

Large cross-free sets in Steiner triple systems

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Abstract

A *cross-free* set of size m in a Steiner triple system (V, \mathcal{B}) is three pairwise disjoint m -element subsets $X_1, X_2, X_3 \subset V$ such that no $B \in \mathcal{B}$ intersects all the three X_i -s. We conjecture that for every admissible n there is an STS(n) with a cross-free set of size $\lfloor \frac{n-3}{3} \rfloor$ which if true, is best possible. We prove this conjecture for the case $n = 18k + 3$, constructing an STS($18k + 3$) containing a cross-free set of size $6k$. We note that some of the 3-bichromatic STSs, constructed by Colbourn, Dinitz and Rosa, have cross-free sets of size close to $6k$ (but cannot have size exactly $6k$).

The constructed STS($18k + 3$) shows that equality is possible for $n = 18k + 3$ in the following result: in every 3-coloring of the blocks of any Steiner triple system STS(n) there is a monochromatic connected component of size at least $\lceil \frac{2n}{3} \rceil + 1$ (we conjecture that equality holds for every admissible n).

The analogue problem can be asked for r -colorings as well, if $r - 1 \equiv 1, 3 \pmod{6}$ and $r - 1$ is a prime power, we show that the answer is the same as in case of complete graphs: in every r -coloring of the blocks of any STS(n), there is a monochromatic connected component with at least $\frac{n}{r-1}$ points, and this is sharp for infinitely many n .

1 Introduction

A hyperwalk in a hypergraph $H = (V, E)$ is a sequence $v_1, e_1, v_2, e_2, \dots, v_{t-1}, e_{t-1}, v_t$ of vertices and edges such that for all $1 \leq i < t$ we have $v_i \in e_i$ and $v_{i+1} \in e_i$.

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We say that $v \sim w$, if there is a hyperwalk from v to w . The relation \sim is an equivalence relation, and the subhypergraphs induced by its classes are called the *connected components* of H . A vertex v that is not covered by any edge forms a trivial component with one vertex v and no edge.

The size of the largest monochromatic component in edge colorings of complete graphs and hypergraphs is well investigated, for a present survey see [5]. For example, in every 3-coloring of the edges of the complete graph K_n there is a monochromatic connected component of size at least $n/2$ and in every 3-coloring of the edges of K_n^3 , the complete 3-uniform hypergraph, there is a monochromatic spanning component. What happens in between, when the blocks of a Steiner triple system (V, \mathcal{B}) are colored? For example, in every coloring of the blocks of STS(9) with 3 colors, there is a monochromatic connected component of size at least 7 but in the 4-coloring of its blocks defined by the four parallel classes, every component in every color has only 3 points. Let $f(n)$ denote the largest m such that in every 3-coloring of the blocks of any STS(n) there is a monochromatic connected component on at least m points. It is easy to see that $f(7) = 6, f(9) = 7$. Our main result is the following.

Theorem 1. $f(6k + 3) \geq 4k + 3$ with equality if k is divisible by 3. Moreover, $f(6k + 1) \geq 4k + 2$.

In fact, the inequalities of Theorem 1 are probably always sharp (one can easily check the cases $k = 1, 2$):

Conjecture 2. For every positive integer k , $f(6k + 1) = 4k + 2, f(6k + 3) = 4k + 3$.

Three pairwise disjoint m -element sets of points, $X_1, X_2, X_3 \subset V$, is a *cross-free set of size m* in a Steiner triple system (V, \mathcal{B}) if no block $B \in \mathcal{B}$ intersects each X_i in exactly one point. To obtain the upper bound in Theorem 1, we need some STS with a cross-free set of size almost $n/3$. In Theorem 4 we construct an STS($6k + 3$) for the case $k \equiv 0 \pmod{3}$ which contains a cross-free set of size $2k$ (and this is best possible).

It is worth noting that constructions of Colbourn, Dinitz and Rosa in [2] provides STS(n)-s with cross-free sets of size asymptotic to $n/3$. They construct 3-bichromatic STSs where all points are partitioned into X_1, X_2, X_3 so that every block intersects *precisely two* of the X_i -s and they can also control the sizes of the X_i s. In particular, they provide 3-bichromatic STS(n)s where the sizes are nearly equal to $n/3$. However, it follows easily that in 3-bichromatic STS(n)s with $|X_1| \leq |X_2| \leq |X_3|$, $n/3 - |X_i|$ tends to infinity with n . Therefore, to achieve a cross-free set of size $2k$ in an STS($6k + 3$) the number of blocks inside the X_i s tends to infinity with n .

