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Relations between the local chromatic number and its directed version

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Abstract

The local chromatic number is a coloring parameter defined as the minimum number of colors that should appear in the most colorful closed neighborhood of a vertex under any proper coloring of the graph. Its directed version is the same when we consider only outneighborhoods in a directed graph. For digraphs with all arcs being present in both directions the two values are obviously equal. Here we consider oriented graphs. We show the existence of a graph where the directed local chromatic number of all oriented versions of the graph is strictly less than the local chromatic number of the underlying undirected graph. We show that for fractional versions the analogous problem has a different answer: there always exists an orientation for which the directed and undirected values coincide. We also determine the supremum of the possible ratios of these fractional parameters, which turns out to be e, the basis of the natural logarithm.

1   Introduction

The local chromatic number of a graph, first considered in [1], is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring. Here *closed neighborhood* of a vertex v means $\{v\} \cup N(v)$, where $N(v)$ is the set of neighbors of v. The number of colors is not restricted, it can be much more than the chromatic number. Formally, denoting the local chromatic number by $\psi(G)$, we have

$$\psi(G) := \min_c \max_{v \in V(G)} |\{c(u) : u \in N(v)\}| + 1,$$

where the minimum is taken over all proper vertex-colorings c of G. Though at first sight one may wonder if this parameter can ever be smaller than the chromatic number itself, it is shown in [1] that there are graphs with local chromatic number 3 and arbitrarily large chromatic number. (It is obvious that $\psi(G) \leq \chi(G)$ always holds, where $\chi(G)$ is the chromatic number. It is also easy to see, that $\psi(G) = 2$ implies $\chi(G) = 2$.)

A generalization to directed graphs was defined in [2]. The directed local chromatic number $\psi_d(D)$ of digraph D is defined by

$$\psi_d(D) := \min_c \max_{v \in V(D)} \{c(u) : u \in N_+(v)\} + 1,$$

where c runs over all proper vertex colorings of D and $N_+(v)$ is the set of out-neighbors of v, i.e., the set of vertices w with an arc of D going from v to w. (A proper vertex-coloring of a digraph is meant to be a proper vertex coloring of the underlying undirected graph).

It is immediate from the definitions that for a directed graph $\vec{G}$ and its underlying undirected graph G we always have $\psi_d(\vec{G}) \leq \psi(G)$ and we automatically have equality if $\vec{G}$ contains both orientations of all edges of G.

We say that $\vec{G}$ is an orientation of G if $\vec{G}$ contains all edges of G with exactly one of the two possible orientations and no other arcs. Whether we can achieve the value $\psi(G)$ as the directed local chromatic number of an orientation of G is a natural question, and was already asked in [9]. We will show in this paper that this is not always the case.

With regard to the fractional versions of these chromatic parameters (also introduced in [2]), we show that every undirected graph G has an orientation the fractional directed local chromatic number of which achieves the fractional local chromatic number of G. The latter happens to coincide with the fractional chromatic number as shown in [2]. We also find the supremum of the fractional (local) chromatic number of the underlying graph of any digraph with a given fractional directed local chromatic number.

Observe the analogy with Shannon capacity of graphs and its generalization, the Sperner capacity of digraphs, cf. [3]. (We mention the connection that the Sperner capacity is always bounded from above by the fractional directed local chromatic number, see [2].) The Sperner capacity of a digraph $\vec{G}$ is at most the Shannon capacity of the underlying graph G and equality is achieved if both orientations of every edge of G is present in $\vec{G}$. Whether the Shannon capacity of an undirected graph can always be

achieved as the Sperner capacity of an orientation is an open question. It was investigated in [6], where it was shown that this is always the case for a non-trivial class of graphs. In the light of that result and no indication that this is not always true, the negative answer in case of the local chromatic number is surprising.

The paper is organized as follows. In Section 2 we show that there exists a graph G such that all orientations $\vec{G}$ of G satisfy $\psi_d(\vec{G}) < \psi(G)$. A lemma (a special case of which is) needed to proving this statement has a somewhat tedious proof, therefore the proof of this lemma is postponed to Section 5. Section 3 is devoted to the analogous question for fractional versions of the above parameters. We show that, in contrast to the result in Section 2, the analogous inequality is always an equality in the fractional case. In Section 4 we give tight bounds for the largest possible ratio of these fractional parameters. As already mentioned, Section 5 contains the proof of Lemma 2.1.

