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Multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects

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Abstract. This paper deals with the multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects. By using critical point theory, a new result is obtained. An example is given to illustrate the main result.

Keywords: critical point theory, boundary value problems, impulsive effects, quasilinear equations.

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1 Introduction

Consider the following problem with impulses

$$\begin{cases} -u''(t) + a(t)u(t) - (|u(t)|^2)''u(t) = f(t, u(t)), & t \in J, \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \dots, m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where $t_0 = 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $J = [0, T] \setminus \{t_1, t_2, \dots, t_m\}$, $f \in C([0, T] \times \mathbb{R}; \mathbb{R})$, $I_j \in C(\mathbb{R}; \mathbb{R})$, $a(t) \in L^\infty[0, T]$, $\Delta(u'(t_j)) = u'(t_j^+) - u'(t_j^-)$ and $u'(t_j^\pm) = \lim_{t \to t_j^\pm} u'(t)$, $j = 1, 2, \dots, m$.

This problem is derived from a class of quasilinear Schrödinger equation. When we look for the standing wave solution whose form is $\Psi(t, x) = e^{-iwt}u(x)$, $w \in \mathbb{R}$ of the following quasilinear Schrödinger equation

$$i\partial_t \Psi = -\Psi'' + W(x)\Psi - (|\Psi|^2)''\Psi - \mu|\Psi|^{q-1}\Psi, \qquad x \in \mathbb{R}, \tag{1.2}$$

where $q > 1$, $\mu > 0$, we can obtain the elliptic equation of the form

$$-u'' + (W(x) - w)u - (|u|^2)''u = \mu|u|^{q-1}u, \qquad x \in \mathbb{R}, \tag{1.3}$$

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which was investigated by some scholars (see [2, 4, 9, 19, 22]).

It is generally known that critical point theory is a classical method to deal with the existence and multiplicity of solutions for differential equations (see [3, 7, 12, 16, 21, 26, 30]). Then a natural question is asked: Can we consider the multiplicity of solutions for second-order quasilinear equations with impulsive effects which are produced by the quasilinear term $(|u|^2)''u$ and u'' by using critical point theory?

Impulsive differential equations can be used to describe many evolution processes (see [5, 10, 11, 14, 17, 27]). Some classical methods and theorems such as fixed point theorems, the method of lower and upper solutions and coincidence degree theory have been widely used to investigate impulsive differential equations (see [1, 8, 13, 15, 20]). Recently, critical point theory has been proved to be an effective tool to investigate boundary value problems for impulsive differential equations. Many valuable results have been obtained by some scholars (see [6, 18, 24, 25, 28, 29]).

In [18], Nieto and O'Regan studied the linear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + \lambda u(t) = \sigma(t), \quad \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = d_j, \quad j = 1, 2, \dots, p, \\ u(0) = u(T) = 0, \end{cases} \tag{1.4}$$

and the nonlinear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = I_j(u(t_j)), \quad j = 1, 2, \dots, p, \\ u(0) = u(T) = 0. \end{cases} \tag{1.5}$$

and got some results by using critical point theory.

In [29], Zhou and Li investigated the nonlinear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + g(t)u(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta(u'(t_j)) = I_j(u(t_j)), \quad j = 1, 2, \dots, p, \\ u(0) = u(T) = 0. \end{cases} \tag{1.6}$$

and obtained the existence of infinitely many solutions by employing the Symmetric Mountain Pass Theorem.

However, there are few articles which considered the multiplicity of standing wave solutions for the impulsive Dirichlet boundary value problem involving the quasilinear term $(|u|^2)''u$. The impulsive effects which brought from the quasilinear term $(|u|^2)''u$ are more complicated than u''.

Motivated by the works mentioned above, in this paper, our purpose is to investigate the multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects (1.1). Moreover, the nonlinearity f does not need to satisfy the Ambrosetti–Rabinowitz condition (see [3]). Furthermore, the impulsive terms $I_j(u)$ need to satisfy the suplinear condition rather than the sublinear condition as those in [18, 23, 28, 29]. By making use of the variant fountain theory (see [30]), the multiplicity of solutions for the problem (1.1) are obtained.

2 Preliminaries

In this section, the following theorem will be needed in the proof of our main results. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j=k}^{\infty} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=0}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$.

