AFFINE IMAGES OF ISOTROPIC MEASURES

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Abstract

Necessary and sufficient conditions are given in order for a Borel measure on the Euclidean sphere to have an affine image that is isotropic. A sharp reverse affine isoperimetric inequality for Borel measures on the sphere is presented. This leads to sharp reverse affine isoperimetric inequalities for convex bodies.

1. Introduction

Basic to Euclidean geometry is the Pythagorean theorem: for each $x \in \mathbb{R}^n$, the square of the Euclidean norm of x may be computed by

$$|x|^2 = |x \cdot e_1|^2 + |x \cdot e_2|^2 + \cdots + |x \cdot e_n|^2,$$

where $e_1, e_2, \ldots, e_n$ are orthogonal unit vectors in $\mathbb{R}^n$, and $x \cdot e_i$ is the standard inner product of x and e_i in $\mathbb{R}^n$. This can be rewritten as

$$|x|^2 = \int_{S^{n-1}} |x \cdot v|^2 \, d\gamma_n(v), \tag{1.1}$$

where $\gamma_n = \frac{1}{2}\sum_{i=1}^n (\delta_{e_i} + \delta_{-e_i})$, and δ_v denotes the delta measure defined on the unit sphere, S^{n-1}, of $\mathbb{R}^n$, by having it concentrated exclusively on the vector $v \in S^{n-1}$. The measure γ_n is a cross-measure. A *cross-measure* is a discrete measure, defined on S^{n-1}, and concentrated equally on the $2n$ points where S^{n-1} intersects the coordinate axes. The geometric significance of the cross-measure lies in the fact that it is the "surface area measure" of the cube in $\mathbb{R}^n$.

Now (1.1) leads to the important concept of *isotropy* of measures, which may be viewed as an extension of the Pythagorean theorem. A

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finite Borel measure μ on the unit sphere S^{n-1} is said to be *isotropic* if for each $x \in \mathbb{R}^n$,

$$|x|^2 = \frac{n}{|\mu|} \int_{S^{n-1}} |x \cdot v|^2 \, d\mu(v), \tag{1.2}$$

where $|\mu| = \mu(S^{n-1})$. This tells us that the inertia of μ in all directions is the same, or equivalently, the ellipsoid of inertia of μ is a sphere. Two basic examples of isotropic measures on S^{n-1} are spherical Lebesgue measure and cross-measures.

Isotropy is an important property of measures that tells us that a measure is, in an important sense, evenly distributed. In 1991, Ball [2] discovered an amazing connection between the isotropy of measures and the Brascamp-Lieb inequality, and used it to establish his celebrated reverse isoperimetric inequality. Ball's work inspired much use of the notion of isotropy of measures in the study of reverse affine isoperimetric inequalities (see, e.g., [2–4, 51, 55, 58], and also the survey [18]). On $\mathbb{R}^n$, isotropic log-concave measures have been intensively investigated in the context of the Kannan-Lovász-Simonovits conjecture [35] and its relatives (see, e.g., [5, 21, 36]).

If μ is a positive Borel measure on S^{n-1} and $A \in \mathrm{GL}(n)$, then the image $A\mu$, of μ under A, is a measure defined, for Borel $\omega \subseteq S^{n-1}$, by

$$A\mu(\omega) = \mu(\langle A^{-1}\omega \rangle), \tag{1.3}$$

where $\langle x \rangle = x/|x|$, for $x \in \mathbb{R}^n \setminus \{0\}$. Any $A\mu$ where $A \in \mathrm{GL}(n)$ will be said to be an *affine image of μ*. That the action defined above is the natural action of $\mathrm{GL}(n)$ on the set of Borel measures on S^{n-1} can be seen by first considering even (i.e., taking the same values on antipodal Borel subsets of S^{n-1}) measures on S^{n-1}. Because these measures are even, we can consider them as measures on S^{n-1} with antipodal points identified, or equivalently, as measures on real projective space, $\mathbb{P}^{n-1}(\mathbb{R})$, where $\mathrm{GL}(n)$ acts naturally sending lines through the origin into their images. One easily sees that the result is definition (1.3).

Problem 1.1. *For a finite Borel measure μ on S^{n-1}, what are necessary and sufficient conditions for the existence of an $A \in \mathrm{SL}(n)$ so that the image $A\mu$ is isotropic?*

If there exists an $A \in \mathrm{SL}(n)$ so that $A\mu$ is isotropic, the measure μ is said to have an affine isotropic image. Problem 1.1 for the cone-volume measure (defined below) of a convex body was posed in [57].

One of the aims of this paper is to extend work of Carlen and Cordero-Erausquin [11] for discrete measures, and Klartag [37] for arbitrary measures, and provide an answer to Problem 1.1.

Perhaps not surprisingly, the problem above is linked to concepts of "concentration of measure." In [8], the authors defined the *subspace concentration condition* of measures (defined below), which limits how concentrated a measure can be in a subspace. (This condition is connected with fully nonlinear partial differential equations.) The authors proved that the subspace concentration condition is both necessary and sufficient for the existence of a solution to the even logarithmic Minkowski problem — which amounts to establishing existence for a Monge-Ampère-type equation in convex geometric analysis. In this paper, we prove that the subspace concentration condition is also necessary and sufficient to answer Problem 1.1.

A finite Borel measure μ on S^{n-1} is said to satisfy the *subspace concentration inequality* if, for every subspace ξ of $\mathbb{R}^n$,

$$\mu(\xi \cap S^{n-1}) \le \frac{1}{n}\,\mu(S^{n-1}) \dim \xi. \tag{1.4}$$

The measure is said to satisfy the *subspace concentration condition* if, in addition to satisfying the subspace concentration inequality (1.4), whenever

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n}\,\mu(S^{n-1}) \dim \xi,$$

for some subspace ξ, then there exists a subspace ξ', that is complementary to ξ in $\mathbb{R}^n$, so that μ is concentrated on $S^{n-1} \cap (\xi \cup \xi')$, or equivalently, so that we also have

$$\mu(\xi' \cap S^{n-1}) = \frac{1}{n}\,\mu(S^{n-1}) \dim \xi'.$$

The measure μ on S^{n-1} is said to satisfy the *strict subspace concentration inequality* if inequality (1.4) is strict for every subspace $\xi \subset \mathbb{R}^n$, such that $0 < \dim \xi < n$.

To answer Problem 1.1, we shall prove that a measure of having an isotropic affine image is the same as the measure of satisfying the subspace concentration condition. This demonstrates the close connection between isotropy and concentration of measure.

One of our goals is to establish the following:

Theorem 1.2. *A finite Borel measure μ on S^{n-1} has an affine isotropic image if and only if μ satisfies the subspace concentration condition.*

For the case of discrete measures, Theorem 1.2 is due to Carlen and Cordero-Erausquin [11]. Klartag [37] established that if a general measure satisfies the strict subspace concentration inequality, then it has an affine isotropic image.

We will use Theorem 1.2 to establish a sharp affine inequality for measures that satisfy the subspace concentration condition. The equality conditions of this inequality characterize cross-measures.

A finite Borel measure μ on S^{n-1} will be said to have *positive subspace mass* if there exists a subspace ξ of co-dimension 1 such that

$\mu(S^{n-1} \cap \xi) > 0$. Spherical Lebesgue measure on S^{n-1}, or any measure that is absolutely continuous with respect to spherical Lebesgue measure, does not have positive subspace mass. On the other hand, every discrete measure on S^{n-1} has positive subspace mass. Intuitively, a cross-measure would have its measure maximally concentrated within subspaces. It is remarkable that there is an affine invariant functional of measures that can be used to demonstrate this. This affine invariant functional may be viewed as a geometric mean of the mass of the measure, while the total mass functional, $|\cdot|$, is the usual (arithmetic) mean.

Let μ be a finite Borel measure on the unit sphere S^{n-1}. Define $U(\mu)$ by

$$U(\mu)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} d\mu(u_1) \cdots d\mu(u_n), \tag{1.5}$$

where the integral is over the subset of the n copies $S^{n-1} \times \cdots \times S^{n-1}$ where the exterior product is non-zero.

Like the total mass $|\mu|$ of μ, the functional U is invariant under $\mathrm{SL}(n)$ transformations; i.e.,

$$|A\mu| = |\mu| \qquad \text{and} \qquad U(A\mu) = U(\mu),$$

for each $A \in \mathrm{SL}(n)$. Thus, we also call U an affine functional.

Note that $U(\mu) \leq |\mu|$. Obviously, $U(\mu) = |\mu|$ if μ is Lebesgue measure on S^{n-1} or any measure that is absolutely continuous with respect to spherical Lebesgue measure, while $U(\mu) < |\mu|$ whenever μ is discrete. The functionals U and $|\cdot|$ will be shown to coincide precisely on those measures that do not have positive subspace mass. We characterize cross-measures, among all measures that satisfy the subspace concentration condition, by establishing an affine inequality between the functional U and the total mass $|\cdot|$.

Theorem 1.3. *Let μ be a finite Borel measure on the unit sphere S^{n-1}.*

1) *Then,*
$$|\mu| \geq U(\mu),$$
with equality if and only if the measure μ does not have positive subspace mass.
2) *If μ satisfies the subspace concentration condition, then*
$$U(\mu) \geq \frac{(n!)^{1/n}}{n} |\mu|,$$
with equality if and only if the central symmetral (2.6) *of μ is an affine image of a cross-measure.*

In the last section of this paper, Theorem 1.2 is applied to solve an open problem posed in [57] regarding the isotropy of the cone-volume measure of a convex body. Theorem 1.3 is applied to partially answer an open problem posed in [52] regarding a reverse affine isoperimetric inequality for polytopes.

2. Preliminaries

The setting for this paper is n-dimensional Euclidean space, $\mathbb{R}^n$. Here $|\cdot|$ will denote the usual norm on $\mathbb{R}^n$, and we shall write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$. As usual, a *proper* subspace of $\mathbb{R}^n$ is one whose dimension is neither 0 nor n.

For $A \in \mathrm{GL}(n)$, the group of invertible linear transformations in $\mathbb{R}^n$, let $\mathrm{tr}A$ and $\det A$ denote the trace and the determinant of A, and write $|A|$ for the absolute value of the determinant of A. Write A^t for the transpose of A, and A^{-t} for the transpose of the inverse of A. The identity in the group $\mathrm{GL}(n)$ will be denoted by I_n.

As usual, the special linear transformation group, $\mathrm{SL}(n)$, is the subgroup of $\mathrm{GL}(n)$ whose elements have determinant 1. We shall refer to the elements of $\mathrm{GL}(n)$ as *affinities* (as opposed to the more cumbersome "non-singular centro-affine transformations".) For a subspace ξ of $\mathbb{R}^n$, we will write $\mathrm{GL}(\xi)$ and $\mathrm{SL}(\xi)$ for the corresponding groups of affinities in ξ. We shall write I_ξ for the identity in $\mathrm{SL}(\xi)$.

If D is a subset of an m-dimensional subspace of $\mathbb{R}^n$, the m-dimensional Lebesgue measure of D will be denoted by $V_m(D)$. If $m = n$ then the subscript will be suppressed. Denote by $\mathrm{P}_\xi : \mathbb{R}^n \to \xi$ the orthogonal projection operator from $\mathbb{R}^n$ to the subspace ξ of $\mathbb{R}^n$.

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact, convex $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : \ y \in K\},$$

and uniquely determines the convex set K. Note that support functions are positively homogeneous of degree one and subadditive. From the definition, it follows immediately that, for $A \in \mathrm{GL}(n)$, the support function of $AK = \{Ax : \ x \in K\}$ is given by

$$h_{AK}(x) = h_K(A^t x),$$

for $x \in \mathbb{R}^n$. A compact, convex subset of $\mathbb{R}^n$ with non-empty interior is called a *convex body*. In general, support functions are continuous, and if the origin is an interior point of the convex body K, then the support function of K is strictly positive when viewed as defined on S^{n-1}. Observe that

$$h_{\mathrm{P}_\xi K} = h_K, \qquad \text{on } S^{n-1} \cap \xi. \tag{2.1}$$

We will make use of the fact that for $A \in \mathrm{GL}(n)$

$$V(AK) = |A|\, V(K). \tag{2.2}$$

More specifically, we will make frequent use of the fact that if $P : \mathbb{R}^n \to \xi \subset \mathbb{R}^n$ is linear, and η is a subspace of $\mathbb{R}^n$ such that $\dim(\eta) = \dim(P\eta) = m$, then for a convex body $D \subset \eta$,

$$V_m(PD) = |P'|V_m(D). \tag{2.3}$$

where P' is the restriction of P to η.

For $A \in \mathrm{GL}(n)$, define the norm $|\cdot|_A$ in $\mathbb{R}^n$ by

$$|x|_A = |A^{-t}x|,$$

for $x \in \mathbb{R}^n$. The unit ball of this norm,

$$\varepsilon(A) = \{x \in \mathbb{R}^n : |x|_A \leq 1\},$$

is an ellipsoid centered at the origin whose volume is equal to $|A|\omega_n$, where ω_n is the volume of the standard unit ball, B^n, in $\mathbb{R}^n$. We shall write $|\cdot|_A^*$ for the dual norm of $|\cdot|_A$; i.e.,

$$|y|_A^* = \max\{x \cdot y : x \in \mathbb{R}^n \text{ and } |x|_A \leq 1\},$$

for $y \in \mathbb{R}^n$.

