ON THE NUMBER OF CONJUGACY CLASSES OF A PERMUTATION GROUP

MARTINO GARONZI AND ATTILA MARÓTI

ABSTRACT. We prove that any permutation group of degree $n \geq 4$ has at most $5^{(n-1)/3}$ conjugacy classes.

1. INTRODUCTION

One of the most important invariants of a finite group G is the number $k(G)$ of its conjugacy classes. This is equal to the number of complex irreducible characters of G. There are many interesting open problems concerning $k(G)$. For example it is not known whether there exists a universal constant $c > 0$ so that $k(G) > c\log(|G|)$ holds for any finite group G (see [16], [9], [8]). In this paper we are interested in giving upper bounds for $k(G)$. Such problems are closely related to the $k(GV)$ theorem (see [19]) and the non-coprime $k(GV)$ problem [3].

One of the important special cases in giving upper bounds for $k(G)$ is the case when G is a permutation group of degree n. Kovács and Robinson [10] proved that $k(G) \leq 5^{n-1}$ and reduced the proposed bound of $k(G) \leq 2^{n-1}$ to the case when G is an almost simple group. This latter bound was later proved by Liebeck and Pyber in [11] for arbitrary finite groups G. Kovács and Robinson in [10] also proved that $k(G) \leq 3^{(n-1)/2}$ for G a solvable permutation group of degree $n \geq 3$. Later Riese and Schmid [18] proved the same bound for $3'$, $5'$ and $7'$-groups, and in [13] the second author obtained the bound $k(G) \leq 3^{(n-1)/2}$ for an arbitrary finite permutation group G of degree $n \geq 3$.

By imposing restrictions on the set of composition factors of the permutation group G, one can obtain stronger bounds on $k(G)$. For example, in [13] it was shown that $k(G) \leq (5/3)^n$ whenever G has no composition factor isomorphic to C_2, and more recently Schmid [20] proved that $k(G) \leq 7^{(n-1)/4}$ for $n \geq 5$ where G has no non-abelian composition factor isomorphic to an alternating group or a group in [1]. However it seems hard to generalize these bounds for arbitrary groups.

The main result of the current paper is the following.

Theorem 1.1. *A permutation group of degree $n \geq 4$ has at most $5^{(n-1)/3}$ conjugacy classes.*

Date: July 23, 2014.

2000 *Mathematics Subject Classification.* 20C99.

Key words and phrases. permutation group, number of conjugacy classes.

The first author wishes to thank the hospitality and financial support of Technische Universität Kaiserslautern where much of this work was carried out. The research of the second author was supported by an Alexander von Humboldt Fellowship for Experienced Researchers and by OTKA K84233.

The direct product of $n/4$ copies of S_4 or D_8 is a permutation group of degree n with exactly $5^{n/4}$ conjugacy classes (whenever n is a multiple of 4). But even more can be said. Pyber has pointed out (see [10] and also [11]) that for each constant $0 < c < 5^{1/4}$ there are infinitely many transitive permutation groups G with $k(G) > c^{n-1}$. In fact, G can be taken to be the transitive 2-group $D_8 \wr C_{n/4} \leq S_n$ whenever n is a power of 2 at least 4. (This can be seen by (1) of Lemma 2.1.)

However, for special subgroups of primitive permutation groups G, one may give better than exponential bounds for $k(G)$. A transitive permutation group G is called primitive if the stabilizer of any point is a maximal subgroup in G. This is equivalent to saying that the only blocks of imprimitivity for G are the singleton sets and the whole set on which G acts. The symmetric group S_n is always primitive and it is easy to see that $k(S_n) = p(n)$, the number of partitions of n. Hardy and Ramanujan [6] and independently but later Uspensky [21] gave an asymptotic formula for $p(n)$ and this is less than exponential. It is a natural question whether $k(G) \leq p(n)$ for any primitive permutation group of degree n. This was shown to be true for sufficiently large n by Liebeck and Pyber [11] and later for all normal subgroups of all primitive groups by the second author in [13]. In this paper we go even further by showing that for any subgroup H of any primitive permutation group G of degree n, apart from the alternating group A_n and S_n, we have $k(H) \leq p(n)$ (see Theorem 3.1). This result is used to give a general upper bound for $k(G)$ for a transitive permutation group G from knowledge of the partition function (see Theorem 4.1). Finally, this result is used to derive Theorem 1.1.

