Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **80**, 1–17; doi: 10.14232/ejqtde.2015.1.80 http://www.math.u-szeged.hu/ejqtde/

Bounded solutions of k-dimensional system of nonlinear difference equations of neutral type

Małgorzata Migda[1], **Ewa Schmeidel**[✉][2] and **Małgorzata Zdanowicz**[2]

[1]Poznan University of Technology, Piotrowo 3A, 60-965 Poznań, Poland
[2]University of Bialystok, K. Ciołkowskiego 1M, 15-245 Białystok, Poland

Received 3 August 2015, appeared 21 November 2015
Communicated by Josef Diblík

Abstract. The k-dimensional system of neutral type nonlinear difference equations with delays in the following form

$$\begin{cases} \Delta\Big(x_i(n) + p_i(n)\,x_i(n-\tau_i)\Big) = a_i(n)\,f_i(x_{i+1}(n-\sigma_i)) + g_i(n), \\ \Delta\Big(x_k(n) + p_k(n)\,x_k(n-\tau_k)\Big) = a_k(n)\,f_k(x_1(n-\sigma_k)) + g_k(n), \end{cases}$$

where $i = 1, \dots, k-1$, is considered. The aim of this paper is to present sufficient conditions for the existence of nonoscillatory bounded solutions of the above system with various $(p_i(n))$, $i = 1, \dots, k$, $k \geq 2$.

Keywords: system of difference equation, k-dimensional, neutral type, nonoscillatory solutions, boundedness, existence.

2010 Mathematics Subject Classification: 39A10, 39A22.

1 Introduction

In this paper we consider a nonlinear difference system of k ($k \geq 2$) equations of the form

$$\begin{cases} \Delta\Big(x_i(n) + p_i(n)\,x_i(n-\tau_i)\Big) = a_i(n)\,f_i(x_{i+1}(n-\sigma_i)) + g_i(n), \\ \Delta\Big(x_k(n) + p_k(n)\,x_k(n-\tau_k)\Big) = a_k(n)\,f_k(x_1(n-\sigma_k)) + g_k(n), \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}_0$, $i = 1, \dots, k-1$, Δ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$. Here $\mathbb{R}$ is a set of real numbers, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\sigma_i, \tau_i \in \mathbb{N}$ for $i = 1, \dots, k$. By n_0 we denote $\max\{\tau_1, \dots, \tau_k, \sigma_1, \dots, \sigma_k\}$, and $\mathbb{N}_0 = \{n_0, n_0+1, \dots\}$. Moreover $a_i = (a_i(n)), g_i = (g_i(n)), p_i = (p_i(n))$ for $i = 1, \dots, k$ are given sequences of real numbers, $x_i = (x_i(n))$ for $i = 1, \dots, k$ are unknown real sequences and functions $f_i \colon \mathbb{R} \to \mathbb{R}$. Throughout this paper X denotes an unknown vector $(x_1, \dots, x_k)$ and $X(n)$ denotes $(x_1(n), \dots, x_k(n)) \in \mathbb{R}^k$. For the elements of $\mathbb{R}^k$ the symbol $|\cdot|$ stands for the maximum norm.

✉Corresponding author. Email: eschmeidel@math.uwb.edu.pl

By $\mathcal{B}$ we denote the Banach space of all bounded sequences in $\mathbb{R}^k$ with the supremum norm, i.e.

$$\mathcal{B} = \left\{ X \colon \mathbb{N} \to \mathbb{R}^k : \|X\| = \sup_{n \in \mathbb{N}} |X(n)| < \infty \right\},$$

and by B the following subset of $\mathcal{B}$

$$B = \{X = (x_1, \dots, x_k) \in \mathcal{B} : x_i \text{ is nonnegative or nonpositive for } i = 1, \dots, k\}.$$

A sequence of real numbers is said to be nonoscillatory if it is either eventually positive or eventually negative. By a solution of system (1.1) we mean a vector X such that its components, i.e. $x_1, \dots, x_k$, satisfy the system (1.1) for sufficiently large n. The solution X of system (1.1) is called nonoscillatory if all its components are nonoscillatory. The solution X of system (1.1) is called bounded if all its components are bounded.

Any higher-order nonlinear neutral difference equation could be rewritten as k-dimensional system of difference equations with one equation of neutral type but not vice-versa. Higher-order nonlinear neutral difference equations have been studied by many authors, see for example [2–4, 8–10, 13–23], and the references cited therein. The theorems presented here generalize and improve the results obtained for three dimensional system in [13].

The following definition and theorems will be used in the sequel.

Definition 1.1 (Uniformly Cauchy subset, [6])**.** A set Ω of sequences in l^∞ is uniformly Cauchy if for every $\varepsilon > 0$, there exists an integer n such that $|X(i) - X(j)| < \varepsilon$ whenever $i, j > n$ for any $X \in \Omega$.

Lemma 1.2 (Arzelà–Ascoli theorem, [1])**.** *A bounded and uniformly Cauchy subset of l^∞ is relatively compact.*

Theorem 1.3 (Krasnoselskii's fixed point theorem, [7])**.** *Let Ω be a bounded closed convex subset of a Banach space and let F, T be maps such that $Fx + Ty \in \Omega$ for every pair $x, y \in \Omega$. If F is a contraction and T is completely continuous, then the equation $Fx + Tx = x$ has a solution in Ω.*

Theorem 1.4 (Schauder's fixed point theorem, [5])**.** *Let Ω be a nonempty, compact and convex subset of a Banach space and let $T \colon \Omega \to \Omega$ be continuous. Then T has a fixed point in M.*

2 Main results

In this section, using the Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, we establish sufficient conditions for the existence of nonoscillatory bounded solutions of system (1.1).

Theorem 2.1. *Assume that for $i = 1, \dots, k$*

$$\sum_{n=1}^{\infty} |a_i(n)| < \infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} |g_i(n)| < \infty, \tag{2.2}$$

$$f_i \colon \mathbb{R} \to \mathbb{R} \text{ is a continuous function} \tag{2.3}$$

and for any closed subset $J \subset \mathbb{R}$

$$\max_{i=1,\dots,k} \sup_{t \in J} \{|f_i(t)|\} > 0. \tag{2.4}$$

Assume also that for each $i = 1, \dots, k$ the terms of sequence p_i are of the same sign for $n \in \mathbb{N}_0$. If for each $i = 1, \dots, k$ there exists a positive real constant c_{p_i} such that

$$0 \le p_i(n) \le c_{p_i} < 1, \qquad n \in \mathbb{N}_0, \tag{2.5}$$

or

$$-1 < -c_{p_i} \le p_i(n) \le 0, \qquad n \in \mathbb{N}_0, \tag{2.6}$$

then system (1.1) has a bounded nonoscillatory solution.

