be negative. Hence we have that  $\varepsilon = 1$ . Consequently,  $\gamma = \delta$  as it was stated.  $\square$

**Theorem 4.** *Let  $d \geq 3$ . Assume  $\text{AccRel}_d$  and  $\text{Ax}\exists\text{UnifOb}$ . There are definable differentiable functions  $S : Q \rightarrow Q$  and  $C : Q \rightarrow Q$  with the following properties:*

- (1)  $C^2 - S^2 = 1$ ,
- (2)  $S(0) = 0$  and  $C(0) = 1$ ,
- (3)  $S(1) > 0$ ,
- (4)  $C(-t) = C(t)$  and  $S(-t) = S(t)$  for all  $t \in Q$ ,
- (5)  $(S')^2 - (C')^2 = 1$ ,
- (6)  $C' = S$  and  $S' = C$ ,
- (7)  $S$  is strictly increasing on  $Q$ ; and  $C$  are strictly increasing on interval  $[0, \infty)$  and strictly decreasing on  $(-\infty, 0]$ .
- (8)  $\text{Ran } S = Q$  and  $\text{Ran } C = [1, \infty)$ .

*Proof.* Let binary relation on observers  $H$  be defined as

$$\begin{aligned} H(m, k) \stackrel{\text{def}}{\iff} & \text{IOb}(m) \wedge \text{Ob}(k) \wedge \text{Dom } \text{lc}_{km}(t) = Q \wedge \text{lc}_{km}(0) = \langle 0, 1, 0 \dots 0 \rangle \\ & \wedge \text{lc}_{km}(1)_1 > 0 \wedge \forall \bar{x} [\bar{x} \in \text{Ran } \text{lc}_{km} \leftrightarrow x_2^2 - x_1^2 = 1 \wedge x_3 = \dots = x_d = 0]. \end{aligned} \quad (9)$$

Let  $\gamma$  be defined as the following relation:

$$\gamma(t) = \bar{x} \stackrel{\text{def}}{\iff} \forall mk \ H(m, k) \wedge \text{lc}_{km}(t) = \bar{x}. \quad (10)$$

By axiom  $\text{Ax}\exists\text{UnifOb}$ , there are such observers  $k$  and  $m$  that  $H(m, k)$  holds. Therefore relation  $\gamma$  is not empty. By Lemma 2,  $\gamma$  is a function and it equals to  $\text{lc}_{km}$  for any observers  $k$  and  $m$  for which relation  $H(m, k)$  holds.

$\text{Dom } \gamma = Q$  by  $\text{Dom } \text{lc}_{km}(t) = Q$ . By Theorem 3,  $\gamma$  is a well-parametrized timelike curve.

Let  $C = \gamma_2$  and  $S = \gamma_1$ . Then  $C : Q \rightarrow Q$  and  $S : Q \rightarrow Q$  are definable differentiable functions since they are coordinate functions of definable differentiable function  $\gamma(t) = \langle S(t), C(t), 0 \dots 0 \rangle$ .

Item (1) holds since  $\text{Ran } \gamma = \{\bar{x} : x_2^2 - x_1^2 = 1 \wedge x_3 = \dots = x_d = 0\}$ , Item (2) holds since  $\gamma(0) = \langle 0, 1, 0 \dots 0 \rangle$ , and Item (3) holds since  $\gamma_1(1) > 0$  by the definition of  $H(m, k)$ . Item (5) holds since  $\gamma' = \langle S', C', 0, \dots, 0 \rangle$  because  $\gamma$  is well-parametrized.

To prove Item (4), let us consider curve  $\delta : t \mapsto \langle -S(t), C(t), 0 \dots, 0 \rangle$ . It is clear that  $\text{Dom } \delta = \text{Dom } \gamma$ . By Item (1),  $\text{Ran } \delta = \text{Ran } \gamma$ . It is clear that  $\delta$  is also a well-parametrized curve by Item (5). Therefore, by Proposition 1,  $\delta(t) = \gamma(\varepsilon t + c)$  for all  $t \in Q$ . By Item (1),  $\delta(0) = \gamma(0)$ . Thus  $c = 0$ . By Chain Rule,  $\delta'(t) = \varepsilon \gamma'(\varepsilon t)$ . Since both  $\delta$  and  $\gamma$  are well-parametrized curves,  $\delta(0) = \gamma(0) = \langle 0, 1, 0, \dots, 0 \rangle$ , and

the tangent line of Hyperbola  $\{\bar{x} : x_1^2 - x_2^2 = 1, x_3 = \dots = x_d\}$  is vertical, we have that  $\gamma'_1(0) = 0$  and  $\gamma'_2(0) = 0$ . Thus  $\delta(0) = \langle -1, 0, \dots, 0 \rangle$ . Hence  $\varepsilon = -1$ . Thus  $\delta(t) = \gamma(-t)$ . Consequently,  $-S(t) = S(-t)$  and  $C(t) = C(-t)$ .

