

Matrix representations of finitely generated Grassmann algebras and some consequences

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ABSTRACT. We prove that the m -generated Grassmann algebra can be embedded into a $2^{m-1} \times 2^{m-1}$ matrix algebra over a factor of a commutative polynomial algebra in m indeterminates. Cayley–Hamilton and standard identities for $n \times n$ matrices over the m -generated Grassmann algebra are derived from this embedding. Other related embedding results are also presented.

1. INTRODUCTION

Let K be a field (of characteristic zero) and consider the free associative K -algebra $K \langle x_1, \dots, x_m \rangle$ generated by the (non-commuting) indeterminates x_1, \dots, x_m . The elements of $K \langle x_1, \dots, x_m \rangle$ are K -linear combinations of monomials of the form $x_{i_1} \cdots x_{i_k}$ with $i_1, \dots, i_k \in \{1, \dots, m\}$ (not necessarily different). The m -generated Grassmann (or exterior) algebra is defined as the factor K -algebra

$$E^{(m)} = K \langle x_1, \dots, x_m \rangle / I(x_1, \dots, x_m),$$

where

$$I(x_1, \dots, x_m) = (x_i x_j + x_j x_i \mid 1 \leq i \leq j \leq m)_K$$

is the (two-sided) K -ideal of $K \langle x_1, \dots, x_m \rangle$ generated by the polynomials $x_i x_j + x_j x_i$ for $1 \leq i \leq j \leq m$. The usual notation for this Grassmann algebra is

$$E^{(m)} = K \langle v_1, \dots, v_m \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \leq m \rangle,$$

where the cosets $v_i = x_i + I(x_1, \dots, x_m)$ are the anticommuting generators of $E^{(m)}$. Clearly, any element of $E^{(m)}$ is a unique K -linear combination of monomials of the form $v_{i_1} \cdots v_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq m$. It follows that $\dim_K E^{(m)} = 2^m$.

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The definition of the countably infinitely generated Grassmann algebra

$$E = K \langle v_1, \dots, v_m, \dots \mid v_i v_j + v_j v_i = 0 \text{ for all } 1 \leq i \leq j \rangle$$

is similar.

For a ring (or K -algebra) R let $M_n(R)$ denote the full $n \times n$ matrix ring (K -algebra) over R with identity $I_n \in M_n(R)$. Since any matrix $A \in M_n(E)$ has a finite number of entries and each entry contains a finite number of generators v_i from E , there exists an integer $m \geq 1$ such that $A \in M_n(E^{(m)})$. It follows that

$$M_n(E) = \bigcup_{m=1}^{\infty} M_n(E^{(m)}).$$

The algebras $E^{(m)}$ and E play a fundamental role in many areas of mathematics.

Our main inspiration is Kemer's pioneering work [K] on the T -ideals of associative algebras, which revealed the importance of the identities satisfied by the full $n \times n$ matrix algebra $M_n(E)$ and by the algebra $M_{n,t}(E)$ of (n, t) -supermatrices (this is a certain K -subalgebra of $M_n(E)$). The prime T -ideals of $K \langle x_1, \dots, x_m, \dots \rangle$ are exactly the T -ideals of the identities satisfied by $M_n(K)$ for $n \geq 1$. The T -prime T -ideals are the prime T -ideals plus the T -ideals of the identities of $M_n(E)$ for $n \geq 1$ and of $M_{n,t}(E)$ for $n - 1 \geq t \geq 1$. Another remarkable result is that, for n sufficiently large, any T -ideal contains the T -ideal of the identities satisfied by $M_n(E)$. Thus the algebras $M_n(E)$ and $M_n(E^{(m)})$ served as the main motivation for the present work.

An additional motivation is the well-known embedding of the skew field \mathbb{H} of the real quaternions into 4×4 real matrices:

$$a + bi + cj + dk \longmapsto \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$. The above definition provides an injective \mathbb{R} -algebra homomorphism $v : \mathbb{H} \rightarrow M_4(\mathbb{R})$. Using the natural extension

$$v_n : M_n(\mathbb{H}) \longrightarrow M_n(M_4(\mathbb{R})) \cong M_{4n}(\mathbb{R}),$$

an $n \times n$ matrix over \mathbb{H} can be viewed as a $4n \times 4n$ matrix over \mathbb{R} . For a quaternionic matrix $A \in M_n(\mathbb{H})$, the Cayley–Hamilton identity for $v_n(A)$ yields the same identity (with real coefficients) of degree $4n$ for A itself.

