Regular graphs are antimagic

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Abstract

In this note we prove - with a slight modification of an argument of Cranston et al. [2] - that k-regular graphs are antimagic for $k \geq 2$.

1 Introduction

Throughout the note graphs are assumed to be simple. Given an undirected graph $G = (V, E)$ and a subset of edges $F \subseteq E$, $F(v)$ denotes the set of edges in F incident to node $v \in V$, and $d_F(v) := |F(v)|$ is the **degree** of v in F. A **labeling** is an injective function $f : E \to \{1, 2, \ldots, |E|\}$. Given a labeling f and a subset of edges F, let $f(F) = \sum_{e \in F} f(e)$. A labeling is **antimagic** if $f(E(u)) \neq f(E(v))$ for any pair of different nodes $u, v \in V$. A graph is said to be **antimagic** if it admits an antimagic labeling.

Hartsfield and Ringel conjectured [4] that all connected graphs on at least 3 nodes are antimagic. The conjecture has been verified for several classes of graphs (see e.g. [3]), but is widely open in general. In [2] Cranston et al. proved that every k-regular graph is antimagic if $k \geq 3$ is odd. Note that 1-regular graphs are trivially not antimagic. We have observed that a slight modification of their argument also works for even regular graphs, hence we prove the following.

Theorem 1. *For $k \geq 2$, every k-regular graph is antimagic.*

It is worth mentioning the following conjecture of Liang [5]. Let $G = (S, T; E)$ be a bipartite graph. A path $P = \{uv, vw\}$ of length 2 with $u, w \in S$ is called an S**-link**.

Conjecture 2. *Let $G = (S, T; E)$ be a bipartite graph such that each node in S has degree at most 4 and each node in T has degree at most 3. Then G has a matching M and a family $\mathcal{P}$ of node-disjoint S-links such that every node $v \in T$ of degree 3 is incident to an edge in $M \cup (\bigcup_{P \in \mathcal{P}} P)$.*

Liang showed that if the conjecture holds then it implies that every 4-regular graph is antimagic. The starting point of our investigations was proving Conjecture 2. As Theorem 1 provides a more general result, we leave the proof of Conjecture 2 for a forthcoming paper [1].

2 Proof of Theorem 1

A **trail** in a graph $G = (V, E)$ is an alternating sequence of nodes and edges $v_0, e_1, v_1, \ldots, e_t, v_t$ such that e_i is an edge connecting v_{i-1} and v_i for $i = 1, 2, \ldots, t$, and the edges are all distinct (but there might be repetitions among the nodes). The trail is **open** if $v_0 \neq v_t$, and **closed** otherwise. The **length** of a trail is the number of edges in it. A closed trail containing every edge of the graph is called an **Eulerian trail**. It is well known that a graph has an Eulerian trail if and only if it is connected and every node has even degree.

Lemma 3. *Given a connected graph $G = (V, E)$, let $T = \{v \in V : d_E(v) \text{ is odd}\}$. If $T \neq \emptyset$, then E can be partitioned into $|T|/2$ open trails.*

Proof. Note that $|T|$ is even. Arrange the nodes of T into pairs in an arbitrary manner and add a new edge between the members of every pair. Take an Eulerian trail of the resulting graph and delete the new edges to get the $|T|/2$ open trails. □

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The main advantage of Lemma 3 is that the edge set of the graph can be partitioned into open trails such that at most one trail starts at every node of V. Indeed, there is a trail starting at v if and only if v has odd degree in G. This is how we see the Helpful Lemma of [2].

Corollary 4 (Helpful Lemma of [2]). *Given a bipartite graph $G = (U, W; E)$ with no isolated nodes in U, E can be partitioned into subsets $E^\sigma, T_1, T_2, \ldots, T_l$ such that $d_{E^\sigma}(u) = 1$ for every $u \in U$, T_i is an open trail for every $i = 1, 2, \ldots, l$, and the endpoints of T_i and T_j are different for every $i \neq j$.*

Proof. Take an arbitrary $E' \subseteq E$ with the property $d_{E'}(u) = 1$ for every $u \in U$. A component of $G - E'$ containing more than one node is called **nontrivial**. If there exists a nontrivial component of $G - E'$ that only contains even degree nodes then let $uw_1 \in E - E'$ be an edge in this component with $u \in U$ and $w_1 \in W$, and let $uw_2 \in E'$. Replace uw_2 with uw_1 in E'. After this modification, the component of $G - E'$ that contains u has an odd degree node, namely w_1. Iterate this step until every nontrivial component of $G - E'$ has some odd degree nodes. Let $E^\sigma = E'$ and apply Lemma 3 to get the decomposition of $E - E^\sigma$ into open trails. □

In what follows we prove that regular graphs are antimagic: for sake of completeness we include the odd regular case, too. We emphasize the differences from the proof appearing in [2].