By Lemma 1,  $S$  and  $C$  are monotonous on interval  $[0, s]$  for all  $0 < s \in Q$ . Hence they are also monotonous on interval  $[0, \infty)$ . By Item (5),  $(\gamma'_1)^2 \geq 1 > 0$ . Therefore, by **CONT**-Darboux Theorem, see [22, §10.3],  $S'(t) > 0$  for all  $t \in Q$  or  $S'(t) < 0$  for all  $t \in Q$ .  $\gamma(1)_1 > \gamma(0)_1$  by Items (2) and (3). Therefore,  $\gamma_t$  is increasing. Thus  $\gamma'_t > 0$ . So  $S$  is strictly increasing on  $Q$ . Item (1),  $C$  is strictly increasing on  $[0, \infty)$  and strictly decreasing on  $(-\infty, 0]$  since  $S$  strictly increasing on  $Q$ .

Now let us prove Item (6). We have  $S'^2 - C'^2 = 1$  by Item (6). By Chain Rule, if we differentiate both sides of this equation, we get that  $2SS' - 2CC' = 0$ . Hence  $C'C = S'S$ . Multiplying  $S'^2 - C'^2 = 1$  by  $S^2$ , we get  $S'^2S^2 - C'^2S^2 = S^2$ . From this we get  $C'^2(C^2 - S^2) = S^2$  by  $C'C = S'S$ . Therefore,  $C'^2 = S^2$  since  $C^2 - S^2 = 1$ . Consequently,  $C'(t) = \pm S(t)$  and  $S'(t) = \pm C(t)$  for all  $t \in Q$ . By Items (1) and (7),  $S'(t) > 0$  and  $C(t) > 0$ . Therefore,  $S' = C$ . If  $t > 0$ , a similar argument show that  $C'(t) = S(t)$ . By (4),  $-C'(t) = C'(-t)$  and  $S(-t) = -S(t)$ . Therefore,  $C'(t) = S(t)$  also if  $t \in (-\infty, 0]$ . By Item (1) and (5),  $C'(0) = 0$  and  $S'(0) = 1$ . Hence  $C'(t) = S(t)$  for all  $t \in Q$ .

By **CONT**-Boltzано Theorem[22, §10.2],  $\text{Ran } S$  and  $\text{Ran } C$  are intervals. Therefore Item (8) holds by Item 1.  $\square$

**Theorem 5.** *Let  $d \geq 3$ . Assume **AccRel<sub>d</sub>** and **Ax $\exists$ UnifOb**. There is a definable differentiable function  $E : Q \rightarrow Q$  with the following properties:*

- (1)  $E(0) = 1$ ,
- (2)  $E(1) > 0$ ,
- (3)  $E(-t)E(t) = 1$ ,
- (4)  $E' = E$ ,
- (5)  $\text{Ran } E = (0, \infty)$ , and
- (6)  $E$  is strictly increasing.

Let the **restriction** of function  $f$  to set  $H$  be defined as

$$f|_H := \{\langle x, y \rangle : x \in \text{Dom } f \cap H \text{ and } y = f(x)\} \quad (11)$$