Proof. For the fixed positive real number r we define a set

$$\Omega_1 = \left\{ X \in B : \frac{1}{8}(1 - c_{p_i})r \le |x_i(n)| \le r,\ i = 1, \dots, k,\ n \in \mathbb{N} \right\}.$$

Clearly Ω_1 is a bounded closed convex subset of the Banach space B. Since condition (2.3) is satisfied, we can take

$$M_f = \max_{i=1,\dots,k} \left\{ |f_i(t)| : |t| \in \left[\frac{1}{8}(1 - c_{p_i})r, r \right] \right\}.$$

From (2.1) and (2.2), there exists such $n_1 \in \mathbb{N}_0$ that

$$\sum_{n=n_1}^{\infty} |a_i(n)| \le \frac{(1 - c_{p_i})r}{8M_f}, \qquad \sum_{n=n_1}^{\infty} |g_i(n)| \le \frac{(1 - c_{p_i})r}{4}.$$

Let I_1, I_2, I_3, I_4 be subsets of the set $\{1, \dots, k\}$ and moreover, $I_i \cap I_j = \emptyset$ for $i \ne j$, $i, j = 1, 2, 3, 4$ and $I_1 \cup I_2 \cup I_3 \cup I_4 = \{1, \dots, k\}$.
We consider four cases

(i)

$$\begin{cases} 0 \le p_i(n) \le c_{p_i} < 1, \\ x_i(n) > 0, \quad \text{for } i \in I_1,\ n \ge n_1, \end{cases}$$

(ii)

$$\begin{cases} -1 < -c_{p_i} \le p_i(n) \le 0, \\ x_i(n) < 0, \quad \text{for } i \in I_2,\ n \ge n_1, \end{cases}$$

(iii)

$$\begin{cases} 0 \le p_i(n) \le c_{p_i} < 1, \\ x_i(n) < 0, \quad \text{for } i \in I_3,\ n \ge n_1, \end{cases}$$

(iv)

$$\begin{cases} -1 < -c_{p_i} \le p_i(n) \le 0, \\ x_i(n) > 0, \quad \text{for } i \in I_4,\ n \ge n_1. \end{cases}$$

Next, we define the maps $F, T\colon \Omega_1 \to B$ where

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}, \qquad T = \begin{bmatrix} T_1 \\ \vdots \\ T_k \end{bmatrix},$$

$$(F_iX)(n) = \begin{cases} (F_iX)(n_1) & \text{for } i = 1, \dots, k,\ 0 \le n < n_1, \\ -p_i(n)x_i(n-\tau_i) + \frac{(1+c_{p_i})r}{2} & \text{for } i \in I_1 \cup I_2,\ n \ge n_1, \\ -p_i(n)x_i(n-\tau_i) + \frac{(1-c_{p_i})r}{2} & \text{for } i \in I_3 \cup I_4,\ n \ge n_1, \end{cases} \tag{2.7}$$

and for $i = 1, \dots, k-1$

$$(T_iX)(n) = \begin{cases} (T_iX)(n_1) & \text{for } 0 \le n < n_1, \\ -\sum\limits_{s=n}^{\infty} a_i(s)\ f_i(x_{i+1}(s-\sigma_i)) - \sum\limits_{s=n}^{\infty} g_i(s) & \text{for } n \ge n_1, \end{cases} \tag{2.8}$$

and

$$(T_kX)(n) = \begin{cases} (T_kX)(n_1) & \text{for } 0 \le n < n_1, \\ -\sum\limits_{s=n}^{\infty} a_k(s)\ f_k(x_1(s-\sigma_k)) - \sum\limits_{s=n}^{\infty} g_k(s) & \text{for } n \ge n_1. \end{cases} \tag{2.9}$$

We will show that F and T satisfy the assumptions of Theorem 1.3. First we prove that if $X, \bar{X} \in \Omega_1$, then $FX + T\bar{X} \in \Omega_1$.

For $n \ge n_1$, $i \in I_1 \cup I_2$ and $i \ne k$ we have

$$\begin{aligned} (F_iX)(n) + (T_i\bar{X})(n) =& -p_i(n)x_i(n-\tau_i) + \frac{(1+c_{p_i})r}{2} \\ & - \sum_{s=n}^{\infty} a_i(s)\ f_i(\bar{x}_{i+1}(s-\sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\ \le& \frac{(1+c_{p_i})r}{2} + \sum_{s=n}^{\infty} |a_i(s)|\ |f_i(\bar{x}_{i+1}(s-\sigma_i))| + \sum_{s=n}^{\infty} |g_i(s)| \\ \le& \frac{1}{2}r + \frac{1}{2}c_{p_i}r + M_f \cdot \frac{(1-c_{p_i})r}{8M_f} + \frac{(1-c_{p_i})r}{4} \\ =& \frac{7}{8}r + \frac{1}{8}c_{p_i}r \le r. \end{aligned}$$

Moreover,

$$\begin{aligned} (F_iX)(n) + (T_i\bar{X})(n) =& -p_i(n)x_i(n-\tau_i) + \frac{(1+c_{p_i})r}{2} \\ & - \sum_{s=n}^{\infty} a_i(s)\ f_i(\bar{x}_{i+1}(s-\sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\ \ge& -|p_i(n)||x_i(n-\tau_i)| + \frac{(1+c_{p_i})r}{2} \\ & - \sum_{s=n}^{\infty} |a_i(s)|\ |f_i(\bar{x}_{i+1}(s-\sigma_i))| - \sum_{s=n}^{\infty} |g_i(s)| \\ \ge& -c_{p_i}r + \frac{1}{2}r + \frac{1}{2}c_{p_i}r - M_f \cdot \frac{(1-c_{p_i})r}{8M_f} - \frac{(1-c_{p_i})r}{4} \\ =& \frac{1}{8}(1-c_{p_i})r. \end{aligned}$$

For $n \geq n_1$ and $i \in I_3 \cup I_4$, and $i \neq k$ we have

$$\begin{aligned}(F_iX)(n) + (T_i\bar{X})(n) =& -p_i(n)x_i(n-\tau_i) + \frac{(1-c_{p_i})r}{2} \\ & - \sum_{s=n}^{\infty} a_i(s)\, f_i(\bar{x}_{i+1}(s-\sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\ \leq& |p_i(n)||x_i(n-\tau_i)| + \frac{(1-c_{p_i})r}{2} \\ & + \sum_{s=n}^{\infty} |a_i(s)|\, |f_i(\bar{x}_{i+1}(s-\sigma_i))| + \sum_{s=n}^{\infty} |g_i(s)| \\ \leq& c_{p_i}r + \frac{1}{2}r - \frac{1}{2}c_{p_i}r + M_f \cdot \frac{(1-c_{p_i})r}{8M_f} + \frac{(1-c_{p_i})r}{4} \\ =& \frac{1}{8}c_{p_i}r + \frac{7}{8}r \leq r.\end{aligned}$$

On the other hand,

$$\begin{aligned}(F_iX)(n) + (T_i\bar{X})(n) =& -p_i(n)x_i(n-\tau_i) + \frac{(1-c_{p_i})r}{2} \\ & - \sum_{s=n}^{\infty} a_i(s)\, f_i(\bar{x}_{i+1}(s-\sigma_i)) - \sum_{s=n}^{\infty} g_i(s) \\ \geq& \frac{(1-c_{p_i})r}{2} - \sum_{s=n}^{\infty} |a_i(s)|\, |f_i(\bar{x}_{i+1}(s-\sigma_i))| - \sum_{s=n}^{\infty} |g_i(s)| \\ \geq& \frac{1}{2}r - \frac{1}{2}c_{p_i}r - M_f \cdot \frac{(1-c_{p_i})r}{8M_f} - \frac{(1-c_{p_i})r}{4} \\ =& \frac{1}{8}(1-c_{p_i})r.\end{aligned}$$

For $i = k$ there is a different definition of the mapping T_k, but all estimations are analogous, and hence omitted.

