Optimal pebbling of grids

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1 Introduction

Graph pebbling has its origin in number theory. It is a model for the transportation of resources. Starting with a pebble distribution on the vertices of a simple connected graph, a *pebbling move* removes two pebbles from a vertex and adds one pebble at an adjacent vertex. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. A vertex is called *reachable* if a pebble can be moved to that vertex using pebbling moves. There are several questions we can ask about pebbling. One of them is: How can we place the smallest number of pebbles such that every vertex is reachable (*optimal pebbling number*)? For a comprehensive list of references for the extensive literature see the survey papers [4, 5, 6]. Results on special grids can be found in [2] where the authors show that $\pi_{\text{opt}}(P_n \square P_2) = \pi_{\text{opt}}(C_n \square P_2) = n$ apart from a few smaller case, and in [11] the author gave upper bounds for the optimal pebbling number of various grids.

In the present paper we give better upper and lower bounds for the optimal pebbling numbers of large grids ($P_n \square P_n$).

Graph rubbling is an extension of graph pebbling. In this version, we also allow a move that removes a pebble each from the vertices v and w that are adjacent to a vertex u, and adds a pebble at vertex u. The basic theory of rubbling and optimal rubbling is developed in [1]. The rubbling number of complete m-ary trees are studied in [3], while the rubbling number of caterpillars are determined in [10]. In [7] the authors gives upper and lower bounds for the rubbling number of diameter 2 graphs.

In the present paper we determine the optimal rubbling number of ladders ($P_n \square P_2$), prisms ($C_n \square P_2$) and Möblus-ladders. We also give upper and lower bounds for the optimal rubbling numbers of large grids ($P_n \square P_n$).

2 Definitions

Throughout the paper, let G be a simple connected graph. We use the notation $V(G)$ for the vertex set and $E(G)$ for the edge set. A *pebble function* on a graph G is a function $p : V(G) \to \mathbb{Z}$ where $p(v)$ is the number of pebbles placed at v. A *pebble distribution* is a nonnegative pebble function. The *size* of a pebble distribution p is the total number of pebbles $\sum_{v \in V(G)} p(v)$. We say that a vertex v is *occupied* if $p(v) > 1$, else it is *unoccupied*.

Consider a pebble function p on the graph G. If $\{v, u\} \in E(G)$ then the *pebbling move* $(v, v \to u)$ removes two pebbles at vertex v, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 2$ and $p'(u) = p(u) + 1$. If $\{w, u\} \in E(G)$ and $v \neq w$, then the *strict rubbling move* $(v, w \to u)$ removes one pebble each at vertices v and w, and adds one pebble at vertex u to create a new pebble function p', so $p'(v) = p(v) - 1$, $p'(w) = p(w) - 1$ and $p'(u) = p(u) + 1$.

A *rubbling move is* either a pebbling move or a strict rubbling move. A *rubbling sequence* is a finite sequence $T = (t_1, \ldots, t_k)$ of rubbling moves. The pebble function obtained from the pebble function p after applying the moves in T is denoted by p_T. The concatenation of the rubbling sequences $R = (r_1, \ldots, r_k)$ and $S = (s_1, \ldots, s_l)$ is denoted by $RS = (r_1, \ldots, r_k, s_1, \ldots, s_l)$.

A rubbling sequence T is *executable* from the pebble distribution p if $p_{(t_1,\ldots,t_i)}$ is nonnegative for all i. A vertex v of G is *reachable* from the pebble distribution p if there is an executable rubbling sequence T such that $p_T(v) \geq 1$. p is a solvable distribution when each vertex is reachable. All the above notions are defined for pebbling as well, just we restrict ourselves to pebbling moves.

The *optimal pebbling* $\pi_{\text{opt}}(G)$ *and rubbling number* $\varrho_{\text{opt}}(G)$ of a graph G is the size of a distribution with the least number of pebbles from which every vertex is reachable using pebbling/rubbling moves. For large graphs it is better to consider the ratio of the optimal pebbling or rubbling number and the number of the vertices of the graph. So the *Optimal Pebbling Density* is $\text{OPD}(G) = \pi_{\text{opt}}(G)/|V(G)|$ and the *Optimal Rubbling Density* is $\text{ORD}(G) = \varrho_{\text{opt}}(G)/|V(G)|$.

Let G and H be simple graphs. Then the *Cartesian product* of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $g = g'$ and $(h, h') \in E(H)$ or if $h = h'$ and $(g, g') \in E(G)$. This graph is denoted by $G \square H$.

P_n and C_n denotes the path and the cycle containing n distinct vertices, respectively. We call $P_n \square P_2$ a *ladder* and $C_n \square P_2$ a *prism*, and $P_{n_1} \square P_{n_2}$ in general a *grid*. It is clear that the prism can be obtained from the ladder by joining the 4 endvertices by two edges to form two vertex disjoint C_n subgraphs. If the four endvertices are joined by two new edges in a switched way to get a C_{2n} subgraph, then a *Möbius-ladder* is obtained.