Theorem 2.1 ([30, Theorem 2.2])**.** *The C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by $\Phi_\lambda(u) = A(u) - \lambda B(u)$, $\lambda \in [1,2]$, satisfies*

(B1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover,

$$\Phi_\lambda(-u) = \Phi_\lambda(u) \text{ for all } (\lambda, u) \in [1,2] \times E.$$

(B2) $B(u) \geq 0$; $B(u) \to +\infty$ as $\|u\| \to +\infty$ on any finite dimensional subspace of E.

(B3) There exist $\rho_k > r_k > 0$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) \quad \text{for all } \lambda \in [1,2]$$

and

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \text{ as } k \to +\infty \quad \text{uniformly for } \lambda \in [1,2].$$

Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \qquad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2),\ b_k(1)] \quad \text{as } n \to +\infty.$$

Particularly, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k, then Φ_1 has infinitely many nontrivial critical points $\{u_k\} \in E \setminus \{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to +\infty$.

In the Sobolev space $H_0^1(0,T)$, consider the inner product

$$\langle u, v \rangle = \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \qquad \forall u, v \in H_0^1(0,T),$$

inducing the norm

$$\|u\|_{H_0^1} = \left(\int_0^T |u(t)|^2 + |u'(t)|^2 dt \right)^{\frac{1}{2}}.$$

By Poincaré's inequality

$$\int_0^T |u(t)|^2 dt \leq \frac{1}{\sqrt{\lambda}} \int_0^T |u'(t)|^2 dt,$$

where $\lambda = \frac{\pi^2}{T^2}$ is the first eigenvalue of the problem $-u'' = \lambda u$ with Dirichlet boundary conditions, the norm $\|u\|_{H_0^1(0,T)}$ and $\|u'\|_{L^2}$ are equivalent.

But, in this paper, we define the following inner product in $H_0^1(0,T)$

$$\langle u, v \rangle_1 = \int_0^T a(t)u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \qquad \forall u, v \in H_0^1(0,T),$$

whose norm is

$$\|u\| = \left(\int_0^T a(t)|u(t)|^2 + |u'(t)|^2 \right)^{\frac{1}{2}}.$$

Throughout our paper, we assume that $\operatorname{ess\,inf}_{t\in[0,T]} a(t) > -\lambda$, which together with Lemma 2.1 in [29] yields that the norm $\|u\|_{H_0^1}$ and $\|u\|$ are equivalent. Thus, by the Sobolev Embedding Theorem, there exists a constant $c > 0$ such that $\|u\|_\infty := \max_{t\in[0,T]} |u(t)| \le c\|u\|$.

For each $u \in H_0^1(0,T)$, u is absolutely continuous and $u' \in L^2(0,T)$. In this case, $\Delta u(t) = u'(t_j^+) - u'(t_j^-) = 0$ may not hold for any $t \in (0,T)$. It leads to the impulsive effects. Thus,

$$\begin{aligned}
-\int_0^T (|u(t)|^2)'' u(t)v(t)dt &= -\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}} (|u(t)|^2)'' u(t)v(t)dt \\
&= -(\sum_{j=0}^{m} 2u'(t_{j+1}^-)u^2(t_{j+1}^-)v(t_{j+1}^-) - 2u'(t_j^+)u^2(t_j^+)v(t_j^+) \\
&\quad - \int_{t_j}^{t_{j+1}} 2u'^2(t)u(t)v(t) + 2u^2(t)u'(t)v'(t)dt) \\
&= \sum_{j=1}^{m} 2\Delta u'(t_j)u^2(t_j)v(t_j) + 2u'(0)u^2(0)v(0) - 2u'(T)u^2(T)v(T) \\
&\quad + \int_0^T 2u'^2(t)u(t)v(t) + 2u^2(t)u'(t)v'(t)dt \\
&= \sum_{j=1}^{m} 2I_j(u(t_j))u^2(t_j)v(t_j) + \int_0^T 2u'^2(t)u(t)v(t) + 2u^2(t)u'(t)v'(t)dt.
\end{aligned}$$

Similarly, we have

$$-\int_0^T u''(t)v(t)dt = \sum_{j=1}^{m} I_j(u(t_j))v(t_j) + \int_0^T u'(t)v'(t)dt.$$

Define the functional $\Phi : H_0^1(0,T) \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m}\int_0^{u(t_j)} (2t^2+1)I_j(t)dt + \int_0^T u'^2(t)u^2(t)dt - \int_0^T F(t,u(t))dt,$$

where $F(t,u) = \int_0^u f(t,s)ds$. Clearly, $\Phi \in C^1(H_0^1(0,T),\mathbb{R})$,

$$\begin{aligned}
\langle \Phi'(u), v\rangle = &\int_0^T u'(t)v'(t) + a(t)u(t)v(t)dt + \int_0^T 2u'^2(t)u(t)v(t) + 2u^2(t)u'(t)v'(t)dt \\
&+ \sum_{j=1}^{m}(2u^2(t_j)+1)I_j(u(t_j))v(t_j) - \int_0^T f(t,u(t))v(t)dt.
\end{aligned}$$

Definition 2.2. A function $u \in H_0^1(0,T)$ is a weak solution of the problem (1.1), if it is a critical point of Φ.