The task is now to prove that F is a contraction mapping. It is easy to see that

$$\begin{aligned}|(F_iX)(n) - (F_i\bar{X})(n)| &\leq |p_i(n)|\, |x_i(n-\tau_i) - \bar{x}_i(n-\tau_i)| \\ &\leq c_{p_i}\, |x_i(n-\tau_i) - \bar{x}_i(n-\tau_i)|,\end{aligned}$$

for any $X, \bar{X} \in \Omega_1$, $i = 1, \ldots, k$ and $n \geq n_1$. Hence

$$\|FX - F\bar{X}\| \leq \max_{i=1,\ldots,k} \{c_{p_i}\} \cdot \|X - \bar{X}\|,$$

where, by (2.5) and (2.6), there is $0 < \max_{i=1,\ldots,k} \{c_{p_i}\} < 1$.

The next step is to show continuity of T. Let $X_j = (x_{1j}, \ldots, x_{kj}) \in \Omega_1$ for $j \in \mathbb{N}$ and for $i = 1, \ldots, k$ there is $x_{ij}(n) \to x_i(n)$ as $j \to \infty$. Since Ω_1 is closed, we have $X = (x_1, \ldots, x_k) \in \Omega_1$. By (2.1), (2.3), (2.8) and Lebesgue's dominated convergence theorem we obtain for $i = 1, \ldots, k-1$

$$\left|(T_iX_j)(n) - (T_iX)(n)\right| \leq \sum_{s=n}^{\infty} |a_i(s)|\, |f_i(x_{i+1\,j}(s-\sigma_i)) - f_i(x_{i+1}(s-\sigma_i))| \to 0 \quad \text{if } j \to \infty,$$

where $n \in \mathbb{N}$. Analogously we conclude for $i = k$. Therefore

$$\|(TX_j) - (TX)\| \to 0 \quad \text{if } j \to \infty,$$

and we see that T is a continuous mapping.

In order to prove that T is completely continuous we can use Lemma 1.2. Hence we have to show that $T\Omega_1$ is uniformly Cauchy (see Definition 1.1). We show transformations for any T_i, $i = 1, \dots, k-1$. Similar arguments apply to T_k.

Let $X \in \Omega_1$. We conclude from the assumptions (2.1), (2.2) and (2.3) that for any given $\varepsilon > 0$ there exists an integer $n_2 > n_1$ such that for $n \geq n_2$ we have

$$\sum_{s=n}^{\infty} |a_i(s)|\,|f_i(x_{i+1}(s-\sigma_i))| + \sum_{s=n}^{\infty} |g_i(s)| < \frac{\varepsilon}{2}.$$

Hence, for $n_4 > n_3 \geq n_2$, we obtain

$$\begin{aligned} |(T_iX)(n_4) - (T_iX)(n_3)| = \Bigg| &\sum_{s=n_4}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) + \sum_{s=n_4}^{\infty} g_i(s) \\ &- \sum_{s=n_3}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) - \sum_{s=n_3}^{\infty} g_i(s) \Bigg| < \varepsilon. \end{aligned}$$

Therefore $T\Omega_1$ is uniformly Cauchy.

By Theorem 1.3, there exists X such that $(FX)(n) + (TX)(n) = X(n)$.

Finally, we verify that X satisfies system (1.1) for $n \geq n_1$. As $(F_iX)(n) + (T_iX)(n) = x_i(n)$, $i = 1, \dots, k$, we have for $i \in I_1 \cup I_2$ and $i \neq k$

$$\begin{aligned} -\,p_i(n)x_i(n-\tau_i) + \frac{(1+c_{p_i})r}{2} - \sum_{s=n}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) - \sum_{s=n}^{\infty} g_i(s) = x_i(n), \\ \Delta\left(x_i(n) + p_i(n)\,x_i(n-\tau_i)\right) = -\Delta \sum_{s=n}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) - \Delta \sum_{s=n}^{\infty} g_i(s), \end{aligned}$$

$$\Delta\left(x_i(n) + p_i(n)\,x_i(n-\tau_i)\right) = a_i(n) f_i(x_{i+1}(n-\sigma_i)) + g_i(n). \tag{2.10}$$

Similarly, we get (2.10) for $i \in I_3 \cup I_4$ and $i \neq k$. In all cases, for $i = k$, the reasoning is also the same as above. The proof is complete. □

Note that for $p_i(n) \equiv 0$, $i = 1, \dots, k$, system (1.1) is not of the neutral type, but Theorem 2.1 is still true.

Example 2.2. Consider a difference system

$$\begin{cases} \Delta\left(x_1(n) + \frac{1}{2n}\,x_1(n-1)\right) = \frac{5n^4-21n^3+22n^2+4n-8}{4n^6-16n^5+10n^4+16n^3-14n^2}\; x_2(n-2) + \frac{1}{n^2}, \\ \Delta\left(x_2(n) - \frac{1}{2n}\,x_2(n-2)\right) = \frac{2n^5-17n^4+43n^3-48n^2+27n-7}{2n^8-4n^7-6n^6+8n^5+8n^4}\; x_3^3(n-1) - \frac{1}{n^2}, \\ \Delta\left(x_3(n) + \frac{1}{2n}\,x_3(n-1)\right) = \frac{3n^3-3n^2+2}{4n^5-2n^4-6n^3}\; x_4(n-1) + \frac{1}{n^3}, \\ \Delta\left(x_4(n) - \frac{1}{2n}\,x_4(n-1)\right) = \frac{2n^2-5n+3}{n^4+n^3}\; x_1^2(n-1). \end{cases}$$

All assumptions of Theorem 2.1 are satisfied. The system above has the bounded (but not unique) solution $X = \left(\left(1+\frac{1}{n}\right), \left(-2+\frac{1}{n^2}\right), \left(-1-\frac{1}{n}\right), \left(2-\frac{1}{n}\right)\right)$ for $n \geq 3$.

Theorem 2.3. *Assume that conditions (2.1), (2.2), (2.3) and (2.4) are satisfied. If there exist positive real numbers $\tilde{c}_{p_i}$, $i = 1, \dots, k$ that*

$$1 < \tilde{c}_{p_i} \le p_i(n), \qquad n \in \mathbb{N}_0, \tag{2.11}$$

or

$$p_i(n) \le -\tilde{c}_{p_i} < -1, \qquad n \in \mathbb{N}_0, \tag{2.12}$$

then system (1.1) has a bounded nonoscillatory solution.

