arXiv:1406.5312v1 [math.OC] 20 Jun 2014

Asymptotic Exponential Arbitrage and Utility-based Asymptotic Arbitrage in Markovian Models of Financial Markets*

Martin Le Doux Mbele Bidima† Miklós Rásonyi ‡

June 23, 2014

Abstract

Consider a discrete-time infinite horizon financial market model in which the logarithm of the stock price is a time discretization of a stochastic differential equation. Under conditions different from those given in [10], we prove the existence of investment opportunities producing an exponentially growing profit with probability tending to 1 geometrically fast. This is achieved using ergodic results on Markov chains and tools of large deviations theory.

Furthermore, we discuss asymptotic arbitrage in the expected utility sense and its relationship to the first part of the paper.

Keywords: Asymptotic exponential arbitrage, Markov chains, large deviations, expected utility.

1 Introduction

In the classical theory of financial markets, absence of arbitrage (riskless profit) is characterized by the existence of suitable "pricing rules": risk-neutral (i.e. equivalent martingale) measures for the discounted price process of the risky asset. This result is often referred to as "the fundamental theorem of asset pricing".

Further developments of arbitrage theory encompass the so-called "large financial markets" (see [7], [6] and the references therein). In these papers the following common feature of numerous models is highlighted: on each finite time horizon $T > 0$, there is no arbitrage opportunity but when T tends to infinity, one may realize riskless profit in the long run. Such trading opportunities are referred to as "asymptotic arbitrage".

An important tool that can be used for the study of asymptotic arbitrage is the theory of large deviations (see [2]), as proposed in [6]. More recently, in [10] we presented the discrete-time versions of some results in [6] about asymptotic arbitrage and, in this framework, we extended

*The authors thank the referee and the associate editor for extremely constructive and helpful reports.

†University of Yaoundé I, Cameroon, e-mail: mbelebidima@gmail.com

‡MTA Alfréd Rényi Institute of Mathematics, Budapest, Hungary and University of Edinburgh, U.K., e-mail: rasonyi@renyi.mta.hu

them by studying "asymptotic exponential arbitrage with geometrically decaying probability of failure", i.e. we discussed the possibility for investors to realize an exponentially growing profit on their long-term investments while controlling (at a geometrically decaying rate) the probability of failing to achieve such a profit. Some of these results were subsequently proved for continuous-time models in [4]. In the present paper we prove results similar to Theorem 5 of [10] using different arguments and technical tools (the large deviation results of [9] instead of those in [8]). In this way we manage to cover some well-known models for asset prices which were untractable in the setting of [10], see Examples 3.14, 3.15 below. We recall now the setting of [10].

Consider a financial market in which two assets are traded: a riskless asset (a bank account or a risk-free bond) with interest rate set to 0, i.e. with price normalized to $B_t := 1$ at all times $t \in \mathbb{N}$; and a single risky asset (such as stock) whose (discounted) price is assumed to evolve as

$$S_t := \exp(X_t), \quad t \in \mathbb{N}, \tag{1}$$

where the logarithm of the stock price, X_t, is an $\mathbb{R}$-valued stochastic process governed by the discrete time difference equation

$$X_t - X_{t-1} = \mu(X_{t-1}) + \sigma(X_{t-1})\varepsilon_t, \quad t \geq 1, \tag{2}$$

starting from a constant $X_0 \in \mathbb{R}$. Here μ, $\sigma : \mathbb{R} \to \mathbb{R}$ are measurable functions (determining the drift and volatility of the stock) and $(\varepsilon_t)_{t\in\mathbb{N}}$ is an $\mathbb{R}$-valued sequence of i.i.d. random variables representing the random driving process of the stock price evolution.

Note that the log-price process X_t is clearly a (discrete-time) Markov chain in the (uncountable) state space $\mathbb{R}$ (see pp. 211–228 in [1]). We suppose that its evolution is modelled on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{N}}$ and $\mathcal{F}_t := \sigma(X_s, 0 \leq s \leq t)$, is the natural filtration of the log-price process X_t of the stock. In the sequel $\mathbb{E}$ denotes expectation with respect to the probability $\mathbb{P}$.

Trading strategies in this market are assumed $\mathbb{F}$-predictable $[0,1]$-valued processes $(\pi_t)_{t\geq 1}$ (i.e. π_t is assumed $\mathcal{F}_{t-1}$-measurable) and no short-selling or borrowing are allowed. This means that, at each time t, investors allocate a proportion $\pi_t \in [0,1]$ of their overall wealth to the stock while the rest remains in the bank account. Hence, given any such strategy, the corresponding wealth process V_t^π of an investor obeys the dynamics

$$\frac{V_t^\pi}{V_{t-1}^\pi} = (1-\pi_t) + \pi_t \frac{S_t}{S_{t-1}}, \quad \text{for all } t \geq 1, \tag{3}$$

where $V_0^\pi := V_0 > 0$ is the investor's initial capital.

Definition 1.1. (Definitions 3 and 4 of [10]) *Let π_t be a trading strategy.*

i) We say that π_t is an asymptotic exponential arbitrage (AEA) in the wealth model (3) *if there is a constant $b > 0$ such that, for all $\epsilon > 0$, there is a time $t_\epsilon \in \mathbb{N}$ satisfying*

$$\mathbb{P}(V_t^\pi \geq e^{bt}) \geq 1 - \epsilon, \ \textit{for all time } t \geq t_\epsilon. \tag{4}$$

ii) We say that π_t generates an asymptotic exponential arbitrage (AEA) with geometrically decaying probability of failure ($GDPF$) if there are constants $b > 0$, and $c > 0$ such that,

$$\mathbb{P}(V_t^\pi \geq e^{bt}) \geq 1 - e^{-ct} \text{ for all large enough } t \geq 1. \tag{5}$$

Clearly, AEA with $GDPF$ implies AEA. The second kind of asymptotic arbitrage above is much more stringent than the first one. Indeed, in (4) above, there is no visible relationship between the tolerance level ϵ and the elapsed time t_ϵ from which the investor starts realizing exponentially growing profit; one may need to wait for a very long time to achieve a desired tolerance level. The concept of AEA with $GDPF$ removes this drawback by allowing investors to control, at a geometrically decaying rate, the probability of failing to achieve an exponentially growing profit in the long term.

We recall the main results of [10] here. Under the following main assumptions: boundedness of the drift and volatility functions μ and σ; σ being bounded away from 0 on compacts and exponential integrability of the ε_ts, we proved Theorem 2 (resp. Theorem 4) of [10] on the existence of AEA (resp. AEA with $GDPF$) in the wealth model (3). Theorem 5 of [10] provided ergodicity-related conditions on X_t which ensured AEA with GDPF.