To see the connection of cross-free sets to $f(n)$, let $G(n)$ be the size of the largest cross-free set present in **some** STS(n).

Lemma 3. $f(n) \leq n - G(n)$.

Proof. Suppose $|X_1| = |X_2| = |X_3| = G(n)$ for a cross-free set $X_1, X_2, X_3 \subset V$ in some STS(n). Then coloring any block B with the smallest i such that $B \subset V \setminus X_i$, we have a 3-coloring of the blocks with one nontrivial monochromatic connected component of size $n - G(n)$ in each color. \square

The next result implies the equality $f(6k + 3) = 4k + 3$ for k divisible by 3 in Theorem 1.

Theorem 4. For $n = 18k + 3$, $G(n) = 6k$.

In fact, Theorem 4 probably can be extended, it would imply Conjecture 2

Conjecture 5. $G(6k + 3) = 2k, G(6k + 1) = 2k - 1$.

It is easy to see that Conjecture 5 is sharp (if true). Indeed, a cross-free set of size $2k + 1$ in an STS($6k + 3$) would mean that there are at most $3 \binom{2k+1}{2}$ blocks and that is less than $\binom{6k+3}{2}/3$. Similarly, a cross-free set of size $2k$ in an STS($6k + 1$) would show that there are at most $3k + 3 \binom{2k}{2}$ blocks, less than $\binom{6k+1}{2}/3$.

One can define $f_r(n)$ similarly for r -colorings of blocks. A lower bound on it can be easily derived from known results.

Proposition 6. $f_r(n) \geq \lceil \frac{n}{r-1} \rceil$.

Proof. Any r -coloring of the blocks of an STS(n) defines an r -coloring of the edges of K_n , by coloring the three pairs defined by a block with the color of the block. In this coloring there is a monochromatic, say red connected component C with at least $\lceil \frac{n}{r-1} \rceil$ vertices, proved first in [4], a more accessible account is the survey [5]. The blocks covering the red edges of C obviously span a red connected component on C . \square

The lower bound of Proposition 6 is trivially sharp for $r = 2$ but also for certain other values of r , starting with $r = 4, 8, 10, 14, \dots$

Proposition 7. $f_r(n) = \frac{n}{r-1}$ for infinitely many n if $r - 1$ is in the form $3^m, p^m, q^{2m}$ where $m \geq 1$, p, q are primes, $p \equiv 1 \pmod{6}, q \equiv -1 \pmod{6}$.

Proof. $f_r(n) \geq \frac{n}{r-1}$ follows from Proposition 6. Suppose $r - 1$ is a prime power in the form $3^m, p^m, q^{2m}$ where $m \geq 1$, p, q are primes, $p \equiv 1 \pmod{6}, q \equiv -1 \pmod{6}$. This implies that $r - 1 \equiv 1 \pmod{6}$ or $r - 1 \equiv 3 \pmod{6}$. Then there exists an affine plane P of order $r - 1$ and we can define an STS($(r - 1)^2$) by substituting each line of P by a copy of an STS($r - 1$). Then the blocks of STS($(r - 1)^2$) can

be naturally colored with r colors according to the r parallel classes of P . In this coloring every component has size $r - 1 = \frac{(r-1)^2}{r-1}$, providing an example with equality. To get infinitely many, we can apply the well-known direct product construction (see [1]) of $\text{STS}(n_1 n_2)$ from $\text{STS}(n_1), \text{STS}(n_2)$. Assume we already know that for some $t \geq 0$ the blocks of $\text{STS}(3^t(r-1)^2)$ can be r -colored so that each color class has $r - 1$ nontrivial components (of size $3^t(r-1)$) and consider the $\text{STS}(3^{t+1}(r-1)^2)$ defined as $\text{STS}(3^t(r-1)^2) \times T$ where T is a single block on three points. Then each component C in each color class of $\text{STS}(3^t(r-1)^2)$ defines a component $C \times T$ in $\text{STS}(3^{t+1}(r-1)^2)$ whose blocks in $C \times T$ can be colored with the same color. This defines a natural r -coloring of the blocks of $\text{STS}(3^{t+1}(r-1)^2)$, preserving the property that each color class has $r - 1$ nontrivial components. \square