Proof. We define a subset Ω_2 of B in the following way

$$\Omega_2 = \left\{ X \in B : \frac{1}{8}(\tilde{c}_{p_i} - 1)r \le |x_i(n)| \le \tilde{c}_{p_i} r, \ i = 1, \dots, k, \ n \in \mathbb{N} \right\}.$$

where r is a fixed positive real number. Obviously Ω_2 is a bounded, closed and convex subset of B. Let us set

$$\tilde{M}_f = \max_{i=1,\dots,k} \left\{ |f_i(t)| : |t| \in \left[\frac{1}{8}(\tilde{c}_{p_i} - 1)r, \tilde{c}_{p_i} r \right] \right\}.$$

From assumptions (2.1) and (2.2), we conclude that there exists $n_5 \in \mathbb{N}_0$ that

$$\sum_{n=n_5}^{\infty} |a_i(n)| \le \frac{(\tilde{c}_{p_i} - 1)r}{8\tilde{M}_f}, \qquad \sum_{n=n_5}^{\infty} |g_i(n)| \le \frac{(\tilde{c}_{p_i} - 1)r}{4}.$$

Let $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4$ be such subsets of the set $\{1, \dots, k\}$ that $\tilde{I}_i \cap \tilde{I}_j = \emptyset$ for $i \ne j$, $i, j = 1, 2, 3, 4$ and $\tilde{I}_1 \cup \tilde{I}_2 \cup \tilde{I}_3 \cup \tilde{I}_4 = \{1, \dots, k\}$.

Since we seek for the nonoscillatory solution, we consider the following cases

(i)
$$\begin{cases} 1 < \tilde{c}_{p_i} \le p_i(n), \\ x_i(n) > 0, \quad \text{for } i \in \tilde{I}_1, \ n \ge n_5, \end{cases}$$

(ii)
$$\begin{cases} p_i(n) \le -\tilde{c}_{p_i} < -1, \\ x_i(n) < 0, \quad \text{for } i \in \tilde{I}_2, \ n \ge n_5, \end{cases}$$

(iii)
$$\begin{cases} 1 < \tilde{c}_{p_i} \le p_i(n), \\ x_i(n) < 0, \quad \text{for } i \in \tilde{I}_3, \ n \ge n_5, \end{cases}$$

(iv)
$$\begin{cases} p_i(n) \le -\tilde{c}_{p_i} < -1, \\ x_i(n) > 0, \quad \text{for } i \in \tilde{I}_4, \ n \ge n_5. \end{cases}$$

We define the maps $F, T \colon \Omega_2 \to B$ in the following way

$$(F_i X)(n) = \begin{cases} (F_i X)(n_5) & \text{for} \quad i = 1, \dots, k, \quad 0 \le n < n_5, \\ -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} & \text{for } i \in \tilde{I}_1 \cup \tilde{I}_2, \ n \ge n_5, \\ -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(\tilde{c}_{p_i}-1)r}{2} & \text{for } i \in \tilde{I}_3 \cup \tilde{I}_4, \ n \ge n_5, \end{cases} \tag{2.13}$$

and for $i = 1, \dots, k-1$

$$(T_iX)(n) = \begin{cases} (T_iX)(n_5) & \text{for } 0 \le n < n_5, \\ -\frac{1}{p_i(n+\tau_i)} \sum\limits_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) - \frac{1}{p_i(n+\tau_i)} \sum\limits_{s=n+\tau_i}^{\infty} g_i(s) & \text{for } n \ge n_5, \end{cases} \tag{2.14}$$

and

$$(T_kX)(n) = \begin{cases} (T_kX)(n_5) & \text{for } 0 \le n < n_5, \\ -\frac{1}{p_k(n+\tau_k)} \sum\limits_{s=n+\tau_k}^{\infty} a_k(s) f_k(x_1(s-\sigma_k)) - \frac{1}{p_k(n+\tau_k)} \sum\limits_{s=n+\tau_i}^{\infty} g_k(s) & \text{for } n \ge n_5. \end{cases} \tag{2.15}$$

Let $X, \bar{X} \in \Omega_2$, $n \ge n_5$. Then also $FX + T\bar{X} \in \Omega_2$. We will present all transformations for the i-th components of F and T, where $i = 1, \dots, k-1$. We have for $i \in \tilde{I}_1 \cup \tilde{I}_2$

$$\begin{aligned} (F_iX)(n) + (T_i\bar{X})(n) = & -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} \\ & - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) \\ & - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\ \le & \frac{(1+\tilde{c}_{p_i})r}{2} + \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)|\, |f_i(x_{i+1}(s-\sigma_i))| \\ & + \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\ \le & \frac{1}{2}\tilde{c}_{p_i} r + \frac{1}{2} r + \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f} + \frac{(\tilde{c}_{p_i}-1)r}{4} \\ = & \frac{7}{8}\tilde{c}_{p_i} r + \frac{1}{8} r \le \tilde{c}_{p_i} r. \end{aligned}$$

On the other hand,

$$\begin{aligned} (F_iX)(n) + (T_i\bar{X})(n) = & -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} \\ & - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) \\ & - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\ \ge & -\frac{|x_i(n+\tau_i)|}{|p_i(n+\tau_i)|} + \frac{(1+\tilde{c}_{p_i})r}{2} \\ & - \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)|\, |f_i(x_{i+1}(s-\sigma_i))| \\ & - \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\ \ge & -r + \frac{1}{2}\tilde{c}_{p_i} r + \frac{1}{2} r - \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f} - \frac{(\tilde{c}_{p_i}-1)r}{4} \\ = & \frac{1}{8}(\tilde{c}_{p_i}-1)r. \end{aligned}$$

Next we have for $i \in \tilde{I}_3 \cup \tilde{I}_4$

$$\begin{aligned}
(F_iX)(n) + (T_i\bar{X})(n) = & -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(\tilde{c}_{p_i}-1)r}{2} \\
& -\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s)\ f_i(\bar{x}_{i+1}(s-\sigma_i)) \\
& -\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\
\leq & \frac{|x_i(n+\tau_i)|}{|p_i(n+\tau_i)|} + \frac{(\tilde{c}_{p_i}-1)r}{2} \\
& + \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)|\ |f_i(\bar{x}_{i+1}(s-\sigma_i))| \\
& + \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\
\leq & r + \frac{1}{2}\tilde{c}_{p_i}r - \frac{1}{2}r + \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f} + \frac{(\tilde{c}_{p_i}-1)r}{4} \\
= & \frac{7}{8}\tilde{c}_{p_i}r + \frac{1}{8}r \leq \tilde{c}_{p_i}r.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(F_iX)(n) + (T_i\bar{X})(n) = & -\frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(\tilde{c}_{p_i}-1)r}{2} \\
& -\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s)\ f_i(\bar{x}_{i+1}(s-\sigma_i)) \\
& -\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s) \\
\geq & \frac{(\tilde{c}_{p_i}-1)r}{2} \\
& - \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |a_i(s)|\ |f_i(\bar{x}_{i+1}(s-\sigma_i))| \\
& - \frac{1}{|p_i(n+\tau_i)|} \sum_{s=n+\tau_i}^{\infty} |g_i(s)| \\
\geq & \frac{1}{2}\tilde{c}_{p_i}r - \frac{1}{2}r - \tilde{M}_f \cdot \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f} - \frac{(\tilde{c}_{p_i}-1)r}{4} \\
= & \frac{1}{8}(\tilde{c}_{p_i}-1)r.
\end{aligned}$$

To see that F is a contraction mapping let us observe that for $i = 1, \ldots, k$

$$\begin{aligned}
|(F_iX)(n) - (F_i\bar{X})(n)| &\leq \frac{1}{|p_i(n+\tau_i)|}\,|x_i(n+\tau_i) - \bar{x}_i(n+\tau_i)| \\
&\leq \frac{1}{\tilde{c}_{p_i}}\,|x_i(n+\tau_i) - \bar{x}_i(n+\tau_i)|.
\end{aligned}$$

Hence

$$\|FX - F\bar{X}\| \leq \frac{1}{\min_{i=1,\ldots,k}\{\tilde{c}_{p_i}\}}\|X - \bar{X}\|,$$

but $\frac{1}{\min_{i=1,\dots,k}\{\tilde{c}_{p_i}\}} < 1$ by (2.11) and (2.12).