In sections 2 and 3 below we continue to consider the same models as in (1), (2), (3). Under a new set of conditions on μ, σ and $(\varepsilon_t)_{t\in\mathbb{N}}$ (see $(A_1), (A_2), (A_3), (A_4)$ below), which are neither stronger nor weaker than the corresponding conditions in [10] recalled above, we show again the existence of AEA with $GDPF$ (see Theorem 2.3 below), using classical large deviations techniques from [2], Markov chains tools from [11] and ergodicity results on Markov chains from [9]. Moreover, the trading strategies generating those arbitrage opportunities will be explicitly constructed; a contribution we already obtained in [10] under different conditions, but it was absent from the inspiring continuous-time work [6]. To get those explicit arbitrage opportunities we will be considering only stationary Markovian strategies, that is; strategies $(\pi_t)_{t\geq 1}$ where $\pi_t = \pi(X_{t-1})$, $t \geq 1$, for some fixed measurable function $\pi : \mathbb{R} \to [0, 1]$.

In section 4, we will discuss the concept of "utility-based" asymptotic arbitrage, that is, asymptotic arbitrage linked to von Neumann-Morgenstern expected utilities (see Chapter 2 of [5]). An optimal investment for an economic agent with utility function U and time horizon T is π_t with final portfolio value V_T^π for which the expected utility $\mathbb{E}U(V_T^\pi)$ is maximal. We do not focus on the construction of optimal strategies but rather on ones that provide (rapidly) increasing expected utilities for the agent as the time horizon tends to infinity. More precisely, we wish to treat questions like: for power utilities U, and given an AEA strategy π_t as in (4), will the investor's expected utility $\mathbb{E}U(V_t^\pi)$ tend to the highest available utility $U(\infty)$? If so, how fast such a convergence will take place? Conversely, if an agent pursues a trading strategy π_t such that her/his expected utility has a convergence rate estimate, will π_t generate AEA (with $GDPF$)? We provide partial answers to these questions in Proposition 4.1, Theorem 4.2 and Theorem 4.4 below.

2 Main theorem on AEA with GDPF

We denote by λ the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. We assume throughout this paper that the Markov chain X_t satisfies the following conditions:

(A_1) The random variables ε_ts have a (common) density γ with respect to λ, and this density is bounded and bounded away from 0 on each compact in $\mathbb{R}$.

(A_2) The drift μ is locally bounded. The volatility σ is positive, bounded away from zero on each compact and it is (globally) bounded.

(A_3) We impose the mean-reverting drift condition

$$\limsup_{|x|\to\infty} \frac{|x+\mu(x)|}{|x|} < 1.$$

(A_4) We assume the following integrability property for the law of the ε_ts:

$$\exists\, \kappa > 0 \text{ such that } \mathbb{E}\big(e^{\kappa\varepsilon_1^2}\big) =: I < \infty, \tag{6}$$

and $\mathbb{E}\varepsilon_1 = 0$ holds[1].

Remark 2.1. These conditions are similar to those of [10]. The main difference is that μ was assumed to be a bounded function in [10] while it may be unbounded in the present paper. In this way we accomodate e.g. autoregressive processes (see Examples 3.14 and 3.15 below), which did not fit the setting of [10]. While we relax boundedness of μ, we need the integrability condition (A_4) on ε_t, which is more stringent than the ones in [10]. Furthermore, (A_3) is a much stronger ergodicity condition on the Markov chain X_t than that of Theorem 5 in [10]. Hence our main result (Theorem 2.3) does not generalize [10] but rather complements it.

Remark 2.2. Analogously to [6], where the exponential of an Ornstein-Uhlenbeck process was considered, our conditions imply that the log-price X_t is ergodic (in a strong sense). It may be argued on ecomonetric grounds that the price increments X_t/X_{t-1} rather than X_t should be assumed ergodic. Just like in [10], we opted for the present setting in order to be consistent with [6]. Very similar arguments could be used to prove analogous results for the case where X_t/X_{t-1} is assumed to be an ergodic Markov chain. We do not pursue this route here.

Consider the following condition:

$$(RC_+) \quad \text{The set } R^+ := \{x \in \mathbb{R} \mid \mu(x) > 0\} \ \text{ satisfies } \lambda(R^+) > 0. \tag{7}$$

We interpret R^+ as representing all states of the stock log-prices X_t whose "drift" is positive. Thus (RC_+) means that the set of states x from which there is a "bright future" (i.e. there is an upward trend for the stock price) has positive Lebesgue measure. This is rather natural: note that short-selling is prohibited in our model hence negative market trends cannot be taken advantage of. We now state the main result of the present article.

[1] This is not a restriction of generality. If we had $\mathbb{E}\varepsilon_1 = m$, we could replace $\mu(x)$ by $\mu'(x) := \mu(x) + \sigma(x)m$ and ε_t by $\varepsilon_t' := \varepsilon_t - m$ and in this way get back to the case $\mathbb{E}\varepsilon_1 = 0$.

Theorem 2.3. *Assume that $(A_1) - (A_4)$ and (RC_+) hold. Then the Markovian strategy $\pi_t^+ := \mathbf{1}_{R^+}(X_{t-1})$ produces an AEA with GDPF.*

The proof will be presented at the end of the next section, after appropriate preparations.

3 Large deviation estimates

Consider the $\mathbb{R}^2$-valued auxiliary process $\Phi_t := (X_{t-1}, X_t)$, $t \geq 0$, consisting of two consecutive values of the log-price process X_t, where X_{-1} is an arbitrarily chosen constant. We present below a set of preliminary results.

Proposition 3.1. *The process Φ_t is a Markov chain with state space $\mathbb{R}^2$.*

Proof. We derive this from [1] pp. 211-228, where the Markov property of any (discrete-time) process Y_t in a Polish state space S is proved when $Y_{t+1} = m(Y_t, \xi_{t+1})$ with $(\xi_t)_t$ a sequence of i.i.d. random variables independent of Y_0 and valued in some measurable space S' and $m : S \times S' \to S$ a measurable function. Clearly, $X_{t+1} = m(X_t, \varepsilon_{t+1})$ for $t \in \mathbb{N}$ with $m(x, y) := x + \mu(x) + \sigma(x)y$, $x, y \in \mathbb{R}$. It follows that we have $\Phi_{t+1} = (X_t, X_{t+1}) = (X_t, m(X_t, \varepsilon_{t+1})) = F(\Phi_t, \xi_{t+1})$, where $\xi_t := (0, \varepsilon_t)$ and F is the measurable function defined on $S \times S' := \mathbb{R}^2 \times \mathbb{R}^2$ by $F((x, y), (a, b)) := (y, m(y, b))$. Since the ε_ts are i.i.d. and independent of X_0, the ξ_ts are also i.i.d. and independent from Φ_0, showing the result. ■

Notice that

$$P(x, A) := P(X_1 \in A | X_0 = x) = \int_A p(x, y) dy, \ x \in \mathbb{R}, \ A \in \mathcal{B}(\mathbb{R}),$$

where

$$p(x, y) := \frac{1}{\sigma(x)} \gamma \left(\frac{y - \mu(x) - x}{\sigma(x)} \right),$$

and this function is bounded away from 0 on each compact in $\mathbb{R}^2$, by (A_1) and (A_2).