Proof of Theorem 1. Note that it suffices to prove the theorem for connected regular graphs. Let $G = (V, E)$ be a connected k-regular graph and let $v^* \in V$ be an arbitrary node. Denote the set of nodes at distance exactly i from v^* by V_i and let q denote the largest distance from v^*. We denote the edge-set of $G[V_i]$ by E_i. Apply Corollary 4 to the induced bipartite graph $G[V_{i-1}, V_i]$ with $U = V_i$ to get E_i^σ and the trail decomposition of $G[V_{i-1}, V_i] - E_i^\sigma$ for every $i = 1, \ldots, q$. The edge set of $G[V_{i-1}, V_i] - E_i^\sigma$ is denoted by E_i'.

Now we define the antimagic labeling f of G as follows. We reserve the $|E_q|$ smallest labels for labeling E_q, the next $|E_q^\sigma|$ smallest labels for labeling E_q^σ, the next $|E_q'|$ smallest labels for labeling E_q', the next $|E_{q-1}|$ smallest labels for labeling E_{q-1}, etc. There is an important difference here between our approach and that of [2] as we switched the order of labeling E_i^σ and E_i', and we don't yet define the labels, we only reserve the intervals to label the edge sets. Next we prove a claim that tells us how to label the edges in E_i'.

Claim 5. *Assume that we have to label the edges of E_i' from interval $s, s+1, \ldots, \ell$ (where $|E_i'| = \ell - s + 1$), and that we are given a trail decomposition of E_i' into open trails. We can label E_i' so that successive labels (in a trail) incident to a node $v_i \in V_i$ have sum at most $s + \ell$, and successive labels (in a trail) incident to a node $v_{i-1} \in V_{i-1}$ have sum at least $s + \ell$.*

Proof. Our proof of this claim is essentially the same as the proof in [2]: we merely restate it for self-containedness. Let $\mathcal{T}$ be the trail decomposition of E_i' into open trails. Take an arbitrary trail $T = u_0, e_1, u_1, \ldots, e_t, u_t$ of length t from $\mathcal{T}$ and consider the following two cases (see Figure 1 for an illustration).

- **Case A:** If $u_0 \in V_{i-1}$ then label $e_1, \ldots, e_t$ by $s, \ell, s+1, \ell-1, \ldots$ in this order. In this case the sum of 2 successive labels is $s + \ell$ at a node in V_i, and it is $s + \ell + 1$ at a node in V_{i-1}.

- **Case B:** If $u_0 \in V_i$ then label $e_1, \ldots, e_t$ by $\ell, s, \ell-1, s+1, \ldots$ in this order. In this case the sum of 2 successive labels is $s + \ell - 1$ at a node in V_i, and it is $s + \ell$ at a node in V_{i-1}.

We prove by induction on $|\mathcal{T}|$. The proof is finished by the following cases.

1. If $\mathcal{T}$ contains a trail of even length, then let T be such a trail (and again t denotes the length of T). If the endpoints of T fall in V_{i-1} then apply Case A. On the other hand, if the endpoints of T fall in V_i then apply Case B. In both cases we use $\frac{t}{2}$ labels from the lower end of the interval, and $\frac{t}{2}$ labels from the upper end, therefore we can label the edges of the trails in $\mathcal{T} - T$ from the (remaining) interval $s + \frac{t}{2}, s + \frac{t}{2} + 1, \ldots, \ell - \frac{t}{2}$, so that the lower bound $s + \frac{t}{2} + \ell - \frac{t}{2} = s + \ell$ holds for the sum of two successive labels at every $v_{i-1} \in V_{i-1}$, and the same upper bound holds at each node $v_i \in V_i$.

