(n, m)-Fold Covers of Spheres

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Abstract

A well known consequence of the Borsuk-Ulam theorem is that if the d-dimensional sphere S^d is covered with less than $d+2$ open sets, then there is a set containing a pair of antipodal points. In this paper we provide lower and upper bounds on the minimum number of open sets, not containing a pair of antipodal points, needed to cover the d-dimensional sphere n times, with the additional property that the northern hemisphere is covered $m > n$ times. We prove that if the open northern hemisphere is to be covered m times then at least $\lceil \frac{d-1}{2} \rceil + n + m$ and at most $d + n + m$ sets are needed. For the case of $n = 1$ and $d \geq 2$, this number is equal to $d + 2$ if $m \leq \lfloor \frac{d}{2} \rfloor + 1$ and equal to $\lfloor \frac{d-1}{2} \rfloor + 2 + m$ if $m > \lfloor \frac{d}{2} \rfloor + 1$. If the closed northern hemisphere is to be covered m times then $d + 2m - 1$ sets are needed, this number is also sufficient. We also present results on a related problem of independent interest. We prove that if S^d is covered n times with open sets, not containing a pair of antipodal points, then there exists a point that is covered at least $\lceil \frac{d}{2} \rceil + n$ times. Furthermore, we show that there are covers in which no point is covered more than $n + d$ times.

1 Introduction

The Lusternik-Schnirelmann [5] version of the Borsuk-Ulam theorem states that in any covering of the d-dimensional sphere S^d with at most $d+1$ open sets, there is a set containing a pair of antipodal points. A natural generalization of this result has been introduced by Stahl [9]; he considered *n-fold covers*, in which every point of S^d must be covered at least n times. He showed that in every n-fold cover of S^d with at most $d+2n-1$ open sets, there is a set containing a pair of antipodal points. Using a construction of Gale [2], it can easily be shown that this bound is tight. For every $n \geq 1$, Gale constructed a set of $d+2n$ points on the d-dimensional unit sphere, with the property that every open half-space that contains the origin contains at least n points of the set. Placing an open hemisphere with its pole at each of these points provides an n-fold cover of S^d with $d + 2n$ open sets, in which no set contains a pair of antipodal points. We refer to this cover as the *Gale n-fold cover* of S^d. An n-fold cover of S^d is said to be *antipodal* if none of its sets contains a pair of antipodal points.

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We consider a variation on this theme. Let $m > n \geq 1$. An (n,m)-*fold* cover of S^d is an n-fold cover of S^d, in which every point of the *open* northern hemisphere is covered m times. An $\overline{(n,m)}$-*fold* cover of S^d is an n-fold cover of S^d, in which every point of the *closed* northern hemisphere is covered m times. Let $f(d,n,m)$ be the minimum number of sets in an antipodal (n,m)-fold cover of S^d with open sets. In a similar way, let $\overline{f}(d,n,m)$ be the minimum number of sets in an antipodal $\overline{(n,m)}$-fold cover of S^d with open sets. Since the case of $d = 0$ is trivial, we will always assume that $d \geq 1$.

In this paper we show lower and upper bounds on $f(d,n,m)$ (Theorem 3.4) and provide the exact value of $\overline{f}(d,n,m)$ (Theorem 3.1). We also compute the exact value of $f(d,1,m)$ (Theorem 3.2 and Proposition 3.3). The search for a lower bound of $f(d,n,m)$ lead us to study the problem of finding a point covered many times in an antipodal n-fold cover of S^d with open sets. Let then $Q(d,n)$ be the maximum integer such that in every antipodal n-fold cover of S^d with open sets there exists a point that is covered $Q(d,n)$ times. This paper is organized as follows. In Section 2 we show upper and lower bounds on $Q(d,n)$ (Theorem 2.2). In Section 3 we give our results on $f(d,n,m)$ and $\overline{f}(d,n,m)$.

2 Bounds on $Q(d,n)$

The problem of determining $Q(d,1)$ has been studied before. Its exact value of $Q(d,1) = \left\lfloor \frac{d}{2} \right\rfloor + 2$ has been settled in a series of papers by Ščepin [6], Izydorek and Jaworowski [3], and Jaworowski [4]. An explicit cover yielding the upper bound for $Q(d,1)$ was given by Simonyi and Tardos [8]. They started by covering S^d with the projections from the origin of the closed facets of a regular $(d+1)$-simplex. Afterwards, they replaced the points of these sets that were covered more than $\left\lfloor \frac{d}{2} \right\rfloor + 1$ times with a new closed set. (Although this gives a cover with closed sets, sufficiently small open neighborhoods of these sets give the desired cover.) At first glance, it seems sensible to use a similar idea to upper bound $Q(d,n)$. However, our attempts of cutting out neighborhoods of often-covered points of an n-fold cover, and placing patches of n sets instead, always produced points that were covered an excessive amount of times. Finally, we decided to use Gale's n-fold cover of S^d.