For $z \in \mathbb{R}^2$ and $A \in \mathcal{B}(\mathbb{R}^2)$, let $Q^t(z, A) := P(\Phi_t \in A | \Phi_0 = z)$ be the t-step transition kernel of the chain Φ_t.

We note that, for $t \geq 2$ and $A \in \mathcal{B}(\mathbb{R}^2)$,

$$Q^t((u, v), A) = \int_{\mathbb{R}^t} 1_A(a_{t-1}, a_t) p(v, a_1) p(a_1, a_2) \dots p(a_{t-1}, a_t) da_1 \dots da_t, \ u, v \in \mathbb{R}. \tag{8}$$

Let λ_2 denote the Lebesgue measure on $\mathcal{B}(\mathbb{R}^2)$.

Proposition 3.2. *The Markov chain Φ_t is ψ-irreducible, i.e. there is a non-trivial measure ψ such that $\psi(A) > 0$ implies that for all z, $Q^t(z, A) > 0$ for some t.*

Proof. It suffices to check that λ_2 is such a measure. If $\lambda_2(A) > 0$ then for $t = 2$ we get from (8) that

$$Q^2((u, v), A) = \int_{\mathbb{R}^2} 1_A(a_1, a_2) p(v, a_1) p(a_1, a_2) da_1 \, da_2 > 0$$

since $p(v,a_1)p(a_1,a_2)$ is (everywhere) positive. ■

We recall two definitions from Chapter 5 of [11] in our specific setting. A set $C_2 \subset \mathbb{R}^2$ is called *small* if

$$Q^t(x,A) \geq \mu(A) \text{ for all } x \in C_2,\ A \in \mathcal{B}(\mathbb{R}^2)$$

with some non-trivial measure μ. The chain Φ_t is *aperiodic* if, for some small set C_2 and corresponding measure μ, the greatest common divisor of the set

$$E_{C_2} := \{n \geq 1 : \text{for all } x \in C_2,\ Q^n(x,A) \geq \delta_n \mu(A) \text{ for some } \delta_n > 0\},$$

is 1.

Proposition 3.3. *If C is a compact interval in $\mathbb{R}$ then $C_2 := \mathbb{R} \times C$ is a small set for the Markov chain Φ_t, and this chain is aperiodic.*

Proof. It suffices to show that for all $u \in \mathbb{R}$ and $v \in C$,

$$Q^i((u,v),A) \geq c_i \lambda_2(A \cap (C \times C))$$

for $i = 2,3$ and appropriate constants $c_2, c_3 > 0$ since this implies $2,3 \in E_{C_2}$. This is true by (8) with

$$c_2 = \inf_{v,a_1 \in C} p(v,a_1) \inf_{a_1,a_2 \in C} p(a_1,a_2), \quad c_3 = \inf_{a_2,a_3 \in C} p(a_2,a_3) \inf_{v,a_1 \in C} p(v,a_1) \inf_{a_1,a_2 \in C} p(a_1,a_2)\lambda(C).$$

■

Now we need certain moment estimates.

Lemma 3.4. *The random variable ε_1 in (6) of Assumption (A_4) satisfies the following property: there is $c > 0$ such that for every real number $a \geq 1$ we have*

$$\mathbb{E}\big(e^{a|\varepsilon|}\big) \leq e^{ca^2}. \tag{9}$$

Proof. Set $\xi := |\varepsilon_1|$. Then we have

$$\begin{aligned}
\mathbb{P}\big(e^{a\xi} > x\big) &= \mathbb{P}\Big(\exp\Big(\kappa\Big[\tfrac{\log(e^{a\xi})}{a}\Big]^2\Big) > \exp\Big(\kappa\Big[\tfrac{\log x}{a}\Big]^2\Big)\Big) \\
&\leq I \exp\Big(-\kappa\big(\log(x)/a\big)^2\Big) \text{ by Markov's inequality} \\
&= I\big(\tfrac{1}{x}\big)^{(\kappa/a^2)\log x},
\end{aligned}$$

see (6) for the definition of I. Since the exponent $(\kappa/a^2)\log x > 2$ provided that $x > e^{2a^2/\kappa}$, we have $\mathbb{E}\big(e^{a\xi}\big) = \int_0^\infty \mathbb{P}\big(e^{a\xi} > x\big)dx \leq e^{2a^2/\kappa} + I\int_{\exp(2a^2/\kappa)}^\infty 1/x^2 dx$. The last integral is less than $\int_1^\infty 1/x^2 dx$, which is finite, thus we conclude the proof of (9) by taking $c = c_1 + (2/\kappa)$ with $c_1 > 0$ large enough. ■

The proof of Theorem 2.3 will be based on results from [9]. In order to apply the results of that paper we will need to verify that the Markov chain Φ_t satisfies condition $(DV3+)$ below. We formulate this condition only in the case where the state space is $\mathbb{R}^d$.

We say that a ψ-irreducible and aperiodic Markov chain Z_t with transition law $R = R(x,A)$ satisfies condition $(DV3+)$ if

(i) There are measurable functions $V, W : \mathbb{R}^d \to [1, \infty)$ and a small set C such that for all $x \in \mathbb{R}^d$,

$$\log(e^{-V} R e^{V})(x) \le -\delta W(x) + b \mathbf{1}_C(x) \tag{10}$$

for some $\delta, b > 0$.

(ii) There exists $t_0 > 0$ such that, for each $r < \|W\|_\infty$, there is a measure β_r with $\beta_r(e^V) < \infty$ and

$$\mathbb{P}_x(Z_{t_0} \in A \text{ and } Z_t \text{ has not quitted } C_W(r) \text{ before } t_0 + 1) \le \beta_r(A) \tag{11}$$

for all $x \in C_W(r)$ and $A \in \mathcal{B}(\mathbb{R}^d)$, where $C_W(r) = \{y \in \mathbb{R}^d : \ W(y) \le r\}$.