Next, let

$$\Phi_\lambda(u) := A(u) - \lambda B(u),$$

where

$$\begin{aligned}
A(u) &:= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m}\int_0^{u(t_j)} (2t^2+1)I_j(t)dt + \int_0^T u'^2(t)u^2(t)dt, \\
B(u) &:= \int_0^T F(t,u(t))dt,
\end{aligned}$$

$\lambda \in [1,2]$. Clearly, the critical points of $\Phi_1(u) = \Phi(u)$ correspond to the weak solutions of the problem (1.1). In $H_0^1(0,T)$, we can choose a completely orthonormal basis e_j and set $X_j = \mathbb{R}e_j$. Thus, Z_k and Y_k can be defined.

3 Main result

Theorem 3.1. *Assume that $F(t,u)$ is even about u and the following conditions are satisfied.*

(H1) $I_j(u)$ are odd about u and $I_j(u)u \geq 0$, $(j = 1,2,\dots,m)$.

(H2) There exist constants $b_j > 0$ and $\gamma_j \in [1,\infty)$ such that $|I_j(u)| \leq b_j\,|u|^{\gamma_j}$.

(H3) $F(t,u) = o(|u|^\nu)$ as $|u| \to 0$ uniformly on $[0,T]$.

(H4) There exist constants $l_1,\ L > 0$ such that

$$|f(t,u)| \leq l_1|u|^p, \qquad |u| \geq L, \quad p \in [0,1), \quad t \in [0,T].$$

(H5) There exist constants $l_2,\ l_3 > 0$ such that

$$F(t,u) \geq l_2|u|^\theta + l_3|u|^\nu, \qquad \theta,\nu \in [1,2), \quad t \in [0,T].$$

Then the problem (1.1) has infinitely many solutions.

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Under the assumptions of Theorem 3.1, there exists a ρ_k small enough such that $a_k(\lambda) := \inf_{u\in Z_k,\|u\|=\rho_k} \Phi_\lambda(u) \geq 0$ and $d_k(\lambda) := \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \to 0$ as $k \to +\infty$ uniformly for any $\lambda \in [1,2]$.*

Proof. Let $\Gamma_k := \sup_{u\in Z_k,\|u\|=1} \|u\|_\infty$. Then $\Gamma_k \to 0$ as $k \to +\infty$. By (H3), for given $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$F(t,u) \leq \epsilon_1|u|^\nu, \qquad |u| \leq \delta_1, \quad t \in [0,T].$$

Based on (H4), we have

$$F(t,u) \leq l_1|u|^{p+1}, \qquad |u| \geq L, \quad t \in [0,T].$$

From the continuity of $F(t,u)$, for $(t,|u|) \in [0,T] \times [\delta_1, L]$, there exists $M > 0$ such that

$$F(t,u) \leq \epsilon_1|u|^\nu + l_1|u|^{p+1} + M.$$

So, we have

$$F(t,u) \leq \epsilon_1|u|^\nu + (M\delta_1^{-1-p} + l_1)|u|^{p+1}, \qquad u \in \mathbb{R}, \quad t \in [0,T]. \tag{3.1}$$

Based on (H1), for any $u \in Z_k$ and $\|u\|$ small enough, we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m}\int_0^{u(t_j)}(2t^2+1)I_j(t)dt + \int_0^T u'^2(t)u^2(t)dt - \lambda\int_0^T F(t,u(t))dt \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\epsilon_1\int_0^T |u|^\nu dt - \lambda(M\delta_1^{-1-p} + l_1)\int_0^T |u|^{p+1}dt \\
&\geq \frac{1}{2}\|u\|^2 - 2\epsilon_1 T\Gamma_k^\nu\|u\|^\nu - 2T\Gamma_k^{p+1}(M\delta_1^{-1-p} + l_1)\|u\|^{p+1} \\
&\geq \frac{1}{8}\rho_k^2 \geq 0,
\end{aligned}$$

where $\|u\| = \rho_k := (16T\Gamma_k^{p+1}(M\delta_1^{-1-p} + l_1) + 16\epsilon_1 T\Gamma_k^\nu)^{\frac{1}{1-p}}$ (without loss of generality, assume that $v \geq p+1$). It is easy to find that $\rho_k \to 0$ as $k \to +\infty$. Thus, we can obtain that $a_k(\lambda) \geq 0$ and $d_k(\lambda) \to 0$ as $n \to +\infty$ uniformly for $\lambda \in [1,2]$.