The proof of the continuity of the mapping T can be performed exactly in the same way as previously.

By virtue of Theorem 1.3, there exists X that $(FX)(n) + (TX)(n) = X(n)$. Finally, we show that X satisfies system (1.1) for $n \geq n_5$. Let $(F_iX)(n) + (T_iX)(n) = x_i(n)$ for $i = 1, \dots, k$. We show all transformations only for $i \in \tilde{I}_1 \cup \tilde{I}_2$ and $i \neq k$, because for the other cases they are analogous. Since

$$\begin{aligned} x_i(n) = &- \frac{x_i(n+\tau_i)}{p_i(n+\tau_i)} + \frac{(1+\tilde{c}_{p_i})r}{2} - \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i)) \\ &- \frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s), \end{aligned}$$

then we have

$$\begin{aligned} \Delta\left(x_i(n) + \frac{x_i(n+\tau_i)}{p_i(n+\tau_i)}\right) = &-\Delta\left(\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i))\right) \\ &-\Delta\left(\frac{1}{p_i(n+\tau_i)} \sum_{s=n+\tau_i}^{\infty} g_i(s)\right). \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{p_i(n+\tau_i+1)}\Delta\Big(x_i(n+\tau_i) + p_i(n+\tau_i)x_i(n)\Big) + \left(\Delta\frac{1}{p_i(n+\tau_i)}\right)\Big(x_i(n+\tau_i) + p_i(n+\tau_i)x_i(n)\Big) \\ &= -\frac{1}{p_i(n+\tau_i+1)}\Delta\left(\sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i))\right) - \frac{1}{p_i(n+\tau_i+1)}\Delta\left(\sum_{s=n+\tau_i}^{\infty} g_i(s)\right) \\ &\quad - \left(\Delta\frac{1}{p_i(n+\tau_i)}\right)\left(\sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i))\right) - \left(\Delta\frac{1}{p_i(n+\tau_i)}\right)\left(\sum_{s=n+\tau_i}^{\infty} g_i(s)\right). \end{aligned}$$

It is easy to notice that

$$-\Delta\left(\sum_{s=n+\tau_i}^{\infty} a_i(s)\, f_i(x_{i+1}(s-\sigma_i))\right) = a_i(n+\tau_i)\, f_i(x_{i+1}(n+\tau_i-\sigma_i)),$$

and

$$-\Delta\left(\sum_{s=n+\tau_i}^{\infty} g_i(s)\right) = g_i(n+\tau_i).$$

Then

$$\Delta\Big(x_i(n+\tau_i) + p_i(n+\tau_i)x_i(n)\Big) = a_i(n+\tau_i)\, f_i(x_{i+1}(n+\tau_i-\sigma_i)) + g_i(n+\tau_i).$$

Now we can transform the last equation into

$$\Delta\Big(x_i(n) + p_i(n)x_i(n-\tau_i)\Big) = a_i(n)\, f_i(x_{i+1}(n-\sigma_i)) + g_i(n).$$

The proof is complete. □

Example 2.4. Now, let us consider a difference system

$$\begin{cases} \Delta\left(x_1(n) + \left(2+\frac{1}{2^n}\right)\, x_1(n-2)\right) = -\frac{13\cdot 8^{n-1}+3\cdot 4^{n-1}}{16^n-4\cdot 8^n+4^{n+1}}\, x_2^2(n-2) + \frac{1}{2^n}, \\ \\ \Delta\left(x_2(n) + \left(-1-\frac{1}{2^n}\right)\, x_2(n-2)\right) = \frac{-3\cdot 4^n-6\cdot 2^n}{2\cdot 8^n+4\cdot 4^n}\, x_3(n-1) - \frac{1}{2^n}, \\ \\ \Delta\left(x_3(n) + \left(1+\frac{1}{2^n}\right)\, x_3(n-1)\right) = \frac{4\cdot 4^n+3\cdot 2^n}{6\cdot 8^n+4\cdot 4^n}\, x_4(n-1), \\ \\ \Delta\left(x_4(n) + \left(-1-\frac{1}{2^n}\right)\, x_4(n-1)\right) = \frac{4\cdot 8^n+3\cdot 4^n}{8\cdot 16^n+32\cdot 8^n+32\cdot 4^n}\, x_1^2(n-2). \end{cases}$$

All assumptions of Theorem 2.3 are satisfied. The sequence

$$X = \left(\left(2+\frac{1}{2^n}\right), \left(-2+\frac{1}{2^n}\right), \left(-1-\frac{1}{2^n}\right), \left(3+\frac{1}{2^n}\right)\right) \quad \text{for } n \geq 2$$

is the bounded solution of the above system.

Now we can formulate the theorem that join both Theorem 2.1 and Theorem 2.3.

Let I_5, I_6, I_7, I_8 be subsets of the set $\{1,\dots,k\}$ such that $I_i \cap I_j = \emptyset$ for $i \neq j$, $i,j = 5,6,7,8$ and $I_5 \cup I_6 \cup I_7 \cup I_8 = \{1,\dots,k\}$.

Theorem 2.5. *Let assumptions* (2.1), (2.2), (2.3) *and* (2.4) *hold. If there exist positive real numbers c_{p_i}, $i \in I_5 \cup I_6$ and $\tilde{c}_{p_i}$, $i \in I_7 \cup I_8$ that satisfy the inequalities*

$$\begin{aligned} 0 \leq p_i(n) \leq c_{p_i} < 1, &\qquad \text{for } i \in I_5,\ n \in \mathbb{N}_0, \\ -1 < -c_{p_i} \leq p_i(n) \leq 0, &\qquad \text{for } i \in I_6,\ n \in \mathbb{N}_0, \\ 1 < \tilde{c}_{p_i} \leq p_i(n), &\qquad \text{for } i \in I_7,\ n \in \mathbb{N}_0, \\ p_i(n) \leq -\tilde{c}_{p_i} < -1, &\qquad \text{for } i \in I_8,\ n \in \mathbb{N}_0, \end{aligned}$$

then system (1.1) *has a bounded nonoscillatory solution.*

Proof. For the fixed positive real number r we define the set

$$\Omega_3 = \Big\{ X \in B : \frac{1}{8}(1-c_{p_i})r \leq |x_i(n)| \leq r,\ i \in I_5 \cup I_6, \\ \frac{1}{8}(\tilde{c}_{p_i}-1)r \leq |x_i(n)| \leq \tilde{c}_{p_i} r,\ i \in I_7 \cup I_8,\ n \in \mathbb{N} \Big\}.$$

Ω_3 is bounded closed convex subset of the Banach space B.