2. Preliminaries

The following lemma collects basic information on the number of conjugacy classes in a subgroup and in a normal subgroup of a finite group.

Lemma 2.1. *Let H be a subgroup and N be a normal subgroup of a finite group G. Then*

(1) $k(H)/|G:H| \leq k(G) \leq k(H) \cdot |G:H|$;
(2) $k(H) \leq \sqrt{|G|k(G)}$; and
(3) $k(G) \leq k(N) \cdot k(G/N)$.

Proof. Statements (1) and (3) can be found in [5] (see also [15]). Statement (2) follows from (1). □

In special cases we will need a straightforward consequence of the Clifford-Gallagher formula [19, Page 18]. The second statement of the following lemma follows from [19, Proposition 8.5d].

Lemma 2.2. *Let $\mathrm{Irr}(N)$ denote the set of complex irreducible characters of a normal subgroup N of a finite group G. Then $S = G/N$ acts on $\mathrm{Irr}(N)$ in a natural way and let $I_S(\theta)$ denote the stabilizer of a character θ in $\mathrm{Irr}(N)$. Then we have*

$$k(G) \leq \sum_{\theta \in \mathrm{Irr}(N)} k(I_S(\theta))/|S : I_S(\theta)|.$$

Moreover if N is a full direct power of a finite group T and S permutes the factors of N transitively and faithfully, then $k(G) \leq k(T \wr S)$.

For a non-negative integer n let the number of partitions of n be denoted by $p(n)$. This is the number of conjugacy classes of the symmetric group S_n. In 1918 Hardy and Ramanujan [6] and independently but later Uspensky [21] proved the following asymptotic formula.

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

In 1937 Rademacher [17] gave a series expansion of $p(n)$, however here we will only need the following lower and upper bounds.

Lemma 2.3. *Let $n \geq 1$ be an integer. Then $e^{2.5\sqrt{n}}/13n < p(n) < e^{\pi\sqrt{2n/3}}$.*

Proof. For the upper bound see [4] and for the lower bound see [13]. □

3. Primitive groups

A transitive permutation group G is called primitive if the stabilizer of any point is a maximal subgroup in G. This is equivalent to saying that the only blocks of imprimitivity for G are the singleton sets and the whole set on which G acts. The symmetric and alternating groups, S_n and A_n, are examples of primitive permutation groups. In this section we will extend Corollary 2.15 (i) of [11] and Theorem 1.3 (i) of [14] to show Theorem 3.1. This result heavily depends on Theorem 1.1 of [12] and also on [2].

Theorem 3.1. *Let G be a primitive permutation group of degree n different from A_n and S_n. Then we have $k(H) \leq p(n)$ for every subgroup H of G.*

Proof. Let G be a primitive permutation group of degree n. If $H \leq G$ are subgroups of $S_m \wr S_r$ in its product action on $n = \binom{m}{k}^r$ points where $m \geq 5$ and S_m acts on k-subsets for some k with $1 \leq k < n$, then $k(H) \leq 2^{mr-1}$ by Theorem 2 of [11]. But for $(k, r) \neq (1, 1)$ we have

$$2^{mr-1} < \frac{e^{2.5\sqrt{\binom{m}{k}^r}}}{13\binom{m}{k}^r} < p(\binom{m}{k}^r) = p(n),$$

where the second inequality follows from Lemma 2.3. Thus we may exclude these cases from the discussion.

By Theorem 1.1 of [12], we then know that $|G| < n^{1+[\log_2(n)]}$ or G is one of the Mathieu groups in their 4-transitive action.

Again by Lemma 2.3, we see that $|G| < n^{1+[\log_2(n)]} < p(n)$ for $n \geq 1500$. Furthermore, by using the exact values of $p(n)$ available in [2], $|G| < p(n)$ is true even for $n \geq 1133$.

If $120 \leq n < 1133$ then $p(n) < |G| < n^{1+[\log_2(n)]}$ holds only if $n = 1024$ and $G = AGL(10, 2)$, $n = 512$ and $G = AGL(9, 2)$, $n = 256$ and $G = AGL(8, 2)$, or $n = 511$, 255, 190, 171, 153, 144, 136, 128, 127, 121, or 120 (this was also obtained by [2]).