□

Lemma 3.3. *Under the assumptions of Theorem 3.1, there exists a r_k small enough such that $b_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0$ for $\lambda \in [1,2]$.*

Proof. Let $M_1 = \max\{b_1, b_2, b_3, \dots\}$. For any $u \in Y_k$, by the equivalence of the norms on the finite-dimensional space Y_k and (H5), we have

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{m}\int_0^{u(t_j)}(2t^2+1)I_j(t)dt + \int_0^T u'^2(t)u^2(t)dt - \lambda\int_0^T F(t,u(t))dt \\
&\le \frac{1}{2}\|u\|^2 + 2M_1\sum_{j=1}^{m} c^{3+\gamma_j}\|u\|^{3+\gamma_j} + M_1\sum_{j=1}^{m} c^{1+\gamma_j}\|u\|^{1+\gamma_j} + c^2c_1^2\|u\|^4 \\
&\quad - \lambda l_2\int_0^T |u|^\theta dt - \lambda l_3\int_0^T |u|^\nu dt \\
&\le \frac{1}{2}\|u\|^2 + 2M_1\sum_{j=1}^{m} c^{3+\gamma_j}\|u\|^{3+\gamma_j} + M_1\sum_{j=1}^{m} c^{1+\gamma_j}\|u\|^{1+\gamma_j} + c^2c_1^2\|u\|^4 \\
&\quad - \lambda l_2c_2\|u\|^\theta - \lambda l_3c_3\|u\|^\nu,
\end{aligned}$$

which together with $\theta, \nu \in [1,2)$ yields that $\Phi_\lambda(u) < 0$ for $\|u\| := r_k < \rho_k$ small enough and $\lambda \in [1,2]$. □

Lemma 3.4. *Under the assumptions of Theorem 3.1, there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that*

$$\Phi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \qquad \Phi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2),\ b_k(1)] \quad \text{as } n \to +\infty.$$

Proof. Clearly, Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Since $F(t,u)$ is even about u and $I_j(u)$ are odd about u, we have $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times H_0^1(0,T)$. Furthermore, by (H5) and the equivalence of the norms on the finite-dimensional space on $H_0^1(0,T)$, there exist two positive constants c_4, c_5 such that $B(u) \ge l_2c_4\|u\|^\theta + l_3c_5\|u\|^\nu$. So, $B(u) \ge 0$, $B(u) \to +\infty$ as $\|u\| \to +\infty$. Thus, (B1) and (B2) are satisfied. By Lemma 3.2 and 3.3, (B3) holds. In view of Theorem 2.1, we can obtain Lemma 3.4. □

Next, we show the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $u(\lambda_n) := u_n \in Y_n$. First, we will prove that $\{u_n\}$ is bounded on $H_0^1(0,T)$. Based on Lemma 3.4, there exist $\lambda_n \to 1$, $u_n \in Y_n$ such that $\Phi'_{\lambda_n}|_{Y_n}(u_n) = 0$, $\Phi_{\lambda_n}(u_n) \to c_k \in [d_k(2),\ b_k(1)]$ as $n \to +\infty$. Thus, we have

$$\begin{aligned}
\Phi_{\lambda_n}(u_n) &= \frac{1}{2}\|u_n\|^2 + \sum_{j=1}^{m}\int_0^{u_n(t_j)}(2t^2+1)I_j(t)dt + \int_0^T u_n'^2u_n^2dt - \lambda_n\int_0^T F(t,u_n)dt \\
&\ge \frac{1}{2}\|u_n\|^2 - 2\int_0^T F(t,u_n)dt.
\end{aligned}$$

