Extremal numbers for odd cycles

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Abstract

We describe the C_{2k+1}-free graphs on n vertices with maximum number of edges. The extremal graphs are unique for $n \notin \{3k-1, 3k, 4k-2, 4k-1\}$. The value of $\mathrm{ex}(n, C_{2k+1})$ can be read out from the works of Bondy [3], Woodall [14], and Bollobás [1], but here we give a new streamlined proof. The complete determination of the extremal graphs is also new.

We obtain that the bound for $n_0(C_{2k+1})$ is $4k$ in the classical theorem of Simonovits, from which the unique extremal graph is the bipartite Turán graph.

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1 Introduction, exact Turán numbers

Given a class of simple graphs $\mathcal{F}$ let us call a graph $\mathcal{F}$-*free* if it contains no copy of F as a (not necessarily induced) subgraph for each $F \in \mathcal{F}$. Let $\mathrm{ex}(n; \mathcal{F})$ denote the maximal number of edges in an $\mathcal{F}$-free graph on n vertices. If the class of graphs $\mathcal{F} = \{F_1, F_2, \dots\}$ consists of a single graph then we write $\mathrm{ex}(n; F)$ instead of $\mathrm{ex}(n; \{F\})$.

Let $T_{n,p}$ denote the *Turán graph*, the complete equi-partite graph, $K_{n_1,n_2,\dots,n_p}$ where $\sum_i n_i = n$ and $\lfloor n/p \rfloor \le n_i \le \lceil n/p \rceil$. By Turán's theorem [12, 13] we have $\mathrm{ex}(n; K_{p+1}) = e(T_{n,p})$; furthermore, $T_{n,p}$ is the unique K_{p+1}-free graph that attains the extremal number. The case $\mathrm{ex}(n; K_3) = \lfloor n^2/4 \rfloor$ was shown earlier by Mantel [10].

There are very few cases when the Turán number $\mathrm{ex}(n; \mathcal{F})$ is known exactly for all n. One can mention the case when $F = M_{\nu+1}$ is a matching of a given size, $\nu + 1$. Erdős and Gallai [6] showed that

$$\mathrm{ex}(n, M_{\nu+1}) = \max\{\binom{2\nu+1}{2}, \binom{\nu}{2} + \nu(n-\nu)\}.$$

For the path of k vertices Erdős and Gallai [6] proved an asymptotic and $\mathrm{ex}(n; P_k)$ was determined for all n and k by Faudree and Schelp [7] and independently by Kopylov [9]. Erdős and Gallai [6] proved an asymptotic for the class of long cycles $\mathcal{C}_{\ge \ell} := \{C_\ell, C_{\ell+1}, C_{\ell+2}, \dots\}$. The exact value of the Turán number $\mathrm{ex}(n; \mathcal{C}_{\ge \ell})$ was determined by Woodall [15] and independently and at the same time by Kopylov [9].

There is one outstanding result which gives infinitely many exact Turán numbers, Simonovits' chromatic critical edge theorem [11]. It states that if $\min\{\chi(F) : F \in \mathcal{F}\} = p + 1 \ge 3$ and there exists an $F \in \mathcal{F}$ with an edge $e \in E(F)$ such that by removing this edge one has $\chi(F - e) \le p$, then there exists an $n_0(\mathcal{F})$ such that $T_{n,p}$ is the only extremal graph for $\mathcal{F}$ for $n \ge n_0$. The authors are not aware of any (non-trivial) further result when $\mathrm{ex}(n, \mathcal{F})$ is known for all n, neither any F for which the value of $n_0(F)$ had been determined, except the case of odd cycle discussed below.

2 The result, the extremal graphs without C_{2k+1}

The aim of this paper is to determine the Turán number of odd cycles for all n and C_{2k+1} together with the extremal graphs. The value of $\mathrm{ex}(n, C_{2k+1})$ can be read out from the works of Bondy [2, 3], Woodall [14], and Bollobás [1] (pp. 147–156) concerning (weakly) pancyclic graphs. For a recent presentation see Dzido [5] who also considered the Turán number of wheels. But here we give a new streamlined proof and a complete description of the extremal graphs.

Since $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ contains no odd cycles, for any $k\geq 1$, $\mathrm{ex}(n;C_{2k+1})\geq \lfloor n^2/4\rfloor$. For C_3 here equality holds for all n with the only extremal graph is $T_{n,2}$ by the Turán-Mantel's theorem. From now on, we suppose that $2k+1\geq 5$. Also for $n\leq 2k$ obviously $\mathrm{ex}(n,C_{2k+1})=\binom{n}{2}$ so we may suppose that $n\geq 2k+1$.

Every edge of an odd cycle is color critical so Simonovits' theorem implies that the complete bipartite graph is the only extremal graph and $\mathrm{ex}(n;C_{2k+1})=e(T_{n,2})=\lfloor n^2/4\rfloor$ for $n\geq n_0(C_{2k+1})$. After choosing the right tools we present a streamlined proof and show that $n_0(C_{2k+1})=4k$ (in case of $2k+1\geq 5$).

