Generalized line graphs: Cartesian products and complexity of recognition *

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Abstract

Putting the concept of line graph in a more general setting, for a positive integer k the k-line graph $L_k(G)$ of a graph G has the K_k-subgraphs of G as its vertices, and two vertices of $L_k(G)$ are adjacent if the corresponding copies of K_k in G share $k-1$ vertices. Then, 2-line graph is just the line graph in usual sense, whilst 3-line graph is also known as triangle graph. The k-anti-Gallai graph $\triangle_k(G)$ of G is a specified subgraph of $L_k(G)$ in which two vertices are adjacent if the corresponding two K_k-subgraphs are contained in a common K_{k+1}-subgraph in G.

We give a unified characterization for nontrivial connected graphs G and F such that the Cartesian product $G \square F$ is a k-line graph. In particular for $k=3$, this answers the question of Bagga (2004), yielding the necessary and sufficient condition that G is the line graph of a triangle-free graph and F is a complete graph (or vice versa). We show that for any $k \geqslant 3$, the k-line graph of a connected graph G is isomorphic to the line graph of G if and only if $G = K_{k+2}$. Furthermore, we prove that the recognition problem of k-line graphs and that of k-anti-Gallai graphs are NP-complete for each $k \geqslant 3$.

Keywords: Triangle graph, k-line graph, anti-Gallai graph, Cartesian product graph.

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1 Introduction

The line graph $L(G)$ of a graph G has vertices representing the edges (K_2-subgraphs) of G and two vertices in the line graph are adjacent if and only if the corresponding edges share a vertex (a K_1 subgraph) in G. The analogous notion in the dimension higher by one is the triangle graph $\mathcal{T}(G)$ of G whose vertices correspond to the triangles (K_3-subgraphs) of G and the vertices representing triangles having a common edge (K_2-subgraph) are adjacent. The natural generalization gives the notion of k-line graph, which together with its specified subgraph, the so-called k-anti-Gallai graph is the main subject of this paper.

1.1 Terminology

All graphs considered here are simple and undirected. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Throughout this paper, a *k-clique* of G will be meant as a complete $K_k \subseteq G$ subgraph. That is, *inclusion-wise maximality is not required.* For the sake of simplicity, if the meaning is clear from the context, we do not distinguish between a clique and its vertex set in notation (e.g., the vertex set of a clique C will also be denoted by C instead of $V(C)$). The *clique number* $\omega(G)$ is the maximum order of a clique contained in G. The *Cartesian product* of two graphs G and F, denoted by $G\Box F$, has the ordered pairs (u, v) as its vertices where $u \in V(G)$ and $v \in V(F)$, and two vertices (u, v) and (u', v') are adjacent if $u = u'$ and v is adjacent to v' or $v = v'$ and u is adjacent to u'. If $v_i \in V(F)$, the *copy* G_i is the subgraph of $G \Box F$ induced by the vertex set $V(G_i) = \{(u_j, v_i) : u_j \in V(G)\}$. The copy F_j for $u_j \in V(G)$ is meant similarly. The *join* $G \vee F$ of two vertex-disjoint graphs is the graph whose vertex set is $V(G) \cup V(F)$ and two vertices u and v of $G \vee F$ are adjacent if and only if either $uv \in E(G) \cup E(F)$, or $u \in V(G)$ and $v \in V(F)$. The *diamond* is a 4-cycle with exactly one chord (or equivalently, the graph $K_4 - e$ obtained from the complete graph K_4 by deleting exactly one edge). Given a graph F, a graph G is said to be *F-free* if it contains no *induced* subgraph isomorphic to F.

Next, we define the two main concepts studied in this paper. For illustration, see Figure 1.

Definition 1. For an integer $k \geqslant 1$, the *k-line graph* $L_k(G)$ of a graph G has vertices representing the k-cliques of G, and two vertices in $L_k(G)$ are adjacent if and only if the represented k-cliques of G intersect in a $(k-1)$-clique.

For $k = 1$ the definition yields $L_1(G) = K_n$ for every graph G of order n. Note that even the K_2-free (edgeless) graph with n vertices has the complete graph K_n as its 1-line graph. The 2-line graph $L_2(G)$ is the line graph of G in the usual sense. The 3-line graph is the triangle graph $\mathcal{T}(G)$.

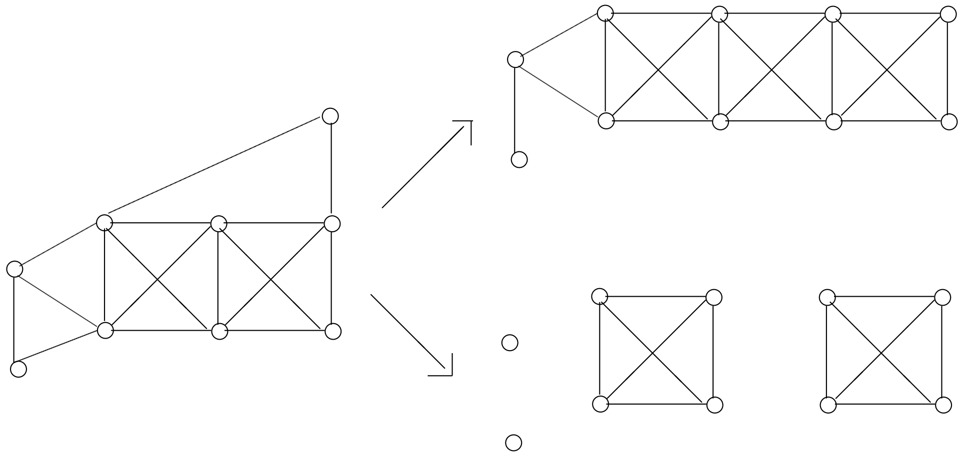

Figure 1: A graph and its 3-line graph and 3-anti-Gallai graph.

Definition 2. For an integer $k \geqslant 1$, the *k-anti-Gallai graph* $\triangle_k(G)$ of a graph G has one vertex for each k-clique of G, and two vertices in $\triangle_k(G)$ are adjacent if and only if the union of the two k-cliques represented by them span a $(k+1)$-clique in G.

Hence, $\triangle_k(G)$ is a subgraph of $L_k(G)$. For every graph G, its 1-anti-Gallai graph is G itself, whilst 2-anti-Gallai graph means anti-Gallai graph (denoted by $\triangle(G)$) in the usual sense. If a vertex c_i of $L_k(G)$ or $\triangle_k(G)$ represents the k-clique C_i of G, we say that c_i is the image of C_i and conversely, C_i is the preimage of c_i. In notation, if the context is clear, the preimage of c_i is denoted either by C_i or by $C(c_i)$. A graph G is called k-line graph or k-anti-Gallai graph if there exists a graph G' such that $L_k(G') = G$ or $\triangle_k(G') = G$ holds, respectively.

1.2 Results

The line graph operator is a classical subject in graph theory. From the rich literature here we mention only the forbidden subgraph characterization given by Beineke in 1970 [6]. The notion of the triangle graph and that of the k-line graph were introduced several times independently by different motivations, and studied from different points of view (see for example [5, 7, 8, 9, 12, 15, 16, 18]). For earlier results on anti-Gallai and k-anti-Gallai graphs we refer the reader to the papers [4, 10, 13] and the book [15]. As relates the most recent works, Anand *et al.* answered a question of Le by showing that the recognition problem of anti-Gallai graphs is NP-complete [2], moreover an application of the anti-Gallai graphs to automate the discovery of ambiguous words is described in [1].