*Proof.* Let  $S : Q \rightarrow Q$  and  $C : Q \rightarrow Q$  be the definable differentiable functions which exist by Theorem 4. Let  $E := C + S$ . Then  $E$  is a definable differentiable function since  $C$  and  $S$  are so. Items (1) and (2) follow directly from Items (2) and (3) of Theorem 4. Item (3) follows from Items (1) and (4) of Theorem 4 because  $E(-t)E(t) = (C(-t) + S(-t))(C(t) + S(t)) = C^2(t) - S^2(t) = 1$ . Item (5) follows from Item (6) of Theorem 4. Item (4) follows because of the following.  $\text{Ran } E|_{[0, \infty)} = [1, \infty)$  because  $E(0) = 1$ ,  $S$  and  $C$  are strictly increasing on  $[0, \infty)$ , and  $\text{Ran } C = [1, \infty)$  by Item (8) of Theorem 4. Hence  $\text{Ran } E|_{(-\infty, 0]} = (0, 1]$  by Item (3). Thus  $\text{Ran } E = (0, \infty)$ . Item (6) follows from Item (6) of Theorem 4 since  $E$  is strictly increasing on  $[0, \infty)$  by Item (7) of Theorem 4 and  $E$  is also strictly increasing on  $(-\infty, 0]$  since  $E(-t)E(t) = 1$  by Item (3).  $\square$

The following first-order logic formula defines that **limit of function**  $f$  is  $A$  at  $x_0$  over every ordered field:

$$\text{Limit}(f, x_0, A) \stackrel{\text{def}}{\iff} \forall \varepsilon > 0 \exists \delta > 0 \forall x \\ 0 < |x - x_0| < \delta \wedge x \in \text{Dom } f \rightarrow |f(x) - A| < \varepsilon. \quad (12)$$

In the case when there is one and only one  $A$  such that  $\text{Limit}(f, x_0, A)$  holds, we write  $\lim_{x \rightarrow x_0} f(x) = A$ . By using the technique of [22, §10], it can be proved that there is at most one  $A$  such that  $\text{Limit}(f, x_0, A)$  holds if  $x_0$  is an accumulation point of  $\text{Dom } f$  (i.e., if for all  $\varepsilon > 0$ , there is  $x \in \text{Dom } f$  such that  $|x - x_0| < \varepsilon$ ).

Let the exponential function of  $\mathbb{R}$  be denoted by  $\exp$ .

**Proposition 2.** *Let  $\langle Q, +, \cdot, \leq \rangle$  be a subfield of  $\mathbb{R}$ .<sup>7</sup> Let  $f : Q \rightarrow Q$  be a differentiable function such that  $f' = f$  and  $f(0) = 1$ . Then  $f = \exp|_Q$ , i.e.,  $f$  is the restriction of the real exponential function to  $Q$ .*

*Proof.* We have that  $f$  is continuous since  $f$  is differentiable, see, e.g., [22, Cor.10.3.5]. Let  $f_*(x) := \lim_{t \rightarrow x} f(t)$ . Function  $f_*$  is well defined since  $f$  is continuous and  $Q$  is dense in  $\mathbb{R}$  (as  $\mathbb{Q} \subseteq Q$ ). Since  $f$  is continuous, we also have that  $f_*$  is an extension of  $f$ , i.e.,  $f_*(x) = f(x)$  if  $x \in Q$ . We are going to show that  $f_* = \exp$ . First we show that  $f'_*(x) = f_*(x)$ . We start by showing that, for all  $x, y \in \mathbb{R}$  and  $\varepsilon_0 > 0$ , there are  $x^*, y^* \in Q$  such that  $|x - x^*| < \varepsilon_0$ ,  $|y - y^*| < \varepsilon_0$ , and

$$\left| \frac{f(x^*) - f(y^*)}{x^* - y^*} - \frac{f_*(x) - f_*(y)}{x - y} \right| < \varepsilon_0. \quad (13)$$

By the triangle inequality,

$$\begin{aligned} & \left| \frac{f_*(x) - f_*(y)}{x - y} - \frac{f(x^*) - f(y^*)}{x^* - y^*} \right| \leq \\ & \left| \frac{f_*(x) - f_*(y)}{x - y} - \frac{f(x^*) - f(y^*)}{x - y} + \frac{f(x^*) - f(y^*)}{x - y} - \frac{x - y}{x^* - y^*} \cdot \frac{f(x^*) - f(y^*)}{x - y} \right| \\ & \leq \left| \frac{f_*(x) - f(x^*)}{x - y} \right| + \left| \frac{f_*(y) - f(y^*)}{x - y} \right| + \left| 1 - \frac{x - y}{x^* - y^*} \right| \cdot \left| \frac{f(x^*) - f(y^*)}{x - y} \right|. \end{aligned} \quad (14)$$