We now recall the results of [9] which we will need in the sequel. Let $W_0 : \mathbb{R}^d \to [1, \infty)$ such that

$$\lim_{r\to\infty} \sup_{x\in\mathbb{R}^d} \Big(\frac{W_0(x)}{W(x)} \mathbf{1}_{\{W(x)>r\}} \Big) = 0. \tag{12}$$

Next, consider the Banach space $L_\infty^{W_0} := \{g : \mathbb{R}^d \to \mathbb{C} : \sup_x \frac{|g(x)|}{W_0(x)} < \infty\}$, equipped with the norm $\|g\|_{W_0} := \sup_x |g(x)|/W_0(x)$.

Theorem 3.5. *Let Z_t satisfy (DV3+) with unbounded W. Then Z_t admits an invariant probability measure ν, the limit*

$$\Lambda(g) := \lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}_z[\exp(\sum_{n=1}^{t} g(Z_n))]$$

exists and it is finite for all $g \in L_\infty^{W_0}$ and for all initial values $Z_0 = z$ (and it is independent of z). Fix $g_0 \in L_\infty^{W_0}$. The function $\theta \to \Lambda(g_0 + \theta g)$ is analytic in θ with Taylor-expansion

$$\Lambda(g_0 + \theta g) = \Lambda(g_0) + \theta \nu(g) + \frac{1}{2}\theta^2 \sigma^2(g) + O(\theta^3),$$

where $\sigma^2(g) := \lim_{t\to\infty}(1/t)\mathrm{var}(g(Z_0) + \ldots + g(Z_{t-1}))$.

Proof. This follows from Theorems 1.2 and 4.3 of [9]. □

Let us define $\Lambda_g(\theta) := \Lambda(\theta g)$ for $\theta \in \mathbb{R}$. Denote

$$\Lambda_g^*(x) := \sup_{\theta\in\mathbb{R}} (\theta x - \Lambda_g(\theta)), \quad x \in \mathbb{R},$$

the Fenchel-Legendre conjugate of $\Lambda_g(\cdot)$.

Corollary 3.6. *Under the conditions of the previous Theorem, if $\sigma^2(g) > 0$ then $\Lambda_g^*(x) > 0$ for all $x \neq \nu(g)$.*

Proof. Λ_g is analytic, a fortiori, it is differentiable. $\Lambda_g(0) = 0$ by the definition of Λ. From the Taylor expansion of the preceding Theorem, $\Lambda_g'(0) = \nu(g)$ and $\Lambda_g''(0) = \sigma^2(g) > 0$ so we get that $\Lambda_g^*(\nu(g)) = \nu(g) \times 0 - \Lambda_g(0) = 0$. By the definition of a conjugate function we always have $\Lambda_g^*(x) \ge 0 \times x - \Lambda_g(0) = 0$ for all $x \in \mathbb{R}$. It follows that $\nu(g)$ is a global minimiser for Λ_g^*. By the differentiability of Λ_g, Λ_g^* is strictly convex on its effective domain. This implies that the global minimiser $\nu(g)$ for Λ_g^* is unique. This uniqueness implies that $\Lambda_g^*(x) > 0$ for all $x \neq \nu(g)$. ■

In order to apply these results to our long-term investment problems we need to establish that Φ_t satisfies $(DV3+)$. First we prove a related statement about X_t.

Proposition 3.7. *The Markov chain X_t satisfies the "drift condition" (10) for $d = 1$, $R(x, A) = P(x, A)$ with a suitable compact interval $C \subset \mathbb{R}$ and $V(x) = W(x) = 1 + qx^2$ with a suitable $q > 0$.*

Proof. Recall that $R\, e^V(x) := \int e^{V(y)} R(x, dy)$, for all $x \in \mathbb{R}$. We have to show

$$P\, e^V(x) \leq e^{V(x) - \delta W(x) + b\mathbf{1}_C(x)} \text{ for all } x \in \mathbb{R} \tag{13}$$

for suitably small $q > 0$ and $C = [-K, K]$ with K suitably large.

Since $Pe^V(x) = \mathbb{E}\big(e^{V(X_1)} \mid X_0 = x\big) = \mathbb{E}\big(e^{V(x+\mu(x)+\sigma(x)\varepsilon_1)}\big)$, it follows from (13) that we need to show,

$$\mathbb{E}\big(e^{1+q(x+\mu(x))^2 + 2q(x+\mu(x))\sigma(x)\varepsilon_1 + q\sigma^2(x)\varepsilon_1^2}\big) \leq e^{(1-\delta)V(x) + b\mathbf{1}_C(x)} \text{ for all } x \in \mathbb{R}. \tag{14}$$

To get this, it is sufficient to prove the two claims below:

Claim 1: For all x with $|x| > K$ with K large enough we have

$$\mathbb{E}\big(e^{1+q(x+\mu(x))^2 + 2q(x+\mu(x))\sigma(x)\varepsilon_1 + q\sigma^2(x)\varepsilon_1^2}\big) \leq e^{(1-\delta)(1+qx^2)} \tag{15}$$

Claim 2: For "small" x (i.e. $|x| \leq K$ we have

$$\sup_{x \in C} \mathbb{E}\big(e^{1+q(x+\mu(x))^2 + 2q(x+\mu(x))\sigma(x)\varepsilon_1 + q\sigma^2(x)\varepsilon_1^2}\big) < G(K), \tag{16}$$

for some positive constant $G(K)$.

Proof of Claim 1. Using Assumption (A_3), for $|x|$ large enough, there is a small $\delta > 0$ such that $(x + \mu(x))^2 \leq (1 - 4\delta)x^2$. Since $1 \leq \delta(1 + qx^2)$ for $|x|$ large enough, it follows that $e^{1+q(x+\mu(x))^2} \leq e^{(1-3\delta)(1+qx^2)}$.