Ky Fan's theorem [1] can be used to prove a lower bound of $\left\lceil \frac{d}{2} \right\rceil + 1$ for $Q(d,1)$. For proving a lower bound of $Q(d,n)$, we will use the following reformulation of Ky Fan's theorem that has been presented in [8].

Theorem 2.1. ***(Ky Fan's Theorem.)***
Let $\mathcal{F}$ be an antipodal cover of S^d. Assume that a linear order is given on $\mathcal{F}$. Then there exist $F_1 < F_2 < \cdots < F_{d+2}$ sets of $\mathcal{F}$ such that

$$F_1 \cap -F_2 \cap F_3 \cap -F_4 \cap \cdots \cap (-1)^{d+1} F_{d+2} \neq \emptyset.$$

□

We now give our bounds on $Q(d,n)$.

Theorem 2.2. $\left\lceil \frac{d}{2} \right\rceil + n \leq Q(d,n) \leq d + n.$

Proof. First we prove the lower bound. Let $\mathcal{F}$ be an antipodal n-fold cover of S^d with open sets. The intersections of all intersecting subsets of $\mathcal{F}$ consisting of n sets form an antipodal 1-fold cover $\mathcal{F}'$ of S^d with open sets. Explicitly, $\mathcal{F}' := \{\bigcap \mathcal{C} : \mathcal{C} \subset \mathcal{F}, |\mathcal{C}| = n \text{ and } \bigcap \mathcal{C} \neq \emptyset\}$. For an arbitrary linear order of $\mathcal{F}$, every set of $\mathcal{F}'$ is of the form $C = \bigcap_{i=1}^{n} F_i$ for some

$F_1 < F_2 < \cdots < F_n$ in $\mathcal{F}$. Hence, we may assign the tuple $v(C) := (F_1, F_2, \ldots, F_n)$ to C and define a linear order on $\mathcal{F}'$, by setting $C_1 < C_2$ if and only if $v(C_1) < v(C_2)$ in the lexicographical order of the tuples $v(C_1)$ and $v(C_2)$. By Ky Fan's theorem there exist sets $C_1 < C_2 < \cdots < C_{d+2}$ of $\mathcal{F}'$ such that $\bigcap_{i=1}^{d+2}(-1)^{i-1}C_i$ is not empty. Since $\mathcal{F}'$ is an antipodal cover, the first coordinates of the tuples associated to consecutive C_i's are different; by the additional assumption that $v(C_1) < \cdots < v(C_{d+2})$, all of these first coordinates are also pairwise different. Let $x \in \bigcap_{j=1}^{\lceil (d+2)/2 \rceil} C_{2j-1}$. This point is in all the first coordinates (sets) of the tuples $v(C_{2j-1})$, and in all the coordinates (sets) of the last tuple. As all of these sets are different, x is in at least $\lceil (d+2)/2 \rceil + n - 1 = \left\lceil \frac{d}{2} \right\rceil + n$ different sets of $\mathcal{F}$.

The upper bound is given by Gale's n-fold cover of S^d. In this cover a point $x \in S^d$ is not covered by precisely those hemispheres (sets) whose poles are separated from x by the hyperplane through the origin and orthogonal to $\vec{x}$. Since there are at least n of these sets and there are $d + 2n$ sets in this cover, x is covered at most $d + n$ times. □

We conjecture that the upper bound of Theorem 2.2 is tight.

Conjeture 2.3. *$Q(d, n) = d + n$ for $n \geq 2$.*

3 Bounds on $f(d, n, m)$ and $\overline{f}(d, n, m)$.

In this section we prove our results for $f(d, n, m)$ and $\overline{f}(d, n, m)$. We start by showing the exact values of $\overline{f}(d, n, m)$ and $f(d, 1, m)$.