We note that the question answered in Section 4 recently turned out to be relevant also for a problem of information transmission, see [8], where a somewhat weaker upper bound was proved independently.

2 Local chromatic number: the oriented and undirected case

For a graph G let

$$\psi_{d,\max} = \max\{\psi_d(\vec{G}) : \vec{G} \text{ is an orientation of } G\}.$$

Here we will construct a graph proving the following statement.

Theorem 1 *There exists a graph G such that*

$$\psi_{d,\max} < \psi(G).$$

To prove this theorem, we will consider the graphs $U(m,k)$ that are the universal graphs for the local chromatic number in the following sense: A graph G has a local k-coloring with at most m colors if and only if G admits a homomorphism to $U(m,k)$. Here a *local k-coloring* is a proper coloring with the vertices of any closed neighborhood receiving at most k distinct colors.

Definition 1 ([1]) *Let $k \le m$ be positive integers and $[m] = \{1, 2, \ldots, m\}$. The graph $U(m,k)$ is defined as follows.*

$$V(U(m,k)) := \{(x, A) : x \in [m], A \subseteq [m], |A| = k - 1, x \notin A\}$$

and

$$E(U(m,k)) := \{\{(x, A), (y, B)\} : x \in B, y \in A\}.$$

The *natural coloring* of $U(m, k)$ colors the vertex $(x, A) \in V(U(m, k))$ with color x. It is easy to see that this is a local k-coloring.

For the proof of Theorem 1 we will use the fact that the natural coloring is basically the only local 3-colorings of $U(5, 3)$. We will prove this statement more generally.

Lemma 2.1 *Let us have $m > k + 1 \geq 4$ and let c be a local k-coloring of the graph $U(m, k)$. Then c is the natural coloring up to permutation of colors.*

As the proof of this lemma is a little tedious, we postpone it to Section 5. Here we present the proof of Theorem 1 assuming that Lemma 2.1 is true for m=5 and k=3.

Proof of Theorem 1 We construct a graph G that has the property given in the statement. Let
$$V(G) = V(U(5, 3)) \cup \{x, y, z\},$$
with $x, y, z \notin V(U(5, 3))$, while the edge set is

$$E(G) = E(U(5,3)) \cup \{\{x,y\}, \{x,z\}, \{y,z\}\} \cup \{\{x, (2, \{1,3\})\}, \{y, (3, \{1,2\})\}, \{z, (1, \{4,5\})\}\}.$$

First we show that $\psi(G) \geq 4$ (in fact it is exactly 4). Assume for contradiction that f is a local 3-coloring of G. Notice that $U(5, 3)$ is a subgraph of G, so by Lemma 2.1 we may assume that f is the natural coloring on $U(5, 3)$, i.e., we assume $f((i, A)) = i$ for every choice of $(i, A) \in V(U(5, 3))$.
The above means that $f(x)$ should be 1 or 3 as otherwise we have too many colors in the neighborhood of $(2, \{1, 3\})$. Similarly $f(y)$ should be 1 or 2 and $f(z)$ should be 4 or 5.
If $f(x) = 1$ then y is connected to vertices of color 1, 3 and 4 or 5, too many distinct colors.
If $f(y) = 1$ then x is connected to vertices of color 1, 2 and 4 or 5, which is also too many. The only remaining possibility is $f(x) = 3$ and $f(y) = 2$, but then the neighborhood of z has too many colors.
The contradiction proves $\psi(G) \geq 4$.

Now we show that G has no orientation $\vec{G}$ that has directed local chromatic number more than 3.

We will focus only on the orientation of the edge $\{x, y\}$. If it is oriented from x to y, then the following coloring g of $\vec{G}$ will show $\psi(G) \leq 3$. Let $g(x) = 1$, $g(y) = 2$ while $U(5, 3)$ gets its natural coloring and $g(z)$ is either 4 or 5. Then the closed outneighborhood of no vertex contains more than 3 colors. For the opposite case when the $\{x, y\}$ edge is oriented from y to x we define the coloring g'. Let $g'(x) = 3, g'(y) = 1$ and $g'(v) = g(v)$ for all vertices $v \neq x, y$. It is easy to check that no closed outneighborhood contains more than 3 colors in this case either. So $\psi_{d,\max}(G) \leq 3$ is strictly smaller than $\psi(G) \geq 4$ and the proof is complete. $\square$

It is annoying that we do not know an example where the gap between $\psi(G)$ and $\psi_{d,\max}(G)$ is more than 1.