Our problem to determine $f(n)$ led to find $G(n)$, the size of the largest cross-free set present in **some** $\text{STS}(n)$. It seems natural and interesting to find or estimate the size $g(n)$ of the largest cross-free set present in **any** $\text{STS}(n)$. Obviously,

$$G(n) \geq g(n) \geq \frac{\alpha(n)}{3}$$

where $\alpha(n)$ is the largest independent set present in *any* $\text{STS}(n)$. For the most recent result and history on $\alpha(n)$ see [3].

Problem 8. *Is $g(n)$ significantly smaller than $G(n)$?*

2 Proof of Theorems 1, 4

We prove first that $f(6k+3) \geq 4k+3$, $f(6k+1) \geq 4k+2$.

Suppose that the blocks of an $\text{STS}(V, \mathcal{B})$ with $|V| = n$ are 3-colored and consider the three components C_1, C_2, C_3 in colors 1, 2, 3 containing a point $v \in V$. There are some cases according to the number of C_i s with points covered only by C_i , we call such points as “private parts” of C_i .

Case 1. No C_i has private part. In this case the sets C_i doubly cover $V \setminus \{v\}$ and v is triply covered. This implies easily that $f(6k+3) \geq 4k+3$ and also $f(6k+1) \geq 4k+2$, unless if the C_i s intersect in one point and all the three doubly covered sets have size $2k$. However, in this case we can have $3k$ blocks covering v and any other block must cover a pair of $(C_i \cap C_j) \setminus \{v\}$. Thus altogether we have at most $3k + 3\binom{2k}{2} < \frac{\binom{6k+1}{2}}{3}$ blocks in $\text{STS}(6k+1)$ and that is a contradiction.

Case 2. Only C_1 has a private part. Now there is no point $w \in V$ that belongs to $(C_2 \cap C_3) \setminus C_1$, otherwise no block can cover wx where x is from the private part of C_1 . Thus in this case C_1 covers V .

Case 3. Two C_i s, say C_1, C_2 have private parts. Now $(C_1 \cap C_3) \setminus C_2$ and $(C_1 \cap C_2) \setminus C_3$ are both empty and any pair of points x, y from the private parts of C_1, C_2 , respectively, must be in a block colored with color 3. Thus the union of the private parts of C_1, C_2 is part of a component C of color 3. We can now apply the argument of Case 1 to the components C, C_1, C_2 .

Case 4. All C_i s have private parts. Now sets covered by precisely two of C_1, C_2, C_3 must be empty and the private parts $X_i \subset C_i$ together with $X_4 = C_1 \cap C_2 \cap C_3$, partition V . Pairs of points $x \in X_1, y \in X_2$ must be in a block of color 3, pairs of points $x \in X_1, y \in X_3$ must be in a block of color 2, pairs of points $x \in X_2, y \in X_3$ must be in a block of color 1, thus the union of any two X_i s is covered by (in fact equal to) a monochromatic component. Observe that every block of our (V, \mathcal{B}) must contain a pair from some of the X_i s, thus

$$s = \sum_{i=1}^4 \binom{|X_i|}{2} \geq \frac{\binom{n}{2}}{3}. \quad (1)$$

First let $n = 6k + 3$, assume w.l.o.g that

$$|X_1| \leq |X_2| \leq |X_3| \leq |X_4|.$$

If $|X_1| \geq 3k + 1 + t$ for some positive integer t then let X_j be the largest among X_2, X_3, X_4 . Then

$$|X_1| + |X_j| \geq 3k + 1 + t + \frac{3k - t + 2}{3} \geq 4k + 3$$

proving what we need. However, if $|X_1| \leq 3k + 1$ then the maximum of s (under the condition that each component has size at most $2k + 2$) is obtained when $|X_1| = 3k + 1, |X_2| = |X_3| = k + 1, |X_4| = k$. But this contradicts to (1). Similar argument works if $n = 6k + 1$, then

$$|X_1| = 3k + 1, |X_2| = |X_3| = |X_4| = k$$

gives the largest s and the contradiction.