If $\varepsilon(A)$ is the ellipsoid associated with $A \in \mathrm{GL}(n)$, then

$$h_{\varepsilon(A)}(x) = |x|_A^* = |Ax|. \tag{2.4}$$

As usual, for $D, D' \subset \mathbb{R}^n$ and real $c, c' \geq 0$, the Minkowski combination $cD + c'D'$ is defined by

$$cD + c'D' = \{cx + c'x' : x \in D \text{ and } x' \in D'\}.$$

Obviously,

$$A(cD + c'D') = cAD + c'AD', \tag{2.5}$$

for $A \in \mathrm{GL}(n)$.

Throughout, a "finite Borel measure" is assumed to be both positive and not identically zero. If μ is a finite Borel measure on the unit sphere S^{n-1}, then $\mathrm{supp}(\mu)$ will denote its support and we will use $\bar{\mu}$ to denote its *central symmetral*; i.e., for each Borel $\omega \subset S^{n-1}$,

$$\bar{\mu}(\omega) = \frac{1}{2}(\mu(\omega) + \mu(-\omega)), \tag{2.6}$$

where $-\omega$ is the antipodal image of ω. From definition (1.3), it follows immediately that the central symmetral of $A\mu$ is equal to $A\bar{\mu}$.

A finite Borel measure μ on the unit sphere S^{n-1} is said to be *isotropic* if for each $u \in S^{n-1}$,

$$\int_{S^{n-1}} |u \cdot v|^2 \, d\mu(v) = \frac{|\mu|}{n}.$$

Or equivalently, if for all i, j,

$$\int_{S^{n-1}} (e_i \cdot v)(e_j \cdot v) \, d\mu(v) = \frac{|\mu|}{n}\delta_{ij}. \tag{2.7}$$

For $x_1, \dots, x_n \in \mathbb{R}^n$, let $[x_1, \dots, x_n]$ denote the $n \times n$ matrix whose columns are the vectors $x_1, \dots, x_n$. We need the following fact (see, e.g., [55]) regarding isotropic measures:

$$\int_{S^{n-1}\times\cdots\times S^{n-1}} |[u_1, \dots, u_n]|^2 \, d\mu(u_1) \cdots d\mu(u_n) = \frac{n!}{n^n} |\mu|^n. \tag{2.8}$$

For quick later reference, we recall that, by (1.3), for a Borel measure μ and $A \in \mathrm{GL}(n)$, the image $A\mu$ is defined for each continuous $f : S^{n-1} \to \mathbb{R}$, by

$$\int_{S^{n-1}} f(v) \, dA\mu(v) = \int_{S^{n-1}} f(\langle Av \rangle) \, d\mu(v), \tag{2.9}$$

where $\langle Av \rangle = Av/|Av|$. Obviously, $(cA)\mu = A\mu$ for $c > 0$. Observe that if μ is isotropic and $O \in \mathrm{O}(n)$, then $O\mu$, the image of μ under O, is isotropic as well. Thus, if Problem 1.1 has an affirmative answer, then a solution may always be chosen to be an element of $\mathrm{SL}(n)$ that is positive definite.

3. Log-John affinities

Measures that have affine images that are isotropic are closely related to solutions of a maximization problem that we call the *log-John problem*. In fact, the existence of an isotropic affine image of a particular measure is equivalent to the measure having a log-John affinity (defined below) associated with it.

For a finite Borel measure μ on S^{n-1}, define $\mathrm{e}_\mu : \mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathbb{R}$ for $P, Q \in \mathrm{GL}(n)$ by

$$\mathrm{e}_\mu(P, Q) = \int_{S^{n-1}} \log \frac{|Pv|}{|Qv|} \, d\mu(v). \tag{3.1}$$

It is easily seen that $\mathrm{e}_\mu : \mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathbb{R}$ is continuous in each argument. We call $\mathrm{e}_\mu(P, Q)$ the *log-eccentricity* of P relative to Q with respect to μ. From the definition we see immediately $\mathrm{e}_\mu(P, Q) = \mathrm{e}_\mu(OP, Q)$ for all $O \in \mathrm{O}(n)$.

We shall make use of the trivial observation that for real $\lambda \neq 0$,

$$\mathrm{e}_\mu(P, \lambda Q) = \mathrm{e}_\mu(P, Q) - |\mu| \log |\lambda|. \tag{3.2}$$

A useful fact, that follows immediately from definition (3.1), is that

$$\mathrm{e}_\mu(P, Q) = \int_{S^{n-1}} \log |Pv| \, d\mu(v) + \mathrm{e}_\mu(I, Q), \tag{3.3}$$

where $I = I_n$ is the identity.

As can be seen in the following lemma, log-eccentricity is a $\mathrm{GL}(n)$-contravariant:

Lemma 3.1. *If μ is a finite Borel measure on S^{n-1}, and $P, Q \in \mathrm{GL}(n)$, then*
$$\mathrm{e}_\mu(PA, QA) = \mathrm{e}_{A\mu}(P, Q),$$
for $A \in \mathrm{GL}(n)$.

Proof. For $A \in \mathrm{GL}(n)$, from (3.1), (2.9), and (3.1) again, we have
$$\begin{aligned}
\mathrm{e}_{A\mu}(P, Q) &= \int_{S^{n-1}} \log \frac{|Pv|}{|Qv|}\, dA\mu(v) \\
&= \int_{S^{n-1}} \log \frac{|P\langle Av\rangle|}{|Q\langle Av\rangle|}\, d\mu(v) \\
&= \int_{S^{n-1}} \log \frac{|PAv|}{|QAv|}\, d\mu(v) \\
&= \mathrm{e}_\mu(PA, QA).
\end{aligned}$$
q.e.d.

For a finite Borel measure μ on S^{n-1}, and a $P \in \mathrm{GL}(n)$, define

$$m(\mu, P) = \sup\big\{\det(Q) : Q \in \mathrm{GL}(n) \text{ and } \mathrm{e}_\mu(P, Q) \geq 0\big\}. \tag{3.4}$$

Observe that $m(\mu, P) > 0$, for all $P \in \mathrm{GL}(n)$. To see this note that if $\det(P) > 0$, then since $\mathrm{e}_\mu(P, P) = 0$, it follows that $m(\mu, P) \geq \det(P) > 0$. If $\det(P) < 0$, then choose an $O \in \mathrm{O}(n)$ so that $\det(OP) > 0$, and since $\mathrm{e}_\mu(P, OP) = \mathrm{e}_\mu(P, P) = 0$, it follows that $m(\mu, P) \geq \det(OP) > 0$.

Does there exist a $Q_0 \in \mathrm{GL}(n)$ such that

$$\det(Q_0) = m(\mu, P) \quad \text{and} \quad \mathrm{e}_\mu(P, Q_0) \geq 0; \tag{3.5}$$

i.e., is the supremum in (3.4) just a maximum? If such a Q_0 exists, then Q_0 will be called a *log-John affinity* of μ relative to P. In this section it will be shown that the existence of a log-John affinity is an intrinsic property of the measure itself and is independent of the choice of P.

From the definition of e_μ, it is obvious that if Q_0 is a log-John affinity of μ relative to P, then λQ_0 is a log-John affinity of μ relative to $\pm\lambda P$, for all $\lambda > 0$. That a log-John affinity, relative to an element of $\mathrm{GL}(n)$, is an affine concept is stated in the following lemma.

Lemma 3.2. *Suppose μ is a finite Borel measure on S^{n-1}, and $Q_{\mu,P}$ is a log-John affinity of μ relative to $P \in \mathrm{GL}(n)$. If $A \in \mathrm{GL}(n)$ with $\det(A) > 0$, then $Q_{\mu,P}A^{-1}$ is a log-John affinity of $A\mu$ relative to PA^{-1}.*

Lemma 3.2 is an immediate consequence of Lemma 3.1 (where P and Q are replaced by PA^{-1} and $Q_{\mu,P}A^{-1}$, respectively).

We now present a maximization problem that will be shown to be equivalent to problem (3.5). For a finite Borel measure μ on S^{n-1} and a fixed $P \in \mathrm{GL}(n)$, define

$$m'(\mu, P) = \sup\big\{\mathrm{e}_\mu(P, Q) : Q \in \mathrm{SL}(n)\big\}. \tag{3.6}$$

Does there exist a $Q_0 \in \mathrm{SL}(n)$ such that

$$\mathrm{e}_\mu(P, Q_0) = m'(\mu, P)? \tag{3.7}$$

We first observe that the normalization $\det Q = 1$ in (3.6) is arbitrary. The existence of a solution to problem (3.7) is independent of the normalization chosen.

The following lemma will show that questions (3.5) and (3.7) are equivalent in that a solution to one is just a scalar multiple of a solution of the other. For this reason we may call Q_0 in (3.7) a *normalized log-John affinity* of μ relative to P.

Lemma 3.3. *Suppose μ is a finite Borel measure on S^{n-1} and $P \in \mathrm{GL}(n)$. If Q_1 is a solution for the maximization problem* (3.5)*, then $Q_2 = |Q_1|^{-\frac{1}{n}} Q_1$ is a solution for the maximization problem* (3.7)*. Conversely, if Q_2 is a solution for the maximization problem* (3.7)*, then $Q_1 = \lambda Q_2$, where $\log \lambda = |\mu|^{-1} \mathrm{e}_\mu(P, Q_2)$, is a solution for the maximization problem* (3.5)*.*

Proof. Suppose Q_1 is a solution of (3.5); i.e. $\det(Q_1) = m(\mu, P) > 0$.

We first observe that $\mathrm{e}_\mu(P, Q_1) = 0$. To see this, suppose $\mathrm{e}_\mu(P, Q_1) = \varepsilon > 0$. Let λ be such that $|\mu| \log \lambda = \varepsilon/2$. From (3.2) we have $\mathrm{e}_\mu(P, \lambda Q_1) = \varepsilon/2 > 0$, and now $\lambda > 1$ produces the affinity λQ_1 that in (3.4) and (3.5) contradicts the maximality of $\det(Q_1)$. Thus, $\mathrm{e}_\mu(P, Q_1) = 0$.

Now $Q_2 = |Q_1|^{-\frac{1}{n}} Q_1$, so that $\det(Q_2) = 1$. Suppose $Q \in \mathrm{SL}(n)$ is arbitrary. Let $R = \lambda Q$, where $\log \lambda = |\mu|^{-1} \mathrm{e}_\mu(P, Q)$.

From (3.2) we have $\mathrm{e}_\mu(P, R) = 0$. From the assumed maximality of $\det(Q_1)$ in (3.4) and (3.5) we have $|R| \le |Q_1|$, or equivalently that $\lambda^n \le |Q_1|$, since $|Q| = 1$.

Now $Q_2 = |Q_1|^{-\frac{1}{n}} Q_1$ and (3.2), together with $\mathrm{e}_\mu(P, Q_1) = 0$, give

$$\mathrm{e}_\mu(P, Q_2) = \frac{|\mu|}{n} \log |Q_1| + \mathrm{e}_\mu(P, Q_1) = \frac{|\mu|}{n} \log |Q_1|,$$

which together with $\lambda^n \le |Q_1|$ and $\log \lambda = |\mu|^{-1} \mathrm{e}_\mu(P, Q)$ gives

$$\mathrm{e}_\mu(P, Q) \le \mathrm{e}_\mu(P, Q_2).$$

Since $Q \in \mathrm{SL}(n)$ is arbitrary, it follows that $Q_2 \in \mathrm{SL}(n)$ is a solution of the maximization problem (3.7).

We now deal with the case where Q_2 is assumed to be a solution of (3.7). Observe that this implies $\det(Q_2) = 1$. Suppose $Q \in \mathrm{GL}(n)$ is such that $\mathrm{e}_\mu(P, Q) \ge 0$ and $\det(Q) > 0$, but is otherwise arbitrary. Since Q_2 is a solution of (3.7), we have,

$$\mathrm{e}_\mu(P, |Q|^{-\frac{1}{n}} Q) \le \mathrm{e}_\mu(P, Q_2),$$

which using (3.2) gives

$$\frac{|\mu|}{n}\log|Q| + \mathrm{e}_\mu(P,Q) \le \mathrm{e}_\mu(P,Q_2).$$

This and the fact that $\mathrm{e}_\mu(P,Q) \ge 0$ shows that

$$\frac{|\mu|}{n}\log|Q| \le \mathrm{e}_\mu(P,Q_2).$$

This, together with $Q_1 = \lambda Q_2$, where $\log\lambda = |\mu|^{-1}\mathrm{e}_\mu(P,Q_2)$, and $\det(Q_2) = 1$, gives

$$\det(Q) \le \lambda^n = \det(Q_1).$$

However, from (3.2) we see that

$$\mathrm{e}_\mu(P,Q_1) = \mathrm{e}_\mu(P,\lambda Q_2) = \mathrm{e}_\mu(P,Q_2) - |\mu|\log\lambda = 0.$$

Therefore, Q_1 is a solution of (3.5). q.e.d.