3 The maximum fractional directed local chromatic number of an orientation

Unlike the case of the integral values discussed in the previous section, there always exists an orientation of any graph for which the fractional relaxation of the directed local chromatic number attains the fractional (undirected) local chromatic number (which is just the fractional chromatic number, see [2]). This is what we prove in this section.

3.1 Linear programming definitions

All fractional graph parameters are defined as the optimum value of certain linear programs. We start by recalling the definition of fractional colorings and the fractional chromatic number.

Let G be a graph. Let $S(G)$ denote the set of independent sets of G. A *fractional coloring* of G is the collection of real weights x_A for independent sets $A \in S(G)$ satisfying the following linear inequalities.

$$\forall A : \ x_A \geq 0$$
$$\forall v \in V(G) : \sum_{v \in A \in S(G)} x_A \geq 1 \tag{1}$$

The fractional chromatic number $\chi^*(G)$ of G is the minimal total weight of a fractional coloring, that is:

$$\chi^*(G) = \min \sum_{A \in S(G)} x_A,$$

where the minimum is taken for all fractional coloring (x_A) of G. (We can write minimum here as it is attained in this and similar LP problems.)

Note that a proper coloring of G can be turned into a fractional coloring by giving weight 1 to the color classes and weight 0 to all other independent sets. This shows $\chi^*(G) \leq \chi(G)$, where the chromatic number $\chi(G)$ of G is the smallest number of colors in a proper vertex coloring of G.

Let $\vec{G}$ be a directed graph and G the underlying undirected graph. By a fractional coloring of $\vec{G}$ we mean a fractional coloring of G. The *local weight* of a fractional coloring (x_A) is

$$1 + \max_{v \in V(\vec{G})} \sum_{A \in S(G) : N_+(v) \cap A \neq \emptyset} x_A,$$

i.e., 1 plus the maximum total weight a vertex "sees". The fractional directed local chromatic number $\psi_d^*(\vec{G})$ is then the minimal local weight of a fractional coloring of $\vec{G}$.

Note that if we turn a directed local k-coloring of $\vec{G}$ into a fractional coloring it has local weight at most k, thus we have $\psi_d^*(\vec{G}) \leq \psi_d(\vec{G})$ as expected. Here a *directed local k-coloring* is a proper coloring with every out-neighborhood receiving at most $k-1$ colors.

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The *fractional local chromatic number* $\psi^*(G)$ of a graph can be defined as $\psi_d^*(\vec{G})$, where $\vec{G}$ is obtained from G by replacing each of its edges by the two arcs representing its two orientations. Note that this is not new graph parameter, but rather we have $\psi^*(G) = \chi^*(G)$ as proved in [2].

To compare the fractional directed local chromatic number of a digraph to the fractional chromatic number of the underlying undirected graph, we will use dual formulation of latter as the fractional clique number. That is, we use that the fractional chromatic number of a graph is the maximum total weight assigned to the vertices satisfying that (i) all weights are non-negative and (ii) the total weight of the vertices of an independent set does not exceed 1.

3.2 Equality for fractional values

We have already mentioned the result $\psi^*(G) = \chi^*(G)$ from [2]. This statement claims that with any fractional coloring of a graph G there will be a vertex $v \in V(G)$ that "sees" a total weight at least $\chi^*(G) - 1$. Here we prove that v can be chosen from a fixed independent set.

Theorem 2 *For a graph G and a vertex $v_0 \in V(G)$ there exists an independent set $A_0 \in S(G)$ containing v_0 such that the following holds. For any fractional coloring $(x_A)_{A \in S(G)}$ of G there is a vertex $v \in A_0$ such that*

$$\sum_{A \in S(G): N(v) \cap A \neq \emptyset} x_A \geq \chi^*(G) - 1.$$

Proof. Consider an optimal fractional clique $(t_v)_{v \in V(G)}$ of G, that is one attaining

$$\sum_{v \in V(G)} t_v = \chi^*(G). \tag{2}$$

Note that $t_v \geq 0$ for all $v \in V(G)$ and

$$\sum_{A \ni v} t_v \leq 1 \tag{3}$$

for every independent set A of G. Let us choose an independent set A_0 containing v_0 for which

$$\sum_{i \in A_0} t_i = 1. \tag{4}$$

Such a set A_0 must exist, otherwise the value of t_{v_0} could be increased showing that the fractional clique $(t_v)_{v \in V(G)}$ does not have maximal total weight.