If G is any of these exceptional cases (with $n \geq 120$) and is not a subgroup of $S_m \wr S_r$ in its product action discussed above, then $k(G)|G| < p(n)^2$, which forces $k(H) < p(n)$ for any subgroup H of G (by (2) of Lemma 2.1). Furthermore if

$n \leq 119$ then we again have $k(G)|G| < p(n)^2$, unless $n = 64$ and $G = AGL(6,2)$, or $n \leq 32$ and G is almost simple or of affine type. Both these statements were derived by [2].

For almost simple primitive groups G of degrees n at most 32 (including the 4-transitive Mathieu groups but excluding A_n and S_n) we can compute the subgroup lattice of G by [2] and so the claim can be checked for all subgroups H of G. Thus we may assume that G is an affine primitive permutation group of degree 64 or at most 32.

We must show that if H is a subgroup of $AGL(m,p)$ with $n = p^m \leq 64$, then $k(H) \leq p(n)$. If $m = 1$ then it is easy to see that $k(H) \leq p = n \leq p(n)$. If $m = 2$ and $p = 5$ or 7, or if $p^m = 27$, then $|AGL(2,p)|k(AGL(2,p)) < p(n)^2$ and we may apply (2) of Lemma 2.1. Thus we may assume that $p = 2$ or 3. The full subgroup lattice of $AGL(m,p)$ can be computed by [2] for all remaining cases except $(m,p) = (5,2)$ and $(m,p) = (6,2)$, and thus the validity of the inequality $k(H) \leq p(n)$ can be checked directly.

Let $m = 5$ and $p = 2$. Any subgroup of $GL(5,2)$ has less than 260 conjugacy classes (this can be obtained by [2] by viewing $GL(5,2)$ as a permutation group on 31 points), and so (3) of Lemma 2.1 gives $k(H) < 260 \cdot 32 < p(32)$ for any subgroup H of $AGL(5,2)$.

Let $m = 6$ and $p = 2$. Put $N = O_2(H)$. The factor group H/N can be viewed as a completely reducible subgroup on a vector space of size 64 (see [11, Page 554]). We claim that $k(H/N) \leq 63$. For this observe that for irreducible linear subgroups T of $GL(V)$ we have $k(T) < |V|$ whenever V is a vector space of size a power of 2 at most 64. (This can be checked by [2] by going through stabilizers of all affine primitive permutation groups of degrees a power of 2 at most 64.) Then, by using part (3) of Lemma 2.1, induction, and noting that a normal subgroup of a completely reducible linear group also acts completely reducibly on the same vector space (Clifford's theorem), we obtain the claim.

Let S be a Sylow 2-subgroup of $AGL(6,2)$ containing N. Suppose that $|S : N| \geq 64$. Then (3) of Lemma 2.1 gives $k(H) \leq |N| \cdot k(H/N) \leq 2^{15} \cdot 63 < 2^{21} < p(64)$. Now suppose that $|S : N| \leq 16$. Then $k(N) \leq |S : N| \cdot k(S) \leq 16 \cdot 1430$, by (1) of Lemma 2.1, and so $k(H) \leq k(N) \cdot k(H/N) \leq 16 \cdot 1430 \cdot 63 < p(64)$. So the only case missing is when $|S : N| = 32$. We would like to bound $k(N)$ in this case. Let S_1 be a maximal subgroup of S containing N. By [2] we know that $k(S_1) \leq 1723$ or $k(S_1) = 1768$. In the first case we have $k(N) \leq 16 \cdot 1723$, and so $k(H) \leq 16 \cdot 1723 \cdot 63 < p(64)$. So suppose that the second case holds. Then let S_2 be a maximal subgroup in S_1 containing N. By [2] again, we know that $k(S_2) \leq 2240$, and so $k(N) \leq 8 \cdot 2240$. This gives $k(H) \leq 8 \cdot 2240 \cdot 63 < p(64)$. □

A straightforward consequence of Theorem 3.1 is the following.