2. Every trail in $\mathcal{T}$ has odd length. If $\mathcal{T}$ contains only one trail then label it using either of the two cases above and we are done. Otherwise let T_1 and T_2 be two trails from $\mathcal{T}$, and let t_i be the length of T_i for both $i = 1, 2$. Label first the edges of T_1 using Case A (starting at the endpoint of T_1 that lies in V_{i-1}). Note that the remaining labels form the interval $s + \frac{t_1+1}{2}, \ldots, \ell - \frac{t_1-1}{2}$. Next label the edges of T_2 using Case B (starting at the endpoint of T_2 that lies in V_i). Note that the sum of successive labels in the trail T_2 becomes $s + \frac{t_1+1}{2} + (\ell - \frac{t_1-1}{2}) - 1 = s + \ell$ at a node in V_i, and it is $s + \frac{t_1+1}{2} + (\ell - \frac{t_1-1}{2}) = s + \ell + 1$ at a node in V_{i-1}, which is fine for us. Finally, the remaining labels form the interval $s + \frac{t_1+1}{2} + \frac{t_2-1}{2}, \ldots, \ell - \frac{t_1-1}{2} - \frac{t_2+1}{2}$, therefore we can label the edges of the trails in $\mathcal{T} - \{T_1, T_2\}$ from the remaining interval so that the lower bound $s + \frac{t_1+1}{2} + \frac{t_2-1}{2} + \ell - \frac{t_1-1}{2} - \frac{t_2+1}{2} = s + \ell$ holds for the sum of two successive labels at every node of V_{i-1}, and the same upper bound holds at every node of V_i.

□

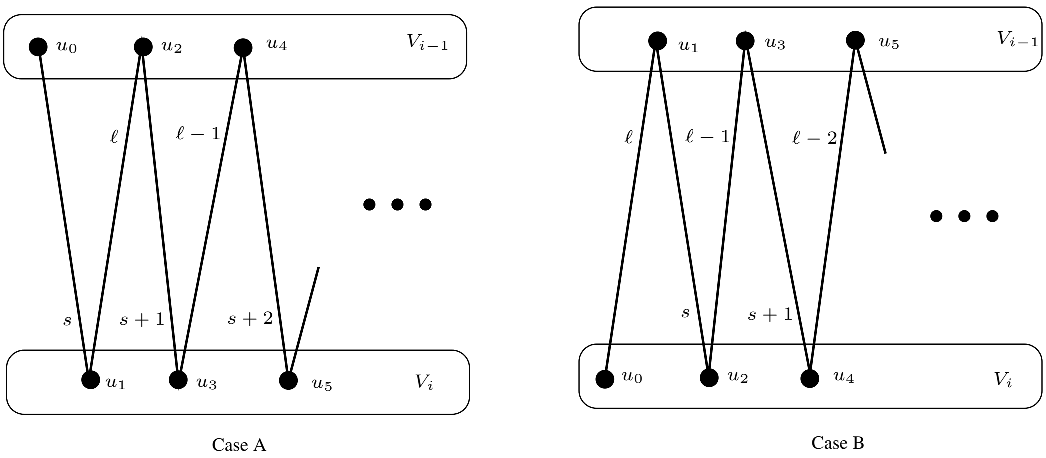

Figure 1: An illustration for labeling trails.

Now we specify how the labels are determined to make sure $f(E(u)) \neq f(E(v))$ for every $u \neq v$. We label the edges of every E_i arbitrarily from their dedicated intervals. Label the edges of every E'_i in the manner described by Claim 5. For any node $v \in V_i$ with $i > 0$, let $\sigma(v)$ denote the unique edge of E_i^σ incident to v. Let $p(v) = f(E(v)) - f(\sigma(v))$ for every $v \in V - v^*$. We label the edges in $E_q^\sigma, E_{q-1}^\sigma, \ldots, E_1^\sigma$ as in [2]: if we already labeled $E_q^\sigma, E_{q-1}^\sigma, \ldots, E_{i+1}^\sigma$ then $p(v_i)$ is already determined for every $v_i \in V_i$. So we order the nodes of V_i in an increasing order according to their p-value and assign the label to their σ edge in this order. This ensures that $f(E(u)) \neq f(E(v))$ for an arbitrary pair $u, v \in V_i$.

We have fully described the labeling procedure. This labeling scheme ensures that $f(E(v_i)) < f(E(v_j))$ if $v_i \in V_i, v_j \in V_j$ and $i \geq j+2$ since G is regular and the edges in $E(v_j)$ get larger labels than those in $E(v_i)$. Similarly, $f(E(v^*)) > f(E(v))$ for every $v \in V - v^*$ for the same reason. It is only left is to show that $f(E(v_i)) \neq f(E(v_{i-1}))$ for arbitrary $v_i \in V_i, v_{i-1} \in V_{i-1}$ and $i \geq 2$.