Theorem 3.1. *$\overline{f}(d, n, m) = d + 2m - 1$ for $m > n$.*

Proof. Let $\mathcal{F}$ be an antipodal $\overline{(n, m)}$-fold cover of S^d with open sets. Note that the intersection of $\mathcal{F}$ with the equator is an antipodal m-fold cover of S^{d-1} with open sets. Therefore, as shown by Stahl [9], $|\mathcal{F}| \geq d + 2m - 1$, which proves the lower bound. For the upper bound, rotate Gale's m-fold cover of S^d so that one of the hemispheres (sets) in this cover coincides with the southern hemisphere. Remove this hemisphere to obtain an antipodal $\overline{(n, m)}$-fold cover of S^d with $d + 2m - 1$ open sets. □

Theorem 3.2. *For $d \geq 2$, $f(d, 1, m)$ is equal to:*

$$f(d, 1, m) = \begin{cases} d + 2 & \text{if } m \leq \left\lfloor \frac{d}{2} \right\rfloor + 1, \\ \left\lfloor \frac{d-1}{2} \right\rfloor + 2 + m & \text{if } m \geq \left\lfloor \frac{d}{2} \right\rfloor + 1. \end{cases}$$

Proof. First we prove the lower bound. Let $\mathcal{F}$ be an antipodal $(1, m)$-fold cover of S^d with open sets. Since $\mathcal{F}$ covers the equator once, there is a point in the equator of S^d covered at least $Q(d-1, 1) = \left\lfloor \frac{d-1}{2} \right\rfloor + 2$ times. Just below this point there is a point in the southern hemisphere covered by the same sets; its antipodal point in the northern hemisphere is covered by at least m other sets. Thus there are at least $\left\lfloor \frac{d-1}{2} \right\rfloor + 2 + m$ sets in $\mathcal{F}$. This lower bound is tight for $m \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$. For $m < \left\lfloor \frac{d}{2} \right\rfloor + 1$, the better lower bound of $d + 2$ follows immediately from Lusternik-Schnirelmann theorem [5] and the fact that $\mathcal{F}$ is a 1-fold cover of S^d.

For the upper bound we carefully construct an antipodal $\left(1, \left\lfloor \frac{d}{2} \right\rfloor + 1\right)$-fold cover of S^d with $d + 2$ open sets. This cover proves the tight upper bound of $d + 2$ for $m \leq \left\lfloor \frac{d}{2} \right\rfloor + 1$.

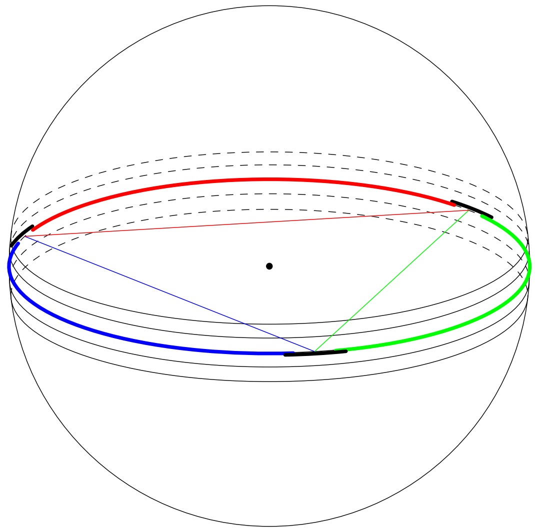

Figure 1: The 1-fold cover of the equator in the proof of Theorem 3.2.

For $m > \lfloor \frac{d}{2} \rfloor + 1$, we add $m - \lfloor \frac{d}{2} \rfloor - 1$ open northern hemispheres to this cover to produce an antipodal $(1, m)$-fold cover of S^d with $\lfloor \frac{d-1}{2} \rfloor + 2 + m$ open sets.

Assume that S^d is the unit sphere centered at the origin. We start by constructing a 1-fold cover of its equator S^{d-1} (the intersection of S^d with the hyperplane $x_{d+1} = 0$) in the following way. Fix a regular d-simplex τ centered at the origin and having its vertices on S^{d-1}. Project its closed facets from the origin to S^{d-1} and let $F'_1, \dots, F'_{d+1}$, be these projections. This produces an antipodal cover of S^{d-1} with $d+1$ closed sets. Let D' be the set of points of S^{d-1} that are covered at least $\lceil \frac{d+2}{2} \rceil = \lceil \frac{d}{2} \rceil + 1$ times in this cover. Note that V' is closed, and since there are only $d+1$ sets F'_i, it does not contain a pair of antipodal points. Choose $\varepsilon_2 > \varepsilon_1 > 0$. Let D be the open ε_2-neighborhood of D', in S^{d-1}. Further, let F_i be the open ε_1-neighborhood of $F'_i \setminus D$, also in S^{d-1}. Choose ε_2 small enough such that none of $F_1, F_2, \dots, F_{d+1}$ and D contain a pair of antipodal points. Choose ε_1 small enough with respect to ε_2, so that every point $x \in S^{d-1}$ has an open neighborhood that is covered by at most $\lceil \frac{d}{2} \rceil$ of the sets F_i. Then $\mathcal{F} := \{F_1, F_2, \dots, F_{d+1}, D\}$ is an antipodal 1-cover of S^{d-1} with $d+2$ open sets. See Figure 1 for an illustration of the $d = 2$ case.