We define two classes of C_{2k+1}-free graphs which could have at least as many edges as $T_{n,2}$ for $n\leq 4k-1$. A *cactus* $B(n;n_1,\dots,n_s)$ (for $n\geq 2$, $s\geq 1$ with $\sum_i(n_i-1)=n-1$) is a connected graph where the 2-connected blocks are complete graphs of sizes $n_1,\dots,n_s$. Let us denote by $g(n,k)$ the largest size of an n-vertex cactus avoiding C_{2k+1}. For this maximum all block sizes should be exactly $2k$ but at most one which is smaller. Write n in the form $n=(s-1)(2k-1)+r$ where $s\geq 1$, $2\leq r\leq 2k$ are integers. Then

$$g(n,k)=(s-1)\binom{2k}{2}+\binom{r}{2}. \tag{1}$$

Note that $g(n,k)>\lfloor n^2/4\rfloor$ for $3\leq n\leq 4k-3$ and we have $g(n,k)=e(T_{n,2})=\lfloor n^2/4\rfloor$ if $n\in\{4k-2,4k-1\}$. Thus the Simonovits threshold $n_0(C_{2k+1})$ is at least $4k$.

For $n\geq k$, define the graph $H_1(n,k)$ on n vertices by its degree sequence; it has k vertices of degree $n-1$ and all other vertices have degree k. Then $H_1(n,k)$ is a complete bipartite graph $K_{k,n-k}$, together with all possible edges added in the first partite set. This graph does not contain the cycle C_{2k+1}. Letting $h_1(n,k)$ denote the size of $H_1(n,k)$,

$$h_1(n,k)=\binom{k}{2}+k(n-k). \tag{2}$$

Note that $h_1(n,k)\leq g(n,k)$ for all $k\leq n$ and here equality holds if n is in the form $n=(s-1)(2k-1)+r$ where $s\geq 1$ and $r\in\{k,k+1\}$.

Theorem 1. *For any $n\geq 1$ and $2k+1\geq 5$,*

$$\mathrm{ex}(n;C_{2k+1})=\begin{cases}\binom{n}{2} & \text{for } n\leq 2k,\\ g(n,k) & \text{for } 2k+1\leq n\leq 4k-1 \text{ and}\\ \lfloor n^2/4\rfloor & \text{for } n\geq 4k-2.\end{cases}$$

Furthermore, the only extremal graphs are K_n for $n\leq 2k$; $B(n;2k,n-2k+1)$ for $2k+1\leq n\leq 4k-1$; $H_1(n,k)$ for $n\in\{3k-1,3k\}$; and the complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ for $n\geq 4k-2$.

3 A lemma on 2-connected graphs without C_{2k+1}

Lemma 2. *Suppose that $n \geq 2k+1 \geq 5$ and G is a 2-connected, C_{2k+1}-free, non-bipartite graph with at least $\lfloor n^2/4 \rfloor$ edges. Then $e(G) \leq \mathrm{ex}(n; C_{2k+1})$ and here equality holds only if $n \in \{3k-1, 3k\}$ and $G = H_1(n,k)$.*

For $5 \leq 2k+1 \leq n$, define the graph $H_2(n,k)$ on n vertices and

$$h_2(n,k) := \binom{2k-1}{2} + 2(n-2k+1)$$

edges, consisting of a complete graph K_{2k-1} containing two special vertices which are connected to all other vertices. Then $H_2(n,k)$ is a 2-connected C_{2k+1}-free graph. For $k=2$ the graphs $H_1(n,k)$ and $H_2(n,k)$ are isomorphic. Recall a result of Kopylov [9] in a form we use it: Suppose that the 2-connected graph G on n vertices contains no cycles of length $2k+1$ or larger and $n \geq 2k+1 \geq 5$. Then

$$e(G) \leq \max\{h_1(n,k), h_2(n,k)\} \tag{3}$$

and this bound is the best possible. Moreover, only the graphs $H_1(n,k)$ and $H_2(n,k)$ could be extremal. For further explanation and background see the recent survey [8].

The other result we need is due to Brandt [4]: Let G be a non-bipartite graph of order n and suppose that

$$e(G) > (n-1)^2/4 + 1, \tag{4}$$

then G contains cycles of every length between 3 and the length of its longest cycle.