Corollary 3.2. *If H is a subnormal subgroup of a primitive permutation group of degree n, then $k(H) \leq p(n)$.*

Proof. If $H = S_n$ then this is clear. If $H = A_n$, then this follows from [14, Lemma 2.3]. Otherwise apply Theorem 3.1. □

4. Transitive groups

In this section we will give an upper bound in terms of the partition function for $k(G)$ when G is a transitive permutation group. This result depends on Theorem 3.1 and is used in the proof of Theorem 1.1.

Theorem 4.1. *Let G be a transitive permutation group of degree n with point stabilizer H. Consider a chain*

$$H = H_0 < H_1 < \ldots < H_t = G$$

with H_i maximal in H_{i+1} for $i = 0, \ldots, t-1$ and call $a_i := |H_i : H_{i-1}|$ for $i = 1, \ldots, t$, so that $a_1 \cdots a_t = |G : H| = n$. Then

$$k(G) \leq \left(p(a_1)^{1/a_1} p(a_2)^{1/a_1 a_2} \cdots p(a_{t-1})^{1/a_1 \cdots a_{t-1}} p(a_t)^{1/a_1 \cdots a_t}\right)^n.$$

Proof. Let G be a minimal counterexample to the statement of the theorem with a fixed chain of subgroups. By Corollary 3.2, we may assume that $t \geq 2$. We now construct a subnormal filtration as in [20]. Let B_0 be the core of H_1 in G, so that G/B_0 is a transitive permutation group of degree n/a_1. Let N be the core of $H = H_0$ in H_1, so that H_1/N is a primitive permutation group of degree a_1. Let $\{x_i\}_{1 \leq i \leq n/a_1}$ be a set of representatives for the right cosets of H_1 in G, with $x_1 = 1$, and define inductively $B_i := B_{i-1} \cap N^{x_i}$ for $i \geq 1$. Then $B_i = B_{i-1} \cap B_1^{x_i}$ and since N is normal in H_1 and H is core-free,

$$B_{n/a_1} \subseteq \bigcap_{i=1}^{n/a_1} N^{x_i} = \bigcap_{g \in G} N^g \subseteq \bigcap_{g \in G} H^g = \{1\}.$$

We obtain a subnormal filtration (grading) $B = B_0 \rhd B_1 \rhd \cdots \rhd B_{n/a_1} = \{1\}$. Observe that $B_i \trianglelefteq B_0$ for all $0 \leq i \leq n/a_1$, this is easily seen by induction on i: since $B_0 \trianglelefteq G$ we have $B_1^{x_i} \trianglelefteq B_0^{x_i} = B_0$ and hence $B_i = B_{i-1} \cap B_1^{x_i} \trianglelefteq B_0$. Let $L := B_0 \cap N$. We have

$$B_i/B_{i+1} = B_i/B_i \cap B_1^{x_{i+1}} = B_i/B_i \cap L^{x_{i+1}} \cong B_i L^{x_{i+1}}/L^{x_{i+1}} \trianglelefteq B_0/L^{x_{i+1}} \cong B_0/L.$$

Since $B_0/L \cong B_0 N/N \trianglelefteq H_1/N$, each B_i/B_{i+1} is isomorphic to a subnormal subgroup of the primitive group H_1/N of degree a_1. By Corollary 3.2, $k(B_i/B_{i+1}) \leq p(a_1)$ for all i. Now consider the chain $H_1/B < H_2/B < \ldots < H_{t-1}/B < H_t/B = G/B$. Each subgroup of the chain is maximal in the following one hence by minimality of G the theorem holds for G/B relative to this chain and hence

$$\begin{aligned}
k(G) &\leq k(B)k(G/B) \leq \Big(\prod_{i=0}^{n/a_1 - 1} k(B_i/B_{i+1})\Big) \cdot k(G/B) \\
&\leq p(a_1)^{n/a_1} \cdot \left(p(a_2)^{(n/a_1)/a_2} \cdots p(a_t)^{(n/a_1)/(a_2 \cdots a_t)}\right) \\
&= p(a_1)^{n/a_1} p(a_2)^{n/a_1 a_2} \cdots p(a_{t-1})^{n/a_1 \cdots a_{t-1}} p(a_t).
\end{aligned}$$

The proof is complete. □

5. Proof of Theorem 1.1

In this section we will prove our main result. The first lemma enables us to deal with cases when n is relatively small.