This finishes the proof of the two inequalities of Theorem 1. It is left to prove that $f(6k + 3) = 4k + 3$ if k is divisible by 3, i.e. to prove Theorem 4. In fact we need to prove only that $G(n) \geq 6k$, however $G(n) < 6k + 1$ follows easily: a partition of V for a STS($18k + 3$) into three sets of size $6k + 1$ cannot be cross-free since then there are at most $t = 3 \binom{6k+1}{2}$ blocks and t is less than the number of blocks required in an STS($18k + 3$).

We construct an STS($18k + 3$) with a cross-free set of size $6k$ as follows. Let H_k be the graph with $6k$ vertices and $4k$ edges, having $2k$ components, k of them a P_4 ,

a path on four vertices, and k of them a single edge. We call the *middle* of H_k the union of the middle edges of the P_4 components in H_k . A *near factor* of a graph with $2m$ (or $2m - 1$) vertices means $m - 1$ pairwise disjoint edges.

Lemma 9. *Let T be the graph containing k vertex disjoint edges on $6k$ vertices. Then the edge set of $G_k = K_{6k} \setminus T$ can be partitioned into $2k$ factors F_1, \dots, F_{2k} and $4k$ near factors E_1, \dots, E_{4k} in such a way that the pairs uncovered by the near factors form a graph isomorphic to H_k and in the isomorphism the middle of H_k corresponds to the pairs of T .*

Based on the lemma, we define an STS($18k + 3$) with a cross-free set of size $6k$. Take three disjoint copies of H_k on vertex sets X_0, X_1, X_2 and define \mathcal{T} as a partial triple system PTS($18k$) on $\cup_{i=0}^2 X_i$ as follows. Partition each X_i into k P_4 paths $a_{6j+1}^i, a_{6j+2}^i, a_{6j+3}^i, a_{6j+4}^i$ and k edges a_{6j+5}^i, a_{6j+6}^i for $j = 0, \dots, k - 1$. This way each X_i spans a copy of H_k .

Now Lemma 9 can be applied with vertex set X_0 to obtain $2k$ factors and $4k$ near factors with the required properties (with respect to the copy of $H_k \subset X_0$). We can extend these factors and near-factors to blocks of \mathcal{T} , using vertices of X_1 as follows. Let a_{6j+4}^1 define blocks with the pairs of the near factor E_{4j+1} with uncovered pair a_{6j+2}^0, a_{6j+3}^0 , $j = 0, \dots, k - 1$. Then a_{6j+1}^1 defines blocks with the pairs of the near factor E_{4j+2} with uncovered pair a_{6j+5}^0, a_{6j+6}^0 , $j = 0, \dots, k - 1$; similarly a_{6j+6}^1 defines blocks with the pairs of the near factor E_{4j+3} with uncovered pair a_{6j+1}^0, a_{6j+2}^0 , $j = 0, \dots, k - 1$; and a_{6j+5}^1 defines blocks with the pairs of the near factor E_{4j+4} with uncovered pair a_{6j+3}^0, a_{6j+4}^0 , $j = 0, \dots, k - 1$. Finally, a_{6j+2}^1, a_{6j+3}^1 define blocks with the pairs of the factors F_{2j+1}, F_{2j+2} , $j = 0, \dots, k - 1$.

The construction of the previous paragraph can be repeated cyclically, defining blocks with one vertex in X_2 and two in X_1 , and a third time defining blocks with one vertex in X_0 and two in X_2 . By Lemma 9, the partial STS \mathcal{T} defined this way covers all pairs of $X_0 \cup X_1 \cup X_2$ except a 3-regular graph U of with the following edges: a_{6j+2}^i, a_{6j+3}^i for $i = 0, 1, 2$ and $j = 0, \dots, k - 1$ (formed by the middle of the three copies of H_k) and the $3 \times 8k$ edges between the pairs X_i, X_j that belong to the uncovered pairs of the $3 \times 4k$ near factors. It can be easily seen that the graph U can be factored into three 1-factors. In fact, these factors are

$$\begin{aligned} & a_{6j+2}^i, a_{6j+3}^i, a_{6j+1}^i, a_{6j+6}^{i-1}, a_{6j+5}^i, a_{6j+4}^{i-1}, \\ & a_{6j+4}^i, a_{6j+2}^{i-1}, a_{6j+5}^i, a_{6j+3}^{i-1}, a_{6j+6}^i, a_{6j+1}^{i-1}, \\ & a_{6j+1}^i, a_{6j+5}^{i-1}, a_{6j+4}^i, a_{6j+3}^{i-1}, a_{6j+6}^i, a_{6j+2}^{i-1}, \end{aligned}$$

where $i = 0, 1, 2$ and $j = 0, \dots, k - 1$ with arithmetic on i, j -s are modulo 3, 6, respectively.