From (3.3) and Lemma 3.3, we get:

Corollary 3.4. *If μ is a finite Borel measure on S^{n-1} and $P \in \mathrm{GL}(n)$, then μ has a log-John affinity relative to P if and only if μ has a log-John affinity relative to the identity I.*

This corollary tells us that the existence of a log-John affinity of a measure does not depend on the particular $P \in \mathrm{GL}(n)$ chosen: If a measure has a log-John affinity relative to one element of $\mathrm{GL}(n)$ then it has a log-John affinity relative to every element of $\mathrm{GL}(n)$. Thus, from this point forward we may simply say that a measure either has a log-John affinity or it does not.

Theorem 3.5. *If μ is a finite Borel measure on S^{n-1} and $P \in \mathrm{GL}(n)$, then the identity is a normalized log-John affinity of μ relative to P if and only if μ is isotropic.*

Proof. First, suppose that the identity I is a normalized log-John affinity of μ relative to P; i.e.,

$$\mathrm{e}_\mu(P,I) = \sup\{\mathrm{e}_\mu(P,Q) : Q \in \mathrm{SL}(n)\}. \tag{3.8}$$

Thus, for each $A \in \mathrm{SL}(n)$,

$$\mathrm{e}_\mu(P,A) \le \mathrm{e}_\mu(P,I).$$

Suppose $L \in \mathrm{GL}(n)$ is arbitrary, but fixed. Since $|L| \ne 0$, there exists an $\varepsilon_o > 0$, such that for all $\varepsilon \in (-\varepsilon_o, \varepsilon_o)$, we can define $A_\varepsilon \in \mathrm{SL}(n)$ by

$$|I + \varepsilon L|^{\frac{1}{n}} A_\varepsilon = I + \varepsilon L.$$

Observe that,

$$|A_\varepsilon v|^2 = \frac{|v + \varepsilon L v|^2}{|I + \varepsilon L|^{\frac{2}{n}}} = \frac{1 + 2\varepsilon(v \cdot Lv) + \varepsilon^2|Lv|^2}{1 + \frac{2\varepsilon}{n}\mathrm{tr}L + \mathrm{O}(\varepsilon^2)},$$

for $v \in S^{n-1}$.

Since $\mathrm{e}_\mu(P, A_\varepsilon)$ attains a maximum at $A_0 = I$, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{S^{n-1}} \log \frac{|Pv|}{|A_\varepsilon v|}\, d\mu(v) = 0.$$

Thus,

$$\int_{S^{n-1}} (v \cdot Lv)\, d\mu(v) - \frac{|\mu|}{n}\mathrm{tr}(L) = 0. \tag{3.9}$$

If in (3.9) we let $L = L_{ij}$, where $L_{ij}e_k = \delta_{jk}e_i$, we get (2.7) and conclude that μ is isotropic.

Now suppose that μ is isotropic. We need to demonstrate that

$$\mathrm{e}_\mu(P, A) \le \mathrm{e}_\mu(P, I),$$

for each $A \in \mathrm{SL}(n)$, or equivalently that

$$\int_{S^{n-1}} \log |Av|\, d\mu(v) \ge 0, \tag{3.10}$$

for each $A \in \mathrm{SL}(n)$.

For each $A \in \mathrm{SL}(n)$, there exists a positive definite $N \in \mathrm{SL}(n)$ and an orthogonal transformation O so that $A = ON$. Thus, one can reduce having to demonstrate (3.10) for all $A \in \mathrm{SL}(n)$ to having to demonstrate

$$\int_{S^{n-1}} \log |Nu|\, d\mu(u) \ge 0 \tag{3.11}$$

for all positive definite $N \in \mathrm{SL}(n)$.

For each positive definite $N \in \mathrm{SL}(n)$, there exists an orthogonal matrix O and a diagonal matrix $\Lambda \in \mathrm{SL}(n)$, with diagonal entries $\lambda_1 \dots \lambda_n$, so that $N = O^t\Lambda O$. However, using (2.9), we have

$$\int_{S^{n-1}} \log |Nu|\, d\mu(u) = \int_{S^{n-1}} \log |\Lambda O u|\, d\mu(u) = \int_{S^{n-1}} \log |\Lambda u|\, dO\mu(u).$$

From the concavity of the log function, we have for $u = (u_1, \dots, u_n) \in S^{n-1}$,

$$2\log|\Lambda u| = \log(\lambda_1^2 u_1^2 + \cdots + \lambda_n^2 u_n^2) \ge u_1^2 \log \lambda_1^2 + \cdots + u_n^2 \log \lambda_n^2. \tag{3.12}$$

Observe that since μ is isotropic and O is orthogonal, $O\mu$ is isotropic, and thus, for all i,

$$\int_{S^{n-1}} u_i^2\, dO\mu(u) = \frac{|\mu|}{n}.$$

Thus, from (3.12) and the fact that $\det \Lambda = 1$, we have

$$\begin{aligned}\int_{S^{n-1}} 2\log|\Lambda u|\, dO\mu(u) &\ge (\log \lambda_1^2 + \cdots + \log \lambda_n^2)\frac{|\mu|}{n} \\ &= \frac{2|\mu|}{n}\log(\lambda_1 \cdots \lambda_n) \\ &= 0.\end{aligned}$$

This establishes (3.11). q.e.d.

The following theorem shows that only finite Borel measures that have a log-John affinity will have affine isotropic images, and conversely any measure that has an affine isotropic image must have a log-John affinity associated with it.

Theorem 3.6. *If μ is a finite Borel measure on S^{n-1}, then μ has a log-John affinity if and only if μ has an affine isotropic image.*

Proof. By Corollary 3.4, μ has a log-John affinity if and only if μ has a log-John affinity, $Q = Q_{\mu,I} \in \mathrm{GL}(n)$, with $\det(Q) > 0$, relative to I. This is equivalent to μ having a log-John affinity $Q/|Q|^{1/n}$ relative to $I/|Q|^{1/n}$. But by Lemmas 3.2 and 3.3, this is equivalent to I being a normalized log-John affinity of $A\mu$ relative to Q^{-1}, where $A = Q/|Q|^{1/n} \in \mathrm{SL}(n)$. However, from Theorem 3.5 we know that I is a normalized log-John affinity of $A\mu$ if and only if $A\mu$ is isotropic.
q.e.d.

We will require the following characterization of measures that have affine isotropic images.

Lemma 3.7. *If μ is a finite Borel measure on S^{n-1}, then μ has an affine image that is isotropic if and only if there exists an $A \in \mathrm{SL}(n)$ so that*

$$|A^{-t}x|^2 = \frac{n}{|\mu|} \int_{S^{n-1}} |x \cdot v|^2 |Av|^{-2}\, d\mu(v), \tag{3.13}$$

for all $x \in \mathbb{R}^n$.

Proof. The isotropic condition (1.2) for $A\mu$ is

$$|x|^2 = \frac{n}{|A\mu|} \int_{S^{n-1}} |x \cdot v|^2\, dA\mu(v),$$

for all $x \in \mathbb{R}^n$. Using definition (2.9) and $|A\mu| = |\mu|$, this can be written as

$$\begin{aligned} |x|^2 &= \frac{n}{|\mu|} \int_{S^{n-1}} |x \cdot Av|^2 |Av|^{-2}\, d\mu(v) \\ &= \frac{n}{|\mu|} \int_{S^{n-1}} |A^t x \cdot v|^2 |Av|^{-2}\, d\mu(v), \qquad (3.14) \end{aligned}$$

for all $x \in \mathbb{R}^n$. This gives the condition (3.13). q.e.d.

Theorem 3.6 and Lemma 3.7 now give:

Theorem 3.8. *If μ is a finite Borel measure on S^{n-1}, then μ has a log-John affinity if and only if there exists an $A \in \mathrm{SL}(n)$ so that*

$$|A^{-t}x|^2 = \frac{n}{|\mu|} \int_{S^{n-1}} |x \cdot v|^2 |Av|^{-2}\, d\mu(v), \tag{3.15}$$

for all $x \in \mathbb{R}^n$.

4. Existence of log-John affinities

Let μ be a finite Borel measure and $h : S^{n-1} \to (0,\infty)$ be continuous. Define

$$\mathrm{e}_\mu(h) = -\int_{S^{n-1}} \log h(v)\, d\mu(v). \tag{4.1}$$

The following lemma was established in [8].

Lemma 4.1. *Suppose μ is a probability measure on S^{n-1} that satisfies the strict subspace concentration inequality. For each positive integer k, let $u_{1,k},\dots,u_{n,k}$ be an orthonormal basis of $\mathbb{R}^n$, and suppose that the n sequences of positive real numbers $\{h_{i,k}\}$ are such that $h_{1,k} \le \cdots \le h_{n,k}$, and such that the product $h_{1,k}\cdots h_{n,k} \ge 1$, and $\lim_{k\to\infty} h_{n,k} = \infty$. If $h_k : S^{n-1} \to (0,\infty)$ is defined by*

$$h_k(v) = \max\{h_{1,k}|v\cdot u_{1,k}|,\dots,h_{n,k}|v\cdot u_{n,k}|\},$$

for all $v \in S^{n-1}$, then the sequence $\mathrm{e}_\mu(h_k)$ is not bounded from below.

The following lemma shows that any finite Borel measure that satisfies the strict subspace concentration inequality has a normalized log-John affinity relative to I.

Lemma 4.2. *If μ is a finite Borel measure on S^{n-1} that satisfies the strict subspace concentration inequality, then there exists a positive definite $A \in \mathrm{SL}(n)$ so that*

$$\sup\big\{\mathrm{e}_\mu(I,Q) : Q \in \mathrm{SL}(n)\big\} = \mathrm{e}_\mu(I,A). \tag{4.2}$$

Proof. Without loss of generality we can assume that μ is a probability measure. Choose a maximizing sequence $Q_k \in \mathrm{SL}(n)$ so that

$$\lim_{k\to\infty} \mathrm{e}_\mu(I,Q_k) = \sup\big\{\mathrm{e}_\mu(I,Q) : Q \in \mathrm{SL}(n)\big\}.$$

Clearly,

$$\lim_{k\to\infty} \mathrm{e}_\mu(I,Q_k) \ge \mathrm{e}_\mu(I,I) = 0. \tag{4.3}$$

For every $Q \in \mathrm{SL}(n)$, there exists a positive definite transformation $P \in \mathrm{SL}(n)$ and an orthogonal transformation $O \in \mathrm{SO}(n)$ so that $Q = OP$. Since $|Qv| = |Pv|$, for $v \in S^{n-1}$, we can assume that the Q_k are positive definite.

Let $h_{1,k},\dots,h_{n,k}$ be the eigenvalues of Q_k, ordered so that $h_{1,k} \le h_{2,k} \le \dots \le h_{n,k}$, with corresponding orthogonal eigenvectors $u_{1,k},\dots, u_{n,k} \in S^{n-1}$. Thus,

$$Q_k u_{i,k} = h_{i,k} u_{i,k}, \tag{4.4}$$

and since $Q_k \in \mathrm{SL}(n)$,

$$h_{1,k}\cdots h_{n,k} = 1. \tag{4.5}$$

For $v \in S^{n-1}$, we have $v = \sum_{i=1}^{n}(v \cdot u_{i,k})u_{i,k}$. This, (4.4), and the fact that $u_{1,k}, \dots, u_{n,k}$ is orthonormal, gives

$$\begin{aligned} |Q_k v|^2 &= \Big| \sum_{i=1}^{n} h_{i,k}(v \cdot u_{i,k})u_{i,k} \Big|^2 \\ &= \sum_{i=1}^{n} h_{i,k}^2 |v \cdot u_{i,k}|^2 \\ &\geq \max_{1 \leq i \leq n} \{h_{i,k}^2 |v \cdot u_{i,k}|^2\} \\ &= h_k(v)^2, \end{aligned}$$

where h_k is as defined in Lemma 4.1. This, definition (3.1), and definition (4.1) show that

$$\mathrm{e}_\mu(I, Q_k) \leq \mathrm{e}_\mu(h_k). \tag{4.6}$$

Assume that the sequence $\{Q_k\}$ is not bounded. Then, for a subsequence,

$$\lim_{k \to \infty} h_{n,k} = +\infty. \tag{4.7}$$

In light of (4.5) and (4.7), Lemma 4.1 tells us that $\{\mathrm{e}_\mu(h_k)\}$ is not bounded from below. Thus, from (4.6) we conclude that the sequence $\{\mathrm{e}_\mu(I, Q_k)\}$ is not bounded from below; this contradicts (4.3). Therefore, the sequence $Q_k \in \mathrm{SL}(n)$ is bounded, and hence it has a subsequence that converges to an $A \in \mathrm{SL}(n)$. Since the Q_k are positive definite (and $|Q_k| = 1$) A is positive definite as well. The continuity of e_μ assures that A is a solution to our maximization problem (4.2). q.e.d.

