2013 UNIT VECTORS IN THE PLANE

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Abstract. Given a norm in the plane and 2013 unit vectors in this norm, there is a signed sum of these vectors whose norm is at most one.

Let B be the unit ball of a norm $\|.\|$ in $\mathbb{R}^d$, that is, B is an 0-symmetric convex compact set with nonempty interior. Assume $V \subset B$ is a finite set. It is shown in [1] that, under these conditions, there are signs $\varepsilon(v) \in \{-1,+1\}$ for every $v \in V$ such that $\sum_{v\in V} \varepsilon(v)v \in dB$. That is, a suitable signed sum of the vectors in V has norm at most d. This estimate is best possible: when $V = \{e_1, e_2, \dots, e_d\}$ and the norm is ℓ_1, all signed sums have ℓ_1 norm d.

In this short note we show that this result can be strengthened when $d = 2$, $|V| = 2013$ (or when $|V|$ is odd) and every $v \in V$ is a unit vector. So from now onwards we work in the plane $\mathbb{R}^2$.

Theorem 1. *Assume $V \subset \mathbb{R}^2$ consists of unit vectors in the norm $\|.\|$ and $|V|$ is odd. Then there are signs $\varepsilon(v) \in \{-1,+1\}$ $(\forall v \in V)$ such that $\|\sum_{v\in V} \varepsilon(v)v\| \le 1$.*

This result is best possible (take the same unit vector n times) and does not hold when $|V|$ is even.

Before the proof some remarks are in place here. Define the convex polygon $P = \text{conv}\{\pm v : v \in V\}$. Then $P \subset B$, and P is again the unit ball of a norm, V is a set of unit vectors of this norm. Thus it suffices to prove the theorem only in this case.

A vector $v \in V$ can be replaced by $-v$ without changing the conditions and the statement. So we assume that $V = \{v_1, v_2, \dots, v_n\}$ and the vectors $v_1, v_2, \dots, v_n, -v_1, -v_2, \dots, -v_n$ come in this order on the boundary of P. Note that n is odd. We prove the theorem in the following stronger form.

Theorem 2. *With this notation* $\|v_1 - v_2 + v_3 - \dots - v_{n-1} + v_n\| \le 1$.

Proof. Note that this choice of signs is very symmetric as it corresponds to choosing every second vertex of P. So the vector $u = 2(v_1 - v_2 + v_3 - \dots - v_{n-1} + v_n)$ is the same (or its negative) when one starts with another vector instead of v_1. Define $a_i = v_{i+1} - v_i$ for

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$i = 1, \dots, n-1$ and $a_n = -v_1 - v_n$ and set $w = a_1 - a_2 + a_3 - \dots + a_n$. It simply follows from the definition of a_i that

$$w = -2(v_1 - v_2 + v_3 - \dots - v_{n-1} + v_n) = -u.$$

Consequently $\|u\| = \|w\|$ and we have to show that $\|w\| \le 2$.

Consider the line L in direction w passing through the origin. It intersects the boundary of P at points b and $-b$. Because of symmetry we may assume, without loss of generality, that b lies on the edge $[v_1, -v_n]$ of P. Then w is just the sum of the projections onto L, in direction parallel with $[v_1, -v_n]$, of the edge vectors $a_1, -a_2, a_3, -a_4, \dots, a_n$. These projections do not overlap (apart from the endpoints), and cover exactly the segment $[-b, b]$ from L. Thus $\|w\| \le 2$, indeed. $\square$

Remark. There is another proof based on the following fact. P is a zonotope defined by the vectors $a_1, \dots, a_n$, translated by the vector v_1. Here the zonotope defined by $a_1, \dots, a_n$ is simply

$$Z = Z(a_1, \dots, a_n) = \left\{ \sum_1^n \alpha_i a_i : 0 \le \alpha_i \le 1 \ (\forall i) \right\}.$$

The polygon $P = v_1 + Z$ contains all sums of the form $v_1 + a_{i_1} + \dots + a_{i_k}$ where $1 \le i_1 < i_2 < \dots < i_k \le n$. In particular with $i_1 = 2, i_2 = 4, \dots, i_k = 2k$

$$v_1 + a_2 + a_4 + \dots a_{2k} = v_1 - v_2 + v_3 - \dots - v_{2k} + v_{2k+1} \in P.$$

This immediately implies a strengthening of Theorem 1 (which also follows from Theorem 2).

Theorem 3. *Assume $V \subset \mathbb{R}^2$ consists of n unit vectors in the norm $\|.\|$. Then there is an ordering $\{w_1, \dots, w_n\}$ of V, together with signs $\varepsilon_i \in \{-1, +1\}$ $(\forall i)$ such that $\|\sum_1^k \varepsilon_i w_i\| \le 1$ for every odd $k \in \{1, \dots, n\}$.*

Of course, for the same ordering, $\|\sum_1^k \varepsilon_i w_i\| \le 2$ for every $k \in \{1, \dots, n\}$. We mention that similar results are proved by Banaszczyk [2] in higher dimension for some particular norms.

In [1] the following theorem is proved. Given a norm $\|.\|$ with unit ball B in R^d and a sequence of vectors $v_1, \dots, v_n \in B$, there are signs $\varepsilon_i \in \{-1, +1\}$ for all i such that $\|\sum_1^k \varepsilon_i w_i\| \le 2d - 1$ for every $k \in \{1, \dots, n\}$. Theorem 1 implies that this result can be strengthened when the v_is are unit vectors in $\mathbb{R}^2$ and k is odd.

Theorem 4. *Assume $v_1, \dots, v_n \in \mathbb{R}^2$ is a sequence of unit vectors in the norm $\|.\|$. Then there are signs $\varepsilon_i \in \{-1, +1\}$ for all i such that $\|\sum_1^k \varepsilon_i w_i\| \le 2$ for every odd $k \in \{1, \dots, n\}$.*

The bound 2 here is best possible as shown by the example of the max norm and the sequence $(-1, 1/2), (1, 1/2), (0, 1), (-1, 1), (1, 1)$.

The **proof** goes by induction on k. The case $k = 1$ is trivial. For the induction step $k \to k+2$ let s be the signed sum of the first k vectors with $\|s\| \leq 2$. There are vectors u and w (parallel with s) such that $s = u + w$, $\|u\| = 1, \|w\| \leq 1$. Applying Theorem 1 to $u, v_{k+1}$and v_{k+2} we have signs $\varepsilon(u), \varepsilon_{k+1}$ and ε_{k+2} with $\|\varepsilon(u)u + \varepsilon_{k+1}v_{k+1} + \varepsilon_{k+2}v_{k+2}\| \leq 1$. Here we can clearly take $\varepsilon(u) = +1$. Then

$$\|s + \varepsilon_{k+1}v_{k+1} + \varepsilon_{k+2}v_{k+2}\| \leq \|u + \varepsilon_{k+1}v_{k+1} + \varepsilon_{k+2}v_{k+2}\| + \|w\| \leq 2$$

finishing the proof. □

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References

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