Lemma 5.1. *If G is a permutation group of degree n all of whose orbits have lengths at most 23 then $k(G) \leq 5^{n/4}$.*

Proof. By induction on n, as in Lemma 3.1 of [14], we may assume that G is transitive. For transitive groups the claim can be checked by [2]. □

By [7] all transitive permutation groups of degree at most 30 are known therefore the 23 in Lemma 5.1 could perhaps be replaced by 30 (or even 31) but it is not clear to what extent this possible improvement could be of help.

Now we proceed to the proof of Theorem 1.1. Many of the computations below have been performed by [2], but we will not point this out in all cases.

Let G be as in the statement of the theorem. It acts faithfully on a set Ω of size n.

We proceed by induction on n. By Lemma 5.1 we can assume that $n \geq 24$. Suppose G is intransitive and let O be a nontrivial orbit of G of size $1 < r < n$. Let N be the kernel of the action of G on O. Then N acts faithfully on $n-r$ points and G/N acts faithfully on r points hence if $r, n-r \geq 4$ then

$$k(G) \leq k(N) \cdot k(G/N) \leq 5^{(n-r-1)/3} \cdot 5^{(r-1)/3} < 5^{(n-1)/3}.$$

If $r \leq 3$ then $k(G/N) \leq r$, and if $n-r \leq 3$ then $k(N) \leq n-r$, from which the result follows likewise. Hence we may assume that G is transitive.

Let H be the stabilizer of $\alpha \in \Omega$ in G. If H is maximal in G then G is a primitive permutation group and thus by Theorem 4.1 and Lemma 2.3 we have $k(G) \leq p(n) \leq e^{\pi\sqrt{2n/3}}$ and this is at most $5^{(n-1)/3}$ for $n \geq 25$.

So assume that H is not maximal in G and let K be such that $H < K < G$. Let $a := |K : H|$ and $b := |G : K|$. Notice that the K-orbit Δ containing α is a block of imprimitivity for the action of G. Let B be the kernel of the action of G on the block system Σ associated to Δ, in other words, B is the normal core of K in G. G/B is a transitive permutation group of degree b. By taking subsequent kernels on the blocks (i.e. arguing as in the proof of Theorem 4.1) we find a subnormal sequence $B_0 = B \trianglerighteq B_1 \trianglerighteq \ldots \trianglerighteq B_b = \{1\}$ such that each factor group B_i/B_{i+1} can be considered as a permutation group of degree a.

If a and b are both at least 4 then we may apply induction and find

$$k(G) \leq k(B) \cdot k(G/B) \leq (5^{(a-1)/3})^b \cdot 5^{(b-1)/3} = 5^{(n-1)/3}.$$

So we may assume that whenever $H < L < G$ either $|G : L| \leq 3$ or $|L : H| \leq 3$.

If both a and b are at most 3 then $n \leq 9$ and the result follows from Lemma 5.1. Assume that $4 \leq a \leq 23$ and $b \leq 3$. Then $k(G/B) \leq 3$ hence since the orbits of B have all size at most 23 by Lemma 5.1 we have $k(G) \leq k(B)k(G/B) \leq 5^{n/4} \cdot 3$ which is at most $5^{(n-1)/3}$ since $n \geq 24$.

We are in one of the following cases.

(1) H is maximal in K and $b = |G : K| \in \{2, 3\}$, $a \geq 24$ (consider the block system associated to K).
(2) K is maximal in G and $a = |K : H| \in \{2, 3\}$.
(3) There exists a subgroup $L < G$ such that $H < K < L < G$ with K maximal in L, $a = |K : H| \in \{2, 3\}$, $c = |G : L| \in \{2, 3\}$, and $q = |L : K| \geq 24/a$ (consider the block system associated to L).

We consider the cases separately. In the following "filtration argument" refers to the argument used in the proof of Theorem 4.1. If $B \leq A$ are subgroups of G, by "filtration associated to A and B" we mean the filtration of the kernel of the action of A on the system of blocks associated to B obtained as in the proof of Theorem 4.1.