From Lemma 4.2 and Lemma 3.3, we have:

Theorem 4.3. *If μ is a finite Borel measure on S^{n-1} that satisfies the strict subspace concentration inequality, then μ has a log-John affinity.*

5. Existence of affine isotropic images — sufficient conditions

Let $\mathcal{E}(\mathbb{R}^n) = \mathcal{E}^n$ be the class of origin-centered ellipsoids in $\mathbb{R}^n$ and $\mathcal{E}_1(\mathbb{R}^n) = \mathcal{E}_1^n$ denote the subclass of $\mathcal{E}^n$ consisting of only the origin-centered ellipsoids having unit volume. (The volume normalization here is chosen for convenience to simplify constants that will arise in proofs below.) For an $E \in \mathcal{E}^n$, define

$$\mathrm{e}_\mu(E) = \mathrm{e}_\mu(h_E) = -\int_{S^{n-1}} \log h_E(u)\, d\mu(u). \tag{5.1}$$

If $\varepsilon(A)$ is the ellipsoid associated with $A \in \mathrm{GL}(n)$, then from (2.4) and definition (3.1) it follows that

$$\mathrm{e}_\mu(\varepsilon(A)) = \mathrm{e}_\mu(I, A).$$

This, Lemma 3.3, and Corollary 3.4 gives:

Lemma 5.1. *If μ is a finite Borel measure on S^{n-1}, then a log-John affinity exists for μ if and only if there exists an ellipsoid $E_0 \in \mathcal{E}_1^n$ so that*

$$\sup\{\mathrm{e}_\mu(E) : E \in \mathcal{E}_1^n\} = \mathrm{e}_\mu(E_0). \tag{5.2}$$

An ellipsoid E_0 from Lemma 5.1 will be called a *log-John ellipsoid* associated with μ. The connection between the existence of a log-John ellipsoid and the existence of log-John affinity (given in Lemma 5.1) is not surprising.

We shall require the following technical fact.

Lemma 5.2. *Suppose that ξ_1, ξ_2 are proper complementary subspaces of $\mathbb{R}^n$. If E_i are origin-centered ellipsoids of co-dimensions $\dim(\xi_i)$ and satisfying $\xi_1 \cap E_1 = \{o\}$ and $\xi_2 \cap E_2 = \{o\}$, then there exists a unique origin-centered ellipsoid E_0 in $\mathbb{R}^n$ that is of maximal volume and has the property that $E_0 + \xi_1 = E_1 + \xi_1$ and $E_0 + \xi_2 = E_2 + \xi_2$.*

Proof. For notational simplicity, throughout this proof abbreviate the unit ball, B^n, in $\mathbb{R}^n$, by B. From (2.5) it follows that the lemma holds if and only it holds after its ingredients are transformed by an element of $\mathrm{GL}(n)$. Thus, we can assume that ξ_1 and ξ_2 are orthogonal, and $\xi_2 \cap (E_1 + \xi_1)$ and $\xi_1 \cap (E_2 + \xi_2)$ are the unit balls $B \cap \xi_2$ and $B \cap \xi_1$, respectively. Obviously,

$$\begin{aligned} B + \xi_1 &= (B \cap \xi_2) + \xi_1 = E_1 + \xi_1, \\ B + \xi_2 &= (B \cap \xi_1) + \xi_2 = E_2 + \xi_2. \end{aligned} \tag{5.3}$$

Let

$$Q = B \cap \xi_1 + B \cap \xi_2. \tag{5.4}$$

Obviously, $B \subset Q$. Let J be the unique ellipsoid of maximal volume contained in Q (the *John ellipsoid* of Q). It is easy to see that if a convex body is invariant under a rotation, so is its John ellipsoid. Since Q is invariant under rotations in ξ_1 and ξ_2, so is J. It follows that the axes of J are in ξ_1 and ξ_2 while $J \cap \xi_1 \subset Q \cap \xi_1 = B \cap \xi_1$ and $J \cap \xi_2 \subset Q \cap \xi_2 = B \cap \xi_2$. This shows that $J \subset B \subset Q$, and since J is the maximal ellipsoid in Q we conclude that $J = B$.

Suppose E_0 is an origin-centered ellipsoid such that

$$E_0 + \xi_1 = E_1 + \xi_1 \quad \text{and} \quad E_0 + \xi_2 = E_2 + \xi_2. \tag{5.5}$$

(Observe that B is an example of one such E_0.) Then, by (5.3), we see that

$$E_0 + \xi_1 = B \cap \xi_2 + \xi_1 \quad \text{and} \quad E_0 + \xi_2 = B \cap \xi_1 + \xi_2.$$

Thus,

$$\mathrm{P}_{\xi_1} E_0 = B \cap \xi_1 \quad \text{and} \quad \mathrm{P}_{\xi_2} E_0 = B \cap \xi_2. \tag{5.6}$$

Obviously, $E_0 \subseteq \mathrm{P}_{\xi_1} E_0 + \mathrm{P}_{\xi_2} E_0$, and thus (5.6) and (5.4), yield

$$E_0 \subseteq \mathrm{P}_{\xi_1} E_0 + \mathrm{P}_{\xi_2} E_0 = B \cap \xi_1 + B \cap \xi_2 = Q.$$

Since $J = B \subset Q$ is the unique largest (in volume) ellipsoid contained in Q, we conclude that $V(E_0) \leq V(B)$, with equality if and only if $E_0 = B$. q.e.d.

The following lemma was established by the authors in [8].

Lemma 5.3. *Suppose μ is a finite Borel measure on S^{n-1} that satisfies the subspace concentration condition. If ξ is a proper subspace of $\mathbb{R}^n$ such that*

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n}\,\mu(S^{n-1}) \dim \xi, \tag{5.7}$$

then the restriction of μ to $S^{n-1} \cap \xi$ satisfies the subspace concentration condition.

The following theorem shows that a measure that satisfies the subspace concentration condition necessarily has a log-John affinity.

Theorem 5.4. *If μ is a finite Borel measure on S^{n-1} that satisfies the subspace concentration condition, then μ has a log-John affinity.*

Proof. By Lemma 5.1, it suffices to prove that there is an $\bar{E} \in \mathcal{E}_1^n$ so that

$$\sup\{\mathrm{e}_\mu(E) : E \in \mathcal{E}_1^n\} = \mathrm{e}_\mu(\bar{E}). \tag{5.8}$$

If μ satisfies the strict subspace concentration inequality, by Lemma 4.2, a solution $\bar{E}$ to the maximization problem (5.8) exists. Thus, what remains is establishing the existence of a solution $\bar{E}$ to the maximization problem (5.8) for the case where μ is concentrated on two complementary subspaces.

We proceed by induction on the dimension of the ambient space, $\mathbb{R}^n$. Since the case $n = 1$ is trivial, we start with $n = 2$. In this case, there exist $u_1, u_2 \in S^1$ such that the measure μ is concentrated on the four points $\{\pm u_1, \pm u_2\}$, and since μ satisfies the subspace concentration condition,

$$\mu(\{\pm u_1\}) = \mu(\{\pm u_2\}) = \frac{1}{2}\mu(S^1). \tag{5.9}$$

Consider the origin-centered parallelogram P with outer unit normals u_i and

$$h_P(\pm u_i) = 1,$$

for both i. Let $w_i \in \mathbb{R}^2$ be the midpoint of the side of P with outer unit normal u_i. Let E_0 be the ellipse of maximum area contained in P. Observe that by considering an SL(2) affinity that transforms P into a square, we can see immediately that E_0 passes through the midpoints

$\pm w_i$ and that the line joining w_i to $-w_i$ is parallel to the sides of P that it does not intersect. Obviously, $h_{E_0}(u_i) = h_P(u_i) = 1$.

Let $\bar{E}_0 = \lambda_0 E_0$ be the volume normalized dilate of E_0; i.e., $\lambda_0 = V(E_0)^{-\frac{1}{2}}$. Then $V(\bar{E}_0) = 1$, and since $h_{E_0}(u_i) = 1$,

$$h_{\bar{E}_0}(\pm u_i) = \lambda_0, \tag{5.10}$$

for both i.

Our aim is to show that $\bar{E}_0$ is a solution of the maximization problem (5.8). To that end, suppose $E \in \mathcal{E}_1^2$. Let

$$h_i = h_E(u_i), \tag{5.11}$$

for both i. Let $\psi \in \mathrm{GL}(2)$ be defined on the basis $\{w_1, w_2\}$ so that

$$\psi(w_i) = h_i w_i, \tag{5.12}$$

for both i. Now, (5.12) tells us that $|\psi| = h_1 h_2$.

Since the lines joining w_i to $-w_i$ are parallel to two sides of any parallelogram with normals $\pm u_1, \pm u_2$, and the w_i are eigenvectors of ψ, it follows that ψ transforms an origin-centered parallelogram with normals $\pm u_1, \pm u_2$ to an origin-centered parallelogram with normals $\pm u_1, \pm u_2$. We deduce that

$$h_{\psi P}(\pm u_i) = |u_i \cdot (h_i w_i)| \quad \text{while} \quad h_P(\pm u_i) = |u_i \cdot w_i|,$$

for both i. However since $h_P(\pm u_i) = 1$, we have $h_{\psi P}(\pm u_i) = h_i$, and since $h_E(\pm u_i) = h_i$, we conclude that $E \subset \psi P$. From the fact that $V(E) = 1$ and $\psi^{-1} E \subset P$, together with the fact that E_0 is the largest (in area) ellipse contained in P, we get

$$1/(h_1 h_2) = |\psi|^{-1} = V(\psi^{-1} E) \le V(E_0) = 1/\lambda_0^2. \tag{5.13}$$

From (5.1), the fact that μ is concentrated on the four points $\{\pm u_1, \pm u_2\}$ together with (5.11), (5.9), (5.13), and (5.10),

$$\begin{aligned}
\mathrm{e}_\mu(E) &= -\int_{S^1} \log h_E(u)\, d\mu(u) \\
&= -\mu(\{\pm u_1\}) \log h_1 - \mu(\{\pm u_2\}) \log h_2 \\
&= -\frac{1}{2}\mu(S^1) \log(h_1 h_2) \\
&\le -\frac{1}{2}\mu(S^1) \log \lambda_0^2 \\
&= -\mu(\{\pm u_1\}) \log \lambda_0 - \mu(\{\pm u_2\}) \log \lambda_0 \\
&= -\int_{S^1} \log h_{\bar{E}_0}(u)\, d\mu(u) \\
&= \mathrm{e}_\mu(\bar{E}_0).
\end{aligned}$$

This establishes the existence of a maximizing ellipse in (5.8) for the case where $n = 2$.

Now, suppose that a solution $\bar{E}$ to the maximization problem (5.8) exists whenever the ambient dimension is less than n. For dimension n, we only need to deal with the case where μ is concentrated on two proper complementary subspaces ξ_1 and ξ_2; i.e., if $\dim\xi_i = m_i$, then $m_1 + m_2 = n$ and $m_i > 0$.

Let μ_i denote the restriction of μ to $\xi_i \cap S^{n-1}$. The subspace concentration condition tells us that

$$\mu_i(\xi_i \cap S^{n-1}) = \frac{m_i}{n}\,\mu(S^{n-1}). \tag{5.14}$$

However, (5.14) and Lemma 5.3 tell us that both μ_i satisfy the subspace concentration condition. Therefore, the inductive hypothesis assures the existence of origin-centered $\bar{E}_i \in \mathcal{E}_1(\xi_i) = \mathcal{E}_1^{m_i}$ of unit m_i-dimensional volume so that

$$\sup\{\mathrm{e}_{\mu_i}(E) : E \in \mathcal{E}_1^{m_i}\} = \mathrm{e}_{\mu_i}(\bar{E}_i). \tag{5.15}$$

By Lemma 5.2, there exists a unique origin-centered ellipsoid E_0 in $\mathbb{R}^n$ that is of maximal volume that satisfies, for both i, the condition that $E_0 + \xi_i^{\perp} = \bar{E}_i + \xi_i^{\perp}$, or equivalently since $\bar{E}_i \subset \xi_i$, that $\mathrm{P}_{\xi_i} E_0 = \bar{E}_i$. Let $\bar{E}_0 = \lambda_0 E_0$, where $\lambda_0 = V(E_0)^{-\frac{1}{n}}$, be the volume normalized dilate of E_0; i.e., $V(\bar{E}_0) = 1$. Since $\mathrm{P}_{\xi_i} E_0 = \bar{E}_i$, from (2.1) we conclude that

$$h_{\mathrm{P}_{\xi_i} E_0} = h_{\bar{E}_i}, \qquad \text{on } S^{n-1} \cap \xi_i, \tag{5.16}$$

for both i.