We now extend $\mathcal{F}$ to an antipodal $\left(1, \lfloor \frac{d}{2} \rfloor + 1\right)$-fold cover of S^d with open sets. Roughly speaking, we first extend all sets of $\mathcal{F}$ to parts of a "belt". Afterwards, we further extend the "facet" sets F_i of $\mathcal{F}$ to cover the northern hemisphere and the set D of $\mathcal{F}$ to cover the southern hemisphere. Let π be the orthogonal projection of $\mathbb{R}^{d+1}$ to the hyperplane $x_{d+1} = 0$. Let $0 < \delta'_1 < \delta'_2 < 1$. Let C''_i be the set of points $x \in \mathbb{R}^{d+1}$ with $\pi(x) \in F_i$, whose last coordinate satisfies $-\delta'_2 < x_{d+1} < \delta'_2$. Similarly, let C''_{d+2} be the set of points $x \in \mathbb{R}^{d+1}$ with $\pi(x) \in D$, whose last coordinate satisfies $-\delta'_2 < x_{d+1} < \delta'_1$. Note that while the facet sets F_i are extended symmetrically to the north and the south, the set D is extended further to the south than to the north.

Next, for each $1 \le i \le d+2$, let C'_i be the set of points $x \in S^d$ such that the infinite ray with apex in the origin and passing through x intersects C''_i. The sets C'_i the parts of "belts" on the sphere. Let $\delta_i = \sin \tan^{-1}(\delta'_i)$ be the corresponding heights of the belt parts,

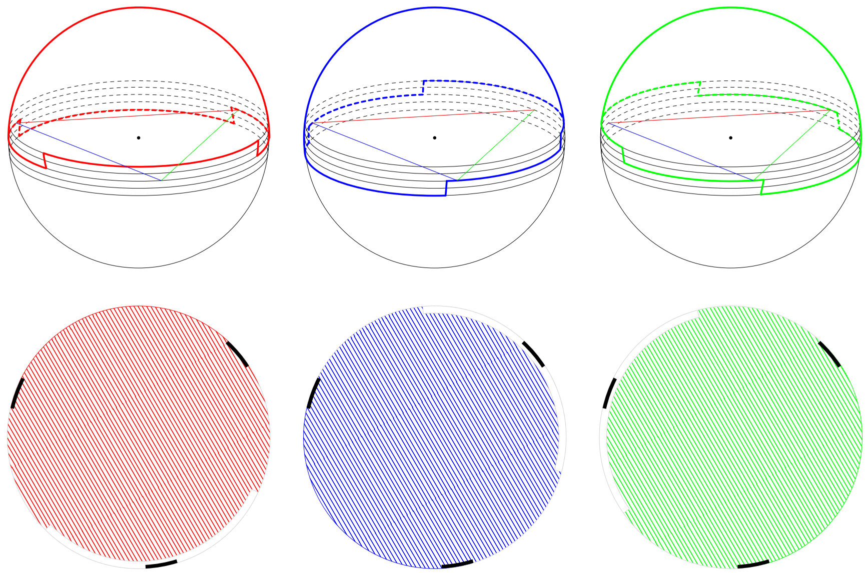

Figure 2: The covering of the northern hemisphere in the proof of Theorem 3.2

for $i = 1, 2$.

Finally, for $1 \leq i \leq d+1$, let C_i be the union of the open northern hemisphere with C'_i, minus the closure of $-C'_i$. Further, let C_{d+2} be the union of the southern hemisphere with C'_{d+2}, minus the closure of $-C'_{d+2}$. See Figures 2 and 3 for an illustration of the northern and southern hemispheres respectively (for the case $d = 2$ and $m = 2$). By construction, all these sets are open and do not contain a pair of antipodal points of S^d. What remains to show is that $\mathcal{C} := \{C_1, \ldots, C_{d+2}\}$ is in fact a $\left(1, \left\lfloor \frac{d}{2} \right\rfloor + 1\right)$-cover of S^d.

We distinguish a few cases with respect to the value of the last coordinate of the points of S^d. The set of points of S^d whose last coordinate is greater than δ_2 are covered $d+1$ times by $C_1, \ldots, C_{d+1}$. The set of points of S^d whose last coordinate is less than or equal to $-\delta_2$ are covered once by C_{d+2}.