By the definition of  $f_*$ , there is a  $\delta$  such that

$$|f_*(x) - f(x^*)| < \frac{\varepsilon|x - y|}{3} \quad \text{and} \quad |f_*(y) - f(y^*)| < \frac{\varepsilon|x - y|}{3} \quad (15)$$

if  $|x - y| < \delta$ . From this, by the triangle inequality, we have that

$$\begin{aligned} |f(x^*) - f(y^*)| & \leq |f(x^*) - f_*(x)| + |f_*(x) - f_*(y)| + |f_*(y) - f(y^*)| \\ & < |f_*(x) - f_*(y)| + \frac{2\varepsilon_0|y - x|}{3}. \end{aligned} \quad (16)$$

Since  $\left| 1 - \frac{x - y}{x^* - y^*} \right|$  can be arbitrarily small if  $|x - x^*|$  and  $|y - y^*|$  are small enough, we can choose  $x^*$  and  $y^*$  such that

$$\left| 1 - \frac{x - y}{x^* - y^*} \right| \cdot \left| \frac{f(x^*) - f(y^*)}{x - y} \right| < \frac{\varepsilon_0}{3}. \quad (17)$$

Therefore, there are  $x^*$  and  $y^*$  arbitrarily close to  $x$  and  $y$  such that

$$\left| \frac{f_*(x) - f_*(y)}{x - y} - \frac{f(x^*) - f(y^*)}{x^* - y^*} \right| < \varepsilon_0. \quad (18)$$

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<sup>7</sup>By Pickert–Hion Theorem, these fields are exactly the fields of Archimedean ordered fields, see, e.g., [9, §VIII], [14, C.44.2].

To prove that  $f'_* = f_*$ , We have to show that, for all  $\varepsilon > 0$ , there is a  $\delta > 0$  such that

$$\left| f_*(x) - \frac{f_*(x) - f_*(y)}{x - y} \right| < \varepsilon \quad (19)$$

if  $|x - y| < \delta$ . By the triangle inequality,

$$\begin{aligned} \left| f_*(x) - \frac{f_*(x) - f_*(y)}{x - y} \right| &\leq |f_*(x) - f(x^*)| \\ &+ \left| f(x^*) - \frac{f(x^*) - f(y^*)}{x^* - y^*} \right| + \left| \frac{f(x^*) - f(y^*)}{x^* - y^*} - \frac{f_*(x) - f_*(y)}{x - y} \right|. \end{aligned} \quad (20)$$

By the definition of  $f_*$ ,

$$|f_*(x) - f(x^*)| < \frac{\varepsilon}{3} \quad (21)$$

if  $|x - x^*|$  small enough. By (18), we have that there, are  $x^*$  and  $y^*$  arbitrarily close to  $x$  and  $y$  such that,

$$\left| \frac{f(x^*) - f(y^*)}{x^* - y^*} - \frac{f_*(x) - f_*(y)}{x - y} \right| < \frac{\varepsilon}{3}. \quad (22)$$

Since  $f' = f$ , we have that

$$\left| f(x^*) - \frac{f(x^*) - f(y^*)}{x^* - y^*} \right| < \frac{\varepsilon}{3} \quad (23)$$

if  $|x^* - y^*|$  is small enough. So, if  $|x - y|$  is small enough and we can choose  $x^*$  and  $y^*$  close enough to  $x$  and  $y$  we have that Eq. (23) holds. Consequently, if  $|x - y|$  is small enough, then Eq. (19) holds, i.e.,  $f_*$  is differentiable and  $f'_* = f_*$ . Therefore, there is a  $c \in \mathbb{R}$  such that  $f_*(x) = c \exp(x)$  for all  $x \in \mathbb{R}$ . We have that  $c = 1$  since  $c = c \exp(0) = f_*(0) = f(0) = 1$ . Therefore,  $f$  is the restriction of function  $\exp$  to  $Q$ ; and this is what we wanted to prove.  $\square$

*Proof of Theorem2.* By Theorem 5, a differentiable function  $E$  is definable in the models of  $\text{AccRel}_d + \text{Ax}\exists\text{UnifOb}$  such that  $E' = E$  and  $E(0) = 1$ . By Proposition 2,  $E$  has to be the restriction of  $\exp$  to the real algebraic numbers. However, this is impossible since then  $E(1)$  is the Euler-number  $e$  which is not an algebraic number.  $\square$

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