We will now show that $\bar{E}_0$ is a solution of the maximization problem (5.8). To that end, suppose $E \in \mathcal{E}_1^n$ is arbitrary but fixed. Define $\lambda_i = V_{m_i}(\mathrm{P}_{\xi_i} E)^{-\frac{1}{m_i}}$, so that $\lambda_i\,\mathrm{P}_{\xi_i} E \in \mathcal{E}_1(\xi_i)$. Thus, from (5.15), we have

$$\mathrm{e}_{\mu_i}(\lambda_i\,\mathrm{P}_{\xi_i} E) \le \mathrm{e}_{\mu_i}(\bar{E}_i). \tag{5.17}$$

Let

$$\begin{aligned} E_1' &= (E + \xi_1^{\perp}) \cap \xi_2^{\perp}, \quad & E_2' &= (E + \xi_2^{\perp}) \cap \xi_1^{\perp}, \\ E_1^o &= (E_0 + \xi_1^{\perp}) \cap \xi_2^{\perp}, \quad & E_2^o &= (E_0 + \xi_2^{\perp}) \cap \xi_1^{\perp}. \end{aligned} \tag{5.18}$$

Then $E_1', E_1^o \subset \xi_2^{\perp}$ and $E_2', E_2^o \subset \xi_1^{\perp}$. It is easily seen that, for both i,

$$\mathrm{P}_{\xi_i} E_i' = \mathrm{P}_{\xi_i} E \quad \text{and} \quad \mathrm{P}_{\xi_i} E_i^o = \mathrm{P}_{\xi_i} E_0 = \bar{E}_i. \tag{5.19}$$

Choose a $\psi \in \mathrm{GL}(n)$ so that, for both i,

$$\psi\xi_i^{\perp} = \xi_i^{\perp} \quad \text{and} \quad \psi E_i' = E_i^o. \tag{5.20}$$

The fact that $\psi\xi_i^{\perp} = \xi_i^{\perp}$, along with (5.18), gives

$$\psi E_1' = (\psi E + \xi_1^{\perp}) \cap \xi_2^{\perp} \quad \text{and} \quad \psi E_2' = (\psi E + \xi_2^{\perp}) \cap \xi_1^{\perp}. \tag{5.21}$$

From (5.21), (5.20), and (5.19) we have

$$\mathrm{P}_{\xi_i}(\psi E) = \mathrm{P}_{\xi_i}(\psi E_i') = \mathrm{P}_{\xi_i} E_i^o = \bar{E}_i. \tag{5.22}$$

Let ψ_1 be defined as ψ on $\xi_1^\perp$ and the identity on $\xi_2^\perp$. Let ψ_2 be defined as the identity on $\xi_1^\perp$ and ψ on $\xi_2^\perp$. From (5.18) we have $E_1 \subset \xi_2^\perp$ and $E_2 \subset \xi_1^\perp$. This and (5.20) gives

$$\psi_1 E_2' = E_2^o \quad \text{and} \quad \psi_2 E_1' = E_1^o. \tag{5.23}$$

From the definition of ψ_i, (2.2) together with (5.23), (2.3), (5.19), and finally using the fact that $V_{m_i}(\bar{E}_i) = 1$, we get

$$\begin{aligned} |\psi| &= |\psi_1|\,|\psi_2| \\ &= \frac{V_{m_1}(E_1^o)V_{m_2}(E_2^o)}{V_{m_1}(E_1')V_{m_2}(E_2')} \\ &= \frac{V_{m_1}(\mathrm{P}_{\xi_1}E_1^o)V_{m_2}(\mathrm{P}_{\xi_2}E_2^o)}{V_{m_1}(\mathrm{P}_{\xi_1}E_1')V_{m_2}(\mathrm{P}_{\xi_2}E_2')} \\ &= \frac{V_{m_1}(\bar{E}_1)V_{m_2}(\bar{E}_2)}{V_{m_1}(\mathrm{P}_{\xi_1}E)V_{m_2}(\mathrm{P}_{\xi_2}E)} \\ &= [V_{m_1}(\mathrm{P}_{\xi_1}E)V_{m_2}(\mathrm{P}_{\xi_2}E)]^{-1}. \end{aligned} \tag{5.24}$$

However by Lemma 5.2, E_0 is the unique origin-centered ellipsoid of maximal volume such that $\mathrm{P}_{\xi_i}E_0 = \bar{E}_i$, for both i. Since from (5.22) we know that $\mathrm{P}_{\xi_i}(\psi E) = \bar{E}_i$, we conclude that $V(\psi E) \le V(E_0)$. It follows from the definition of the λ_i, (5.24), (2.2) together with the fact that $V(E) = 1$, the fact that $V(\psi E) \le V(E_0)$, and the definition of λ_0, that

$$\begin{aligned} \lambda_1^{-m_1}\lambda_2^{-m_2} &= V_{m_1}(\mathrm{P}_{\xi_1}E)V_{m_2}(\mathrm{P}_{\xi_2}E) \\ &= 1/|\psi| = 1/V(\psi E) \ge 1/V(E_0) = \lambda_0^n. \end{aligned} \tag{5.25}$$

The fact that the measure μ is concentrated on the two $\xi_i \cap S^{n-1}$, together with (2.1), (5.17), (5.14), (5.16), (5.25), and finally the fact that $\lambda_0 E_0 = \bar{E}_0$, gives

$$\begin{aligned}
\mathrm{e}_\mu(E) &= -\int_{\xi_1\cap S^{n-1}} \log h_{\mathrm{P}_{\xi_1} E}\, d\mu_1 - \int_{\xi_2\cap S^{n-1}} \log h_{\mathrm{P}_{\xi_2} E}\, d\mu_2\\
&= -\int_{\xi_1\cap S^{n-1}} (\log \lambda_1^{-1} + \log h_{\lambda_1 \mathrm{P}_{\xi_1} E})\, d\mu_1\\
&\qquad - \int_{\xi_2\cap S^{n-1}} (\log \lambda_2^{-1} + \log h_{\lambda_2 \mathrm{P}_{\xi_2} E})\, d\mu_2\\
&\le \mu_1(\xi_1 \cap S^{n-1}) \log \lambda_1 + \mu_2(\xi_2 \cap S^{n-1}) \log \lambda_2\\
&\qquad - \int_{\xi_1\cap S^{n-1}} \log h_{\bar{E}_1}\, d\mu_1 - \int_{\xi_2\cap S^{n-1}} \log h_{\bar{E}_2}\, d\mu_2\\
&= \mu(S^{n-1}) \log(\lambda_1^{\frac{m_1}{n}} \lambda_2^{\frac{m_2}{n}}) - \int_{S^{n-1}} \log h_{E_0}\, d\mu\\
&\le -\mu(S^{n-1}) \log \lambda_0 - \int_{S^{n-1}} \log h_{E_0}\, d\mu\\
&= -\int_{S^{n-1}} \log h_{\lambda_0 E_0}\, d\mu\\
&= -\int_{S^{n-1}} \log h_{\bar{E}_0}\, d\mu\\
&= \mathrm{e}_\mu(\bar{E}_0).
\end{aligned}$$

This establishes (5.8) by induction. q.e.d.

6. Existence of affine isotropic images — necessary conditions

We begin by recalling that from Lemma 5.1, a finite Borel measure μ has a log-John affinity if and only if there exists a $\varphi_0 \in \mathrm{SL}(n)$ such that

$$\inf_{\varphi\in \mathrm{SL}(n)} \left\{ \int_{S^{n-1}} \log |\varphi u|\, d\mu(u) \right\} = \int_{S^{n-1}} \log |\varphi_0 u|\, d\mu(u). \tag{6.1}$$

It turns out that a finite Borel measure that violates any of the subspace concentration inequalities will not have a log-John affinity. This fact is contained in the following lemma.

Lemma 6.1. *If μ is a finite Borel measure on S^{n-1} for which there exists a proper subspace ξ so that*

$$\mu(\xi \cap S^{n-1}) > \frac{1}{n}\, \mu(S^{n-1}) \dim \xi,$$

then μ does not have a log-John affinity.

Proof. Let $m = \dim \xi$. For $t \in (0,1)$, let $\varphi_t \in \mathrm{SL}(n)$ be so that $\varphi_t|_\xi = t^{n-m} I_\xi$ and $\varphi_t|_{\xi^\perp} = t^{-m} I_{\xi^\perp}$. Observe that $|\varphi_t| \le t^{-m}$ on S^{n-1}.

Thus,

$$\begin{aligned}
&\int_{S^{n-1}} \log|\varphi_t u|\,d\mu(u)\\
&\qquad = \int_{S^{n-1}\cap\xi} \log|\varphi_t(u)|\,d\mu(u) + \int_{S^{n-1}\setminus\xi} \log|\varphi_t(u)|\,d\mu(u)\\
&\qquad \le \int_{S^{n-1}\cap\xi} \log|t^{n-m}u|\,d\mu(u) + \int_{S^{n-1}\setminus\xi} \log|t^{-m}|\,d\mu(u)\\
&\qquad = [(n-m)\mu(\xi\cap S^{n-1}) - m\mu(S^{n-1}\setminus\xi)]\log t.
\end{aligned}$$

However,

$$(n-m)\mu(\xi\cap S^{n-1}) - m\mu(S^{n-1}\setminus\xi) = n\mu(\xi\cap S^{n-1}) - m\mu(S^{n-1}) > 0,$$

by hypothesis. Thus, we conclude

$$\lim_{t\to 0}\int_{S^{n-1}} \log|\varphi_t u|\,d\mu(u) = -\infty,$$

and that the infimum in (6.1) is not attained. q.e.d.

The subspace concentration condition requires that equality in the subspace concentration inequalities can only occur in pairs — in pairs of complementary subspaces. The critical nature of this condition is demonstrated in the following lemma.

Lemma 6.2. *Suppose μ is a finite Borel measure on S^{n-1}. If there exists a proper subspace ξ so that*

$$\mu(\xi\cap S^{n-1}) = \frac{1}{n}\,\mu(S^{n-1})\dim\xi,$$

and there does not exist a subspace ξ' complementary to ξ such that μ is concentrated on $S^{n-1}\cap(\xi\cup\xi')$, then μ does not have a log-John affinity.

Proof. Without loss of generality, we may assume $\mu(S^{n-1}) = 1$. Let $m = \dim\xi$. From the hypothesis, we have

$$\mu(S^{n-1}\cap\xi) = m/n \qquad\text{and}\qquad \mu(S^{n-1}\setminus\xi) = (n-m)/n. \tag{6.2}$$

Abbreviate,

$$M_0 = \int_{S^{n-1}\setminus\xi} \log|\,\mathrm{P}_{\xi^\perp}u|\,d\mu(u).$$

The continuous function $u\mapsto \log|\,\mathrm{P}_{\xi^\perp}u|$ on the open set $S^{n-1}\setminus\xi$ tends to $-\infty$ near ξ. Thus, M_0 may not be finite.

We first consider the case where $M_0 = -\infty$, and we will show that if this were the case μ would have no log-John affinity.

For $t \in (0,1)$, let $\varphi_t \in \mathrm{SL}(n)$ be so that $\varphi_t|_\xi = t^{n-m} I_m$ and $\varphi_t|_{\xi^\perp} = t^{-m} I_{n-m}$. From (6.2), we have

$$\begin{aligned}
&\int_{S^{n-1}} \log|\varphi_t u|\, d\mu(u) \\
&\qquad = \int_{S^{n-1}\cap\xi} \log|t^{n-m}u|\, d\mu(u) \\
&\qquad\quad + \int_{S^{n-1}\setminus\xi} \log(|t^{n-m}\,\mathrm{P}_\xi u|^2 + |t^{-m}\,\mathrm{P}_{\xi^\perp} u|^2)^{\frac12}\, d\mu(u) \\
&\qquad = \frac{m}{n}\log t^{n-m} + \frac{n-m}{n}\log t^{-m} \\
&\qquad\quad + \int_{S^{n-1}\setminus\xi} \log(|t^{n}\,\mathrm{P}_\xi u|^2 + |\,\mathrm{P}_{\xi^\perp} u|^2)^{\frac12}\, d\mu(u) \\
&\qquad = \int_{S^{n-1}\setminus\xi} \log(|t^{n}\,\mathrm{P}_\xi u|^2 + |\,\mathrm{P}_{\xi^\perp} u|^2)^{\frac12}\, d\mu(u).
\end{aligned}$$

This, an application of the Reverse Fatou Lemma, and our assumption that $M_0 = -\infty$, gives

$$\begin{aligned}
&\lim_{t\to 0}\int_{S^{n-1}} \log|\varphi_t u|\, d\mu(u) \\
&\qquad\qquad = \lim_{t\to 0}\int_{S^{n-1}\setminus\xi} \log(|t^{n}\,\mathrm{P}_\xi u|^2 + |\,\mathrm{P}_{\xi^\perp} u|^2)^{\frac12}\, d\mu(u) \\
&\qquad\qquad = -\infty.
\end{aligned}$$

Thus, the infimum of $\int_{S^{n-1}} \log|\varphi u|\, d\mu(u)$, over all $\varphi \in \mathrm{SL}(n)$, is $-\infty$, which means that μ would have no log-John affinity if it were the case that $M_0 = -\infty$.

We turn to the case where M_0 is finite, and we will assume that M_0 is finite throughout the rest of the proof.