A similar approach to get a Cayley–Hamilton identity of degree $2n$ for a matrix $A \in M_n(E^{(2)})$ is based on embedding the two-generated exterior algebra $E^{(2)}$ into a 2×2 matrix algebra over a certain commutative ring (see [SzvW]).

In order to present a Cayley–Hamilton identity of degree $2^{m-1}n$ for a matrix in $M_n(E^{(m)})$, first we consider the so called constant trace (CT-)representations of an arbitrary K -algebra. In Section 2 we derive a Cayley–Hamilton identity for $n \times n$ matrices over any algebra having a CT-representation. Then in Section 3, using induction, we present a CT-representation (K -embedding)

$$\varepsilon^{(m)} : E^{(m)} \longrightarrow M_{2^{m-1}}(K[z_1, \dots, z_m]/(z_1^2, \dots, z_m^2)).$$

Notice that we cannot expect similar results for the infinitely generated Grassmann algebra E . Since E does not satisfy any of the standard identities, it follows that E does not embed into any full matrix algebra over a commutative ring.

Finally in Section 4 we use a certain factor of a skew polynomial algebra to give a broad generalization of the embedding process for $E^{(m)}$.

2. CT-REPRESENTATIONS AND CAYLEY–HAMILTON IDENTITIES

Let ${}_K R$ be an arbitrary and ${}_K \Omega$ be a commutative (associative) algebra over K (notice that $K \subseteq Z(R)$ and $K \subseteq \Omega$). For an integer $t \geq 1$, we consider representations of R over Ω which are injective K -algebra homomorphisms (K -embeddings) $\varepsilon : R \rightarrow M_t(\Omega)$. We call ε a constant trace (CT-) representation if $\text{tr}(\varepsilon(r)) \in K$ for all $r \in R$ (here $\text{tr}(\varepsilon(r))$ is the sum of the diagonal entries of the $t \times t$ matrix $\varepsilon(r) \in M_t(\Omega)$).

2.1.Theorem. *Let $\varepsilon : R \rightarrow M_t(\Omega)$ be a CT-representation of R over Ω . If $A \in M_n(R)$ is an $n \times n$ matrix, then A satisfies a Cayley–Hamilton identity of the form*

$$A^{tn} + c_1 A^{tn-1} + \cdots + c_{tn-1} A + c_{tn} I_n = 0,$$

where $c_i \in K$, $1 \leq i \leq tn$.

Proof. Let

$$\varepsilon_n : M_n(R) \rightarrow M_n(M_t(\Omega)) \cong M_{tn}(\Omega)$$

be the natural extension of ε . For any matrix $A = [a_{i,j}]$ in $M_n(R)$, the trace of the $tn \times tn$ matrix $B = \varepsilon_n(A)$ is the sum of the traces of the diagonal $t \times t$ blocks:

$$\text{tr}(B) = \sum_{i=1}^n \text{tr}(\varepsilon(a_{i,i})).$$

Since ε is a CT-representation, we have $\text{tr}(\varepsilon(a_{i,i})) \in K$ for each $1 \leq i \leq n$. It follows that $\text{tr}(B) \in K$. For the coefficients of the characteristic polynomial

$$\det(zI - B) = c_0 z^{tn} + c_1 z^{tn-1} + \cdots + c_{tn-1} z + c_{tn} \in \Omega[z]$$

of B , the following recursion holds: $c_0 = 1$ and

$$c_k = -\frac{1}{k} (c_{k-1} \text{tr}(B) + c_{k-2} \text{tr}(B^2) + \cdots + c_1 \text{tr}(B^{k-1}) + c_0 \text{tr}(B^k))$$

for $1 \leq k \leq tn$ (Newton formulae, see [R]). In view of

$$\text{tr}(B^k) = \text{tr}((\varepsilon_n(A))^k) = \text{tr}(\varepsilon_n(A^k)) \in K,$$

we deduce that $c_i \in K$ for each $0 \leq i \leq tn$. Thus $\det(zI - B) \in K[z]$ and the Cayley–Hamilton identity for $B \in M_{tn}(\Omega)$ is of the form

$$B^{tn} + c_1 B^{tn-1} + \cdots + c_{tn-1} B + c_{tn} I_n = 0.$$

It follows that

$$\begin{aligned} (\varepsilon_n(A))^{tn} + c_1 (\varepsilon_n(A))^{tn-1} + \cdots + c_{tn-1} \varepsilon_n(A) + c_{tn} I_n &= \\ \varepsilon_n(A^{tn} + c_1 A^{tn-1} + \cdots + c_{tn-1} A + c_{tn} I_n) &= 0 \end{aligned}$$

holds in $M_{tn}(\Omega)$, thus the injectivity of ε_n gives the desired identity. \square