Now let $(x_A)_{A \in S(G)}$ be any fractional coloring of G. We have

$$
\begin{aligned}
\chi^*(G) - 1 = \quad & \sum_{v \notin A_0} t_v & \text{by (2) and (4)} \\
\leq \quad & \sum_{v \notin A_0} t_v \sum_{A \ni v} x_A & \text{by (1)} \\
= \quad & \sum_{A \in S(G)} x_A \sum_{v \in A \setminus A_0} t_v & \\
\leq \quad & \sum_{A \in S(G)} x_A (1 - \sum_{v \in A_0 : N(v) \cap A = \emptyset} t_v) & \text{see below} \\
= \quad & \sum_{A \in S(G)} x_A \sum_{v \in A_0 : N(v) \cap A \neq \emptyset} t_v & \text{by (4)} \\
= \quad & \sum_{v \in A_0} t_v \sum_{A : N(v) \cap A \neq \emptyset} x_A & \\
\leq \quad & \max_{v \in A_0} \sum_{A : N(v) \cap A \neq \emptyset} x_A & \text{by (3),}
\end{aligned}
$$

where the marked inequality follows from (3) applied to the independent set $A \cup \{v \in A_0 \mid N(v) \cap A = \emptyset\}$.

Comparing the first and last lines above proves the theorem. $\qquad\square$

Theorem 3 *Let G be a finite undirected graph. Then*

$$
\max_{\vec{G}} \psi_d^*(\vec{G}) = \chi^*(G),
$$

where the maximum is taken over all orientations $\vec{G}$ of G.

Proof. It is clear that $\chi^*(G) = \psi^*(G)$ is an upper bound on the left hand side. We need to give an orientation $\vec{G}$ of G satisfying $\psi_d^*(\vec{G}) \geq \chi^*(G)$.

Consider any independent set A_0 of G that satisfies the statement of Theorem 2. Let us obtain $\vec{G}$ from G by orienting each edge connecting A_0 to its complement in the direction leaving A_0 and orienting the remaining edges of G arbitrarily.

Now consider any fractional coloring $(x_A)_{A \in S(G)}$ of $\vec{G}$. This is a fractional coloring of G and by the choice of A_0 there is a vertex $v \in A_0$ with $\sum_{A : N(v) \cap A \neq \emptyset} x_A \geq \chi^*(G) - 1$. Here $N(v)$ refers to neighborhood of v in G, but the orientation we chose makes this equal to the outneighborhood $N_+(v)$ of v in $\vec{G}$. Thus we also have $\sum_{A : N_+(v) \cap A \neq \emptyset} x_A \geq \chi^*(G) - 1$ finishing the proof. $\qquad\square$

4 The smallest fractional directed local chromatic number

In this section we determine the supremum of $\chi^*(G)$ given a fixed bound on $\psi_d^*(G)$ and, as a consequence, we also find the supremum of the ratio $\frac{\chi^*(G)}{\psi_d^*(G)}$. Here G is a directed graph (note that we dropped the arrow from the notation $\vec{G}$ used in the earlier sections). Recall that $\chi^*(G)$ and $\psi_d^*(G)$ are the fractional chromatic number and the fractional directed local chromatic number, respectively, and the former is defined as the fractional chromatic number of the underlying undirected graph, and is also equal to the fractional

local chromatic number of this undirected graph ([2]). We also note that the boundedness of the above ratio was independently proved in [8], where the somewhat weaker upper bound, $\frac{5}{4}e^2$, was presented.

Theorem 4 *(a) For every finite, loopless directed graph G we have*

$$\chi^*(G) \leq \frac{k^k}{(k-1)^{k-1}} < ek,$$

where $k = \psi_d^*(G) > 1$ and e is the basis of the natural logarithm.
  (b) For every $k \geq 2$ and $\varepsilon > 0$ there exists a loopless finite directed graph G with $\psi_d^*(G) \leq k$ and

$$\chi^*(G) > \frac{k^k}{(k-1)^{k-1}} - \varepsilon.$$

If k is an integer, then the above graph can be chosen to further satisfy $\psi_d(G) = k$.

Note that for graphs G with no edges we have $\chi^*(G) = \psi_d^*(G) = 1$, while non-trivial graphs have $\psi_d^*(G) \geq 2$. Therefore Theorem 4 establishes these tight results:

$$\sup\{\chi^*(G) \mid \psi_d^*(G) \leq k\} \quad = \quad \begin{cases} \frac{k^k}{(k-1)^{k-1}} & \text{for } k \geq 2 \\ 1 & \text{for } 1 \leq k < 2, \end{cases}$$

$$\sup \frac{\chi^*(G)}{\psi_d^*(G)} \quad = \quad e.$$

Before proving the theorem above we give simple alternative definitions for both graph parameters concerned.