3 Optimal rubbling number of the ladder, the n-prism and Möbius-ladder

Our main result is the following formula for the optimal rubbling number of ladders:

Theorem 1. *Let $n = 3k + r$ such that $0 \leq r < 3$ and $n, r \in \mathbb{N}$, so $k = \left\lfloor \frac{n}{3} \right\rfloor$.*

$$\varrho_{opt}(P_n \square P_2) = \begin{cases} 1 + 2k & if\ r = 0, \\ 2 + 2k & if\ r = 1, \\ 2 + 2k & if\ r = 2. \end{cases}$$

So $\text{ORD}(P_n \square P_2) \approx \frac{1}{3}$.

To show that the above values are upper bounds for $\varrho_{opt}(P_n \square P_2)$ it is enough to give a solvable distribution. It is not too hard to show that these are really solvable. Such distributions are shown on Figure 1.

We also prove that this is best posibble.

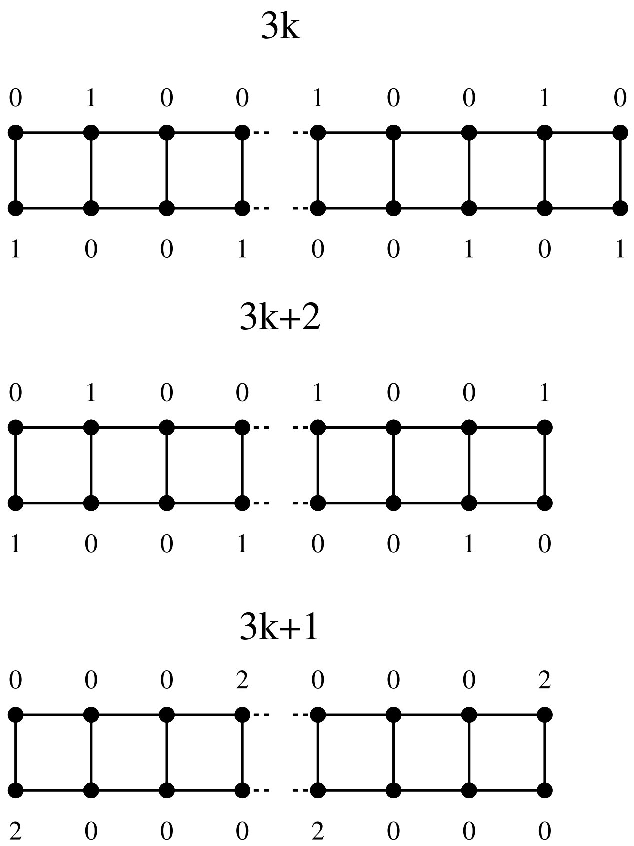

Figure 1: Optimal distributions.

4 Optimal pebbling and rubbling numbers of large grids

We turn our attention to larger grids now, in the following we assume that n is large enough (say ≥ 100). Shiue [11] proved that the analogue of Graham's conjecture for optimal pebbling is true: $\pi_{opt}(G_1 \square G_2) \leq \pi_{opt}(G_1)\pi_{opt}(G_2)$. Since in [9] it was proved that $\pi_{opt}(P_n) = \lceil 2n/3 \rceil$, this implies that $OPD(P_n \square P_n) \leq \frac{4}{9} + o(1)$. In [12] the authors gave a construction showing that $OPD(P_n \square P_n) \leq \frac{4}{13} + o(1)$. Our first result is better construction.

Theorem 2.
$$\pi_{opt}(P_n \square P_n) \leq \frac{2}{7}n^2 + O(n),$$
so $OPD(P_n \square P_n) \leq \frac{2}{7} + o(1)$.

We conjecture that this is a sharp bound. Applying the well known weight argument, it is fairly easy to obtain that $OPD(P_n \square P_n) \geq \frac{1}{9}$. The authors in [12] claim $OPD(P_n \square P_n) \geq \frac{1}{6}$. Unfortunately, we belive that their proof contains an error, may be it can be corrected easily, but we do not see how. However, they introduced an interesting notion: excess weight. Using this notion, but following a different approach we proved the following lower bound.

Theorem 3. $OPD(P_n \square P_n) \geq \frac{4}{25}$.

For the optimal rubbling number of large grids we do not know any previous results. We give a construction to prove:

Theorem 4.

$$\varrho_{opt}(P_n \square P_n) \leq \frac{1}{5}n^2 + O(n),$$

so $ORD(P_n \square P_n) \leq \frac{1}{5} + o(1)$.

We conjecture that this is a sharp bound. A simlilar argument to the one we used fo pebbling also gives a nontrivial lower bound for the optimal rubbling number.

Theorem 5. $ORD(P_n \square P_n) \geq \frac{5}{37}$.

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