3. CT-REPRESENTATION OF $E^{(m)}$

For $m = 1$ we have a natural isomorphism $E^{(1)} = K \langle v_1 \mid v_1^2 = 0 \rangle \cong K[z]/(z^2)$ of K -algebras. If $m = 2$, then the assignments

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v_1 \mapsto \begin{bmatrix} z_1 & 0 \\ 0 & -z_1 \end{bmatrix}, \quad v_2 \mapsto \begin{bmatrix} 0 & z_2 \\ z_2 & 0 \end{bmatrix}$$

define the following CT-representation $\varepsilon^{(2)} : E^{(2)} \rightarrow M_2(K[z_1, z_2]/(z_1^2, z_2^2))$:

$$\varepsilon^{(2)}(c_0 + c_1 v_1 + c_2 v_2 + c_3 v_1 v_2) = \begin{bmatrix} c_0 + c_1 z_1 + (z_1^2, z_2^2) & c_2 z_2 + c_3 z_1 z_2 + (z_1^2, z_2^2) \\ c_2 z_2 - c_3 z_1 z_2 + (z_1^2, z_2^2) & c_0 - c_1 z_1 + (z_1^2, z_2^2) \end{bmatrix},$$

where $c_0, c_1, c_2, c_3 \in K$ and (z_1^2, z_2^2) is the ideal of the commutative polynomial ring $K[z_1, z_2]$ generated by the monomials z_1^2, z_2^2 .

3.1.Theorem. *For some integers $m, t \geq 2$, let $\varepsilon^{(m)} : E^{(m)} \rightarrow M_t(\Omega)$ be a CT-representation of $E^{(m)}$ over a commutative K -algebra Ω . Then the assignments*

$$1 \mapsto \begin{bmatrix} I_t & 0 \\ 0 & I_t \end{bmatrix}, \quad v_i \mapsto \begin{bmatrix} \varepsilon^{(m)}(v_i) & 0 \\ 0 & -\varepsilon^{(m)}(v_i) \end{bmatrix} \quad \text{for } 1 \leq i \leq m,$$

$$\text{and } v_{m+1} \mapsto \begin{bmatrix} 0 & \widehat{z} I_t \\ \widehat{z} I_t & 0 \end{bmatrix} \quad (\text{with } \widehat{z} = z + (z^2) \text{ in } \Omega[z]/(z^2))$$

define a CT-representation $\varepsilon^{(m+1)} : E^{(m+1)} \rightarrow M_{2t}(\Omega[z]/(z^2))$.

Proof. Any element of $E^{(m+1)}$ can be uniquely written as $g + h v_{m+1}$, where

$$g = \sum_{1 \leq i_1 < \dots < i_k \leq m} c_{i_1, \dots, i_k} v_{i_1} \cdots v_{i_k} \quad \text{and } h = \sum_{1 \leq j_1 < \dots < j_l \leq m} d_{j_1, \dots, j_l} v_{j_1} \cdots v_{j_l}$$

are in $E^{(m)}$ with $c_{i_1, \dots, i_k}, d_{j_1, \dots, j_l} \in K$. For $1 \leq i_1 < \dots < i_k \leq m$ and $1 \leq j_1 < \dots < j_l \leq m$, our assignment gives

$$v_{i_1} \cdots v_{i_k} \mapsto \begin{bmatrix} \varepsilon^{(m)}(v_{i_1} \cdots v_{i_k}) & 0 \\ 0 & (-1)^k \varepsilon^{(m)}(v_{i_1} \cdots v_{i_k}) \end{bmatrix}$$

and

$$v_{j_1} \cdots v_{j_l} v_{m+1} \mapsto \begin{bmatrix} 0 & \varepsilon^{(m)}(v_{j_1} \cdots v_{j_l}) \widehat{z} \\ (-1)^l \varepsilon^{(m)}(v_{j_1} \cdots v_{j_l}) \widehat{z} & 0 \end{bmatrix}.$$