In this paper we study three related topics. The first one concerns a question of Bagga [5] asking for a characterization of graphs G for which $G \,\square\, K_n$ is a triangle graph. As a complete solution in a much more general setting, in Section 2 we give a necessary and sufficient condition for a Cartesian product $G \,\square\, F$ to be a 3-line graph. Then, in Section 3, this result is generalized by establishing a unified characterization for $G \,\square\, F$ to be a k-line graph, for every $k \geqslant 2$.

We also study the algorithmic hardness of recognition problems. Due to the forbidden subgraph characterization in [6], the 2-line graphs can be recognized in polynomial time. In contrast to this, we prove in Section 4 that the analogous problem is NP-complete for the triangle graphs. Then, in Section 5 the same hardness is established for k-line graphs for each fixed $k \geqslant 4$. Via some lemmas and a constructive reduction, we obtain that recognizing k-anti-Gallai graphs is also NP-complete for each $k \geqslant 3$. The latter result solves a problem raised by Anand *et al.* [2], extending their theorem from $k = 2$ to larger values of k.

In Section 6, graphs with $L_k(G) \cong L(G)$ are identified for each $k \geqslant 3$. Finally, in the concluding section we put some remarks and formalize a problem which remains open.

1.3 Some basic facts

Here we list some basic statements, which can be found in [15] or can be proved directly from the definitions.

Observation 1 ([15])**.** *Every k-line graph is $K_{1,k+1}$-free.*

Observation 2. *Every clique K_n of a k-line graph $L_k(G)$ either corresponds to n k-cliques of G sharing a fixed $(k-1)$-clique, or corresponds to n k-cliques contained in a common K_{k+1}. In particular, every clique of order n in a triangle graph $\mathcal{T}(G)$ corresponds to n triangles of G which are either incident with a fixed edge, or contained in a common K_4.*

Proof. Let $c_1, \ldots, c_n$ be the vertices of an n-clique K_n of $L_k(G)$ and $C_1, \ldots, C_n$ be the corresponding k-cliques in G. Moreover, let $v_1, \ldots, v_k \in V(G)$ be the vertices which induce C_1. Since c_2 is adjacent to c_1 in $L_k(G)$, the k-clique C_2 has precisely one vertex outside C_1. We assume without loss of generality that $C_2 = \{u, v_2, v_3, \ldots, v_k\}$. Now, suppose that there exists a vertex in K_n, say c_3, such that its preimage C_3 does not contain some vertex from the set $C_1 \cap C_2 = \{v_2, v_3, \ldots, v_k\}$; say, v_k is omitted. In this case, since c_3 is adjacent to both c_1 and c_2, the k-clique C_3 must be induced by $\{u, v_1, v_2, \ldots, v_{k-1}\}$. Then, for any further vertex c_i, the preimage must be of the form $C_i = \{u, v_1, v_2, \ldots, v_k\} \setminus \{v_{j_i}\}$ for some $2 \leqslant j_i \leqslant k-1$. This proves that if not all the intersections $C_i \cap C_j$ are the same, then each of the k-cliques $C_1, \ldots, C_n$ is contained in the $(k+1)$-clique $\{u, v_1, v_2, \ldots, v_k\}$. $\square$

Observation 3. *If G is the k-line graph of a K_{k+1}-free graph, then for every $k' > k$, G is also the k'-line graph of a $K_{k'+1}$-free graph.*

Proof. Let $G = L_k(H)$ for a K_{k+1}-free graph H. Consider the join $H' = H \vee K_{k'-k}$. Since H is K_{k+1}-free, H' is $K_{k'+1}$-free and every k'-clique of H' originates from a k-clique of H extended by the $k' - k$ new vertices. Additionally, two k'-cliques of H' intersect in a $K_{k'-1}$ if and only if the corresponding k-cliques of H meet in a K_{k-1}. Consequently, $L_{k'}(H') = L_k(H) = G$. $\square$

2 Cartesian product and triangle graphs

In this section we solve a problem posed in [5] by Bagga.

Theorem 4. *The Cartesian product $G \square F$ of two nontrivial connected graphs is a triangle graph if and only if F is a complete graph and G is the line graph of a triangle-free graph (or vice versa).*

Before proving the theorem we verify a lemma.

Lemma 5. *If G contains a diamond as an induced subgraph then $G \square K_n$ is not a triangle graph for $n \geqslant 2$.*

Proof. To prove the lemma we apply the following result from [5].

($*$) If H is a triangle graph with $K_4 - e$ as an induced subgraph, then there exists a vertex x in H such that x is adjacent to three vertices of one triangle of $K_4 - e$ and nonadjacent to the fourth vertex.

Let G be a graph which contains a diamond induced by the vertices u_1, u_2, u_3 and u_4, where (u_1, u_4) is the non-adjacent vertex pair. Assume for a contradiction that there exists a graph H whose triangle graph is $G \square K_n$ for some $n \geqslant 2$. Let $v_1 \in V(K_n)$. Then, $(u_1, v_1), (u_2, v_1), (u_3, v_1), (u_4, v_1)$ is an induced diamond in $G \square K_n$. Since $G \square K_n$ is a triangle graph, by ($*$), it must contain a vertex (u_5, v_1) which is adjacent to all vertices of one of the triangles in the diamond and not adjacent to the fourth vertex. Let (u_4, v_1) be the vertex which is not adjacent to (u_5, v_1). Let t_i be the triangle in H corresponding to the vertex (u_i, v_1) in $G \square K_n$ for $i = 1, \ldots, 5$. Then t_1, t_2, t_3 and t_5 must be the triangles of a K_4 and t_4 is a triangle which shares the edge which is common to the triangles t_2 and t_3. Let $v_2 \in V(K_n) \setminus \{v_1\}$ (it exists, since $n \geqslant 2$). Then (u_1, v_2) is adjacent to (u_i, v_1) only for $i = 1$. Therefore, the triangle in H corresponding to the vertex (u_1, v_2) must share an edge with t_1 and not with any other t_i for $i = 2, \ldots, 5$. But, each edge of t_1 is shared with at least one among t_2, t_3 and t_5, which gives a contradiction. Therefore, $G \square K_n$ is not a triangle graph. □

Proof of Theorem 4. If both G and F are non-complete graphs, then $G \square F$ contains an induced $K_{1,4} \subset P_3 \square P_3$ and hence, by Observation 1, it is not a triangle graph. So we can assume that $F = K_n$ for some $n \geqslant 2$.

If G is not a line graph, then by the theorem of Beineke [6], G contains one of the nine forbidden subgraphs as an induced subgraph (see Figure 2). If it is $K_{1,3}$, then $G \square K_n$ contains an induced $K_{1,4}$, which is forbidden for triangle graphs. In the case of any of the remaining eight graphs, G contains an induced diamond and hence, by Lemma 5, $G \square K_n$ cannot be a triangle graph.

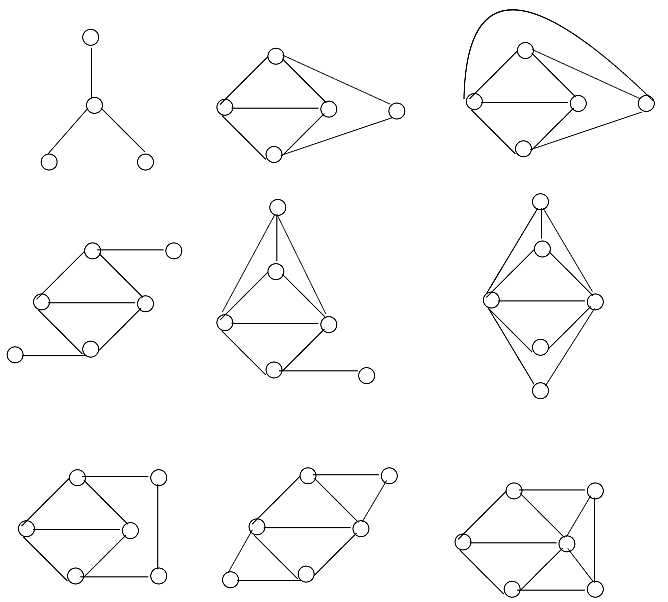

Figure 2: Forbidden subgraphs for line graph.