Lemma 4.1 *Let $G = (V, E)$ be a (directed or undirected) graph. We have*

$$\chi^*(G) = (\sup \min_{v \in V} P[v \in I])^{-1},$$

where the supremum is over random variables I, whose values are independent sets of G.

Proof. The fractional chromatic number $\chi^*(G)$ is defined as the minimum of the total weight s of fractional colorings $(x_A)_{A \in S(G)}$ of G. The $\leq$ direction of the lemma is proved by considering the random variable I that takes an independent set A with probability x_A/s. Now let I be a random variable taking values from $S(G)$ and let $c = \min_{v \in V} P[v \in I]$. For the reverse direction consider the fractional coloring given by $x_A = P[I = A]/c$. $\square$

Let $G = (V, E)$ be a directed graph, C be an arbitrary finite set (the set of colors) and $r \geq 1$ an integer. We denote by $\binom{C}{r}$ the family of subsets of C of size r. We call a function $\chi : V \to \binom{C}{r}$ an *r-multi-coloring* of G if for all $c \in C$ the set $\{v \in V \mid c \in f(v)\}$ is independent in G. Note that 1-multi-coloring is a proper coloring and in general r-multi-coloring is a homomorphism to the appropriate Kneser graph. An r-multi-coloring χ of

G is an *h-local r-multi-coloring* if $|\bigcup_{w \in N^+(v)} \chi(w)| \leq h - r$ for all $v \in V$. The following easy lemma was already used in [2].

Lemma 4.2 *The fractional directed local chromatic number $\psi_d^*(G)$ of a directed graph G is the infimum of the fractions h/r such that a h-local r-multi-coloring of G exists.*

Proof of Theorem 4. (a) In light of Lemmas 4.1 and 4.2 to prove the upper bound on the fractional chromatic number it is enough consider an h-local r-multi-coloring χ of a directed graph G and define a random variable I taking independent sets as values that satisfies

$$P[v \in I] \geq \frac{(h/r - 1)^{h/r-1}}{(h/r)^{h/r}}$$

for every vertex $v \in V$.

Let us "select" each of the colors used by χ independently and with the same probability $1 - \gamma$ to be set later. Let C' stand for the set of these selected colors. Let $I = \{v \in V \mid \chi(v) \cap C' \neq \emptyset, \forall w \in N^+(v) : \chi(w) \cap C' = \emptyset\}$. Here I consists of the vertices $v \in V$ with at least one of their r colors selected but satisfying that none of their out-neighbors have any selected colors. Clearly, this is an independent set. Let us fix a vertex v. We have $v \in I$ if one of the colors in $\chi(v)$ is selected but none of the colors in $S = \cup_{w \in N^+(v)} \chi(w)$ is selected. Here S and $\chi(v)$ are disjoint by the definition of multi-coloring, so these events are independent and we have $P[v \in I] = (1 - \gamma^r)\gamma^{|S|}$. We have $|S| \leq h - r$ as χ is an h-local r-multi-coloring, so $P[v \in I] \geq (1 - \gamma^r)\gamma^{h-r}$. Setting $\gamma = (1 - r/h)^{1/r}$ gives the desired bound and finishes the proof of the upper bound.

(b) Let now $m \geq k$ be positive integers and C a set of m colors. We consider the directed analogue of the universal graph $U(m, k)$ used above. The directed graph $U_d(m, k)$ is defined as follows. The vertex set consists of pairs (c, H) consisting of $c \in C$ and $H \subset C$ satisfying $c \notin H$ and $|H| = k - 1$. We set a directed edge from (c, H) to (c', H') if $c' \in H$. It is easy to see that $U_d(m, k)$ is the universal graph for directed local k-coloring using m colors, that is, a directed graph has such a coloring if and only if it has a homomorphism to $U_d(m, k)$. We will not use this universality but we use that the *natural coloring* χ_0 mapping the vertex (x, H) to x is a directed local k-coloring showing $\psi_d(G) \leq k$. In fact, we have equality here since $U_d(m, k)$ contains a complete directed subgraph on the k vertices $(x, H \setminus \{x\})$ for some k-element subset H of C.