Finally, \mathcal{T} is extended to an STS($18k+3$) by extending each factor of U to a block with one of three new points A, B, C which also forms the last block. This finishes the proof of Theorem 4 and with it Theorem 1. \square

Proof of Lemma 9. The required partition is constructed from the *standard factorization* of K_{6k} on vertex set $\{1, 2, \dots, 6k-1\} \cup \infty$ where (for $i = 1, 2, \dots, 6k-1$) factor F_i contains (i, ∞) and $\{(i-j, i+j) : 1 \leq j \leq 3k-1\}$ with mod $6k-1$ arithmetic.

We shall keep $2k-1$ of the factors F_i and define the near factors E_1, \dots, E_{4k} by deleting one edge from each of the other $4k$ factors so that the deleted edges form a graph isomorphic to H_k . The factor formed by the middle of H_k is left uncovered and all other edges of H_k form a new factor F^* which is added as the $2k$ -th factor in the partition. To define the construction, it is enough to specify the set of $4k$ pairs (all from different F_i) which form a graph Z_k isomorphic to H_k . The construction is the simplest for $k \equiv 1 \pmod{2}$ so we describe that first.

Suppose that $k \equiv 1 \pmod{2}$ and set $W = \{(1, 3), (2, 4), (3, 5), (5, \infty)\}$. Moreover, for $m \in \{6, 12, \dots, 6(k-2)\}$ let $L_m = A_m \cup B_m$ be the following set of eight pairs on twelve consecutive numbers:

$$A_m = \{(m, m+2), (m+2, m+4), (m+4, m+6), (m+1, m+3)\},$$

$$B_m = \{(m+5, m+7), (m+7, m+9), (m+9, m+11), (m+8, m+10)\}.$$

It is immediate to check that W, A_m, B_m are all define (6-vertex) graphs with a P_4 component and a K_2 component. Thus the graph Z_k defined by W (for $k=1$) and by $W \cup_{m=6}^{6(k-2)} L_m$ (for odd $k > 1$) is isomorphic to H_k . Moreover, since all edges (apart from $(5, \infty)$) of Z_k are in the form $(j, j+2)$ and $j \neq 4$, each edge of Z_k belongs to different F_i .

The case $k \equiv 0 \pmod{2}$ is slightly more involved, we use another type of components C_m, D_m (beside W) to define Z_k .

$$C_m = \{(m, m+1), (m, m+2), (m+2, m+4), (m+3, m+5)\},$$

$$D_m = \{(m, m+2), (m+1, m+2), (m+1, m+3), (m+4, m+5)\}.$$

For $k=2$ we use W followed by C_6 to define Z_2 . For $k > 2$ start with W , then $\frac{k}{2}$ copies of C_m ($m = 6, 12, \dots, 3k$) then $\frac{k-2}{2}$ copies of D_m ($m = 3k+6, \dots, 6(k-1)$). To check here that each edge of Z_k belongs to different F_i , note that ‘‘jumping pairs’’ $(j, j+2)$ are obviously from different F_i (from F_{j+1}). The same is true for the ‘‘consecutive pairs’’ $(j, j+1)$. To check consecutive pairs against jumping pairs, notice that for $m = 6, 12, \dots, 3k$ the pair $(m, m+1)$ of C_m belongs to F_{3k+m} , a starting point of the D -block opposite to C_m thus it is not skipped by any jumping pair. Similarly, for

$m = 3k + 6, \dots, 6(k - 1)$, the pairs $(m + 1, m + 2)$ and $(m + 4, m + 5)$ in D_m belong to F_{m+2-3k} and F_{m+5-3k} , respectively, and they are not skipped in their opposite C -blocks. \square

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