By the same way as Lemma 3.2, we have

$$\Phi_{\lambda_n}(u_n) \ge \frac{1}{2}\|u_n\|^2 - 2Tc^{p+1}(M\delta_1^{-1-p} + l_1)\|u_n\|^{p+1} - 2\epsilon_1Tc^\nu\|u_n\|^\nu,$$

which implies that $\{u_n\}$ is bounded on $H_0^1(0,T)$. Then there exists a subsequence of $\{u_n\}$ (for simplicity denoted again by $\{u_n\}$) such that $u_n \rightharpoonup u$ in $H_0^1(0,T)$ and $u_n \to u$ uniformly in

$C[0, T]$. Thus,

$$\begin{aligned}
&\langle \Phi'_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u), u_n - u \rangle \to 0,\\
&\int_0^T (u_n'^2(t)u_n(t) - u'^2(t)u(t))(u_n(t) - u(t))dt \to 0,\\
&\sum_{j=1}^m (I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \to 0,\\
&\sum_{j=1}^m (I_j(u_n(t_j))u_n^2(t_j) - I_j(u(t_j))u^2(t_j))(u_n(t_j) - u(t_j)) \to 0,\\
&\int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t))dt \to 0,
\end{aligned}$$

as $n \to +\infty$. Moreover,

$$\begin{aligned}
&\int_0^T (u_n^2(t)u_n'(t) - u^2(t)u'(t))(u_n'(t) - u'(t))dt\\
&\quad = \int_0^T \left[(u_n^2(t) - u^2(t))u_n'(t) + u^2(t)(u_n'(t) - u'(t))\right](u_n'(t) - u'(t))dt\\
&\quad = \int_0^T u_n'(t)(u_n'(t) - u'(t))(u_n^2(t) - u^2(t))dt + \int_0^T u^2(t)|u_n'(t) - u'(t)|^2 dt.
\end{aligned}$$

Since

$$\int_0^T u_n'(t)(u_n'(t) - u'(t))(u_n^2(t) - u^2(t))dt \to 0$$

as $n \to +\infty$, we have

$$\begin{aligned}
&\langle \Phi'_{\lambda_n}(u_n) - \Phi'_{\lambda_n}(u), u_n - u \rangle\\
&\quad = \|u_n - u\|^2 + 2\int_0^T (u_n'^2(t)u_n(t) - u'^2(t)u(t))(u_n(t) - u(t))dt\\
&\qquad +2\int_0^T (u_n^2(t)u_n'(t) - u^2(t)u'(t))(u_n'(t) - u'(t))dt\\
&\qquad + \sum_{j=1}^m (I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j))\\
&\qquad +2\sum_{j=1}^m (I_j(u_n(t_j))u_n^2(t_j) - I_j(u(t_j))u^2(t_j))(u_n(t_j) - u(t_j))\\
&\qquad -\lambda_n \int_0^T (f(t, u_n(t)) - f(t, u(t)))(u_n(t) - u(t))dt\\
&\quad = \|u_n - u\|^2 + 2\int_0^T u^2(t)|u_n'(t) - u'(t)|^2 dt + o(1),
\end{aligned}$$

which implies that $u_n \to u$ in $H_0^1(0, T)$. Then Φ_1 has infinitely many nontrivial critical points $\{u^k\} \in H_0^1(0, T) \setminus \{0\}$ satisfying $\Phi_1(u^k) \to 0^-$ as $k \to +\infty$. Thus, the problem (1.1) has infinitely many solutions. □

Example 3.5.

$$\begin{cases}
-u''(t) + a(t)u(t) - (|u(t)|^2)''u(t) = f(t, u(t)), \quad t \in J,\\
\Delta(u'(t_j)) = I_j(u(t_j)), \quad j = 1,\\
u(0) = u(1) = 0,
\end{cases}$$

where $a(t) = 1$, $t_1 = \frac{1}{2}$, $F(t,u) = |u|\ln(1+|u|^{\frac{1}{2}}) + |u|^{\frac{5}{4}}(\sin|u|^{\frac{1}{2}}+3)$, $I_j(u) = u^3$, $\nu = 1$. Clearly, the conditions of (H1), (H2), (H3) and (H5) are satisfied. Moreover,

$$|f(t,u)| \leq \ln(1+|u|^{\frac{1}{2}}) + \frac{|u|^{\frac{1}{2}}}{2(1+|u|^{\frac{1}{2}})} + 5|u|^{\frac{1}{4}} + \frac{1}{2}|u|^{\frac{3}{4}} \leq 2|u|^{\frac{4}{5}}, \qquad |u| \geq L,$$

where L should be large enough. Thus, (H4) holds. Then Example 3.5 has infinitely many solutions.

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