Thus

$$\varepsilon^{(m+1)}(g + h v_{m+1}) = \begin{bmatrix} \varepsilon^{(m)}(g_0 + g_1) & \varepsilon^{(m)}(h_0 + h_1) \widehat{z} \\ \varepsilon^{(m)}(h_0 - h_1) \widehat{z} & \varepsilon^{(m)}(g_0 - g_1) \end{bmatrix},$$

where $g = g_0 + g_1$ and $h = h_0 + h_1$ are the unique presentations as sums of an even and an odd element (with respect to the natural \mathbb{Z}_2 -grading $E^{(m)} = E_0^{(m)} \oplus E_1^{(m)}$). Straightforward verification shows that $\varepsilon^{(m+1)} : E^{(m+1)} \rightarrow M_{2t}(\Omega[z]/(z^2))$ is an injective homomorphism of K -algebras. In view of

$$\text{tr}(\varepsilon^{(m+1)}(g + h v_{m+1})) = \text{tr}(\varepsilon^{(m)}(g_0 + g_1)) + \text{tr}(\varepsilon^{(m)}(g_0 - g_1)) \in K,$$

we deduce that $\varepsilon^{(m+1)}$ is a CT-representation of $E^{(m+1)}$. \square

3.2.Corollary. *For any integer $m \geq 2$ there exists a CT-representation*

$$\varepsilon^{(m)} : E^{(m)} \rightarrow M_{2^{m-1}}(K[z_1, \dots, z_m]/(z_1^2, \dots, z_m^2))$$

of $E^{(m)}$, where (z_1^2, \dots, z_m^2) is the ideal of $K[z_1, \dots, z_m]$ generated by the monomials z_1^2, \dots, z_m^2 .

Proof. Starting from $\varepsilon^{(2)} : E^{(2)} \longrightarrow M_2(K[z_1, z_2]/(z_1^2, z_2^2))$ and using

$$(K[z_1, \dots, z_m]/(z_1^2, \dots, z_m^2))[z]/(z^2) \cong K[z_1, \dots, z_m, z_{m+1}]/(z_1^2, \dots, z_m^2, z_{m+1}^2),$$

iteration of the construction in Theorem 3.1 gives the desired CT-representation. \square

3.3.Proposition. *If $\varepsilon : E^{(m)} \longrightarrow M_t(\Omega)$ is a (not necessarily CT-) representation of $E^{(m)}$ for some integers $m, t \geq 2$ over a commutative K -algebra Ω , then $m \leq 2t - 1$.*

Proof. Let $S_m(x_1, \dots, x_m) = \sum_{\pi \in \text{Sym}(m)} \text{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(m)}$ be the standard polynomial in $K \langle x_1, \dots, x_m \rangle$. The well-known Amitsur–Levitzki theorem (see [R]) asserts that $S_{2t} = 0$ is a polynomial identity on $M_t(\Omega)$. The existence of the embedding ε ensures that $S_{2t} = 0$ is also an identity on $E^{(m)}$. On the other hand

$$S_m(v_1, \dots, v_m) = m! v_1 \cdots v_m \neq 0$$

shows that $S_m = 0$ is not a polynomial identity on $E^{(m)}$. If $2t \leq m$, then $S_m = 0$ follows from $S_{2t} = 0$ (in any algebra). Thus we have $2t \not\leq m$. \square

3.4.Theorem (“Cayley–Hamilton”). *If $A \in M_n(E^{(m)})$ is an $n \times n$ matrix, then A satisfies an identity of the form*

$$A^{2^{m-1}n} + c_1 A^{(2^{m-1}n)-1} + \cdots + c_{(2^{m-1}n)-1} A + c_{2^{m-1}n} I_n = 0,$$

where $c_i \in K$, $1 \leq i \leq 2^{m-1}n$.

Proof. A combination of Theorem 2.1 and Corollary 3.2 gives the identity. \square

3.5.Remark. In particular, we know the following about embeddability of $E^{(3)}$ into matrix algebras. By Corollary 3.2, $E^{(3)}$ admits a CT-embedding into a 4×4 matrix algebra. On the other hand, it has no CT-embedding into any 2×2 matrix algebra. The simple reason is that for $c_1, c_2 \in K$, the 2×2 Cayley–Hamilton identity

$$(v_1 + v_2 v_3)^2 + c_1(v_1 + v_2 v_3) + c_2 = 0$$

never holds in $E^{(3)}$, although such an identity should hold in $E^{(3)}$ by Theorem 2.1 if it CT-embeds into a 2×2 matrix algebra. We could not decide whether $E^{(3)}$ has a CT-embedding into a 3×3 matrix algebra.