To bound the fractional chromatic number of $U_d(m, k)$ we use the simple observation that for any graph $G = (V, E)$ we have $\chi^*(G) \geq |V|/\alpha(G)$, where $\alpha(G)$ is the size of the largest independent set in G. We need a lower bound only, but remark that for vertex-transitive graphs such as $U_d(m, k)$ equality holds.

Let I be a largest independent set of $U_d(m, k)$ and let $C' = \{\chi_0(v) \mid v \in I\}$ be the set of their colors under the natural coloring. For a vertex $v = (c, H) \in I$ we must have

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$c \in C'$ and $H \cap C' = \emptyset$. All the vertices of $U_d(m,k)$ satisfying this form an independent set of size $(m-l)\binom{l}{k-1}$, where $l = m - |C'|$. As $U_d(m,k)$ has $m\binom{m-1}{k-1}$ vertices we have

$$\chi^*(U_d(m,k)) \geq \frac{m\binom{m-1}{k-1}}{\max_l((m-l)\binom{l}{k-1})} \geq \frac{m(m-1)^{k-1}}{\max_l((m-l)l^{k-1})} \geq (1-\frac{1}{m})^{k-1}\frac{k^k}{(k-1)^{k-1}}.$$

The last inequality above follows from realizing that $l = m - m/k$ maximizes the denominator. The lower bound obtained here is exactly the desired bound except the multiplicative error term $(1 - 1/m)^{k-1}$. The effect of this error term can be made arbitrarily small by choosing m large enough. This proves the last statement of the theorem.

It remains to prove the first statement of part (b), namely the tightness of our upper bound on the fractional chromatic number for graphs with a non-integer fractional directed local chromatic number. We do this similarly to our proof for the integer case but have to consider universal graphs for h-local r-multi-colorings.

Let $m \geq h$ and $r \leq h/2$ be positive integers, let C be a set of m colors and consider the directed graph $U_d(m,h,r)$ whose vertices are pairs (Q, H) satisfying $Q \in \binom{C}{r}$, $H \in \binom{C}{h-r}$ and $Q \cap H = \emptyset$. We have a directed edge from (Q,H) to (Q', H') if $Q' \subseteq H$. Clearly, $U_d(m,h,r)$ is the universal graph for having h-local r-multi-coloring using m colors. (Note that the undirected version $U(m,h,r)$ of $U_d(m,h,r)$ is just the graph denoted $U_r(m,h)$ in [2].) The *natural multi-coloring* maps a vertex (Q,H) to Q and this is clearly an h-local r-multi-coloring and shows $\psi_d^*(U_d(m,h,r)) \leq h/r$.

Let I be a largest independent set in $U_d(m,h,r)$ and set $\mathcal{H} = \{H | \exists Q : (Q,H) \in I\} \subseteq \binom{C}{h-r}$. Let $\mathcal{S}$ be the size r shadow of $\mathcal{H}$, that is $\mathcal{S} = \cup_{H \in \mathcal{H}}\binom{H}{r}$. Let us find the value l (not necessarily integer) such that $|\mathcal{H}| = \binom{l}{h-r}$. By the general form of the Kruskal-Katona theorem [4] (exercise 13.31) we have $|\mathcal{C}| \geq \binom{l}{r}$. A vertex $(Q,H) \in I$ satisfies $Q \notin \mathcal{S}$ and $H \in \mathcal{H}$. Thus we have

$$\alpha(U_d(m,h,r)) = |I| \leq \left(\binom{m}{r} - \binom{l}{r}\right)\binom{l}{h-r}.$$

We use the inequalities $\binom{m}{r} - \binom{l}{r} \leq (m^r - l^r)/r!$ and $\binom{l}{h-r} \leq l^{h-r}/(h-r)!$ plus calculus to obtain:

$$\alpha(U_d(m,h,r)) \leq \frac{(m^r - l^r)l^{h-r}}{r!(h-r)!} \leq \frac{(h/r-1)^{h/r-1}}{(h/r)^{h/r}} \cdot \frac{m^r}{r!(h-r)!}.$$

Finally using that $U_d(m,h,r)$ has $n = \binom{m}{r}\binom{m-r}{h-r} \geq (m-h)^h/(r!(h-r)!)$ vertices we obtain

$$\chi^*(U_d(m,h,r)) \geq \frac{n}{\alpha(U_d(m,h,r))} \geq (1-h/m)^h \cdot \frac{(h/r)^{h/r}}{(h/r-1)^{h/r-1}}.$$