Let $n_6 = \max\{c_1, c_5\}$. From assumptions (2.1) and (2.2) we have

$$\begin{aligned} \sum_{n=n_6}^{\infty} |a_i(n)| &\leq \frac{(1-c_{p_i})r}{8M_f}, &\qquad i \in I_5 \cup I_6, \\ \sum_{n=n_6}^{\infty} |g_i(n)| &\leq \frac{(1-c_{p_i})r}{4}, &\qquad i \in I_5 \cup I_6, \\ \sum_{n=n_6}^{\infty} |a_i(n)| &\leq \frac{(\tilde{c}_{p_i}-1)r}{8\tilde{M}_f}, &\qquad i \in I_7 \cup I_8, \\ \sum_{n=n_6}^{\infty} |g_i(n)| &\leq \frac{(\tilde{c}_{p_i}-1)r}{4}, &\qquad i \in I_7 \cup I_8, \end{aligned}$$

where

$$M_f = \max_{i\in I_5\cup I_6}\left\{|f_i(t)| : |t| \in \left[\frac{1}{8}(1-c_{p_i})r, r\right]\right\},$$
$$\tilde{M}_f = \max_{i\in I_7\cup I_8}\left\{|f_i(t)| : |t| \in \left[\frac{1}{8}(\tilde{c}_{p_i}-1)r, \tilde{c}_{p_i}r\right]\right\}.$$

We can now proceed analogously as in the proof of Theorem 2.1 and Theorem 2.3. Repeating reasoning in these proofs we define for $n \geq n_6$ the maps $F, T\colon \Omega_3 \to B$ by formulas (2.7)–(2.8) for $i \in I_5 \cup I_6$ and (2.13)–(2.15) for $i \in I_7 \cup I_8$. The rest of the proof also runs as in Theorem 2.1 and Theorem 2.3. □

In the next theorem we consider the case $p_i(n) \equiv 1$, $i = 1, \ldots, k$ and get even better result than in the previous theorems.

Theorem 2.6. *Assume that conditions (2.1), (2.2), (2.3) and (2.4) are satisfied. If $p_i(n) \equiv 1$, $i = 1, \ldots, k$ then for any real constants $d_1, \ldots, d_k$ there exists a solution X of system (1.1) that $\lim_{n\to\infty} X(n) = (d_1, \ldots, d_k)$.*

Proof. Let $d_i \in \mathbb{R}$, $i = 1, \ldots, k$ and let ε be any positive real number. There exists a constant $M > 0$ such that

$$|f_i(t)| \leq M \quad \text{for} \quad t \in [d_i - \varepsilon, d_i + \varepsilon], \qquad i = 1, \ldots, k.$$

Let us denote

$$S_{a_i}(n) = \sum_{j=n}^{\infty} |a_i(j)|, \qquad S_{g_i}(n) = \sum_{j=n}^{\infty} |g_i(j)|, \qquad i = 1, \ldots, k.$$

By (2.1) and (2.2) there exists such an index $n_7 \geq n_0$ that for $n \geq n_7$ we have

$$S_{a_i}(n) \leq \frac{\varepsilon}{2M}, \quad \text{and} \quad S_{g_i}(n) \leq \frac{\varepsilon}{2}, \qquad i = 1, \ldots, k.$$

We define a subset Ω_5 of $\mathcal{B}$ by

$$\Omega_5 = \left\{X \in \mathcal{B} : X(0) = \cdots = X(n_7 - 1) = D \text{ and } |X(n) - D| \leq M|S_A(n)| + |S_G(n)| \text{ for } n \geq n_7\right\},$$

where $D = (d_1, \ldots, d_k)$, $S_A = (S_{a_1}, \ldots, S_{a_k})$, $S_G = (S_{g_1}, \ldots, S_{g_k})$. It is easy to check, that Ω_5 is the convex subset of $\mathcal{B}$. It can be also shown that Ω_5 is compact (see, for example, the proof of Theorem 1 in [12] or Lemma 4.7 in [11]). Now, for $n \geq 0$, we define a map

$$T\colon\ \Omega_5 \to \mathcal{B},$$

as follows, for $i = 1, \ldots, k-1$

$$(T_iX)(n) = \begin{cases} d_i, & \text{for} \quad n < n_7, \\ d_i - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} g_i(s), & \\ \quad \text{for } n \geq n_7 \ \text{ and } \tau_i > 0, & \\ d_i - \frac{1}{2} \sum\limits_{s=n}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) - \frac{1}{2} \sum\limits_{s=n}^{\infty} g_i(s), & \\ \quad \text{for } n \geq n_7 \ \text{ and } \tau_i = 0, & \end{cases}$$

and

$$(T_kX)(n) = \begin{cases} d_k, \quad \text{for} \quad n < n_7, \\ d_k - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+(2j-1)\tau_k}^{n+2j\tau_k-1} a_k(s) f_k\left(x_1(s-\sigma_k)\right) - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+(2j-1)\tau_k}^{n+2j\tau_k-1} g_k(s), \\ \qquad \text{for } n \geq n_7 \text{ and } \tau_k > 0, \\ d_k - \frac{1}{2}\sum\limits_{s=n}^{\infty} a_k(s) f_k\left(x_1(s-\sigma_k)\right) - \frac{1}{2}\sum\limits_{s=n}^{\infty} g_k(s), \\ \qquad \text{for } n \geq n_7 \text{ and } \tau_k = 0. \end{cases}$$

We will show that $T(\Omega_5) \subseteq \Omega_5$. It is obvious that

$$\sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} |a_i(s)| \leq \sum_{s=n}^{\infty} |a_i(s)|, \qquad i = 1, \dots, k, \tag{2.16}$$

$$\sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} |g_i(s)| \leq \sum_{s=n}^{\infty} |g_i(s)|, \qquad i = 1, \dots, k. \tag{2.17}$$

Moreover, if $X \in \Omega_5$, then $|x_i(n) - d_i| \leq h$ for all $n \in \mathbb{N}$, $i = 1, \dots, k$. Hence $|f_i(x_{i+1}(n))| \leq M$, $i = 1, \dots, k-1$ and also $|f_k(x_1(n))| \leq M$ for every $X \in \Omega_5$, $n \in \mathbb{N}$. Therefore and by (2.16) and (2.17), for $n \geq n_7$ and $\tau_i > 0$, we get

$$|(T_iX)(n) - d_i| \leq M \sum_{s=n}^{\infty} |a_i(s)| + \sum_{s=n}^{\infty} |g_i(s)| = MS_{a_i}(n) + S_{g_i}(n), \tag{2.18}$$

for $i = 1, \dots, k-1$. The same estimation holds for $i = k$.