To each point $x \in S^d$ whose last coordinate satisfies $-\delta_2 < x_{d+1} \leq \delta_2$, we assign the point x' in the equator such that $\pi(x)$ lies on the infinite ray from the origin to x'. Suppose that the last coordinate of x is greater than zero. By our choice of ε_1 and ε_2, there is an open neighborhood of x' in S^{d-1} that is covered by at most $\left\lceil \frac{d}{2} \right\rceil$ sets of $-F_1, \ldots, -F_{d+1}$. Assume without loss of generality that $-F_1, -F_2, \ldots, -F_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ do not cover this neighborhood, then at least $C_1, C_2, \ldots, C_{\left\lfloor \frac{d}{2} \right\rfloor + 1}$ cover x. Therefore $\mathcal{C}$ covers the open northern hemisphere $\left\lfloor \frac{d}{2} \right\rfloor + 1$ times. If the last coordinate of x is greater than $-\delta_2$ and at most zero, then x is covered by the extensions of the sets in $\mathcal{F}$ that cover x'. Thus, x is covered at least once and $\mathcal{C}$ is a $\left(1, \left\lfloor \frac{d}{2} \right\rfloor + 1\right)$-cover of S^d. □

Proposition 3.3. *$f(1, 1, m) = 2 + m$.*

Proof. For the upper bound, note that an antipodal $(1, m)$-cover of S^1 with $2 + m$ open sets can be obtained by adding $m - 1$ open northern hemispheres to an antipodal 1-cover of S^1

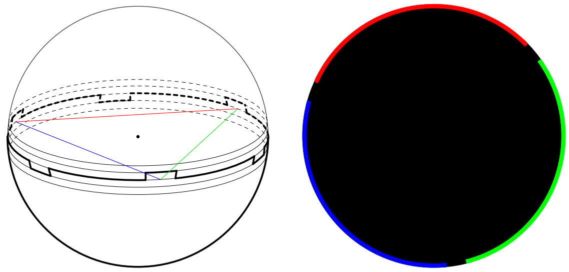

Figure 3: The covering of the southern hemisphere in the proof of Theorem 3.2

with three open sets. For the lower bound, suppose first that there exist two points x and y on the open upper hemisphere that are not covered by the same m sets. Then on the path from x to y on the northern hemisphere, there exists a point z that is covered by at least $m+1$ different sets. As its antipodal point $-z$ needs at least one set to be covered as well, this gives a total of at least $m+2$ sets. Hence, assume that all points on the open northern hemisphere are covered by the same m sets. But then none of the two points on the equator can be covered by any of these sets, again resulting in a total of at least $m+2$ sets. □

In the remaining part of this section, we present lower and upper bounds for $f(d,n,m)$ for arbitrary values of n and m.

Theorem 3.4. $\left\lceil \frac{d-1}{2} \right\rceil + n + m \leq f(d,n,m) \leq d + n + m.$

Proof. Let $\mathcal{F}$ be an antipodal (n,m)-fold cover of S^d with open sets. Since $\mathcal{F}$ covers the equator once, by Theorem 2.2 there is a point in the equator of S^d covered at least $\left\lceil \frac{d-1}{2} \right\rceil + n$ times. Just below this point there is a point in the southern hemisphere covered by the same sets; its antipodal point in the northern hemisphere is covered by at least m other sets. Thus in total there are at least $\left\lceil \frac{d-1}{2} \right\rceil + n + m$ sets in $\mathcal{F}$. For the upper bound take Gale's n-fold cover of S^d with $d+2n$ sets. Add $m-n$ open northern hemispheres to obtain an antipodal (n,m)-fold cover of S^d with $d+n+m$ open sets. □

In the case where $m - n < \left\lceil \frac{d}{2} \right\rceil$, the following proposition gives an improvement of the lower bound from Theorem 3.4. We omit the proof, which is a direct application of Stahl's result in combination with the fact that every (n,m)-fold cover is also an n-fold cover.

Proposition 3.5. $f(d,n,m) \geq d + 2n.$

In the proof of the lower bound of $f(d,n,m)$ in Theorem 3.4 we used the lower bound of $Q(d,n)$ given in Theorem 2.2. Hence, any improvement on the lower bound of $Q(d,n)$ immediately improves the lower bound of $f(d,n,m)$. Assuming that Conjecture 2.3 holds, the proof of Theorem 3.4 would leave a gap of only one between the lower and upper bounds of $f(d,n,m)$.

Acknowledgments.

We are indebted to László Zádori [10] who asked whether $f(2,1,2)$ equals four or five, a question that started this piece of research.

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