By (A_2) there is $M > 0$ such that, for all x, $\sigma(x) \leq M$. If we choose q such that $qM^2 < \kappa/2$ then it is enough to show that $\mathbb{E}\big(e^{2q|x+\mu(x)|M|\varepsilon_1| + (\kappa/2)\varepsilon_1^2}\big) \leq e^{2\delta qx^2}$. By the Cauchy-Schwarz inequality, it suffices to prove

$$\sqrt{\mathbb{E}\big(e^{4q|x+\mu(x)|M|\varepsilon_1|}\big)}\sqrt{\mathbb{E}\big(e^{\kappa\varepsilon_1^2}\big)} \leq e^{2\delta qx^2} \tag{17}$$

By (6), the second term on the left-hand side of (17) is the constant $\sqrt{I}$. This is smaller than $e^{\delta qx^2}$ for large enough $|x|$. So, since again by (A_3), $4q|x + \mu(x)|M \leq 4qM|x|$ for $|x|$ large enough, it remains to show $\sqrt{\mathbb{E}\big(e^{4qM|x||\varepsilon_1|}\big)} \leq e^{\delta qx^2}$ for large $|x|$, or, equivalently,

$$\mathbb{E}\big(e^{4qM|x||\varepsilon_1|}\big) \leq e^{2\delta qx^2} \text{ for large } |x|. \tag{18}$$

Applying Lemma 3.4, the left-hand side of (18) is smaller than $e^{16cq^2M^2|x|^2}$ for some fixed constant $c > 0$. Hence, if one chooses q small enough such that $16q^2M^2c < 2\delta q$ and $qM^2 < \kappa/2$ then (18) holds, showing Claim 1.

Proof of Claim 2. By Assumption (A_2), μ is bounded above on any compact $C = [-K, K]$ by some positive constant $A = A(K)$ and the function $x \mapsto (x + \mu(x))^2$ is also bounded on C by some positive constant $B = B(K)$. Applying Cauchy-Schwarz Inequality and (6),

$$\begin{aligned}
\mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon_1+q\sigma^2(x)\varepsilon_1^2}\big) &\leq \mathbb{E}\big(e^{1+qB+2q(K+A)M|\varepsilon_1|+(\kappa/2)\varepsilon_1^2}\big) \\
&\leq e^{(1+qB)}\sqrt{\mathbb{E}\big(e^{4q(K+A)M|\varepsilon_1|}\big)}\sqrt{\mathbb{E}\big(e^{\kappa\varepsilon_1^2}\big)} \\
&= e^{(1+qB)}\sqrt{I}\sqrt{\mathbb{E}\big(e^{4q(K+A)M|\varepsilon_1|}\big)}
\end{aligned}$$

We then choose K large enough such that $4q(K + A)M \geq 1$ and we get, by Lemma 3.4, that for all $x \in C = [-K, K]$,

$$\mathbb{E}\big(e^{1+q(x+\mu(x))^2+2q(x+\mu(x))\sigma(x)\varepsilon+q\sigma^2(x)\varepsilon^2}\big) \leq e^{(1+qB)}\sqrt{I}\sqrt{e^{16c'q^2(K+A)^2M^2}},$$

for a fixed constant $c' > 0$. This holds for all $x \in C$, hence (16) holds true when taking the supremum over C of the left-hand side of this latter inequality. ■

Proposition 3.8. *The Markov chain Φ_t satisfies $(DV3+)\,(i)$.*

Proof. We follow Proposition 4.1 of [9] and deduce this statement from Proposition 3.7 above. Recall $V(x) = W(x) = 1 + qx^2$, C and $\delta > 0$ from that Proposition. Take $C_2 := \mathbb{R} \times C$. For $x, y \in \mathbb{R}$ define $V_2(x, y) := V(y) + (\delta/2)W(x)$ and $W_2(x, y) := (1/2)(W(x) + W(y))$. Then

$$\begin{aligned}
\log e^{-V_2}Qe^{V_2}(x, y) &= -V(y) - (\delta/2)W(x) + \log\int_{\mathbb{R}} e^{\frac{\delta}{2}W(y)+V(z)}P(y, dz) \\
&\leq -V(y) - (\delta/2)W(x) + (\delta/2)W(y) + [V(y) - \delta W(y) + b1_C(y)] \\
&\leq -\delta W_2(x, y) + b1_{C_2}(x, y),
\end{aligned}$$

showing that (10) is true with V_2, W_2. As C_2 has been shown to be small in Proposition 3.3, we conclude. ■

Proposition 3.9. *The chain Φ_t satisfies condition $(DV3+)\,(ii)$ as well.*

Proof. Consider $V_2(x, y), W_2(x, y)$, defined in the previous Proposition. We choose $t_0 := 2$, and let $r < \|W\|_\infty = \infty$.

It suffices to prove existence of a measure β_r on $\mathcal{B}(\mathbb{R}^2)$ such that,

$$\beta_r(e^{V_2}) < \infty \text{ and } Q^2\big((x, y), D \cap C_W(r)\big) \leq \beta_r(D), \tag{19}$$

for all $(x, y) \in C_W(r)$ and all $D \in \mathcal{B}(\mathbb{R}^2)$.

Let H denote the projection of $C_W(r)$ on the first coordinate (which is the same as its projection on the second coordinate). By (A_1) and (A_2), the function $p(x, y)$ is bounded on $H \times H$ by a constant J. Hence

$$Q^2\big((x, y), D \cap C_W(r)\big) \leq \int_{D\cap C_W(r)} p(y, a_1)p(a_1, a_2)da_1da_2 \leq J^2\lambda_2(D \cap C_W(r)) =: \beta_r(D).$$

Finally, it is clear that $\beta_r(e^{V_2}) < \infty$ as it is the Lebesgue-integral of a continuous function on a compact of $\mathbb{R}^2$. ■

Corollary 3.10. *The Markov chain Φ_t has an invariant probability measure ν equivalent to λ_2.*

Proof. Theorem 3.5 implies that Φ_t has an invariant probability measure, say, ν.

Furthermore, from (8), $\mathbb{P}(\Phi_2 \in \cdot | \Phi_0 = (x, y))$ is λ_2-absolutely continuous for each $(x, y) \in \mathbb{R}^2$, hence we get $\nu \ll \lambda_2$. On the other hand, from the definitions of recurrent and positive chains on pages 186 and 235 of [11], it follows by Proposition 10.1.1 and Theorem 10.4.9 of the same reference that $\nu \sim \psi$, where ψ is a maximal irreducibility measure. Hence $\psi \gg \lambda_2$ by Proposition 4.2.2 (*ii*) in [11], so we get $\nu \gg \lambda_2$. It follows that $\nu \sim \lambda_2$, as required. ■

We now proceed to a proper investigation of asymptotic arbitrage exponential opportunities in the wealth model (3). Inspecting again the dynamics of the investor's wealth process V_t^π in this model, for any Markovian strategy π_t, we may express it in the form

$$V_t^\pi = V_0 \exp\big(\sum_{n=1}^{t} f(\Phi_n)\big) = V_0 \exp\Big(t \frac{\sum_{n=1}^t f(\Phi_n)}{t}\Big), \text{ for all } t \geq 1, \tag{20}$$

where the function f is defined by

$$f(x, y) := \log\big((1 - \pi(x)) + \pi(x) \exp(y - x)\big), \quad x, y \in \mathbb{R}, \tag{21}$$

and $\Phi_t = (X_{t-1}, X_t)$, $t \in \mathbb{N}$, is the Markov chain in consideration. We will need to insure that, for any Markovian strategy π_t, the sequence of random variables $\log(V_t^\pi / V_0) = \sum_{n=1}^t f(\Phi_n)$ satisfies a large deviation principle (*LDP*) hypotheses. That is, we will need that the limit $\Lambda_f(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(e^{\theta \sum_{n=1}^t f(\Phi_n)})$ exists, for each $\theta \in \mathbb{R}$, with Λ_f satisfying the remaining conditions in Gärtner-Ellis Theorem as stated in Theorem 2.3.6 in [2].