For $n \geq n_7$, $\tau_i = 0$ we have

$$\begin{aligned} |(T_iX)(n) - d_i| &= \left| \frac{1}{2} \sum_{s=n}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) + \frac{1}{2} \sum_{s=n}^{\infty} g_i(s) \right| \\ &\leq MS_{a_i}(n) + S_{g_i}(n), \quad i = 1, \dots, k-1, \end{aligned}$$

and similarly for $i = k$. This gives $T(X) \in \Omega_5$ for every $X \in \Omega_5$ and $T(\Omega_5) \subseteq \Omega_5$. Similarly as in the proof of Theorem 2.1, it can be shown that T is continuous.

By Schauder's fixed point theorem there exists $X \in \Omega_5$ such that $T(X) = X$, which is a solution of system (1.1). In fact, for $n \geq n_7$, $\tau_i > 0$ and $i = 1, \dots, k-1$ we have

$$x_i(n) = d_i - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)\tau_i}^{n+2j\tau_i-1} g_i(s).$$

Hence

$$\begin{aligned} x_i(n) + x_i(n-\tau_i) = 2d_i &- \sum_{j=1}^{\infty} \sum_{s=n+2(j-1)\tau_i}^{n+2j\tau_i-1} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) \\ &- \sum_{j=1}^{\infty} \sum_{s=n+2(j-1)\tau_i}^{n+2j\tau_i-1} g_i(s) \\ &= 2d_i - \sum_{s=n}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) - \sum_{s=n}^{\infty} g_i(s), \qquad i = 1, \dots, k-1. \end{aligned}$$

Therefore

$$\begin{aligned}\Delta\left(x_i(n)+x_i(n-\tau_i)\right) = &-\sum_{s=n+1}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right)\\ &+\sum_{s=n}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right)\\ &-\sum_{s=n+1}^{\infty} g_i(s)+\sum_{s=n}^{\infty} g_i(s), \qquad i=1,\dots,k-1,\end{aligned}$$

and finally

$$\Delta\left(x_i(n)+x_i(n-\tau_i)\right)=a_i(n) f_i\left(x_{i+1}(n-\sigma_i)\right)+g_i(n), \qquad i=1,\dots,k-1.$$

In the case $\tau_i=0$ we obtain

$$\begin{aligned}\Delta\left(x_i(n)+x_i(n)\right) &= 2\Delta x_i(n)\\ &= 2\Delta\left(d_i-\frac{1}{2}\sum_{s=n}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right)-\frac{1}{2}\sum_{s=n}^{\infty} g_i(s)\right)\\ &= a_i(n) f_i\left(x_{i+1}(n-\sigma_i)\right)+g_i(n), \qquad i=1,\dots,k-1.\end{aligned}$$

The same reasoning applies to the case $i=k$. It is clear that X fulfills system (1.1) for $n \geq n_7$. By (2.1) and (2.2) sequences S_{a_i} and S_{g_i}, $i=1,\dots,k$, tend to zero. From (2.18) we get $\lim_{n\to\infty} X(n)=D$, that is our claim. □

Example 2.7. Let us consider the following system

$$\begin{cases}\Delta\left(x_1(n)+x_1(n-1)\right) = -\frac{11\cdot 3^n}{12\cdot 9^n-108\cdot 3^n+243}\, x_2^2(n-2)+\frac{1}{3^n},\\ \Delta\left(x_2(n)+x_2(n-2)\right) = \frac{20}{6\cdot 3^n+81}\, x_3(n-3),\\ \Delta\left(x_3(n)+x_3(n-2)\right) = \frac{23}{9\cdot 3^n-9}\, x_4(n-1)-\frac{1}{3^n},\\ \Delta\left(x_4(n)+x_4(n-3)\right) = \frac{35\cdot 3^n}{12\cdot 9^n+108\cdot 3^n+243}\, x_1^2(n-2)+\frac{7}{3^n}.\end{cases}$$

All assumptions of Theorem 2.6 are satisfied. It is easy to check that

$$X=\left(\left(2+\frac{1}{3^n}\right),\left(-2+\frac{1}{3^n}\right),\left(-2-\frac{1}{3^n}\right),\left(3-\frac{1}{3^n}\right)\right)$$

for $n \geq 3$ is the solution of the above system having the property $\lim_{n\to\infty} X(n)=(2,-2,-2,3)$.

In the theorem below we consider the case $p_i(n)\equiv -1$, $i=1,\dots,k$.

Theorem 2.8. *Let conditions* (2.3) *and* (2.4) *be satisfied and assume*

$$\sum_{n=1}^{\infty} n|a_i(n)|<\infty, \qquad i=1,\dots,k, \tag{2.19}$$

$$\sum_{n=1}^{\infty} n|g_i(n)|<\infty, \qquad i=1,\dots,k. \tag{2.20}$$

If $p_i(n)\equiv -1$, $i=1,\dots,k$, then for any real constants $d_1,\dots,d_k$ there exists a solution X of system (1.1) *that $\lim_{n\to\infty} X(n)=(d_1,\dots,d_k)$.*

Proof. We can now proceed analogously to the proof of Theorem 2.6. Let $d_i \in \mathbb{R}$, $i = 1, \dots, k$ and let ε be any positive real number. There exists a constant $M > 0$ such that

$$|f_i(t)| \le M \quad \text{for } t \in [d_i - \varepsilon, d_i + \varepsilon], \qquad i = 1, \dots, k.$$

Write

$$S_{a_i}(n) = \sum_{j=n}^{\infty} j|a_i(j)|, \qquad S_{g_i}(n) = \sum_{j=n}^{\infty} j|g_i(j)|, \qquad i = 1, \dots, k.$$

If the sequences $a_1, \dots, a_k$ and $g_1, \dots, g_k$ satisfy (2.19) and (2.20), then immediately satisfy (2.1) and (2.2) consequently. Hence, for $n \ge n_7$, we have

$$S_{a_i}(n) \le \frac{\varepsilon}{2M}, \quad \text{and} \quad S_{g_i}(n) \le \frac{\varepsilon}{2}, \qquad i = 1, \dots, k.$$

We define a map

$$T : \Omega_5 \to \mathcal{B},$$

in the following way, for $i = 1, \dots, k-1$

$$(T_i X)(n) = \begin{cases} d_i & \text{for } n < n_7 \\ d_i - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+j\tau_i}^{\infty} a_i(s) f_i\left(x_{i+1}(s - \sigma_i)\right) - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+j\tau_i}^{\infty} g_i(s) & \text{for } n \ge n_7, \end{cases}$$

and

$$(T_k X)(n) = \begin{cases} d_k & \text{for } n < n_7, \\ d_k - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+j\tau_k}^{\infty} a_k(s) f_k\left(x_1(s - \sigma_k)\right) - \sum\limits_{j=1}^{\infty} \sum\limits_{s=n+j\tau_k}^{\infty} g_k(s) & \text{for } n \ge n_7. \end{cases}$$