Claim 6. *For arbitrary $v_i \in V_i, v_{i-1} \in V_{i-1}$ and $i \geq 2$ we have*

(i) $p(v_i) \leq \frac{k-2}{2}(s+\ell)+\ell$ and $p(v_{i-1}) \geq \frac{k-2}{2}(s+\ell)+s$, if k is even, and

(ii) $p(v_i) \leq \frac{k-1}{2}(s+\ell)$ and $p(v_{i-1}) \geq \frac{k-1}{2}(s+\ell)$, if k is odd.

Proof. Assume first that k is even. In this case $p(v)$ is the sum of an odd number of labels. We pair up all but one of these labels using the trail decomposition of E'_i to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E'_i$. Let $t = d_{E'_i}(v_i)$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_i) \leq \frac{t}{2}(s+\ell)+(k-1-t)s \leq \frac{k-2}{2}(s+\ell)+\ell$.
   (b) If t is odd then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t-1}{2}(s+\ell)+\ell$ by Claim 5, giving $p(v_i) \leq \frac{t-1}{2}(s+\ell)+\ell+(k-1-t)s \leq \frac{k-2}{2}(s+\ell)+\ell$.
2. Now take a node $v_{i-1} \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_i$. Let again $t = d_{E'_i}(v_{i-1})$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t}{2}(s+\ell)+(k-1-t)\ell \geq \frac{k-2}{2}(s+\ell)+s$.
   (b) If t is odd then $\sum_{e \in E'_i \cap E(v_{i-1})} f(e) \geq \frac{t-1}{2}(s+\ell)+s$ by Claim 5, giving $p(v_{i-1}) \geq \frac{t-1}{2}(s+\ell)+s+(k-1-t)\ell \geq \frac{k-2}{2}(s+\ell)+s$.

This concludes the proof of (i).

Although the proof of (ii) can be found in [2], we also present it here to make the paper self contained. The proof is very similar to the even case. So assume that k is odd. In this case $p(v)$ is the sum of an even number of labels. We pair up these labels using the trail decomposition of E'_i to get the bounds needed.

1. Take a node $v_i \in V_i$. Note that $f(e) < s$ for every $e \in E(v_i) - E'_i$. Let $t = d_{E'_i}(v_i)$.
   (a) If t is even then $\sum_{e \in E'_i \cap E(v_i)} f(e) \leq \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_i) \leq \frac{t}{2}(s+\ell)+(k-1-t)s \leq \frac{k-1}{2}(s+\ell)$.

(b) If t is odd then $\sum_{e\in E'_i\cap E(v_i)} f(e) \le \frac{t-1}{2}(s+\ell)+\ell$ by Claim 5, giving $p(v_i) \le \frac{t-1}{2}(s+\ell)+\ell+(k-1-t)s \le \frac{k-1}{2}(s+\ell)$.

2. Now take a node $v_{i-1} \in V_{i-1}$. Note that $f(e) > \ell$ for every $e \in E(v_{i-1}) - E'_i$. Let again $t = d_{E'_i}(v_{i-1})$.

   (a) If t is even then $\sum_{e\in E'_i\cap E(v_{i-1})} f(e) \ge \frac{t}{2}(s+\ell)$ by Claim 5, giving $p(v_{i-1}) \ge \frac{t}{2}(s+\ell)+(k-1-t)\ell \ge \frac{k-1}{2}(s+\ell)$.

   (b) If t is odd then $\sum_{e\in E'_i\cap E(v_{i-1})} f(e) \ge \frac{t-1}{2}(s+\ell)+s$ by Claim 5, giving $p(v_{i-1}) \ge \frac{t-1}{2}(s+\ell)+s+(k-1-t)\ell \ge \frac{k-1}{2}(s+\ell)$.

This concludes the proof of (ii), and we are done. □

The assignment of the labels implies $f(\sigma(v_i)) < s$ and $f(\sigma(v_{i-1})) > \ell$ for $v_i \in V_i$ and $v_{i-1} \in V_{i-1}$. Claim 6 yields $f(E(v_i)) < f(E(v_{i-1}))$, finishing the proof of Theorem 1. □

Remark 7. Observe that the proof of Theorem 1 does not really use the regularity of the graph, it merely relies on the fact that the degree of a node $v_i \in V_i$ is not smaller than that of a node $v_j \in V_j$ where $i < j$. Hence the following result immediately follows.

Theorem 8. *Assume that a connected graph $G = (V, E)$ ($|V| \ge 3$) has a node $v^* \in V$ of maximum degree such that $d_E(v_i) \ge d_E(v_j)$ whenever $v_i \in V_i, v_j \in V_j$ and $i < j$, where V_ℓ denotes the set of nodes at distance exactly ℓ from v^*. Then G is antimagic.*

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