Define the function $W_0 : \mathbb{R}^2 \to [1, \infty)$ by $W_0(x, y) := 1 + |x| + |y|$, for all $x, y \in \mathbb{R}$. Clearly, W_0 satisfies (12) with $d = 2$ and $W = W_2$.

Lemma 3.11. *The function f belongs to the space $L_\infty^{W_0}$.*

Proof. For all $x, y \in \mathbb{R}$, since $\pi(x) \in [0, 1]$, we have $1 - \pi(x) + \pi(x) \exp(y - x) \leq 1 + \exp(y - x)$. It follows that $f(x, y) \leq |x| + |y| + 1$.

On the other hand, for $0 \leq a \leq 1/2$, we have $1 - a + a \exp(y - x) \geq 1/2$. And for $a > 1/2$, we have $1 - a + a \exp(y - x) \geq (1/2) \exp(y - x)$. To sum up, we obtain that $f(x, y) \geq \log(1/2) - |x| - |y|$ for all $x, y \in \mathbb{R}$.

Hence $|f(x, y)| \leq c(1 + |x| + |y|)$, for some constant $c > 0$, and the claim follows. ■

Proposition 3.12. *Let π_t be any Markovian strategy in the wealth model* (3)*. Then $\Lambda_f(\theta) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_{(X_{-1}, X_0)}\big(e^{\theta \sum_{n=1}^t f(\Phi_n)}\big)$, $\theta \in \mathbb{R}$ is a well-defined analytic function so the averages $\frac{1}{t} \log(V_t^\pi / V_0) = \frac{1}{t} \sum_{n=1}^t f(\Phi_n)$ satisfy a large deviations estimate with good rate function Λ_f^* (the convex conjugate of Λ_f).*

Proof. By Theorem 3.5, Λ_f verifies the conditions of the Gärtner-Ellis Theorem 2.3.6 in [2] (analyticity implies essential smoothness). Applying this theorem we conclude. ■

Lemma 3.13. *If* (RC_+) *is satisfied then the Markovian strategy* $\pi^+(x) := \mathbf{1}_{R^+}(x)$ *is such that*

$$\nu(f) = \mathbb{E}\Big(\log\big((1-\pi^+(\tilde{X}_0)) + \pi^+(\tilde{X}_0)\exp(\tilde{X}_1 - \tilde{X}_0)\big)\Big) > 0, \tag{22}$$

where the pair of random variables $(\tilde{X}_0, \tilde{X}_1)$ *has distribution* ν.

Proof. Since ν is a probability measure on $\mathcal{B}(\mathbb{R}^2)$ and is invariant for the chain $\Phi_t = (X_t, X_{t+1})$, there is a pair of $\mathbb{R}$-valued random variables $(\tilde{X}_0, \varepsilon_1)$ such that ε_1 is independent of $\tilde{X}_0$ and, defining $\tilde{X}_1 = \tilde{X}_0 + \mu(\tilde{X}_0) + \sigma(\tilde{X}_0)\varepsilon_1$, the pair $(\tilde{X}_0, \tilde{X}_1)$ has distribution ν. For all $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}\big(\tilde{X}_1 \mid \tilde{X}_0 = x\big) &= \mathbb{E}\big(x + \mu(x) + \sigma(x)\varepsilon_1 \mid \tilde{X}_0 = x\big) \\ &= x + \mu(x) + \sigma(x)\mathbb{E}(\varepsilon_1 \mid \tilde{X}_0 = x) \\ &= x + \mu(x) + \sigma(x)\mathbb{E}(\varepsilon_1) \\ &= x + \mu(x), \end{aligned}$$

since ε_1 independent of $\tilde{X}_0$. It follows that if $x \in R_+$ then we have

$$\mathbb{E}(\tilde{X}_1 \mid \tilde{X}_0 = x) > x. \tag{23}$$

Consider now the explicitly defined Markovian strategy $\pi^+(x) := \mathbf{1}_{R^+}(x)$ for $x \in \mathbb{R}$, which is constructed as follows: at each time $t > 0$, we invest all the current wealth in the stock if the log-market price of risk is above 0, and we put everything in the bank account otherwise. Given this strategy, consider the corresponding $f(x,y) = \log\big((1-\pi^+(x)) + \pi^+(x)\exp(y-x)\big)$, $(x,y) \in \mathbb{R}^2$. Since by definition $\nu(f) = \int_{\mathbb{R}^2} f(x,y)\nu(dx,dy)$, we have $\nu(f) = \mathbb{E}\Big(\log\big((1-\pi^+(\tilde{X}_0)) + \pi^+(\tilde{X}_0)\exp(\tilde{X}_1 - \tilde{X}_0)\big)\Big)$. Next, by Corollary 3.10, ν has a λ_2-a.e. positive density with respect to λ_2, hence its $\tilde{X}_0$-marginal, denoted by η, has a λ-a.e. positive density $\ell(x)$. Therefore we obtain that

$$\begin{aligned} \nu(f) &= \int_{\mathbb{R}} \mathbb{E}\big(\log((1-\pi^+(x)) + \pi^+(x)\exp(\tilde{X}_1 - x)) \mid \tilde{X}_0 = x\big)\eta(dx) \\ &\geq \int_{R^+} \mathbb{E}\big(\log\exp(\tilde{X}_1 - x) \mid \tilde{X}_0 = x\big)\eta(dx) \\ &= \int_{R^+} \mathbb{E}\big(\tilde{X}_1 - x \mid \tilde{X}_0 = x\big)\ell(x)\lambda(dx) \\ &> 0, \end{aligned}$$

by (23), showing the lemma. ∎

Proof of Theorem 2.3. If f is ν-a.s. constant then the statement is trivial. If not, then $\sigma^2(f) > 0$ by the argument of Theorem 5 in [10]. Proposition 3.12 says that $\frac{1}{t}\log(V_t^{\pi^+}/V_0)$ satisfies a large deviations principle with good rate function Λ_f^*. In particular, applying the upper large deviations inequality (2.3.7) of Theorem 2.3.6 in [2] we get