For $m = 3$ and $t = 2$ the inequality $m \leq 2t - 1$ in Proposition 3.3 becomes equality, thus Proposition 3.3 does not contradict the existence of a non CT-representation $\varepsilon : E^{(3)} \longrightarrow M_2(\Omega)$. Indeed, the assignments

$$1 \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, v_1 \longmapsto \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}, v_2 \longmapsto \begin{bmatrix} y & 2y \\ -2y & -y \end{bmatrix}, v_3 \longmapsto \begin{bmatrix} -2z & -z \\ z & 2z \end{bmatrix}$$

define a K -embedding $E^{(3)} \longrightarrow M_2(K[x, y, z]/(x^2, y^2, z^2))$. Since

$$v_1 v_2 v_3 \longmapsto \begin{bmatrix} 3xyz & 0 \\ 0 & 3xyz \end{bmatrix},$$

this embedding is not a CT-representation.

3.6.Remark. Any matrix $A \in M_n(E)$ (here $\text{char}(K) = 0$ is essential) satisfies left and right Cayley–Hamilton identities of the form

$$A^{n^2} + g_1 A^{n^2-1} + \cdots + g_{n^2-1} A + g_{n^2} I_n = 0,$$

$$A^{n^2} + A^{n^2-1} h_1 + \cdots + A h_{n^2-1} + I_n h_{n^2} = 0,$$

and a central Cayley–Hamilton identity of the form

$$A^{2n^2} + u_1 A^{2n^2-1} + \cdots + u_{2n^2-1} A + u_{2n^2} I_n = 0,$$

where $g_i, h_i \in E$, $1 \leq i \leq n^2$ and $u_j \in E_0$, $1 \leq j \leq 2n^2$ (see [Sz1]).

3.7.Theorem. *The standard identity $S_{2^m n} = 0$ of degree $2^m n$ is a polynomial identity on $M_n(E^{(m)})$.*

Proof. Since the natural extension

$$\left(\varepsilon^{(m)}\right)_n : M_n(E^{(m)}) \longrightarrow M_n(M_{2^{m-1}}(K[z_1, \dots, z_m]/(z_1^2, \dots, z_m^2)))$$

of $\varepsilon^{(m)}$ (in Corollary 3.2) is a K -embedding and

$$M_n(M_{2^{m-1}}(K[z_1, \dots, z_m]/(z_1^2, \dots, z_m^2))) \cong M_{2^{m-1}n}(K[z_1, \dots, z_m]/(z_1^2, \dots, z_m^2))$$

satisfies $S_{2^m n} = 0$ by the Amitsur–Levitzki theorem, the proof is complete. \square

If $n = 1$, then Theorems 3.4 and 3.7 are far from being sharp.

3.8.Remark. Using Theorem 5.5 of Domokos [D] and the fact that $S_{m+2} = 0$ is an identity on $E^{(m)}$, we obtain that $S_{(m+1)n^2+1} = 0$ is an identity on $M_n(E^{(m)})$.

The left regular representation of $E^{(m)}$ is an embedding

$$\lambda^{(m)} : E^{(m)} \longrightarrow \text{End}_K(E^{(m)}) \cong M_{2^m}(K)$$

of K -algebras, where $\text{End}_K(E^{(m)})$ is the algebra of all K -linear maps of the 2^m dimensional vector space ${}_K E^{(m)}$ and $\lambda^{(m)}(g) : E^{(m)} \longrightarrow E^{(m)}$ is the left multiplication by $g \in E^{(m)}$. The size of the matrix algebra is 2^{m-1} in Corollary 3.2, half of the size 2^m provided by the above left regular representation. On the other hand, the base field K is replaced by a much bigger base ring (K -algebra) in Corollary 3.2. Since we cannot derive Theorems 3.4 and 3.7 from the regular representation, our half sized embedding is really better in many ways than the regular representation.

If we keep K as the base field, then the next theorem gives a lower bound for the matrix size of any possible embedding of $E^{(m)}$.