Let G be the line graph of a graph H which contains a triangle. Let $T = (u_1, u_2, u_3)$ be a triangle in H. If $H \neq K_3$, there exists a vertex u_4 adjacent to some u_i in T (not necessarily to u_i alone). But, then $G = L(H)$ contains a diamond and hence by Lemma 5, $G \square K_n$ is not a triangle graph. If H is K_3 itself, then $H = L(K_{1,3})$ also.

Conversely, let G be the line graph of a triangle-free graph H. Then $\mathcal{T}(H \vee \overline{K_n}) = G \square K_n$, completing the proof of the theorem. $\square$

Concerning edgeless graphs, note that $G \square K_1 \cong G$, and hence the characterization problem of graphs G such that $G \square K_1$ is a triangle graph is equivalent to characterizing triangle graphs.

3 Cartesian product and k-line graphs

As proved in Section 2, if both G and F are nontrivial connected graphs, $G \square F$ is a triangle graph if and only if one of G and F is a line graph of a K_3-free graph and the other one is a complete graph. We will see that a direct analogue of this theorem is not valid for k-line graphs in general. For instance, one can observe that the grid graph $P_n \square P_m$ is a k-line graph for every n, m and $k \geqslant 4$.

Our main result in this section is the following theorem, which gives a necessary and sufficient condition for a product $G \square F$ of non-edgeless graphs to be a k-line graph. Recall that k-line graphs were defined for $k = 1$, too.

Theorem 6. *For every $k \geqslant 2$, the product $G \square F$ of two non-edgeless connected graphs is a k-line graph if and only if there exist positive integers k_1 and k_2 such that G is the k_1-line graph of a K_{k_1+1}-free graph, F is the k_2-line graph of a K_{k_2+1}-free graph and $k_1 + k_2 \leqslant k$ holds.*

If F is a complete graph then it is the k_2-line graph of a K_{k_2+1}-free graph for every $k_2 \geqslant 1$. (For example, $K_n = L_{k_2}(K_{k_2-1} \vee nK_1)$ for all $k_2 \geqslant 1$, where the degenerate case $k_2 = 1$ simply means that $K_n = L_1(nK_1)$.) Hence, in this particular case of Theorem 6, the existence of an appropriate $k_1 \leqslant k-1$ is required. By Observation 3, this is equivalent to the claim that G is the $(k-1)$-line graph of a K_k-free graph. Thus, we obtain:

Corollary 7. *For every two integers $n \geqslant 2$ and $k \geqslant 2$, the product $G \,\square\, K_n$ is a k-line graph if and only if G is the $(k-1)$-line graph of a K_k-free graph.*

Theorem 6 will be proved at the end of this section. First we need some lemmas.

Lemma 8.

(*i*) *If H contains a K_{k+1} subgraph, then for the corresponding $(k+1)$-clique $\mathcal{C}$ of $L_k(H)$, each vertex $c \in V(L_k(H)) \setminus \mathcal{C}$ is either adjacent to none of the vertices of $\mathcal{C}$ or c is adjacent to exactly two vertices of $\mathcal{C}$.*

(*ii*) *Assume that $k \geqslant 2$ and no component of the k-line graph $L_k(H)$ is isomorphic to K_{k+1}. Then, H is K_{k+1}-free if and only if $L_k(H)$ is diamond-free.*

Proof. First, assume that H contains a K_{k+1}, which is induced by the vertex set $V = \{v_1, \ldots, v_{k+1}\} \subseteq V(H)$, and consider the corresponding $(k+1)$-clique $\mathcal{C}$ in $L_k(H)$ whose vertices $c_1, \ldots, c_{k+1}$ represent the k-subsets of V. If there is a further vertex c^* adjacent to at least one vertex of $\mathcal{C}$, then in H the k-clique C^*, which is the preimage of c^*, intersects V in exactly $k-1$ vertices. Without loss of generality, we assume that for every c_i the preimage is $C_i = V \setminus \{v_i\}$, moreover that $C^* \cap V = \{v_1, \ldots, v_{k-1}\}$. Then, $|C^* \cap C_k| = |C^* \cap C_{k+1}| = k-1$, but $|C^* \cap C_i| = k-2$ holds for every $1 \leqslant i \leqslant k-1$. Therefore, c^* is adjacent to exactly two vertices of $\mathcal{C}$. This verifies (i). Concerning (ii), note that if the $(k+1)$-clique $\mathcal{C}$ is not a component of $L_k(H)$ then such a c^* surely exists, and if $k \geqslant 2$, vertices c_1, c_k, c_{k+1}, c^* induce a diamond in $L_k(H)$.

For the other direction of (ii), assume that H is K_{k+1}-free, and in $L_k(H)$ vertices c_1, c_2, c_3, c_4 induce a diamond where c_2 and c_4 are nonadjacent. By Observation 2, the preimages C_1, C_2, C_3 of c_1, c_2, c_3 are three different k-cliques of H sharing a fixed $(k-1)$-clique. Then, $|C_1 \cap C_2 \cap C_3| = k-1$ and similarly, $|C_1 \cap C_3 \cap C_4| = k-1$ hold. By the adjacency of c_1 and c_3, $|C_1 \cap C_3| = k-1$ is valid as well. Hence, the vertex sets $C_1 \cap C_2 \cap C_3$ and $C_1 \cap C_3 \cap C_4$ must be the same, contradicting the non-adjacency of c_2 and c_4. Thus, we conclude that for a K_{k+1}-free H, the k-line graph contains no induced diamond. $\square$

Lemma 9. *If the Cartesian product $G \,\square\, F$ of two non-edgeless graphs is the k-line graph of H, then H is K_{k+1}-free.*

Proof. Suppose for a contradiction that H contains a complete subgraph X induced by vertices $x_1, \ldots, x_{k+1}$. Then, in $L_k(H) = G \,\square\, F$ the vertices $c_1, \ldots, c_{k+1}$, representing the k-cliques $C_1, \ldots, C_{k+1} \subset X$, form a complete subgraph. Hence, all these vertices

$c_1, \ldots, c_{k+1}$ must belong either to the same copy of G or to the same copy of F. Assume without loss of generality that $c_i = (v_1, u_i)$ for every $1 \leqslant i \leqslant k+1$, and let v_j be a neighbor of v_1 in G. Then, the vertex (v_j, u_1) is adjacent to only one vertex (namely, to (v_1, u_1)) from the $(k+1)$-clique. This contradicts Lemma 8(i) and hence, H is K_{k+1}-free. $\square$

Lemma 10. *If G is the k-line graph of H, and G contains an induced cycle $c_1c_2c_3c_4$, then the corresponding k-cliques C_1, C_2, C_3, C_4 of H satisfy $C_2 \setminus C_1 = C_3 \setminus C_4$.*

Proof. As the 1-line graph $L_1(H)$ is a complete graph, it contains no induced four-cycle. Hence, we can assume that $k \geqslant 2$. Let $C_1 \setminus C_2 = \{v_1\}$, $C_2 \setminus C_1 = \{v_2\}$, $C_1 \setminus C_4 = \{z_1\}$ and $C_4 \setminus C_1 = \{z_2\}$. We observe that $v_1 \neq z_1$ and $v_2 \neq z_2$, as any $v_i = z_i$ would mean the adjacency of c_2 and c_4. Now, suppose for a contradiction that $v_2 \notin C_3$. Since c_1c_2 and c_2c_3 are edges in G and $v_2 \in C_2$, but $v_2 \notin C_1$ and $v_2 \in C_3$, it follows that, $C_3 \cap C_2 = C_2 \setminus \{v_2\} = C_1 \cap C_2$. This implies $|C_1 \cap C_3| = k-1$, which contradicts $c_1c_3 \notin E(G)$. Therefore, $v_2 \in C_3$ holds and since $v_2 \notin C_4$, the desired equality $C_3 \setminus C_4 = \{v_2\} = C_2 \setminus C_1$ follows. $\square$

In view of Lemma 10, we can give a structural characterization for graphs whose k-line graph is the Cartesian product $G \,\square\, F$ of two non-edgeless graphs. We will use the following notions.