$$\limsup_{t\to\infty} \frac{1}{t}\log\mathbb{P}\Big(\frac{1}{t}\log(V_t^{\pi^+}/V_0) < \nu(f)/2\Big) \leq -\inf_{x\in(-\infty,\nu(f)/2]} \Lambda_f^*(x), \tag{24}$$

where $\nu(f) > 0$ by Lemma 3.13. By Corollary 3.6, $\Lambda_f^*(\nu(f)/2) > 0$ and $\nu(f)$ is the unique minimiser of Λ_f^*. By strict convexity, Λ_f^* is decreasing on $(-\infty, \nu(f)]$. These imply that the right hand side of (24) is equal to $-\Lambda_f^*(\nu(f)/2)$. Hence

$$\mathbb{P}\big(V_t^{\pi^+} \geq e^{\log(V_0)+\nu(f)t/2}\big) \geq 1 - e^{-t\Lambda_f^*(\nu(f)/2)} \text{ for large } t, \tag{25}$$

and the result follows. ■

The following two examples fall outside the scope of [10] but can be treated using Theorem 2.3 above.

Example 3.14. Stable autoregressive process.

Consider the model $S_t := e^{X_t}$ where X_t is a stable autoregressive process (that is, the discrete-time version of the Ornstein-Uhlenbeck process):

$$X_{t+1} = \alpha X_t + \varepsilon_{t+1}, \text{ for all } t \geq 1, \tag{26}$$

where $0 < |\alpha| < 1$, X_0 is constant and the ε_t are i.i.d. $\mathcal{N}(0, 1)$.

In this typical example, the drift and volatility functions are identified as $\mu(x) = (\alpha - 1)x$ and $\sigma(x) = 1$, for all $x \in \mathbb{R}$. All the assumptions $(A_1) - (A_4)$ on μ, σ and on the ε_ts trivially hold.

Next, $R^+ = (-\infty, 0)$. Obviously, (RC_+) holds. It follows that the trading opportunity $\pi_t^+ = \pi^+(X_{t-1})$, with $\pi^+ := \mathbf{1}_{R^+}$ realizes an AEA with $GDPF$, by Theorem 2.3. ■

Example 3.15. A Cox-Ingersoll-Ross–type process.

In mathematical finance the process H_t described by the stochastic differential equation

$$dH_t = -\beta H_t dt + \sigma \sqrt{|H_t|} dW_t \tag{27}$$

is often used to model stochastic volatility or the short rate in bond markets, here W_t is a Brownian motion. We present here a slight modification of the discretization of this model. The modifications are necessary, since the volatility of H_t is neither bounded above nor bounded away from 0.

Let us define the log-price process by

$$X_{t+1} = \alpha X_t + \sigma \min\{\max\{\sqrt{|X_t|}, c_1\}, c_2\}\varepsilon_t, \quad t \geq 1,$$

where $|\alpha| < 1$, $\sigma > 0$, $0 < c_1 < c_2$ are given constants and ε_t is $\mathcal{N}(0, 1)$. It is easy to check that this process also satisfies the conditions of Theorem 2.3.

4 Utility-based Asymptotic Arbitrage

We consider risk-averse investors with initial capital $V_0 = x \in (0, \infty)$ who express their preferences in terms of a utility function $U : (0, \infty) \to \mathbb{R}$, where U belongs to the subclass of Hyperbolic Absolute Risk Aversion ($HARA$) utility functions $U(x) = x^\alpha$, with $0 < \alpha < 1$, or $U(x) = -x^\alpha$, with $\alpha < 0$, for all $x \in (0, \infty)$. The parameter α is related to risk-aversion: the larger $-\alpha$ is, the more afraid investors become of losses, see [5].

As mentioned in the introductory section, in this paper we do not intend to solve the finite horizon utility maximization problem which is well-discussed in the literature and which consists in finding the maximal expected utility $u(x) := \sup_\pi \mathbb{E}U(V_T^\pi)$ together with an optimal strategy $(\pi_t^*)_{1 \leq t \leq T}$ verifying $u(x) = \mathbb{E}U(V_T^{\pi^*})$. Instead, we focus on trading opportunities that provide (rapidly) increasing expected utilities for the agents as the time horizon tends to infinity, in the spirit of [3] and [6].

Consider first the subclass of power utility functions $U(x) := x^{\alpha}$, with $0 < \alpha < 1$, for $x \in (0, \infty)$.

Proposition 4.1. *If a trading strategy π_t realizes an AEA then there is a constant $b > 0$ such that*

$$\mathbb{E}U(V_t^{\pi}) \geq e^{\alpha bt}, \text{ for all large enough } t. \tag{28}$$

Proof. By definition of AEA, there are a constant $b > 0$ and a time $t_{1/2}$ such that $\mathbb{P}(V_t^{\pi} \geq e^{bt}) \geq 1/2$ for all time $t \geq t_{1/2}$. It follows that $\mathbb{E}U(V_t^{\pi}) \geq \mathbb{E}U(e^{bt})\mathbf{1}_{\{V_t^{\pi} \geq e^{bt}\}} \geq (1/2)e^{\alpha bt} \geq e^{\alpha b' t}$ for any $b' < b$ and for all t large enough, as required. ∎

Next, suppose that investors choose from the second subclass of power utility functions $U(x) := -x^{\alpha}$, with $\alpha < 0$, for all $x \in (0, \infty)$. These functions express larger risk-aversion and are thought to be more realistic. We derive the key result of this section below. We remark that, despite the short proof, the following theorem relies on all the heavy machinery of the paper [9] as well as on our preliminary results established in Section 2 and it is, in fact, highly non-trivial.

Theorem 4.2. *Assume that the log-price X_t satisfies $(A_1) - (A_4)$ as well as (RC_+). Let π_t^+ be the strategy defined in Theorem 2.3. Then there is $\alpha_0 < 0$ such that for any risk-aversion coefficient $0 > \alpha > \alpha_0$, the expected utility of the corresponding investor's wealth converges to 0 at an exponential rate. That is, with the power utility $U(x) := -x^{\alpha}$, $x \in (0, \infty)$, we have,*

$$|\mathbb{E}U(V_t^{\pi^+})| \leq Ke^{-ct}, \text{ for all large enough } t, \tag{29}$$

for some constants $K = K(\alpha), c = c(\alpha) > 0$.