Proof of Lemma 2: The inequality $e(G) \leq \mathrm{ex}(n, C_{2k+1})$ follows from the definition. Suppose that here equality holds. Apply Brandt's theorem (4). We obtain that G contains cycles of all lengths $3, 4, \ldots, \ell$ where ℓ stands for the longest cycle length in G. It follows that $\ell \leq 2k$. Kopylov's theorem (3) implies that

$$\max\{g(n,k), \lfloor n^2/4 \rfloor\} \leq \mathrm{ex}(n, C_{2k+1}) = e(G) \leq \max\{h_1(n,k), h_2(n,k)\}.$$

Since $g(n,k) > h_2(n,k)$ except for $(n,k) \in \{(5,2),(6,2)\}$ and $g(n,k) > h_1(n,k)$ except if n is in the form $n = (s-1)(2k-1)+r$ where $s \geq 2$ and $r \in \{k, k+1\}$ we obtain that $e(G) = h_1(n,k)$, n should be in this form, and $G = H_1(n,k)$.

Finally, $h_1(n,k) < \lfloor n^2/4 \rfloor$ for $n \geq 4k$ so we obtain that indeed $n \in \{3k-1, 3k\}$. $\square$

4 The proof of Theorem 1

Suppose that G is an extremal C_{2k+1}-free graph, $e(G) = \mathrm{ex}(n, C_{2k+1})$. Then G is connected. Consider the cactus-like block-decomposition of G, $V(G) = V_1 \cup V_2 \cup \cdots \cup V_s$, where the induced subgraphs $G[V_i]$ are either edges or maximal 2-connected subgraphs of G. Let $n_i := |V_i|$, we have $n - 1 = \sum_i (n_i - 1)$, and each $n_i \geq 2$. We have $e(G[V_i]) = \mathrm{ex}(n_i, C_{2k+1})$ otherwise one can replace $G[V_i]$ by an extremal graph of the same order n_i and obtain another C_{2k+1}-free graph of size larger than $e(G)$. Therefore $e(G[V_i]) \geq \lfloor n_i^2/4 \rfloor$ and there are three types of blocks
— complete graphs (if $n_i \leq 2k$),
— bipartite blocks with $e(G[V_i]) = \lfloor n_i^2/4 \rfloor$. Finally,
— if $n_i \geq 2k+1$ and $G[V_i]$ is not bipartite then Lemma 2 implies that $n_i \in \{3k-1, 3k\}$ and $G[V_i] = H_1(n_i, k)$.

We may rearrange the graphs $G[V_i]$ and the sets V_i such a way that they share a common vertex $v \in \cap V_i$ and otherwise the sets $V_i \setminus \{v\}$ are pairwise disjoint. The obtained new graph G^* also C_{2k+1}-free and extremal, it has the same size and order as G has.

If $s = 1$ then we are done. Suppose $s \geq 2$. If all blocks are complete graphs, then $e(G) \leq g(n, k)$. Since $g(n, k) < e(T_{n,2})$ for $n > 4k - 1$ we get that $n \leq 4k - 1$ and G^* (and G) has only two blocks and at least one of them is of size $2k$.

Finally, suppose that there are two blocks V_i and V_j, $|V_i| = a$ and $|V_j| = b$, such that $G[V_i]$ and $G[V_j]$ are not both complete subgraphs. We claim that in this case one can remove the edges of $G[V_i]$ and $G[V_j]$ from G^* and place a copy of $T_{a+b-1,2}$ or some other graph L onto $V_i \cup V_j$ such that the obtained new graph is C_{2k+1}-free and it has more edges than $e(G)$, a contradiction.

Indeed, if $G[V_i]$ is a large bipartite graph, $a := n_i \geq 2k+1$, $G[V_i] = T_{a,2}$ and $G[V_j]$ is a complete bipartite graph, too, then we can increase $e(G^*)$ since

$$e(T_{a,2}) + e(T_{b,2}) \leq \frac{1}{4}a^2 + \frac{1}{4}b^2 < \lfloor \frac{1}{4}(a+b-1)^2 \rfloor = e(T_{a+b-1,2}). \tag{5}$$

In the remaining cases the inequalities concerning the number of edges of $e(L)$ are just elementary high school algebra. If $G[V_i] = T_{a,2}$ and $G[V_j] = H_1(b, k)$ or K_b then we can replace them again by a complete bipartite graph $T_{a+b-1,2}$. From now on, we may suppose that each block is either a complete graph (of size at most $2k$) or an $H_1(a, k)$. If $G[V_i] = H_1(a, k)$ for some $a \in \{3k-1, 3k\}$ and $G[V_j] = H_1(b, k)$ (with $b \in \{3k-1, 3k\}$) or $G[V_j] = K_b$ with $k \leq b \leq 2k$ then we replace $G[V_i] \cup G[V_j]$ again by a $T_{a+b-1,2}$. Finally, if $G[V_i] = H_1(a, k)$ for some $a \in \{3k-1, 3k\}$ and $G[V_j] = K_b$ with $2 \leq b \leq k$ then we replace $G[V_i] \cup G[V_j]$ by two complete graphs of sizes $2k$

and $a + b - 2k$ and use $e(H_1(a,k)) + e(K_b) < e(B(a+b-1; 2k, a+b-2k))$ to get a contradiction. This completes the proof of the claim and Theorem 1. □

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