If $L_k(H) = G \,\square\, F$, consider a copy G_i and the k-cliques of H which are represented by the vertices of G_i. A vertex contained in all these k-cliques is called a *universal vertex*, otherwise it is *non-universal* (with respect to G_i). Formally, let

$$U_i^G(H) = \bigcap\{C(v_j, u_i) : v_j \in V(G)\}, \text{ and } X_i^G(H) = \bigcup\{C(v_j, u_i) : v_j \in V(G)\} \setminus U_i^G(H).$$

Analogously, the sets U_i^F and X_i^F of universal and non-universal vertices with respect to the copy F_i are also introduced.

Lemma 11. *Let G and F be two connected graphs and let $L_k(H) = G \,\square\, F$.*

(i) *For every two copies G_i and G_j of G the non-universal vertices are the same: $X_i^G(H) = X_j^G(H)$.*

(ii) *Each copy G_i is the k-line graph of the subgraph induced by $U_i^G(H) \cup X_i^G(H)$.*

Proof. First assume that u_i and u_j are adjacent vertices in F. For every two vertices v_a and v_b of G there is a path between (v_a, u_i) and (v_b, u_i) in G_i. This path together with the corresponding vertices of G_j induces a chain of 4-cycles. Hence, the repeated application of Lemma 10 implies that

$$C(v_a, u_i) \setminus C(v_a, u_j) = C(v_b, u_i) \setminus C(v_b, u_j) \quad \text{and} \quad C(v_a, u_j) \setminus C(v_a, u_i) = C(v_b, u_j) \setminus C(v_b, u_i)$$

for every two vertices v_a and v_b of G. As follows, there are fixed vertices $w_{i,j} \in U_i^G(H)$ and $w_{j,i} \in U_j^G(H)$ such that the preimage of any vertex of copy G_j can be obtained from the preimage of the corresponding vertex of copy G_i by replacing $w_{i,j}$ with $w_{j,i}$ in the k-clique. Thus, $U_j^G(H) = (U_i^G(H) \setminus \{w_{i,j}\}) \cup \{w_{j,i}\}$ and for the non-universal vertices $X_i^G(H) = X_j^G(H)$ holds. Since F is connected, the latter equality is valid for nonadjacent vertices $u_i, u_j \in V(F)$ as well. This verifies (i).

To prove (ii), observe that $U_i^G(H) \neq U_j^G(H)$ and $|U_i^G(H)| = |U_j^G(H)|$ for every pair i, j. Therefore, neither $U_i^G(H) \setminus U_j^G(H)$ nor $U_j^G(H) \setminus U_i^G(H)$ is empty. Consequently, every k-clique in the subgraph induced by $U_i^G(H) \cup X_i^G(H)$ has its image in copy G_i. This proves (ii). $\square$

Proof of Theorem 6. To prove sufficiency, let $G = L_{k_1}(G')$ and $F = L_{k_2}(F')$ where G' is K_{k_1+1}-free, F' is K_{k_2+1}-free, and $k_1 + k_2 \leqslant k$. Then, the join $H' = G' \vee F'$ is $K_{k_1+k_2+1}$-free and every $(k_1 + k_2)$-clique of H' originates from a k_1-clique of G' and from a k_2-clique of F'. Thus, the vertices of $L_{k_1+k_2}(H')$ correspond to the pairs (v_i, u_j) with $v_i \in V(G)$ and $u_j \in V(F)$. Moreover two vertices (v_i, u_j) and (v_k, u_ℓ) in $L_{k_1+k_2}(H')$ are adjacent if and only if

- either $v_i = v_k$ and the k_2-cliques $C(u_j)$ and $C(u_\ell)$ share a $(k_2 - 1)$-clique in F', that is $u_j u_\ell \in E(F)$,
- or $u_j = u_\ell$ and the k_1-cliques $C(v_i)$ and $C(v_k)$ share a $(k_1 - 1)$-clique in G', that is $v_i v_k \in E(G)$.

Therefore, $L_{k_1+k_2}(H') = G \,\square\, F$, and by Observation 3, $L_k(H' \vee K_{k-k_1-k_2}) = G \,\square\, F$ also holds for every $k > k_1 + k_2$.

To prove necessity, suppose that $H = G \,\square\, F$ is the k-line graph of H'. By Lemma 9, H' is K_{k+1}-free. Consider the sets $A = U_1^G(H')$ and $B = U_1^F(H')$ of universal vertices in copies G_1 and F_1.

Claim A. G is the $(k - |A|)$-line graph of a $K_{k-|A|+1}$-free graph and F is the $(k - |B|)$-line graph of a $K_{k-|B|+1}$-free graph.

Proof. By Lemma 11(ii), $G_1 \cong G$ is the k-line graph of the subgraph $G^* \subset H'$ induced by $A \cup X_1^G(H')$. If the universal vertices of G^* are deleted, we obtain $G' = G^* - A$. While constructing G' from G^*, every k-clique is shrunk into a $(k - |A|)$-clique and two cliques share exactly $k - |A| - 1$ vertices if and only if the corresponding vertices of G are adjacent. This proves that $G = L_{k-|A|}(G')$. It is clear that G' is $K_{k-|A|+1}$-free. The analogous argument for F yields $F' = F^* - B$ such that $F = L_{k-|B|}(F')$ and F' is $K_{k-|B|+1}$-free.

Claim B. $|A| + |B| \geqslant k$.

Proof. Assume to the contrary that $|A| + |B| < k$. Then, there exists a vertex $z \in C(v_1, u_1) \setminus (A \cup B)$. As the graph $G_1 \cong G$ is connected and z cannot be contained in all preimages $C(v_\ell, u_1)$, there exist adjacent vertices (v_i, u_1) and (v_j, u_1) such that $z \in C(v_i, u_1)$ and $z \notin C(v_j, u_1)$. This means $C(v_i, u_1) \setminus C(v_j, u_1) = \{z\}$. Similarly for F_1, there exist indices m and n such that $C(v_1, u_m) \setminus C(v_1, u_n) = \{z\}$ holds. By Lemma

10, for the 4-cycle induced by $\{(v_i, u_n), (v_i, u_m), (v_j, u_m), (v_j, u_n)\}$ the following equalities hold:

$$C(v_1, u_m) \setminus C(v_1, u_n) = C(v_i, u_m) \setminus C(v_i, u_n) = \{z\},$$
$$C(v_i, u_1) \setminus C(v_j, u_1) = C(v_i, u_n) \setminus C(v_j, u_n) = \{z\}.$$

They give a contradiction on the question whether z is in $C(v_i, u_n)$ or not. This proves $|A| + |B| \geqslant k$.

Denoting $k_1 = k - |A|$ and $k_2 = k - |B|$, $k_1 + k_2 \leqslant k$ follows by Claim B. Then, Claim A proves the necessity of the condition given for G and F in the theorem. □

The proof of Theorem 6 also verifies the following statement.