Proof. Recall section 2, in particular, Corollary 3.10, Proposition 3.12 and (21). When f is constant ν-a.s., the statement is trivial. Otherwise we may assume $\sigma^2(f) > 0$ (see the proof of Theorem 2.3). In section 2 we obtained that $\Lambda_f(0) = 0$, $\Lambda_f'(0) = \nu(f) > 0$. Since by analyticity of Λ_f, Λ_f' is continuous, there exists $\alpha_0 < 0$ such that $\Lambda_f(\alpha) < 0$ for $\alpha \in (\alpha_0, 0)$. Theorem 3.1 of [9] implies that for some constant d_{α}, we have

$$\frac{-\mathbb{E}e^{\alpha(f(\Phi_1)+\ldots+f(\Phi_n))}}{e^{n\Lambda_f(\alpha)}} = \frac{\mathbb{E}U(V_n^{\pi})/V_0^{\alpha}}{e^{n\Lambda_f(\alpha)}} \to d_{\alpha}, \ n \to \infty, \tag{30}$$

showing the statement. ∎

It seems that, in general, we should not expect more than this result (i.e. we cannot get such a theorem for all $\alpha < 0$). To illustrate this, we now construct an example where there is AEA with $GDPF$ but, for some $\alpha < 0$, we have $\mathbb{E}(V_t^{\pi})^{\alpha} \to -\infty$ as $t \to \infty$.

Example 4.3. Consider the log-price X_t governed by the equation $X_{t+1} = X_t + \varepsilon_{t+1}$, $t \in \mathbb{N}$, with $X_0 = 0$, where ε_t are i.i.d. random variables in $\mathbb{R}$ with common distribution chosen such that $\mathbb{E}e^{-\varepsilon_1} > 1$ and $\mathbb{E}\varepsilon_1 > 0$. For example $\varepsilon_1 \sim \mathcal{N}(1/4, 1)$ will do. We identify the drift and volatility as $\mu \equiv 0$ and $\sigma \equiv 1$.

Choose the trading strategy $\pi_t \equiv 1$ for all t and let $V_0 = 1$. Then we have $V_t := \exp(\varepsilon_1 + \cdots + \varepsilon_t)$ for all $t \geq 1$. Since $1/5 < 1/4 = \mathbb{E}\varepsilon_1$, by Cramér's theorem (see e.g. [2]), there are a constant $c > 0$ and $t_0 > 0$ such that for all $t \geq t_0$, we have $\mathbb{P}(V_t \geq e^{t/5}) \geq 1 - e^{-ct}$. Hence there is AEA with $GDPF$.

However, for $\alpha = -1$, we have, by independence,

$$\mathbb{E}U(V_t) = \mathbb{E}(-V_t^{-1}) = -\mathbb{E}\exp\{-(\varepsilon_1 + \cdots + \varepsilon_t)\} = -(\mathbb{E}e^{-\varepsilon_1})^t \to -\infty$$

as $t \to \infty$. ■

Finally we investigate what happens if a risk-averse agent produces expected utility tending to $0 = U(\infty)$ exponentially fast as $t \to \infty$. It turns out that his/her strategy produces AEA with $GDPF$. Indeed, following the footsteps of Proposition 2.2 in [6] we get the following result.

Theorem 4.4. *Consider the power utility $U(x) = -x^\alpha$ for some $\alpha < 0$. Let π_t be such that $|\mathbb{E}U(V_t^\pi)| \leq Ke^{-ct}$ for all large enough t, for some constants $c, K > 0$. Then π_t provides an AEA with $GDPF$.*

Proof. We may and will assume $K = 1$. We need to find constants $b > 0$, $c' > 0$ such that $\mathbb{P}(V_t^\pi \geq e^{bt}) \geq 1 - e^{-c't}$ for all large enough t. Choose $b > 0$ such that $c + \alpha b > 0$, then we have

$$\begin{aligned} \mathbb{P}(V_t^\pi < e^{bt}) &= \mathbb{P}\big(|U(V_t^\pi)| > |U(e^{bt})|\big) \\ &\leq \frac{\mathbb{E}|U(V_t^\pi)|}{|U(e^{bt})|} \text{ by Markov's inequality.} \end{aligned} \tag{31}$$

But $\mathbb{E}|U(V_t^\pi)| = |\mathbb{E}U(V_t^\pi)| \leq e^{-ct}$ and $|U(e^{bt})| = e^{\alpha bt}$ imply $\mathbb{P}(V_t^\pi < e^{bt}) \leq e^{-(c+\alpha b)t}$ for all t. Hence the result follows taking $c' := c + \alpha b$. ■

To conclude, if an economic agent with $HARA$ utility risk-aversion coefficient $\alpha < 0$ achieves an expected utility that converges exponentially fast to 0, then his/her strategy provides AEA with $GDPF$, too. Conversely, under the stringent conditions of Section 2, one is able to construct strategies producing AEA with $GDPF$ which also provide expected utilities tending to 0 exponentially fast for α not too negative (i.e. for not too risk-averse investors).

References

[1] R.N. Bhattacharya and E.C. Waymire. *Stochastic Processes with Applications*, John Wiley & Sons, Inc., New York, 1990.

[2] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*, 2nd edition, Springer, 1998.

[3] N. Dokuchaev. Mean-Reverting Market Models: Speculative Opportunities and Non-Arbitrage, *Appl. Math. Finance*, Vol. 14, 319–337, 2007.

[4] K. Du and A. D. Neufeld. A Note on Asymptotic Exponential Arbitrage with Exponentially Decaying Failure Probability. *J. Appl.Probab.*, Vol. 50, 801–809, 2013.

[5] H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*, 2nd edition, Walter de Gruyter, 2004.

[6] H. Föllmer and W. Schachermayer. Asymptotic Arbitrage and Large Deviations, *Math. Financ. Econ.*, Vol. 1, 213–249, 2007.

[7] Y.M. Kabanov and D.O. Kramkov. Asymptotic Arbitrage in Large Financial Markets, *Finance Stochast.*, Vol. 2, 143–172, 1998.

[8] I. Kontoyiannis and S. Meyn. Spectral Theory and Limit Theorems for Geometrically Ergodic Markov Processes, *Ann. Appl. Probab.*, Vol. 13, 304–362, 2003.

[9] I. Kontoyiannis and S. Meyn. Large Deviations Asymptotics and the Spectral Theory of Multiplicative Regular Processes, *Electronic. Journ. Probab.*, Vol. 10, 61–123, 2005.

[10] M.L.D. Mbele Bidima and M. Rásonyi. On Long-term Arbitrage Opportunities in Markovian Models of Financial Markets, *Ann. Oper. Res.*, Vol. 200, 131–146, 2012.

[11] S. Meyn and R. Tweedie. *Markov Chains and Stochastic Stability*, 2nd edition, Cambridge University Press, 2009.