Case 1. By Theorem 4.1, since $p(b) \leq b$ we have $k(G) \leq p(a)^b b$. Thus it is sufficient to show that $p(a)^b b \leq 5^{(ab-1)/3}$, i.e. $p(a) \leq ((5^{(ab-1)/3})/b)^{1/b}$. For this it is sufficient to show that $p(a) \leq ((5^{(2a-1)/3})/3)^{1/3}$ for $a \geq 24$. If $a \geq 55$ this follows from the bound $p(a) \leq e^{\pi\sqrt{2n/3}}$ (Lemma 2.3), and if $24 \leq a \leq 54$ it follows by inspection.

Case 2. In this case G/B is a primitive group of degree b. Applying the filtration argument used in the proof of Theorem 4.1, since $p(a) \leq a$ we find $k(G) \leq a^b k(G/B)$ and it is enough to prove that $a^b k(G/B) \leq 5^{(ab-1)/3}$, i.e. (*) $k(G/B) \leq ((5^{(ab-1)/3})/a^b) = (5^{(a-1/b)/3}/a)^b$. Recall that $ab = n \geq 24$. If $a = 3$ then $b \geq 8$, now $p(b) \leq (5^{(3-1/8)/3}/3)^b$ follows from the bound $p(b) \leq e^{\pi\sqrt{2b/3}}$ (Lemma 2.3) if $b \geq 34$ and by inspection if $8 \leq b \leq 33$. Suppose now $a = 2$, so that $b \geq 12$. If $b = 12$ let S be a block stabilizer, then $|G : S| = b$ and S is a permutation group on 24 points having at least 2 orbits hence by Lemma 5.1 we have $k(G) \leq 12 \cdot k(S) \leq 12 \cdot 5^6$ and this is less than $5^{23/3}$. Let $b \in \{13, 14, 15\}$. Then using the fact that any primitive group of degree b different from S_b has at most $k(A_b)$ conjugacy classes we see that (*) holds unless $G/B \cong S_b$. If B is not elementary abelian of rank b then the filtration argument implies $k(G) \leq a^{b-1}k(G/B) \leq 5^{(ab-1)/3}$. So assume that $B \cong C_2^b$ and $G/B \cong S_b$. Then by the Clifford-Gallagher formula (Lemma 2.2) $k(G) \leq k(C_2 \wr S_b)$ which is at most $5^{(n-1)/3}$ by [2]. If $16 \leq b \leq 55$ then (*) holds by inspection using $k(G/B) \leq p(b)$, and if $b \geq 56$ it follows from the bound $p(b) \leq e^{\pi\sqrt{2b/3}}$ (Lemma 2.3).

Case 3. By Theorem 4.1, since $p(a) \leq a$ and $p(c) \leq c$ we have $k(G) \leq a^b p(q)^c c$ where $b = qc$. We want to prove that $k(G) \leq 5^{(n-1)/3}$ where $n = ab = aqc$. If $a = 3$ then it is sufficient to prove that $3^b p(q)^c c \leq 5^{(aqc-1)/3}$ for $q \geq 8$. Raising both sides to the power $1/c$ and rearranging, using the fact that $c^{1/c} \leq 1.5$ we see that it is sufficient to prove that $p(q) \leq \frac{1}{1.5}(5^{\frac{1}{3}(3-1/16)}/3)^q$ for $q \geq 8$. If $q \geq 31$ this follows from the bound $p(q) \leq e^{\pi\sqrt{2q/3}}$ (Lemma 2.3), and the case $8 \leq q \leq 30$ is checked by inspection.

Now assume that $a = 2$ and $q \geq 16$. We prove that (**) $2^{cq} \cdot p(q)^c \cdot c \leq 5^{(2cq-1)/3}$. Raising both sides of (**) to the power $1/c$ and rearranging we see that it is enough to prove that $p(q) \leq \frac{1}{1.5}(5^{\frac{1}{3}(2-1/32)}/2)^q$, and for this it is enough to prove that $p(q) \leq \frac{1}{1.5}(1.43)^q$. If $q \geq 60$ this follows from the bound $p(q) \leq e^{\pi\sqrt{2q/3}}$ (Lemma 2.3), and if $16 \leq q \leq 59$ inequality (**) can be checked by inspection.