Corollary 12. *If G is K_{k_1+1}-free and F is K_{k_2+1}-free, then*

$$L_{k_1}(G) \,\square\, L_{k_2}(F) \cong L_{k_1+k_2}(G \vee F).$$

4 NP-completeness of recognizing triangle graphs

As is well-known, the line graphs can be recognized in polynomial time due to the forbidden subgraph characterization by Beineke [6]. Also linear-time algorithms were designed for solving this problem [14, 17]. Here we prove that triangle graphs (that is, 3-line graphs) are hard to recognize.

Theorem 13. *The following problems are* NP*-complete:*

(i) Recognizing triangle graphs.

(ii) Deciding whether a given graph is the triangle graph of a K_4-free graph.

Moreover, both problems remain NP*-complete on the class of connected graphs.*

Before proving Theorem 13, we verify two lemmas which give necessary conditions for graphs to be anti-Gallai or triangle graphs of some K_4-free graph, respectively.

Lemma 14. *Assume that F is a connected non-trivial graph and F is the anti-Gallai graph of F'. Then F' is K_4-free if and only if every edge of F is contained in exactly one triangle, or equivalently*

($\star$) every maximal clique of F is a triangle and any two triangles share at most one vertex.

Proof. If F' contains a K_4 subgraph then $F = \triangle(F')$ contains an induced $K_6 - 3K_2$ and ($\star$) does not hold. If F' is K_4-free, any three pairwise adjacent vertices in $\triangle(F')$ correspond to three edges of F' which form a triangle. Hence no edge of $\triangle(F')$ belongs to more than one triangle. Additionally, by definition, in an anti-Gallai graph every edge corresponds to two preimage-edges of a triangle; hence, every edge of $\triangle(F')$ is contained in a K_3. Since F is assumed to be connected and non-edgeless, it contains no isolated vertices. This proves that every maximal clique of F is a K_3 and any two triangles have at most one vertex in common. □

Lemma 15. *Assume that G is a connected graph which is not isomorphic to K_4, moreover $G = \mathcal{T}(G')$. Then G' is K_4-free if and only if*

($\star\star$) *each vertex of G is contained in at most three maximal cliques and these cliques are pairwise edge-disjoint.*

Proof. In this proof, the vertex of G whose preimage is a triangle abc in G' will be denoted by t_{abc}.

First suppose that G' contains a K_4 induced by the vertices x, y, z, u. Clearly, the four triangles of the K_4 correspond to a 4-clique in $\mathcal{T}(G')$. By our condition $\mathcal{T}(G')$ is not a 4-clique, hence there exists a triangle in G', containing exactly two vertices from x, y, z, u. Say, this triangle is xyw. In $\mathcal{T}(G')$, the vertex originated from xyz is contained in both cliques induced by the vertex sets $\{t_{xyz}, t_{xyu}, t_{yzu}, t_{xzu}\}$ and $\{t_{xyz}, t_{xyu}, t_{xyw}\}$, respectively. Maybe the second clique is not maximal, but since there is no edge between t_{yzu} and t_{xyw}, there are two different maximal cliques with the common edge $t_{xyz}t_{xyu}$. This shows that ($\star\star$) does not hold.

For the converse, suppose that G' is K_4-free. Then by Observation 2, each clique of $\mathcal{T}(G')$ corresponds to triangles sharing a fixed edge in G'. Thus, a vertex $t_{xyz} \in V(\mathcal{T}(G))$ can be contained only in those maximal cliques which correspond to the three edges of its preimage-triangle (one or two of these cliques might be missing) and any two of these maximal cliques have t_{xyz} as the only common vertex, hence ($\star\star$) holds. $\square$

While proving that the recognition problem of triangle graphs is NP-complete, we will use the following notion. The *triangle-restriction* of a graph is obtained if the edges not contained in any triangles and the possibly arising isolated vertices are deleted. Every graph has a triangle-restriction, and the application of this operator changes neither the anti-Gallai graph (if it is connected)[1], nor the triangle graph. A graph is called *triangle-restricted* if each edge and each vertex of it belongs to a triangle. The *clique graph* $\mathcal{K}(G)$ of a graph G is the intersection graph of the set of all *maximal* cliques of G.[2]

Proof of Theorem 13. The decision problems are clearly in NP. The NP-completeness of (*ii*) will be reduced from the following theorem recently proved by Anand et al. [2]: Deciding whether a connected graph F is the anti-Gallai graph of some K_4-free graph is an NP-complete problem.

Consider an instance F to decide whether it is the anti-Gallai graph of a K_4-free graph. In the first step, we check the necessary condition ($\star$); if it does not hold, F is not the anti-Gallai graph of any K_4-free graphs. From now on, suppose that ($\star$) holds for F. Then every maximal clique of F is a triangle and the clique graph $G = \mathcal{K}(F)$ is exactly the triangle-intersection graph of F. If F is connected then so is G, and $G \cong K_4$ holds if and only if F is the union of four triangles sharing exactly one vertex. Hence,

[1]If some edges of a graph F' are not contained in any triangles, their images in the anti-Gallai graph are isolated vertices. The deletion of these edges from F' results in the deletion of all isolated vertices from the anti-Gallai graph.

[2]That is, the vertices of $\mathcal{K}(G)$ correspond to the maximal cliques of G and two vertices of $\mathcal{K}(G)$ are adjacent if the corresponding cliques share at least one vertex.

from now on we assume that $G \not\cong K_4$. In addition, if F fulfills property ($\star$), then its triangle-intersection graph G fulfills property ($\star\star$).

Next, we prove that F is the anti-Gallai graph of a K_4-free triangle-restricted graph H if and only if G is the triangle graph of H.

Assume that $F = \triangle(F')$. By ($\star$), F' is K_4-free, hence its triangles are in one-to-one correspondence with the triangles of F and by ($\star$) this yields a one-to-one correspondence with the vertices of G. Moreover two triangles in F' share an edge if and only if the corresponding triangles share a vertex in F; and if and only if the corresponding vertices in the clique graph G are adjacent. Therefore, $G = \mathcal{T}(F')$.

To prove the other direction, assume that $G = \mathcal{T}(G')$. Since G satisfies ($\star\star$), G' must be K_4-free. We can choose G' to be triangle-restricted. Now, for every vertex $t \in V(G)$, if t is contained in only two maximal cliques, then in addition the 1-element vertex set $\{t\}$ will also be considered as a 'maximal clique' of G. Similarly, if t is contained in only one clique of G, then $\{t\}$ is also taken as a 'maximal clique' with multiplicity 2. Then the edges of G' are in one-to-one correspondence with the maximal cliques of G. These maximal cliques are in one-to-one correspondence with the vertices of F, where the one-element cliques of G represent vertices contained in only one triangle of F. Also, two edges of G' belong to a common triangle if and only if the corresponding maximal cliques have a common vertex (which represents the triangle); that is, if and only if the two vertices of F, represented by the cliques, are adjacent. This proves $F = \triangle(G')$.

Checking ($\star$) and constructing G from F takes polynomial time. So, the recognition problem of the anti-Gallai graphs can be reduced to that of the triangle graphs in polynomial time. Hence, the recognition problem of triangle graphs is NP-complete and this remains valid on the class of connected graphs satisfying ($\star\star$). $\square$

5 Recognizing generalized line graphs and anti-Gallai graphs

In this section we turn to the recognition problems of general k-line graphs and k-anti-Gallai graphs. In sharp contrast to the linear-time recognizability of k-line graphs for $k \leqslant 2$, by Theorem 13 the analogous problem is NP-complete for $k = 3$. Also, anti-Gallai graphs are hard to recognize as proved by Anand et al. via a reduction from 3-SAT [2]. Now, we complete these results by proving that the recognition problems of k-line graphs and k-anti-Gallai graphs are NP-complete for each $k \geqslant 3$.