We will prove that $T(\Omega_5) \subseteq \Omega_5$. It is easy to observe that

$$\sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} |a_i(s)| \le \sum_{s=n}^{\infty} s\,|a_i(s)|\,, \qquad i = 1, \dots, k, \tag{2.21}$$

$$\sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} |g_i(s)| \le \sum_{s=n}^{\infty} s\,|g_i(s)|\,, \qquad i = 1, \dots, k. \tag{2.22}$$

By (2.21) and (2.22) for $n \ge n_7$ we get

$$\begin{aligned} |(T_i X)(n) - d_i| &= \left| \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} a_i(s) f_i\left(x_{i+1}(s - \sigma_i)\right) + \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} g_i(s) \right| \\ &\le M \sum_{s=n}^{\infty} s\,|a_i(s)| + \sum_{s=n}^{\infty} s\,|g_i(s)| = M S_{a_i}(n) + S_{g_i}(n) \end{aligned} \tag{2.23}$$

for $i = 1, \dots, k-1$. Analogously we get this for $i = k$. Hence $T(X) \in \Omega_5$ for any $X \in \Omega_5$ and $T(\Omega_5) \subseteq \Omega_5$. Reasoning similarly as in the proof of Theorem 2.1, it can be shown that T is continuous.

By Schauder's fixed point theorem there exists $X \in \Omega_5$ such that $T(X) = X$ and it is a solution of system (1.1). For $n \ge n_7$ we have

$$x_i(n) = d_i - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} a_i(s) f_i\left(x_{i+1}(s - \sigma_i)\right) - \sum_{j=1}^{\infty} \sum_{s=n+j\tau_i}^{\infty} g_i(s),$$

for $i = 1, \dots, k-1$, and

$$x_i(n-\tau_i) = d_i - \sum_{j=1}^{\infty} \sum_{s=n+(j-1)\tau_i}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) - \sum_{j=1}^{\infty} \sum_{s=n+(j-1)\tau_i}^{\infty} g_i(s).$$

Since

$$x_i(n) - x_i(n-\tau_i) = -\sum_{s=n}^{\infty} a_i(s) f_i\left(x_{i+1}(s-\sigma_i)\right) - \sum_{s=n}^{\infty} g_i(s),$$

we have

$$\Delta\left(x_i(n) - x_i(n-\tau_i)\right) = a_i(n) f_i\left(x_{i+1}(n-\sigma_i)\right) + g_i(n), \qquad i = 1, \dots, k-1.$$

The same conclusion can be drawn for $i = k$.

Finally we see that X satisfies system (1.1) for $n \geq n_7$. By (2.19) and (2.20) sequences S_{a_i} and S_{g_i}, $i = 1, \dots, k$, tend to zero. From (2.23) we get

$$\lim_{n\to\infty} X(n) = D. \qquad \square$$

Acknowledgements

This work was partially supported by the Ministry of Science and Higher Education of Poland (04/43/DS PB/0084).

References

[1] R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan, *Discrete oscillation theory*, Hindawi Publishing Corporation, New York, 2005. MR2179948; url

[2] R. P. Agarwal, S. R. Grace, Oscillation of higher-order nonlinear difference equations of neutral type, *Appl. Math. Lett.* **12**(1999), No. 8, 77–83. MR1751347; url

[3] R. P. Agarwal, E. Thandapani, P. J. Y. Wong, Oscillation of higher-order neutral difference equation, *Appl. Math. Lett.* **10**(1997), No. 1, 71–78. MR1429478; url

[4] Y. Bolat, O. Akin, Oscillatory behaviour of a higher-order nonlinear neutral type functional difference equation with oscillating coefficients, *Appl. Math. Lett.* **17**(2004), 1073–1078. MR2087757; url

[5] A. Burton, *Stability by fixed point theory for functional differential equations*, Dover Publications, 2006. MR2281958

[6] S. S. Cheng, W. T. Patula, An existence theorem for a nonlinear difference equation, *Nonlinear Anal.* **20**(1993), 193–203. MR1202198; url

[7] L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation theory for functional differential equations*, CRC Press, 1995. MR1309905

[8] R. Jankowski, E. Schmeidel, Almost oscillation criteria for second order neutral difference equation with quasidifferences, *Int. J. Difference Equ.* **9**(2014), No. 1, 77–86. url

[9] R. Jankowski, E. Schmeidel, Asymptotically zero solution of a class of higher nonlinear neutral difference equations with quasidifferences, *Discrete Contin. Dyn. Syst. Ser. B* **19**(2014), No. 8, 2691–2696. MR3275022; url

[10] M. Liu, Z. Guo, Solvability of a higher-order nonlinear neutral delay difference equation, *Adv. Difference Equ.* **2010**, Art. ID 767620, 14 pp. MR2727276; url

[11] J. Migda, Approximative solutions of difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 13, 1–26. MR3183611

[12] J. Migda, Asymptotic behavior of solutions of nonlinear difference equations, *Math. Bohem.* **129**(2004), No. 4, 349–359. MR2102609

[13] M. Migda, E. Schmeidel, M. Zdanowicz, Existence of nonoscillatory bounded solutions of three dimensional system of neutral difference equations, *Appl. Anal. Discrete Math.* **9**(2015), No. 2, 271–284. url

[14] M. Migda, E. Schmeidel, Convergence of solutions of higher order neutral difference equations with quasi-differences, *Tatra Mt. Math. Publ.* **63**(2015), 205–213. url

[15] M. Migda, Oscillation criteria for higher order neutral difference equations with oscillating coefficient, *Fasc. Math.* **44**(2010), 85–93. MR2722634

[16] M. Migda, J. Migda, Oscillatory and asymptotic properties of solutions of even order neutral difference equations, *J. Difference Equ. Appl.* **15**(2009), No. 11–12, 1077–1084. MR2569136 ; url

[17] M. Migda, G. Zhang, Monotone solutions of neutral difference equations of odd order, *J. Difference Equ. Appl.* **10**(2004), No. 7, 691–703. MR2064816 ; url

[18] N. Parhi, A. K. Tripathy, Oscillation of a class of nonlinear neutral difference equations of higher order, *J. Math. Anal. Appl.* **284**(2003), 756–774. MR1998666; url

[19] E. Thandapani, R. Karunakaran, I. M. Arockiasamy, Bounded nonoscillatory solutions of neutral type difference systems, *Electron. J. Qual. Theory Differ Equ.*, Spec. Ed. I, **2009**, No. 25, 1–8. MR2558850

[20] Z. Wang, J. Sun, Asymptotic behavior of solutions of nonlinear higher-order neutral type difference equations, *J. Difference Equ. Appl.* **12**(2006), 419-432. MR2241385; url

[21] A. Zafer, Oscillatory and asymptotic behavior of higher order difference equations, *Math. Comput. Modelling* **21**(1995), No. 4, 43–50. MR1317929 ; url

[22] Y. Zhou, Y. Q. Huang, Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations, *J. Math. Anal. Appl.* **280**(2003), No. 1, 63–76. MR1972192; url

[23] Y. Zhou, B. G. Zhang, Existence of nonoscillatory solutions of higher-order neutral delay difference equations with variable coefficients, *Comput. Math. Appl.* **45**(2003), 991–1000. MR2000572; url