3.9.Theorem. *If $\lambda : E^{(m)} \longrightarrow M_t(K)$ is an embedding of K -algebras, then $3 \cdot 2^{m-2} = 2^{m-1} + 2^{m-2} \leq \left\lfloor \frac{t^2}{4} \right\rfloor + 1$.*

Proof. Consider the K -subalgebra $E_0^{(m)}[v_m]$ of $E^{(m)}$ generated by the centre $E_0^{(m)}$ and the generator v_m . Clearly, $E_0^{(m)}[v_m]$ is commutative, and any element of $E_0^{(m)}[v_m]$ is of the form $g_0 + h_0 v_m$, where $g_0 \in E_0^{(m)}$ and $h_0 \in E_0^{(m-1)}$. It follows that $\dim_K E_0^{(m)}[v_m] = 2^{m-1} + 2^{m-2}$. Since $\lambda(E_0^{(m)}[v_m]) \subseteq M_t(K)$ is a commutative subalgebra of $M_t(K)$, we can apply Schur’s inequality (see [M]):

$$2^{m-1} + 2^{m-2} = \dim_K E_0^{(m)}[v_m] = \dim_K \lambda(E_0^{(m)}[v_m]) \leq \left\lfloor \frac{t^2}{4} \right\rfloor + 1. \quad \square$$

3.10.Remark. Since M. Domokos and M. Zubor recently proved (see [DZ]) that $\dim_K C \leq 3 \cdot 2^{2r-2}$ for any commutative K -subalgebra $C \subseteq E^{(2r)}$ of $E^{(2r)}$, the argument in the above proof cannot be improved if $m = 2r$ is even.

4. SKEW POLYNOMIAL RINGS AND EMBEDDINGS

For a ring (K -algebra) endomorphism $\sigma : R \rightarrow R$, consider the skew polynomial ring (K -algebra) $R[w, \sigma]$ in the skew indeterminate w . The elements of $R[w, \sigma]$ are left polynomials of the form $f(w) = a_0 + a_1 w + \cdots + a_k w^k$ with $a_0, a_1, \dots, a_k \in R$. Besides the obvious addition, we have the following multiplication rule in $R[w, \sigma]$: $wr = \sigma(r)w$ for all $r \in R$ and

$$(a_0 + a_1 w + \cdots + a_k w^k)(b_0 + b_1 w + \cdots + b_l w^l) = c_0 + c_1 w + \cdots + c_{k+l} w^{k+l},$$

where

$$c_m = \sum_{i+j=m, i \geq 0, j \geq 0} a_i \sigma^i(b_j).$$

If σ is an involution, then w^2 is a central element of $R[w, \sigma]$: we have $\sigma(\sigma(r)) = r$ and $w^2 r = w\sigma(r)w = \sigma(\sigma(r))w^2 = rw^2$ for all $r \in R$, moreover w^2 commutes with the powers of w . Thus the ideal (w^2) of $R[w, \sigma]$ generated by w^2 can be written as $(w^2) = R[w, \sigma]w^2 = w^2 R[w, \sigma]$. Consider the factor ring (K -algebra) $R[w, \sigma]/(w^2)$, then for any element $f(w) \in R[w, \sigma]$ there exists exactly one left polynomial of the form $r + sw \in R[w, \sigma]$ in the residue class $f(w) + (w^2)$. Hence the elements of $R[w, \sigma]/(w^2)$ can be represented by linear left polynomials with coefficients in R , and the multiplication in $R[w, \sigma]/(w^2)$ is the following:

$$(r + sw)(p + qw) = rp + (rq + s\sigma(p))w,$$

where $r, s, p, q \in R$.

If $R = E = E_0 \oplus E_1$ (or $R = E^{(m)} = E_0^{(m)} \oplus E_1^{(m)}$) is the natural \mathbb{Z}_2 -grading, then $\sigma(g) = \sigma(g_0 + g_1) = g_0 - g_1$ defines a natural involution (here $g = g_0 + g_1$ is the unique presentation as a sum of an even and an odd element). It is easy to see that $E[w, \sigma]/(w^2) \cong E$ and $E^{(m)}[w, \sigma]/(w^2) \cong E^{(m+1)}$ as K -algebras.

We note that the idea of considering $R[w, \sigma]/(w^2)$ comes from [SSz].