Theorem 16. *The following problems are* NP*-complete for every fixed* $k \geqslant 3$*:*

(i) *Recognizing* k*-line graphs.*

(ii) *Deciding whether a given graph is the* k*-line graph of a* K_{k+1}*-free graph.*

Moreover both problems remain NP*-complete on the class of connected graphs.*

Proof. Clearly, problems (i) and (ii) are in NP. Moreover, by Theorem 13, both problems are NP-complete for $k = 3$, already on the class of connected graphs. Therefore, we can proceed by induction on k.

For the inductive step, assume that $k \geqslant 4$ and that (ii) is NP-complete for $k-1$ on the class of connected graphs. Let G be a connected graph and construct the Cartesian product $H = G \,\Box\, K_2$, which is also connected. Due to Corollary 7, G is a $(k-1)$-line graph of a K_k-free graph if and only if H is a k-line graph of a K_{k+1}-free graph. Therefore, (ii) is NP-complete for every $k \geqslant 3$. On the other hand, by Lemma 9 a graph of the form $G \,\Box\, K_2$ is a k-line graph if and only if it is a k-line graph of a K_{k+1}-free graph. Hence, the above reduction also proves the NP-completeness of (i) for every $k \geqslant 3$. $\square$

Before proving the same hardness for the recognition problem of k-anti-Gallai graphs, we state a lemma. Note that part (i) gives the same condition (namely diamond-freeness) for $\triangle_k(G)$ as Lemma 8 does for $L_{k+1}(G)$ to ensure that G is K_{k+2}-free.

Lemma 17. *For every $k \geqslant 2$, every graph G and its k-anti-Gallai graph $\triangle_k(G)$ satisfy the following relations:*

(i) *G is K_{k+2}-free if and only if $\triangle_k(G)$ is diamond-free.*

(ii) *G is K_{k+2}-free if and only if each maximal clique of $\triangle_k(G)$ is either an isolated vertex or a $(k+1)$-clique. Moreover any two maximal cliques intersect in at most one vertex.*

Proof. First, assume that G contains a $(k+2)$-clique induced by the vertex set $V = \{v_1, \ldots, v_{k+2}\}$. Consider the following k-cliques:

$$C_1 = V \setminus \{v_1, v_2\}, \quad C_2 = V \setminus \{v_2, v_3\}, \quad C_3 = V \setminus \{v_1, v_3\}, \quad C_4 = V \setminus \{v_1, v_4\}.$$

Any two of these k-cliques are in a common $(k+1)$-clique except the pair (C_2, C_4). Therefore, in the k-anti-Gallai graph the corresponding vertices c_1, c_2, c_3 and c_4 induce a diamond. In addition, the two maximal cliques containing $c_1c_2c_3$ and $c_1c_3c_4$, respectively, must be different and intersect in more than one vertex.

To prove the other direction of (i) and (ii), assume that G is K_{k+2}-free. First, consider an edge $c_ic_j \in E(\triangle_k(G))$. The union $C_i \cup C_j$ of the represented k-cliques induces a K_{k+1} subgraph in G, whose k-clique subgraphs are represented by vertices forming a K_{k+1} subgraph in $\triangle_k(G)$. Hence, every edge of $\triangle_k(G)$ belongs to a $(k+1)$-clique. Now, suppose that $c_ic_jc_\ell$ is a triangle in $\triangle_k(G)$. The union $C_i \cup C_j$ of the preimage cliques induces a K_{k+1} in G. Also $C_i \cup C_j \cup C_\ell$ induces a complete subgraph as every two of its vertices are contained in a common clique. Since G is K_{k+2}-free, $C_i \cup C_j \cup C_\ell$ is a $(k+1)$-clique as well, and $C_\ell \subset C_i \cup C_j$ must hold. This implies that in the k-anti-Gallai graph, every two adjacent vertices c_i, c_j together with all their common neighbors form a $(k+1)$-clique. As follows concerning (i), $\triangle_k(G)$ contains no induced diamond. Furthermore, each edge belongs to exactly one maximal clique of $\triangle_k(G)$ and this must be a $(k+1)$-clique. These complete the proof of (i) and (ii). $\square$

Theorem 18. *The following problems are NP-complete for every fixed $k \geqslant 3$:*

(i) *Recognizing k-anti-Gallai graphs.*

(*ii*) *Recognizing k-anti-Gallai graphs on the class of connected and diamond-free graphs.*

(*iii*) *Deciding whether a given connected graph is a k-anti-Gallai graph of a K_{k+2}-free graph.*

Proof. The membership in NP is obvious for each of (*i*)–(*iii*). As diamond-freeness can be checked in polynomial time, statements (*ii*) and (*iii*) imply each other by Lemma 17(*i*). It is also clear that (*ii*) implies (*i*). Then, it is enough to prove (*iii*). For $k = 2$, problem (*iii*) was proved to be NP-complete in [2].

We proceed by induction on k. Consider a generic connected instance G and an integer $k \geqslant 3$.

For each fixed k, the condition given in Lemma 17(*ii*) can be checked in polynomial time. If it does not hold, G cannot be a k-anti-Gallai graph of any K_{k+2}-free graph. From now on we suppose that every maximal clique of G is of order $k+1$ and any two maximal cliques have at most one vertex in common. For such a G we construct the following graph G_e and prove that G is a k-anti-Gallai graph if and only if G_e is a $(k+1)$-anti-Gallai graph.

Construction of G_e. Take two disjoint copies G^1 and G^2 of G with vertex sets $V(G^j) = \{c_i^j : c_i \in V(G)\}$ $(j = 1, 2)$, moreover one vertex b_s for each $(k+1)$-clique B_s of G. Besides the edges of G^1 and G^2 take all edges of the form $b_s c_i^j$ for which $c_i \in B_s$ and $j = 1, 2$ hold.

As G consists of $(k+1)$-cliques such that any two of them intersect in at most one vertex, G_e consists of $(k+2)$-cliques such that any two of them intersect in at most one vertex.

Claim C. If $G = \triangle_k(G')$ then $G_e = \triangle_{k+1}(G' \vee 2K_1)$.

Proof. Let $\mathcal{B}$ denote the set of k-cliques of G'. Corresponding to the relation $G = \triangle_k(G')$, we have a bijection $\phi : \mathcal{B} \mapsto V(G)$ such that every k-clique of G' is mapped to the vertex representing it in $\triangle_k(G')$.

Partition the set $\mathcal{A}$ of $(k+1)$-cliques of the join $G'_e = G' \vee \{z_1, z_2\}$ into three subsets. An $A \in \mathcal{A}$ is said to be of Type 1 or 2 or 3 if it contains z_1, or z_2, or none of them, respectively. (Since z_1 and z_2 are nonadjacent, no clique contains both of them.)

Now, define a bijection $\varphi : \mathcal{A} \mapsto V(G_e)$ as follows. For every $A \in \mathcal{A}$,

$$\varphi(A) = \begin{cases} (\phi(A \setminus \{z_1\})^1 & \text{if } A \text{ is of Type 1,} \\ (\phi(A \setminus \{z_2\})^2 & \text{if } A \text{ is of Type 2,} \\ \quad b_\ell & \text{if } A \text{ is of Type 3, and } A \text{ is the } (k+1)\text{-clique } B_\ell \text{ of } G'. \end{cases}$$

To prove Claim C, we show that two $(k+1)$-cliques A_1 and A_2 of G'_e are contained in a common K_{k+2} if and only if $\varphi(A_1)$ and $\varphi(A_2)$ are adjacent in G_e.