Define the finite Borel measure $\tilde\mu$ on $S^{n-1}\cap\xi^\perp$ by letting

$$\int_{S^{n-1}\cap\xi^\perp} f\, d\tilde\mu = \int_{S^{n-1}\setminus\xi} f(\langle \mathrm{P}_{\xi^\perp} u\rangle)\, d\mu(u), \tag{6.3}$$

for each continuous $f : S^{n-1}\cap\xi^\perp \to \mathbb{R}$. (Recall that $\langle \mathrm{P}_{\xi^\perp} u\rangle = \mathrm{P}_{\xi^\perp} u/|\,\mathrm{P}_{\xi^\perp} u|$.) By choosing $f = |\cdot|$ in (6.3), and using (6.2), we have

$$\tilde\mu(S^{n-1}\cap\xi^\perp) = \mu(S^{n-1}\setminus\xi) = (n-m)/n, \tag{6.4}$$

for the measure $\tilde\mu$.

Suppose $\tilde{A} \in \mathrm{SL}(\xi^{\perp})$. From the definition of M_0 and definition (6.3) we see that

$$\begin{aligned}\int_{S^{n-1}\setminus\xi} \log|\tilde{A}(\mathrm{P}_{\xi^\perp}u)|\,d\mu(u) &= \int_{S^{n-1}\setminus\xi} \log|\,\mathrm{P}_{\xi^\perp}u|\,d\mu(u)\\ &\quad + \int_{S^{n-1}\setminus\xi} \log|\tilde{A}\langle \mathrm{P}_{\xi^\perp}u\rangle|\,d\mu(u)\\ &= M_0 + \int_{S^{n-1}\cap\xi^\perp} \log|\tilde{A}u|\,d\tilde{\mu}(u),\end{aligned}$$

and hence is finite. From this and the fact that on $S^{n-1}\setminus\xi$ the function $u \mapsto \log|\tilde{A}(\mathrm{P}_{\xi^\perp}u)|$ is continuous and is negative close to ξ, we conclude that

$$\int_{S^{n-1}\setminus\xi} \big|\log|\tilde{A}(\mathrm{P}_{\xi^\perp}u)|\big|\,d\mu(u) \quad < \ \infty. \tag{6.5}$$

For each $t > 0$ and $A \in \mathrm{SL}(\xi)$, consider $\varphi_t \in \mathrm{SL}(n)$ defined so that we have

$$\varphi_t|_\xi = t^{n-m}A \qquad \text{and} \qquad \varphi_t|_{\xi^\perp} = t^{-m}\tilde{A}. \tag{6.6}$$

From (6.6) and (6.2), we get

$$\begin{aligned}&\int_{S^{n-1}} \log|\varphi_t u|\,d\mu(u)\\ &\quad= \int_{S^{n-1}\cap\xi} \log|t^{n-m}Au|\,d\mu(u)\\ &\qquad + \int_{S^{n-1}\setminus\xi} \log(|t^{n-m}A\,\mathrm{P}_\xi u|^2 + |t^{-m}\tilde{A}\,\mathrm{P}_{\xi^\perp}u|^2)^{\frac{1}{2}}\,d\mu(u)\\ &\quad= \frac{m}{n}\log t^{n-m} + \int_{S^{n-1}\cap\xi} \log|Au|\,d\mu(u) + \frac{n-m}{n}\log t^{-m}\\ &\qquad + \int_{S^{n-1}\setminus\xi} \log(|t^{n}A\,\mathrm{P}_\xi u|^2 + |\tilde{A}\,\mathrm{P}_{\xi^\perp}u|^2)^{\frac{1}{2}}\,d\mu(u)\\ &\quad= \int_{S^{n-1}\cap\xi} \log|Au|\,d\mu(u)\\ &\qquad + \int_{S^{n-1}\setminus\xi} \log(|t^{n}A\,\mathrm{P}_\xi u|^2 + |\tilde{A}\,\mathrm{P}_{\xi^\perp}u|^2)^{\frac{1}{2}}\,d\mu(u),\end{aligned}$$

where for $u \in S^{n-1}$, we have written $u = \mathrm{P}_\xi u + \mathrm{P}_{\xi^\perp}u$.

From this, (6.5), Lebesgue's Dominated Convergence Theorem, and (6.5), we get

$$\begin{aligned}\lim_{t\to 0}\int_{S^{n-1}} \log|\varphi_t u|\,d\mu(u) &= \int_{S^{n-1}\cap\xi} \log|Au|\,d\mu(u) + \int_{S^{n-1}\setminus\xi} \log|\tilde{A}(\mathrm{P}_{\xi^\perp}u)|\,d\mu(u)\\ &= \int_{S^{n-1}\cap\xi} \log|Au|\,d\mu(u) + \int_{S^{n-1}\cap\xi^\perp} \log|\tilde{A}u|\,d\tilde{\mu}(u) + M_0. \qquad (6.7)\end{aligned}$$

From (6.1), we see that if $\int_{S^{n-1}} \log|\varphi u|\,d\mu(u)$ is not bounded from below, for $\varphi\in \mathrm{SL}(n)$, then there exists no log-John affinity of μ and we are done.

We turn to the case where $\int_{S^{n-1}} \log|\varphi u|\,d\mu(u)$ is bounded from below, for $\varphi\in\mathrm{SL}(n)$. However in this case (6.7) tells us that

$$M_1 = \inf_{\substack{A\in\mathrm{SL}(\xi)\\ \tilde{A}\in\mathrm{SL}(\xi^\perp)}} \left\{\int_{S^{n-1}\cap\xi} \log|Au|\,d\mu(u) + \int_{S^{n-1}\cap\xi^\perp} \log|\tilde{A}u|\,d\tilde{\mu}(u)\right\}$$

is finite. From (6.7), we see that

$$\inf_{\varphi\in\mathrm{SL}(n)} \left\{\int_{S^{n-1}} \log|\varphi u|\,d\mu(u)\right\} \le M_1 + M_0. \qquad (6.8)$$

If we could show that for all $\varphi\in\mathrm{SL}(n)$,

$$\int_{S^{n-1}} \log|\varphi u|\,d\mu(u) > M_1 + M_0, \qquad (6.9)$$

then (6.8) would allow us to conclude that the infimum in (6.8) can not be attained, for any $\varphi\in\mathrm{SL}(n)$, and thus, the measure μ has no log-John affinity, as desired. We proceed to establish (6.9).

Suppose $\varphi\in\mathrm{SL}(n)$ is arbitrary. If $\varphi\xi\neq\xi$, let $O\in\mathrm{SO}(n)$ be an orthogonal transformation such that $O\varphi\xi=\xi$, and observe that $|O\varphi u| = |\varphi u|$, for all $u\in S^{n-1}$. Hence, we may assume that our $\varphi\in\mathrm{SL}(n)$ is so that

$$\varphi\xi = \xi \quad\text{and}\quad \det(\varphi|_\xi) > 0.$$

By writing $x\in\mathbb{R}^n$ as $x = \mathrm{P}_\xi x + \mathrm{P}_{\xi^\perp}x$, we see that

$$\varphi x = \varphi\,\mathrm{P}_\xi x + \varphi\,\mathrm{P}_{\xi^\perp}x = \varphi\,\mathrm{P}_\xi x + \mathrm{P}_\xi\varphi\,\mathrm{P}_{\xi^\perp}x + \mathrm{P}_{\xi^\perp}\varphi\,\mathrm{P}_{\xi^\perp}x.$$

Thus, φ can be decomposed, on $\mathbb{R}^n$, as

$$\varphi = A\,\mathrm{P}_\xi + B\,\mathrm{P}_{\xi^\perp} + \tilde{A}\,\mathrm{P}_{\xi^\perp}, \qquad (6.10)$$

where $A\in\mathrm{GL}(\xi)$ is given by $A=\varphi|_\xi$ and $\tilde{A}\in\mathrm{GL}(\xi^\perp)$ is given by $\tilde{A} = \mathrm{P}_{\xi^\perp}\varphi|_{\xi^\perp}$, and where $B = \mathrm{P}_\xi\varphi|_{\xi^\perp}$ is a linear transformation from $\xi^\perp$ to ξ. So, $\det A>0$, and $|A|\,|\tilde{A}| = |\varphi| = 1$. Therefore, there exist

$A_0 \in \mathrm{SL}(\xi)$ and $\tilde{A}_0 \in \mathrm{SL}(\xi^\perp)$ such that $A = t^{n-m} A_0$ and $\tilde{A} = t^{-m}\tilde{A}_0$ for $t = |A|^{\frac{1}{m(n-m)}}$.

We write $u = \mathrm{P}_\xi u + \mathrm{P}_{\xi^\perp} u$, and from (6.10), (6.2), (6.5), and the definition of M_1, we get

$$\begin{aligned}
&\int_{S^{n-1}} \log|\varphi u|\, d\mu(u) \\
&\quad = \int_{S^{n-1}\cap\xi} \log|t^{n-m} A_0 u|\, d\mu(u) \\
&\qquad + \int_{S^{n-1}\setminus\xi} \log(|A\,\mathrm{P}_\xi u + B\,\mathrm{P}_{\xi^\perp} u|^2 + |t^{-m}\tilde{A}_0\,\mathrm{P}_{\xi^\perp} u|^2)^{\frac{1}{2}}\, d\mu(u) \\
&\quad \geq \frac{m}{n}\log t^{n-m} + \int_{S^{n-1}\cap\xi} \log|A_0 u|\, d\mu(u) \\
&\qquad + \frac{n-m}{n}\log t^{-m} + \int_{S^{n-1}\setminus\xi} |\tilde{A}_0\,\mathrm{P}_{\xi^\perp} u|\, d\mu(u) \\
&\quad = \int_{S^{n-1}\cap\xi} \log|A_0 u|\, d\mu(u) + \int_{S^{n-1}\setminus\xi} \log|\tilde{A}_0\,\mathrm{P}_{\xi^\perp} u|\, d\mu(u) \\
&\quad = \int_{S^{n-1}\cap\xi} \log|A_0 u|\, d\mu(u) + \int_{S^{n-1}\cap\xi^\perp} \log|\tilde{A}_0 u|\, d\tilde{\mu}(u) + M_0 \\
&\quad \geq M_1 + M_0,
\end{aligned}$$

with equality implying

$$\mu(\{u \in S^{n-1}\setminus\xi : |A\,\mathrm{P}_\xi u + B\,\mathrm{P}_{\xi^\perp} u| > 0\}) = 0. \tag{6.11}$$

Let η be the subspace defined by

$$\eta = \{x \in \mathbb{R}^n : \, A\,\mathrm{P}_\xi x + B\,\mathrm{P}_{\xi^\perp} x = 0\}.$$

For $x \in \eta$, let $y = \mathrm{P}_{\xi^\perp} x \in \xi^\perp$. Then $\mathrm{P}_\xi x = -A^{-1}By$. Thus, $x = \mathrm{P}_{\xi^\perp} x + \mathrm{P}_\xi x = y - A^{-1}By$. Conversely, since $A^{-1}B : \xi^\perp \to \xi$, it is easily seen that $y - A^{-1}By \in \eta$ for each $y \in \xi^\perp$. Thus,

$$\eta = \{y - A^{-1}By : \, y \in \xi^\perp\}.$$

Suppose $y - A^{-1}By = 0$ for $y \in \xi^\perp$. Since $A^{-1}By \in \xi$ while $y \in \xi^\perp$, we conclude $y = 0$. Therefore, the subspace η is a non-singular linear image of $\xi^\perp$, and thus is $(n-m)$-dimensional. But obviously, ξ and η only meet at the origin and are thus complementary subspaces. Since, by hypothesis, there is no subspace ξ' complementary to ξ so that μ is concentrated on $(\xi \cup \xi') \cap S^{n-1}$, we have

$$0 < \mu\left(S^{n-1}\setminus(\xi\cup\eta)\right) = \mu\left(\{u \in S^{n-1}\setminus\xi : \, |A\,\mathrm{P}_\xi u + B\,\mathrm{P}_{\xi^\perp} u| > 0\}\right).$$

Therefore (6.11) cannot hold and there is strict inequality in (6.11), which in turn yields (6.9) and shows that μ has no log-John affinity. q.e.d.

The following theorem shows that the subspace concentration condition is a necessary condition for the existence of a log-John affinity.

Theorem 6.3. *If μ is a finite Borel measure on S^{n-1} that has a log-John affinity, then μ satisfies the subspace concentration condition.*

Proof. By Lemma 6.1, μ satisfies the subspace concentration inequalities. If there is a proper subspace ξ so that

$$\mu(\xi \cap S^{n-1}) = \frac{1}{n}\,\mu(S^{n-1}) \dim \xi,$$

then by Lemma 6.2, there is a subspace ξ' complementary to ξ so that μ is concentrated on $(\xi \cup \xi') \cap S^{n-1}$. Therefore, μ satisfies the subspace concentration condition. q.e.d.

Theorem 6.4. *A finite Borel measure on the unit sphere has an affine isotropic image if and only if it satisfies the subspace concentration condition.*

The proof now follows from Theorems 3.6, 5.4, and 6.3.