4.1.Theorem("Fundamental Embedding"). *For an involution $\sigma : R \rightarrow R$, putting*

$$\mu(r + sw + (w^2)) = \begin{bmatrix} r + (z^2) & sz + (z^2) \\ \sigma(s)z + (z^2) & \sigma(r) + (z^2) \end{bmatrix}$$

(with $r, s \in R$) gives a K -embedding $\mu : R[w, \sigma]/(w^2) \rightarrow M_2(R[z]/(z^2))$.

Proof. We only have to prove the multiplicative property of μ :

$$\begin{aligned} \mu((r + sw + (w^2))(p + qw + (w^2))) &= \mu(rp + (rq + s\sigma(p))w + (w^2)) = \\ &= \begin{bmatrix} rp + (z^2) & (rq + s\sigma(p))z + (z^2) \\ \sigma(rq + s\sigma(p))z + (z^2) & \sigma(rp) + (z^2) \end{bmatrix} = \\ &= \begin{bmatrix} r + (z^2) & sz + (z^2) \\ \sigma(s)z + (z^2) & \sigma(r) + (z^2) \end{bmatrix} \cdot \begin{bmatrix} p + (z^2) & qz + (z^2) \\ \sigma(q)z + (z^2) & \sigma(p) + (z^2) \end{bmatrix} = \\ &= \mu(r + sw + (w^2))\mu(p + qw + (w^2)). \quad \square \end{aligned}$$

The following is a broad generalization of Theorem 3.1.

4.2.Theorem. *Let $\varepsilon : R \longrightarrow M_t(\Omega)$ be a CT-representation of R over a commutative K -algebra Ω for some integer $t \geq 2$. If $\sigma : R \longrightarrow R$ is an involution, then there exists an induced CT-representation $\varepsilon^* : R[w, \sigma]/(w^2) \longrightarrow M_{2t}(\Omega[z]/(z^2))$ of the factor $R[w, \sigma]/(w^2)$.*

Proof. Consider the natural extension

$$\bar{\varepsilon}_2 : M_2(R[z]/(z^2)) \longrightarrow M_2((M_t(\Omega)[z])/(z^2)) \cong M_{2t}(\Omega[z]/(z^2))$$

of $\bar{\varepsilon} : R[z]/(z^2) \longrightarrow (M_t(\Omega)[z])/(z^2)$, where $\bar{\varepsilon}(r + sz + (z^2)) = \varepsilon(r) + \varepsilon(s)z + (z^2)$ for $r, s \in R$. Using the map $\mu : R[w, \sigma]/(w^2) \longrightarrow M_2(R[z]/(z^2))$ in Theorem 4.1, the composition $\varepsilon^* = \bar{\varepsilon}_2 \circ \mu$ gives the induced CT-representation. Indeed, $\text{tr}(\varepsilon^*(r + sw + (w^2))) = (\text{tr}(\varepsilon(r)) + (z^2)) + (\text{tr}(\varepsilon(\sigma(r))) + (z^2)) \in K + (z^2)$. \square

Let $\sigma : R \longrightarrow R$ be a (K -algebra) endomorphism with $\sigma^t = 1$ (such a σ is an automorphism). Now w^t is a central element of $R[w, \sigma]$ and the ideal (w^t) of $R[w, \sigma]$ can be written as $(w^t) = R[w, \sigma]w^t = w^t R[w, \sigma]$. We close this section by mentioning the following generalization of Theorem 4.1 (see [Sz2]), which seems to have applications in Galois theory.

4.3.Theorem. *For an endomorphism $\sigma : R \longrightarrow R$ with $\sigma^t = 1$, putting*

$$\mu(r_0 + r_1 w + \cdots + r_{t-1} w^{t-1} + (w^t)) = [\sigma^{i-1}(r_{j-i})z^{j-i} + (z^t)]_{t \times t}$$

gives an embedding $\mu : R[w, \sigma]/(w^t) \longrightarrow M_t(R[z]/(z^t))$, where the difference $j-i \in \{0, 1, \dots, t-1\}$ is taken modulo t , and the element $\sigma^{i-1}(r_{j-i})z^{j-i} + (z^t)$ of the factor $R[z]/(z^t)$ is in the (i, j) position of the $t \times t$ matrix $[\sigma^{i-1}(r_{j-i})z^{j-i} + (z^t)]_{t \times t}$. The trace of $[\sigma^{i-1}(r_{j-i})z^{j-i} + (z^t)]_{t \times t}$ is in $R^\sigma + (z^t)$, where $R^\sigma = \{r \in R \mid \sigma(r) = r\}$ is the fixed ring of σ .

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