- Type-1 cliques are mapped onto $V(G^1)$. In addition, two cliques A_1, A_2 of Type 1 are contained in a common $(k+2)$-clique in G'_e if and only if $A_1 \setminus \{z_1\}$ and $A_2 \setminus \{z_1\}$ are in a common $(k+1)$-clique in G'; or equivalently, if and only if $(\phi(A_1 \setminus \{z_1\})^1$ and $(\phi(A_2 \setminus \{z_1\})^1$ are adjacent in G^1. Similarly, Type-2 cliques are mapped onto $V(G^2)$ and the adjacencies in G^2 correspond to the adjacencies required in $\triangle_{k+1}(G'_e)$.

- If A_1 is of Type 1 and A_2 is of Type 3, their images are adjacent in $\triangle_{k+1}(G'_e)$ if and only if $A_1 \setminus \{z_1\} \subset A_2$; that is, if the $(k+1)$-clique A_2 contains the k-clique $A_1 \setminus \{z_1\}$ in G'. This corresponds to the adjacency defined in Construction of G_e. The analogous property holds for cliques of Type 2 and Type 3.

- Since z_1 and z_2 are nonadjacent, no two cliques, one of Type 1 and the other of Type 2, belong to a common K_{k+2} in G'_e. Correspondingly, by the construction, there is no edge between $V(G^1)$ and $V(G^2)$ in G_e. Finally, as G' is K_{k+2}-free, no two $(k+1)$-cliques of Type 3 are in a common $(k+2)$-clique in G'_e. This corresponds to the fact that $V(G_e) \setminus (V(G^1) \cup V(G^2))$ is an independent vertex set.

These observations prove that $G_e = \triangle_{k+1}(G'_e)$.

Concerning the following claim, let us recall that Construction of G_e is applied for a $(K_{k+2}$, diamond)-free graph G, and yields a $(K_{k+3}$, diamond)-free G_e.

Claim D. If $G_e = \triangle_{k+1}(F')$ then there exist two vertices $z_1, z_2 \in V(F')$ such that $G = \triangle_k(F' - \{z_1, z_2\})$.

Proof. By Lemma 17, F' must be K_{k+3}-free. Consider a $(k+2)$-clique D_ℓ of G_e. This contains exactly one vertex from $V(G_e) \setminus (V(G^1) \cup V(G^2))$, say b_ℓ, and assume that the other vertices of D_ℓ are from $V(G^1)$. The preimages of the vertices of D_ℓ are exactly the $(k+1)$-clique subgraphs of a $(k+2)$-clique A_ℓ of F'. There is a unique vertex u_ℓ, called complementing vertex of D_ℓ, such that $u_\ell \in A_\ell$. Moreover it is not contained in the preimage $C(b_\ell)$ but is contained in the preimage of each further vertex of D_ℓ. For this vertex, $A_\ell \setminus C(b_\ell) = \{u_l\}$ holds, and $C(c_i^1) \setminus C(b_\ell) = \{u_l\}$ is valid for every $c_i^1 \in D_\ell$.

Assume for a contradiction that there exist two different complementing vertices for the $(k+2)$-cliques meeting $V(G^1)$. By the connectivity of G^1, there exist two $(k+2)$-cliques, say D_1 and D_2, intersecting in a vertex c_i^1 with complementing vertices $u_1 \neq u_2$. Then, consider the vertices $b_1 \in D_1$, $b_2 \in D_2$, the induced 4-cycle $c_i^1 b_1 c_i^2 b_2$ in G_e and the preimage $(k+1)$-cliques C_i^1, B_1, C_i^2, B_2. Since $c_i^1 b_1, c_i^1 b_2 \in E(G_e)$, there exist vertices x and y in F' such that

$$B_1 = C_i^1 \setminus \{u_1\} \cup \{x\}, \quad B_2 = C_i^1 \setminus \{u_2\} \cup \{y\}.$$

By our assumption, $u_1 \neq u_2$. Moreover, $x \neq y$ must be valid, since $x = y$ would imply for $B_1 \cup B_2 = C_i^1 \cup \{x\}$ to be a $(k+2)$-clique, contradicting $b_1 b_2 \notin E(G_e)$. Further, if any two vertices coincide from the remaining pairs of u_1, u_2, x, y, it would contradict the above definition of x and y. Hence, u_1, u_2, x, y are four different vertices and the intersection $M = C_i^1 \cap B_1 \cap B_2 = B_1 \cap B_2$ is a $(k-1)$-clique. Observe that the vertices in $M \cup \{u_1, u_2, x, y\}$ are pairwise adjacent as contained together in at least one of the $(k+2)$-cliques $C_i^1 \cup B_1$ and $C_i^1 \cup B_2$, the only exception is the pair x, y. They are surely nonadjacent, as otherwise $M \cup \{u_1, u_2, x, y\}$ would be a forbidden $(k+3)$-clique in F'.

Next, consider the k-element intersections $C_i^2 \cap B_1$ and $C_i^2 \cap B_2$. Both of them must contain the entire M and one further vertex from $\{x, u_2\}$ and $\{y, u_1\}$, respectively. But all the four possible choices are forbidden. The choice (u_2, u_1) would mean $C_i^1 = C_i^2$; any of the choices (x, u_1) or (u_2, y) would imply that $C_i^1 \cup C_i^2$ is a $(k+2)$-clique, contradicting $c_i^1 c_i^2 \notin E(G_e)$; and finally, the choice (x, y) contradicts the non-adjacency of x and y. By this contradiction we conclude that all $(k+2)$-cliques intersecting G^1 have the same complementing vertex, say z_1, and the similar result for G^2 with the common complementing vertex z_2 also follows. Deleting these vertices from F', we obtain the graph F'' for which $\triangle_k(F'') = G$ holds.

Via Claims C and D, we have proved that G is the k-anti-Gallai graph of some K_{k+2}-free graph if and only if G_e is the $(k+1)$-anti-Gallai graph of some K_{k+3}-free graph. We conclude that if problem (iii) is NP-complete for an integer $k \geqslant 2$, the same hardness follows for $k+1$. Moreover, the reduction takes time polynomial in terms of $|V(G)|$ and k. This proves (iii) from which (i) and (ii) directly follow. □

6 Graphs with isomorphic line graph and k-line graph

Lemma 19. *If K_n is a subgraph of $L_k(G)$ for $n \geqslant k+2$, then the k-cliques in G corresponding to these n vertices in $L_k(G)$ share $k-1$ vertices.*

Proof. By Observation 2, the vertices of K_n either correspond to n k-cliques of G contained in a common K_{k+1}, or correspond to n k-cliques sharing a fixed $(k-1)$-clique. The former case is impossible if $n \geqslant k+2$. Hence, the statement follows. □

Theorem 20. *Let G be a connected graph. Then, for any $k \geqslant 3$, $L_k(G) \cong L(G)$ holds if and only if $G = K_{k+2}$.*

Proof. We begin with the remark that K_{k+2} indeed satisfies $L_k(K_{k+2}) \cong L(K_{k+2})$. Isomorphism can be established by the vertex-complementarity of edges (2-cliques) and k-cliques. Two edges of K_{k+2} share a vertex (and hence are adjacent in $L(K_{k+2})$) if and only if their complementing k-tuples (k-cliques) share exactly $k-1$ vertices (and hence are adjacent in $L_k(K_{k+2})$).

The rest of the proof is devoted to the "only if" part. Let G be a connected graph such that $L_k(G) \cong L(G)$. Let $t = \omega(L_k(G)) = \omega(L(G))$. If $t \geqslant k+2$, then by Lemma 19, the k-cliques in G corresponding to the t vertices in $L_k(G)$ that induce a t-clique, share $k-1$ vertices.

Therefore, $\Delta(G) \geqslant k-2+t$ and hence $\omega(L(G)) \geqslant k-2+t > t$ for $k \geqslant 3$, which contradicts $\omega(L(G)) = t$.