7. Affine inequalities for measures

If $T \subset S^{n-1}$, then for notational simplicity, we shall write

$$(T)^j = \underbrace{T \times \cdots \times T}_{j}.$$

For a finite Borel measure μ on S^{n-1}, we define the invariant $U(\mu)$ as an integral over a subset of $(S^{n-1})^n$:

$$U(\mu)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} d\mu(u_1) \cdots d\mu(u_n). \tag{7.1}$$

From (1.3), we see that the total measure $|\mu| = \mu(S^{n-1})$ is invariant under $\mathrm{SL}(n)$-transformations; i.e., $|A\mu| = |\mu|$, for all $A \in \mathrm{SL}(n)$. The invariant U is also $\mathrm{SL}(n)$ invariant. Indeed, from (2.9) we get

$$U(A\mu) = U(\mu),$$

for all $A \in \mathrm{SL}(n)$.

Obviously, $U(\mu) \leq |\mu|$. If μ is absolutely continuous with respect to spherical Lebesgue measure on S^{n-1}, then $U(\mu) = |\mu|$. If μ is discrete, then $U(\mu) < |\mu|$. The following theorem shows that the invariant U captures the concentration of measures in subspaces.

Theorem 7.1. *If μ is a finite Borel measure on S^{n-1}, then*

$$U(\mu) \leq |\mu|,$$

with equality if and only if μ does not have positive subspace mass.

Proof. First, suppose that the measure μ does not have positive subspace mass. Then

$$\mu(\xi_i \cap S^{n-1}) = 0,$$

for each proper subspace ξ_i of $\mathbb{R}^n$ with $\dim \xi_i = i$.

For $0 < i < n$, let Ω_i be the set of points $(u_1, \ldots, u_n) \in (S^{n-1})^n$ such that there are exactly i unit vectors that are linearly independent among $u_1, \ldots, u_n$. Then,

$$\{(u_1, \ldots, u_n) \in (S^{n-1})^n : u_1 \wedge \cdots \wedge u_n = 0\} = \bigcup_{i=1}^{n-1} \Omega_i.$$

For $1 \le j_1 < \cdots < j_i \le n$ define

$$A_{j_1,\ldots,j_i} = \{(u_1, \ldots, u_n) \in \Omega_i : \, u_{j_1} \wedge \cdots \wedge u_{j_i} \neq 0\},$$

so that

$$\Omega_i = \bigcup_{1 \le j_1 < \cdots < j_i \le n} A_{j_1,\ldots,j_i}.$$

Clearly, if $1 \le j_1 < \cdots < j_i \le n$ and $1 \le j_1' < \cdots < j_i' \le n$, then

$$\int_{A_{j_1,\ldots,j_i}} d\mu(u_1) \cdots d\mu(u_n) = \int_{A_{j_1',\ldots,j_i'}} d\mu(u_1) \cdots d\mu(u_n).$$

Then

$$\begin{aligned}
&\int_{\Omega_i} d\mu(u_1) \cdots d\mu(u_n) \\
&\quad \le \sum_{1 \le j_1 < \cdots < j_i \le n} \int_{A_{j_1,\ldots,j_i}} d\mu(u_1) \cdots d\mu(u_n) \\
&\quad = \binom{n}{i} \int_{A_{1,\ldots,i}} d\mu(u_1) \cdots d\mu(u_n) \\
&\quad = \binom{n}{i} \int_{(S^{n-1})^n} \mathbf{1}_{A_{1,\ldots,i}} \, d\mu(u_1) \cdots d\mu(u_n) \\
&\quad = \binom{n}{i} \int_{(S^{n-1})^i} \int_{(S^{n-1})^{n-i}} \mathbf{1}_{A_{1,\ldots,i}} \, d\mu(u_{i+1}) \cdots d\mu(u_n) \, d\mu(u_1) \cdots d\mu(u_i) \\
&\quad = \binom{n}{i} \int_{u_1 \wedge \cdots \wedge u_i \neq 0} \int_{(\xi_i \cap S^{n-1})^{n-i}} d\mu(u_{i+1}) \cdots d\mu(u_n) \, d\mu(u_1) \cdots d\mu(u_i) \\
&\quad = \binom{n}{i} \int_{u_1 \wedge \cdots \wedge u_i \neq 0} \mu(\xi_i \cap S^{n-1})^{n-i} d\mu(u_1) \cdots d\mu(u_i) = 0,
\end{aligned}$$

where $\mathbf{1}_{A_{1,\ldots,i}}$ is the characteristic function of $A_{1,\ldots,i}$ and ξ_i is the subspace spanned by $u_1, \ldots, u_i$. Thus,

$$\int_{u_1 \wedge \cdots \wedge u_n = 0} d\mu(u_1) \cdots d\mu(u_n) = 0. \tag{7.2}$$

This and (7.1) give

$$U(\mu)^n = \int_{(S^{n-1})^n} d\mu(u_1)\cdots d\mu(u_n) = |\mu|^n.$$

Conversely, assume that $U(\mu) = |\mu|$; then (7.2) holds. For a subspace ξ_{n-1} of $\mathbb{R}^n$ of co-dimension 1, we have

$$\begin{aligned}\mu(\xi_{n-1}\cap S^{n-1})^n &= \int_{(\xi_{n-1}\cap S^{n-1})^n} d\mu(u_1)\cdots d\mu(u_n)\\ &\le \int_{u_1\wedge\cdots\wedge u_n=0} d\mu(u_1)\cdots d\mu(u_n)\\ &= 0.\end{aligned}$$

Thus, the measure μ does not have positive subspace mass. q.e.d.

Theorem 7.2. *If μ is a finite Borel measure on S^{n-1} that has an isotropic affine image, then*

$$U(\mu) \ge \frac{(n!)^{1/n}}{n}|\mu|, \tag{7.3}$$

with equality if and only if $\bar{\mu}$, the central symmetral of μ, is an affine image of a cross-measure.

Proof. Since the measure μ has an isotropic affine image, there is an $A \in \mathrm{SL}(n)$ so that $A\mu$ is isotropic. From (2.8), and the fact that the total mass is invariant under $\mathrm{SL}(n)$-transformations, we know

$$\int_{(S^{n-1})^n} |[u_1,\dots,u_n]|^2\, dA\mu(u_1)\cdots dA\mu(u_n) = \frac{n!}{n^n}|A\mu|^n = \frac{n!}{n^n}|\mu|^n. \tag{7.4}$$

From (7.1) and (7.4), we have

$$\begin{aligned}U(A\mu)^n &= \int_{u_1\wedge\cdots\wedge u_n\neq 0} dA\mu(u_1)\cdots dA\mu(u_n)\\ &\ge \int_{u_1\wedge\cdots\wedge u_n\neq 0} |[u_1,\dots,u_n]|^2\, dA\mu(u_1)\cdots dA\mu(u_n)\\ &= \int_{(S^{n-1})^n} |[u_1,\dots,u_n]|^2\, dA\mu(u_1)\cdots dA\mu(u_n)\\ &= \frac{n!}{n^n}|\mu|^n,\end{aligned}$$

with equality if and only if it is the case that whenever $u_1,\dots,u_n \in \mathrm{supp}(A\mu)$ are linearly independent, we have $|[u_1,\dots,u_n]| = 1$. Therefore, any linearly independent $u_1,\dots,u_n$ in $\mathrm{supp}(A\mu)$ are orthogonal. This, and the fact that $A\mu$ is isotropic, implies that the central symmetral of $A\mu$ is a cross-measure, and thus $\bar{\mu}$ is an affine image of a cross-measure. q.e.d.

Observe that the first statement of Theorem 1.3 is Theorem 7.1. The second statement of Theorem 1.3 follows from Theorems 7.2 and 1.2.

8. Applications to cone-volume measures of convex bodies

If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then the cone-volume measure, V_K, of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel $\omega \subset S^{n-1}$, by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x)\, d\mathcal{H}^{n-1}(x),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

In recent years, cone-volume measures have appeared in, e.g., [43, 45, 62, 63, 66, 73]. *Firey's Question* asks if a body whose cone-volume measure is proportional to spherical Lebesgue measure on S^{n-1} must be a ball. This fundamental question was answered, in the affirmative, by Andrews [1] in $\mathbb{R}^3$. An answer to the Firey's Question in $\mathbb{R}^n$, for $n > 3$, is one of the major open problems in geometric analysis.

As an aside, we note that the cone-volume measure is (up to a factor of n) the L_0-surface area measure within the rapidly evolving L_p-Brunn-Minkowski theory (see e.g. [7, 8, 10, 13, 14, 18, 19, 22, 23, 25–29, 32, 33, 38–51, 53–55, 57–59, 61, 64–67, 69–73, 75, 76, 78]). For $p = 0$, the L_p-Brunn-Minkowski theory is more commonly called the log-Brunn-Minkowski theory.

The total mass of the cone-volume measure of the body K is obviously the volume of K; i.e.,

$$|V_K| = V(K). \tag{8.1}$$

It was shown in [8] that the affine image of the cone-volume measure of a convex body is the cone-volume measure of the affine image of the body; i.e., if K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and $A \in \mathrm{SL}(n)$, then

$$AV_K = V_{A^{-t}K}. \tag{8.2}$$

This is an easy consequence of definition (2.9) and (1.10) in [57].

Now (8.2) allows us to rewrite a basic question posed in [57] as follows:

Problem 8.1. *For a given convex body K that contains the origin in its interior, is there an $A \in \mathrm{SL}(n)$ so that*

$$|x|^2 = \frac{n}{V(K)} \int_{S^{n-1}} |x \cdot v|^2 \, dV_{AK}(v),$$

for all $x \in \mathbb{R}^n$; i.e., does the cone-volume measure of a convex body have an affine isotropic image?

We shall show that for arbitrary convex bodies (that contain the origin in their interiors) the answer is negative. We will give an affirmative answer to this question for convex bodies that are origin-symmetric.

Lemma 8.2. *For $n \geq 2$, there exists a polytope T in $\mathbb{R}^n$ that contains the origin in its interior whose cone-volume measure has no affine isotropic image.*

Proof. Let T be a simplex that contains the origin very close to one of its vertices. The cone-volume measure of T is discrete and is concentrated mostly at one point. Therefore, it cannot satisfy the subspace concentration condition. By Theorem 1.2, a measure on S^{n-1} has an affine isotropic image if and only if it satisfies the subspace concentration condition. We conclude that the cone-volume measure of T has no affine isotropic image. q.e.d.

The following lemma was proved in [8]. For polytopes, the inequality-part of the subspace concentration condition of Lemma 8.3 was established by He, Leng, and Li [30], with a shorter proof provided by Xiong [77].

Lemma 8.3. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then the cone-volume measure V_K satisfies the subspace concentration condition.*

From (8.2), Lemma 8.3, together with Theorems 5.4 and 3.8, we get:

Theorem 8.4. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then K has an $\mathrm{SL}(n)$-image whose cone-volume measure is isotropic; i.e., there exists an $A \in \mathrm{SL}(n)$ so that*

$$|x|^2 = \frac{n}{V(K)} \int_{S^{n-1}} |x \cdot v|^2 \, dV_{AK}(v), \tag{8.3}$$

for all $x \in \mathbb{R}^n$.

The $\mathrm{SL}(n)$-invariant U was defined in [52]. For a convex body K in $\mathbb{R}^n$ that contains the origin in its interior, define $U(K)$, as an integral over a subset of $(S^{n-1})^n$, by

$$U(K)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} dV_K(u_1) \cdots dV_K(u_n); \tag{8.4}$$

i.e.,

$$U(K) = U(V_K), \tag{8.5}$$

in the notation of the previous section.

Obviously, $U(K) \leq V(K)$. When K is a polytope, we have $U(K) < V(K)$.

The following theorem characterizes equality in the inequality $U(K) \leq V(K)$. It is an immediate consequence of Theorem 7.1, (8.5), and (8.1).

Theorem 8.5. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then*

$$U(K) \le V(K),$$

with equality if and only if the cone-volume measure V_K does not have positive subspace mass.

The affine invariant U can be viewed as a variant of volume, V, that measures the effect of positive subspace mass of the cone-volume measure of K. The polytopal case of the following problem was posed in [52].

Problem 8.6. *Suppose K is a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Is it the case that*

$$U(K) \ge \frac{(n!)^{1/n}}{n} V(K), \tag{8.6}$$

with equality if and only if K is a parallelotope?

When K is an origin-symmetric polytope, He, Leng, and Li [30] established inequality (8.6), and later Xiong [77] gave a simplified proof. Xiong [77] proved (8.6) for polytopes in two and three dimensions. Here, we establish (8.6) under a condition.

Theorem 8.7. *Suppose K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior. If K has an affine image whose cone-volume measure is isotropic, then*

$$U(K) \ge \frac{(n!)^{1/n}}{n} V(K), \tag{8.7}$$

with equality if and only if K is a parallelotope.

Theorem 8.7 follows immediately from Theorem 7.2, together with (8.2), (8.1), (8.5), and the fact that the central symmetral of V_K is an affine image of a cross-measure if and only if K is a parallelotope. (Note that if $u_1, \dots, u_n$ are linearly independent unit vectors, then a convex body in $\mathbb{R}^n$ whose outer unit normals is a subset of $\{\pm u_1, \dots, \pm u_n\}$ must be a parallelotope.)

In light of Lemma 8.3, Theorems 6.4 and 7.2 give an affirmative answer to Problem 8.6 for origin-symmetric convex bodies.