To finish the proof of the theorem let $s \geq 2$ be an arbitrary real. Take positive integers $m \geq h$ and r such that $2 \leq h/r \leq s$. Consider the graph $U_d(m,h,r)$. It satisfies

$$\psi_d^*(U(m,h,r)) \leq h/r \leq s,$$

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$$\chi^*(U_d(m,h,r)) \geq (1 - h/m)^h (h/r)^{h/r}/(h/r - 1)^{h/r-1}.$$

If we choose h/r close enough to s and m large enough this last value will be arbitrarily close to $s^s/(s-1)^{s-1}$. □

Remark 1. Part (a) of Theorem 4 can also be proven using the proof method of Theorem 5 in [2] stating $\psi^*(G) = \chi^*(G)$. On the one hand we can use that G having an h-local r-multi-coloring with m colors is equivalent to G admitting a homomorphism to $U_d(m,h,r)$. On the other hand we can also use the fact that if G admits a homomorphism to another graph H, then $\chi^*(G) \leq \chi^*(H)$. Thus it is enough to determine the fractional chromatic number of $U_d(m,h,r)$ (and maximize its value in m while h/r is fixed), to get the largest possible fractional chromatic number of a graph with fractional directed local chromatic number h/r. Using the vertex-transitivity of the graphs $U_d(m,h,r)$ this can be done by determining their independence number and using that for a vertex-transitive graph F the fractional chromatic number $\chi^*(F)$ is equal to $\frac{|V(F)|}{\alpha(F)}$. This way a more precise calculation in the proof of part (b) can actually lead to a proof for part (a), too. ◇

Remark 2. As we have seen the fractional chromatic and the fractional local chromatic numbers agree for undirected graphs, but the former may be larger by a factor up to e if we consider directed graphs. However, if we consider another variant of these chromatic parameters for directed graphs, originally introduced by Neumann-Lara [5], then the two parameters agree for directed graphs as well. For the definitions of these variants one uses acyclic subsets (i.e., subsets of the vertices that induce acyclic graphs) the same way as one uses independent sets for the variants we studied. ◇

5   Proof of Lemma 2.1

In this section we do not distinguish colorings that differ only in the permutations of colors, i.e., we identify colorings that induce the same partition on the vertex set.

For the proof of Lemma 2.1 we will need the following other lemma. Notice that the graph $U(k+1,k)$ is a k-chromatic graph, thus it is not an example for having strictly larger chromatic number than local chromatic number. On the other hand, all the graphs $U(m,k)$ with $m > k+1 > 3$ give such examples.

As opposed to $U(m,k)$ with $m > k+1 > 3$, the graph $U(k+1,k)$ have local k-colorings different from the natural coloring. Indeed any proper k-coloring is also a local k-coloring. Our next lemma states that for $k \geq 3$ there are no other local k-colorings.

Lemma 5.1 *For every $k \geq 3$ the graph $U(k+1,k)$ has $k+2$ different local k-colorings: its natural coloring and the k-colorings obtained from the natural coloring by keeping the color of all but a single color class and recoloring the vertices of this last color class the only way to obtain a proper k-coloring.*

Proof. Let f be a local k-coloring of $U(k+1,k)$. For simplicity we write $(a,\overline{b})$ for the vertex $(a, [k+1] \setminus \{a,b\})$ and $f(a,\overline{b})$ for $f((a,\overline{b}))$. Let m stand for the number of colors f uses.

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The vertices $(i, \overline{k+1})$ for $i \in [k]$ form a k-clique. As f is proper it assigns distinct colors to these vertices. We may therefore assume $f(i, \overline{k+1}) = i$ for $i \in [k]$. As f is a local k-coloring all the neighbors of the vertices in this clique must also receive colors from $[k]$. Thus the $m - k$ colors outside $[k]$ that f uses must appear on vertices receiving color $k + 1$ in the natural coloring (all other vertices are neighbors of our clique).

By symmetry all the $k + 1$ color classes of the natural coloring must contain $m - k$ color classes of the coloring f, so we must have $m \geq (k + 1)(m - k)$. This means that $m = k$ or $m = k+1$. Furthermore, in case $m = k+1$ all $k+1$ color classes must live inside a single color class of the natural coloring, meaning that f is equivalent to the natural coloring.

It remains to consider the case $m = k$, thus f is a proper k-coloring.