Now assume that $a = 2$ and either $13 \leq q \leq 15$ or $(q, c) = (12, 3)$. Every nontrivial subnormal subgroup of any primitive group of degree q is a primitive group of degree q, a primitive group of degree q which is not the full symmetric group S_q has at most $k(A_q)$ conjugacy classes, and we have $k(A_{12}) = 43$, $k(A_{13}) = 55$, $k(A_{14}) = 72$, $k(A_{15}) = 94$. Moreover, the ratio $5^{(n-1)/3}/(2^{cq} \cdot p(q)^c \cdot c)$ is less than 2. Thus we may assume that the kernel of the action of G on the system of blocks associated to the primitive group K/H_K is a direct product $C_2^{cq} = C_2^b$, indeed if this is not the case then using the filtration argument we see that $k(G) \leq 2^{cq-1} \cdot p(q)^c \cdot c \leq 5^{(n-1)/3}$. Consider the filtration $\mathcal{F}_1$ associated to L and K. The two factors of this filtration are isomorphic to subnormal subgroups of the primitive group L/K_L of degree q. Consider the filtration $\mathcal{F}_2$ associated to L and H. By the Clifford-Gallagher formula (Lemma 2.2) a fixed factor of $\mathcal{F}_2$ has at most $k(S_2 \wr A)$ conjugacy classes, where A is a permutation group of degree q isomorphic to a factor of $\mathcal{F}_1$. If no factor of $\mathcal{F}_1$ is isomorphic to S_q then it is enough to show that $c \cdot k(A_q)^c \cdot 2^{cq} \leq 5^{(n-1)/3}$ which is true, and if there is a factor of $\mathcal{F}_1$ isomorphic to S_q then since $k(S_2 \wr S_{13}) = 1770$, $k(S_2 \wr S_{14}) = 2665$ and $k(S_2 \wr S_{15}) = 3956$ by the Clifford-Gallagher formula (Lemma 2.2) it is enough to show that $c \cdot k(S_2 \wr S_q) \cdot 2^{q(c-1)} \cdot p(q)^{c-1} \leq 5^{(n-1)/3}$ which is true.

Now assume that $(a, q, c) = (2, 12, 2)$. K is the stabilizer of a block of size 2 (there are 24 such blocks). It acts on the 24 points of a block system consisting of 12 blocks of size 2 intransitively, hence if N denotes the kernel of this action we deduce $k(K/N) \leq 5^{24/4} = 5^6$. Now look at the (faithful) action of N on the remaining 24 points. If this action is intransitive then $k(N) \leq 5^{24/4}$ by Lemma 5.1. If it is transitive then there is an induced transitive action of N on the second block system of twelve blocks of size 2. Since any transitive group of degree 12 has at most $p(12) = 77$ conjugacy classes (by [2]), by Theorem 4.1 we deduce $k(N) \leq 2^{12} \cdot 77$ and even $k(N) \leq 2^{11} \cdot 77$, in which case $k(G) \leq |G : K| \cdot k(K/N) \cdot k(N) \leq 24 \cdot 5^6 \cdot 2^{11} \cdot 77 \leq 5^{47/3}$, unless the kernel of the action of N on the 12 blocks of size 2 is a full direct product C_2^{12}. Suppose this is the case. Let R be the kernel of the transitive action of N on the twelve blocks of size 2 of the second block system. If $k(N/R) \notin \{65, 77\}$ then $k(N/R) \leq 55$ and $k(G) \leq |G : K| \cdot k(K/N) \cdot k(N) \leq 24 \cdot 5^6 \cdot 2^{12} \cdot 55 \leq 5^{47/3}$, so now assume $k(N/R) \in \{65, 77\}$. It can be checked by [2] that $k(S_2 \wr N/R) \in \{1165, 1265, 1960, 2210\}$. By the Clifford-Gallagher formula (Lemma 2.2), $k(N) \leq k(S_2 \wr N/R) \leq 2210$ hence $k(G) \leq |G : K| \cdot k(K/N) \cdot k(N) \leq 24 \cdot 5^6 \cdot 2210 \leq 5^{47/3}$.

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Dipartimento di Matematica, Università degli studi di Padova, Via Trieste 63, 35100 Padova, Italy

E-mail address: mgaronzi@math.unipd.it

Fachbereich Mathematik, Universität Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany

E-mail address: maroti@mathematik.uni-kl.de