Theorem 8.8. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$U(K) \ge \frac{(n!)^{1/n}}{n} V(K), \tag{8.8}$$

with equality if and only if K is a parallelotope.

In view of the above applications, the following problem is of significant interest.

Problem 8.9. *If K is a convex body in $\mathbb{R}^n$ whose centroid is at the origin, does the cone-volume measure of K satisfy the subspace concentration condition?*

We note that an affirmative answer to Problem 8.9 implies an affirmative answer to Problem 8.6. For two and three dimensional polytopes, an affirmative answer to Problem 8.9 has been given by Xiong [77].

Added in proof: After this paper was submitted for publication, an affirmative answer to Problem 8.9, for polytopes, was given by Henk and Linke [31].

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References

[1] B. Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), 151–161, MR1714339, Zbl 0936.35080.

[2] K.M. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. **44** (1991), 351–359, MR1136445, Zbl 0694.46010.

[3] K.M. Ball, *An elementary introduction to modern convex geometry* Flavors of geometry, (Silvio Levy, ed.), Cambridge University Press, 1997, 1–58, MR1491097, Zbl 0901.52002.

[4] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. **134** (1998), 335–361, MR1650312, Zbl 0901.26010.

[5] F. Barthe, D. Cordero-Erausquin, *Invariances in variance estimates*, Proc. Lond. Math. Soc. **106** (2013), 33–64, MR3020738, Zbl 06139421.

[6] F. Barthe, O. Guedon, S. Mendelson, and A. Naor, *A probabilistic approach to the geometry of the l_p^n-ball*, Ann. Probab. **33** (2005), 480–513, MR2123199, Zbl 1071.60010.

[7] J. Bastero and M. Romance, *Positions of convex bodies associated to extremal problems and isotropic measures*, Adv. Math. **184** (2004), 64–88, MR2047849, Zbl 1053.52011.

[8] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc. (JAMS) **26** (2013), 831–852, MR3037788, Zbl 1272.52012.

[9] K. J. Böröczky, E. Lutwak, D. Yang, G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math. **231** (2012), 1974–1997, MR2964630, Zbl 1258.52005.

[10] S. Campi and P. Gronchi, *The L^p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141, MR1901248, Zbl 1002.52005.

[11] E. Carlen and D. Cordero-Erausquin, *Subadditivity of the entropy and its relation to Brascamp-Lieb type inequalities*, Geom. Funct. Anal. **19** (2009), 373–405, MR2545242, Zbl 1231.26015.

[12] E. A. Carlen, E. H. Lieb, M. Loss, *A sharp analog of Young's inequality on S^N and related entropy inequalities*, J. Geom. Anal. **14** (2004), 487–520, MR2077162, Zbl 1056.43002.

[13] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differential Equations **36** (2009), 419–436, MR2551138, Zbl 1202.26029.

[14] K. S. Chou and X. J. Wang, *The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math. **205** (2006), 33–83, MR2254308, Zbl 1245.52001.

[15] N. Dafnis and G. Paouris, *Small ball probability estimates, Ψ_2-behavior and the hyperplane conjecture*, J. Funct. Anal. **258** (2010). 1933–1964, MR2578460, Zbl 1189.52004.

[16] B. Fleury, O. Guédon, and G. Paouris, *A stability result for mean width of L_p-centroid bodies*, Adv. Math. **214** (2007), 865–877, MR2349721, Zbl 1132.52012.

[17] R. J. Gardner, *Geometric Tomography*, 2nd edition, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, MR2251886, Zbl 1102.52002.

[18] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405, MR1898210, Zbl 1019.26008.

[19] R. J. Gardner, D. Hug, and W. Weil, *Operations between sets in geometry*, J. European Math. Soc. (JEMS) **15** (2013), 2297–2352, MR3120744.

[20] P. M. Gruber *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften, **336**. Springer, Berlin, 2007, MR2335496, Zbl 1139.52001.

[21] O. Guedon, E. Milman, *Interpolating thin-shell and sharp large-deviation estimates for isotropic log-concave measures*, Geom. Funct. Anal. **21** (2011), 1043–1068, MR2846382, Zbl 1242.60012.

[22] C. Haberl, *L_p intersection bodies*, Adv. Math. **217** (2008) 2599–2624, MR2397461, Zbl 1140.52003.

[23] C. Haberl, *Star body valued valuations*, Indiana Univ. Math. J. **58** (2009), 2253–2276, MR2583498, Zbl 1183.52003.

[24] C. Haberl, *Minkowski valuations intertwining with the special linear group*, J. Eur. Math. Soc. (JEMS) **14** (2012), 1565–1597, MR2966660, Zbl 1270.52018.

[25] C. Haberl and M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not. **2006**, Article ID 10548, 29 pp., MR2250020, Zbl 1115.52006.

[26] C. Haberl and L. Parapatits, *Valuations and surface area measures*, J. Reine Angew. Math. DOI: 10.1515/crelle-2012-0044.

[27] C. Haberl and L. Parapatits, *The centro-affine Hadwiger theorem* J. Amer. Math. Soc. (JAMS), (in press).

[28] C. Haberl and F. E. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26, MR2545028, Zbl 1185.52005.

[29] C. Haberl and F. E. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641–658, MR2530600, Zbl 1180.46023.

[30] B. He, G. Leng, and K. Li, *Projection problems for symmetric polytopes*, Adv. Math. **207** (2006), 73–90, MR2264066, Zbl 1111.52012.

[31] M. Henk and E. Linke, *Cone-volume measures of polytopes*, Adv. Math. (in press).

[32] C. Hu, X.-N. Ma, and C. Shen, *On the Christoffel-Minkowski problem of Firey's p-sum*, Calc. Var. Partial Differential Equations **21** (2004), 137–155, MR2085300, Zbl 1161.35391.

[33] D. Hug, E. Lutwak, D. Yang, and G. Zhang, *On the L_p Minkowski problem for polytopes*, Discrete Comput. Geom. **33** (2005), 699–715, MR2132298, Zbl 1078.52008.

[34] F. John, *Polar correspondence with respect to a convex region*, Duke Math. J. **3** (1937), 355–369, MR1545993, Zbl 0017.03701.

[35] R. Kannan, L. Lovász, and M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom. **13** (1995), 541–559, MR1318794, Zbl 0824.52012.

[36] B. Klartag, *A Berry-Esseen type inequality for convex bodies with an unconditional basis*, Probab. Theory Related Fields **145** (2009), 1–33, MR2520120, Zbl 1171.60322.

[37] B. Klartag, *On nearly radial marginals of high-dimensional probability measures*, J. Eur. Math. Soc. (JEMS) **12** (2010), 723–754, MR2639317, Zbl 1200.28015.

[38] M. Ludwig, *Projection bodies and valuations*, Adv. Math. **172** (2002), 158–168, MR1942402, Zbl 1019.52003.

[39] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), 239–256, MR1932331, Zbl 1015.52012.

[40] M. Ludwig, *Ellipsoids and matrix-valued valuations*, Duke Math. J. **119** (2003), 159–188, MR1991649, Zbl 1033.52012.

[41] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213, MR2159706, Zbl 1077.52005.

[42] M. Ludwig, *Intersection bodies and valuations*, Amer. J. Math. **128** (2006), 1409–1428, MR2275906, Zbl 1115.52007.

[43] M. Ludwig, *General affine surface areas*, Adv. Math. **224** (2010), 2346–2360, MR2652209, Zbl 1198.52004.

[44] M. Ludwig, *Valuations on Sobolev spaces*, Amer. J. Math. **134** (2012), 827–842, MR2931225, Zbl 1255.52013.

[45] M. Ludwig and M. Reitzner, *A classification of $SL(n)$ invariant valuations*, Ann. of Math. **172** (2010), 1219–1267, MR2680490, Zbl 1223.52007.

[46] M. Ludwig, J. Xiao, and G. Zhang, *Sharp convex Lorentz-Sobolev inequalities*, Math. Ann. **350** (2011), 169–197, MR2785767, Zbl 1220.26020.

[47] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150, MR1231704, Zbl 0788.52007.

[48] E. Lutwak, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294, MR1378681, Zbl 0853.52005.

[49] E. Lutwak and V. Oliker, *On the regularity of solutions to a generalization of the Minkowski problem*, J. Differential Geom. **41** (1995), 227–246, MR1316557, Zbl 0867.52003.

[50] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132, MR1863023, Zbl 1034.52009.

[51] E. Lutwak, D. Yang, and G. Zhang, *A new ellipsoid associated with convex bodies*, Duke Math. J. **104** (2000), 375–390, MR1781476, Zbl 0974.52008.

[52] E. Lutwak, D. Yang, and G. Zhang, *A new affine invariant for polytopes and Schneider's projection problem*, Trans. Amer. Math. Soc. **353** (2001), 1767–1779, MR1813595, Zbl 0971.52011.

[53] E. Lutwak, D. Yang, and G. Zhang, *The Cramer-Rao inequality for star bodies*, Duke Math. J. **112** (2002), 59–81, MR1890647, Zbl 1021.52008.

[54] E. Lutwak, D. Yang, and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17–38, MR1987375, Zbl 1073.46027.

[55] E. Lutwak, D. Yang, and G. Zhang, *Volume inequalities for subspaces of L_p*, J. Differential Geom. **68** (2004), 159–184, MR2152912, Zbl 1119.52006.

[56] E. Lutwak, D. Yang, and G. Zhang, *Moment-entropy inequalities*, Ann. Probab. **32** (2004), 757–774, MR2039942, Zbl 1053.60004.

[57] E. Lutwak, D. Yang, and G. Zhang, *L_p John ellipsoids*, Proc. London Math. Soc. **90** (2005), 497–520, MR2142136, Zbl 1074.52005.

[58] E. Lutwak, D. Yang, and G. Zhang, *Volume inequalities for isotropic measures*, Amer. J. Math. **129** (2007), 1711–1723, MR2369894, Zbl 1134.52010.

[59] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16, MR1601426, Zbl 0906.52003.

[60] G. Maresch and F. E. Schuster, *The sine transform of isotropic measures*, Int. Math. Res. Not. (IMRN) **2012**, 717–739, MR2889155, Zbl 1244.52010.

[61] M. Meyer and E. Werner, *On the p-affine surface area*, Adv. Math. **152** (2000), 288–313, MR1764106, Zbl 0964.52005.

[62] A. Naor, *The surface measure and cone measure on the sphere of l_p^n*, Trans. Amer. Math. Soc. **359** (2007), 1045–1079, MR2262841, Zbl 1109.60006.

[63] A. Naor and D. Romik, *Projecting the surface measure of the sphere of l_p^n* , Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 241–261, MR1962135, Zbl 1012.60025.

[64] G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. **16** (2006), 1021–1049, MR2276533, Zbl 1114.52004.

[65] G. Paouris, *Small ball probability estimates for log-concave measures*, Trans. Amer. Math. Soc. **364** (2012), 287–308, MR2833584, Zbl 1248.60027.

[66] G. Paouris and E. Werner, *Relative entropy of cone measures and L_p centroid bodies*, Proc. London Math. Soc, **104** (2012), 253–286, MR2880241, Zbl 1246.52008.

[67] D. Ryabogin and A. Zvavitch, *The Fourier transform and Firey projections of convex bodies*, Indiana Univ. Math. J. **53** (2004), 667–682, MR2086696, Zbl 1062.52004.

[68] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, 2nd Edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2013.

[69] F. E. Schuster and M. Weberndorfer, *Volume inequalities for asymmetric Wulff shapes* J. Differential Geom. **92** (2012), 263–283, MR2998673, Zbl 1264.53010.

[70] C. Schütt and E. Werner, *Surface bodies and p-affine surface area*, Adv. Math. **187** (2004), 98–145, MR2074173, Zbl 1089.52002.

[71] A. Stancu, *The discrete planar L_0-Minkowski problem*, Adv. Math. **167** (2002), 160–174, MR1901250, Zbl 1005.52002.

[72] A. Stancu, *On the number of solutions to the discrete two-dimensional L_0-Minkowski problem*, Adv. Math. **180** (2003), 290–323, MR2019226 , Zbl 1054.52001.

[73] A. Stancu, *Centro-affine invariants for smooth convex bodies*, Int. Math. Res. Not. (IMRN) **2012**, 2289–2320, MR2923167, Zbl 1250.52009.

[74] A. C. Thompson, *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996, MR1406315, Zbl 0868.52001.

[75] E. Werner, *On L_p-affine surface areas*, Indiana Univ. Math. J. **56** (2007), 2305–2323, MR2360611, Zbl 1132.52008.

[76] E. Werner and D.-P. Ye, *New L_p affine isoperimetric inequalities*, Adv. Math. **218** (2008), 762–780, MR2414321, Zbl 1155.52002.

[77] G. Xiong, *Extremum problems for the cone volume functional of convex polytopes*, Adv. Math. **225** (2010), 3214–3228, MR2729006, Zbl 1213.52011.

[78] V. Yaskin and M. Yaskina, *Centroid bodies and comparison of volumes* Indiana Univ. Math. J. **55** (2006), 1175–1194, MR2244603, Zbl 1102.52005.

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