For $i \neq j$, $i, j \in [k]$ we have $f(i, \overline{j}) \in \{i, j\}$ as $(i, \overline{j})$ is connected to $(l, \overline{k+1})$ of color l for $l \in [k] \setminus \{i, j\}$. We call a vertex $(i, \overline{j})$ *special* if $f(i, \overline{j}) = j$ (even if $i = k + 1$). If a vertex $(i, \overline{j})$ for $i \in [k]$ is not special we have $f(i, \overline{j}) = i$.

Note that the vertices $(i, \overline{j})$ for a fixed j form a clique, thus at most one of them can be special (having color j) or f is not proper.

If all the vertices $(k + 1, \overline{i})$ are special, then these are all the special vertices and f is obtained from the natural coloring by recoloring its last color class to the remaining colors.

Otherwise we have $f(k + 1, \overline{i}) = j$ for some $i, j \in [k]$, $i \neq j$. To make f proper all the vertices $(j, \overline{l})$ with $l \in [k] \setminus \{j\}$ must be special. This means that no more vertices can be special except possibly a single vertex $(x, \overline{j})$ for some x. Indeed $(k+1, \overline{j})$ must be special, moreover $f(k + 1, \overline{l}) = j$ for all $l \in [k]$ as $(k + 1, \overline{l})$ is connected to vertices of all colors but j, namely to the special vertices $(j, \overline{l'})$ if $l \neq j$ and to the non-special vertices $(l', \overline{l''})$ with $l', l'' \in [k] \setminus \{j\}$ if $l = j$. This makes f equivalent to the k-coloring obtained from the natural coloring by recoloring color class j. $\qquad \square$

Proof of Lemma 2.1. We use the phrase *class* for the color classes of the natural coloring in $U(m, k)$. We write $c(x, A)$ for the color $c((x, A))$ of the vertex (x, A).

First we consider the $m = k + 2$ special case.

For $i \in [k + 2]$ consider the subgraph G_i of $U(m, k)$ induced by the vertices (x, A) with $i \notin A \cup \{x\}$. Clearly, G_i is isomorphic to $U(k + 1, k)$, and thus the restriction to G_i of the local k-coloring c must be one of the few colorings described in Lemma 5.1. In particular, the intersection of a class with G_i either receives a single color by c (a monochromatic intersection) or each vertex of the intersection receives different colors (a colorful intersection). Furthermore, $k + 1$ classes intersect G_i but at most one of these intersections is colorful.

Let us fix a color class of the natural coloring of $U(m, k)$. It intersects $k + 1$ of the $k + 2$ subgraphs G_i. If at least two of these intersections are monochromatic, then all of them have to be, so c assigns the same color to the entire class. Otherwise at least k of the intersections are colorful. Since altogether there are at most $k + 2 < 2k$ colorful intersections, we must have at most a single class to which c assigns multiple colors.

If all the classes are monochromatic, then c must assign distinct colors to them or it is not a proper coloring. This makes c equivalent to the natural coloring as needed. Otherwise we have a single non-monochromatic class with k or $k + 1$ of its non-empty intersections with the subgraphs G_i colorful. All that other classes are monochromatic and must receive distinct colors, thus we assume that c equals to the natural coloring outside the only exceptional class.

Each vertex in this exceptional class is contained in exactly two of the subgraphs G_i, and since more than half these intersections are colorful there must be a vertex (x, A) that is in two distinct subgraphs G_i and G_j with the intersection of either of them with the exceptional class being colorful. Lemma 5.1 determines the color of (x, A) from the coloring of G_i outside the exceptional class: it must be j. The contradiction finishing the proof of the $m = k + 2$ case of the lemma comes from observing that a similar argument using the restriction of c to G_j gives $c(x, A) = i$.

Finally, let us consider the case $m > k + 2$. For any subset $H \subset [m]$ with $|H| = k + 2$ the vertices (x, A) satisfying $A \cup \{x\} \subset H$ induce a subgraph G_H of G isomorphic to $U(k + 2, k)$. Thus c must be equivalent to the natural coloring on G_H. This means that $c(x, A) = c(x, B)$ whenever $|A \cup B| \leq k+1$ as in this case (x, A) and (x, B) are in a common subgraph G_H. Now consider two arbitrary vertices (x, A) and (x, B) from the same class. Clearly there exists a sequence $A = A_0, A_1, \ldots, A_k = B$ such that $c(x, A_{i-1}) = c(x, A_i)$ for the above reason for all $i \in [k]$. Thus we have all $c(x, A) = c(x, B)$ and all the classes are monochromatic at c. To make c proper it must assign distinct colors to to distinct classes, thus c is equivalent to the natural coloring. This finishes the proof of the lemma. $\square$

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