Therefore, $t \leqslant k+1$, which means $\omega(L(G)) \leqslant k+1$. Thus $\Delta(G) \leqslant k+1$, from which $\omega(G) \leqslant k+2$ clearly follows. Moreover, since, $L_k(G)$ is not the null graph, $\omega(G) \geqslant k$ also holds. Hence, we have $k \leqslant \omega(G) \leqslant k+2$. In the rest of the proof we consider the three possible values of ω.

Case 1. $\omega(G) = k$

We saw earlier that a 4-clique in $L_k(G)$ would require in G either a $(k+1)$-clique or a $(k-1)$-clique with four external neighbors. Hence in the current situation with $\Delta(G) \leqslant k+1$ and $\omega(G) = k$ we must have $\omega(L_k(G)) \leqslant 3$, which also means $\omega(L(G)) \leqslant 3$

and therefore G has $\Delta(G) \leqslant 3$. Then $\omega(G) \leqslant 4$, that is $k = 3$ or 4. We show that these cases cannot occur.

For $k = 3$, the condition $\Delta(G) \leqslant 3$ implies that $\Delta(L_3(G)) \leqslant 1 < 2 \leqslant \Delta(L(G))$ holds, hence $L_3(G) \ncong L(G)$. For $k = 4$, $\Delta(G) \leqslant 3$ yields $G \cong K_4$, thus $L_4(K_4) = K_1 \ncong L(K_4)$. Consequently, $L_k(G) \ncong L(G)$ if $\omega(G) = k$.

Case 2. $\omega(G) = k + 1$

Subcase A: G contains only one K_{k+1}

Since $\Delta(G) \leqslant k+1$, G contains no $(k-1)$-clique with $k+1$ common external neighbors. Hence, $L_k(G)$ also has only one K_{k+1}. Therefore, $L(G)$ also contains only one K_{k+1} and hence G has only one vertex of degree $k + 1$. It means that only one vertex of K_{k+1} in G has a neighbor outside the K_{k+1}, so that $L_k(G) = L_k(K_{k+1}) = K_{k+1}$ (for G is connected, $L(G)$ is connected and hence $L_k(G)$ is connected). But, $L(G) \neq K_{k+1}$. Therefore, G must contain more than one K_{k+1}.

Subcase B: G contains two K_{k+1} *subgraphs which share* $l > 0$ *vertices*

In this case, $\Delta(G) \geqslant 2k - l + 1$. By $\Delta(G) \leqslant k + 1$ we see that $l = k$, therefore $K_{k+2} - e \subseteq G$. Since $L_k(G)$ is connected and k vertices in $K_{k+2} - e$ already attained the maximum possible degree $k + 1$, $L_k(G) = L_k(K_{k+2} - e)$ must hold. But then the number of vertices in $L_k(G)$ is strictly less than the number of vertices in $L(G)$, which is a contradiction.

Subcase C: G contains a K_{k+1} *which does not share vertices with any other* K_{k+1}

Either this reduces to subcase A or, by connectivity, $k-1$ vertices of this K_{k+1} belong to a k-clique outside this K_{k+1}. In that case, these $k-1$ vertices cannot have any further neighbors (since they attained the maximum degree $k + 1$) and there are only two more vertices in the K_{k+1} under consideration. If $k > 3$ then, since $L_k(G)$ is connected, there are no further k-cliques in G and hence $L_k(G)$ is K_{k+1} together with a vertex having exactly two neighbors in K_{k+1}, which is obviously not isomorphic to $L(G)$. If $k = 3$, using the same arguments, we can see that $L_k(G)$ is a K_4 with one or two further vertices having exactly two neighbors in K_4. In either case, $L_k(G) \neq L(G)$.

Case 3. $\omega(G) = k + 2$

In this case $\Delta(G) \leqslant k + 1$ together with connectivity implies $G = K_{k+2}$, and this is exactly what had to be proven. ☐

7 Concluding remarks

Concerning the algorithmic complexity of recognizing k-line graphs, the problem is trivial for $k = 1$, solvable in linear-time for $k = 2$, and as we have proved, it becomes NP-complete for each $k \geqslant 3$. It is worth noting that there is a further jump between the behavior of 2- and 3-line graphs, which likely is in connection with the jump occurring in time complexity. Namely, this further difference is in the uniqueness of preimages.

As a matter of fact, each connected line graph different from K_3 has a unique preimage if we disregard isolated vertices. In other words, viewing the situation from the side of preimages, the line graphs of two non-isomorphic graphs containing no isolated vertices and no K_3-components surely are non-isomorphic. The similar statement is not true for triangle graphs, even if we suppose that every edge of the preimage is contained in a triangle. For example, there are seven essentially different graphs whose triangle graph is the 8-cycle (see Figure 3). Additionally, the number of non-isomorphic pre-images of an n-cycle goes to infinity as $n \to \infty$ [3].

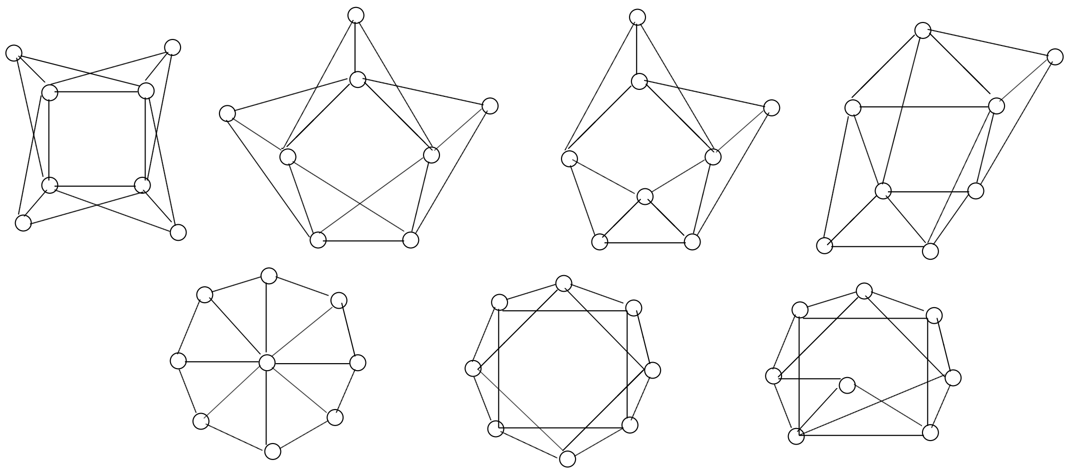

Figure 3: The seven triangle restricted graphs whose triangle graph is C_8.

Finally, we make some remarks on an open problem closely related to the results of this paper.

For an integer $k \geqslant 1$, the vertices of the *k-Gallai graph* $\Gamma_k(G)$ represent the k-cliques of G, moreover its two vertices are adjacent if and only if the corresponding k-cliques of G share a $(k-1)$-clique but they are not contained in a common $(k+1)$-clique. By this definition, the 1-Gallai graph $\Gamma_1(G)$ is exactly the complement $\overline{G}$ of G, whilst 2-Gallai graph means Gallai graph in the usual sense, as introduced by Gallai in [11]. Obviously, $\Gamma_k(G) \subseteq L_k(G)$ holds. Moreover $\triangle_k(G)$ and $\Gamma_k(G)$ together determine an edge partition of the k-line graph. Our theorems together with the earlier results from [6] and [2] determine the time complexity of recognition problems of the k-line graphs and the k-anti-Gallai graphs for each fixed $k \geqslant 1$. Since $G = \Gamma_1(\overline{G})$, every graph is a 1-Gallai graph. But for each $k \geqslant 2$ this recognition problem remains open.

Problem 21. *Determine the time complexity of the recognition problem of k-Gallai graphs for $k \geqslant 2$.*

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