Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **87**, 1–8; doi: 10.14232/ejqtde.2015.1.87 http://www.math.u-szeged.hu/ejqtde/

Existence of solutions for a Kirchhoff type problem involving the fractional p-Laplacian operator

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Received 23 April 2015, appeared 30 November 2015
Communicated by Michal Fečkan

Abstract. This paper is concerned with the existence of solutions to a Kirchhoff type problem involving the fractional p-Laplacian operator. We obtain the existence of solutions by Ekeland's variational principle.

Keywords: Kirchhoff type problem, fractional p-Laplacian, Ekeland variational principle.

2010 Mathematics Subject Classification: 35J20, 35J92.

1 Introduction

Great attention has been focused on studying fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and for concrete applications, since they naturally arise in many different contexts. For an elementary introduction on this topic and for a quite extensive list of related references we refer to [9].

In this paper, we are interested in the following problem

$$\begin{cases} -M\left(\|u\|_Z^p\right)\mathcal{L}_K u(x) = f(x,u) + |u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $p > 1$, Ω is an open bounded set in $\mathbb{R}^N$, $p^* = \frac{Np}{N-ps}$ if $N > ps$, and $p^* = +\infty$ if $N \le ps$, is the fractional critical exponent, with $s \in (0,1)$ fixed, $\|\cdot\|_Z$ is a norm which is defined in (2.3), M and f are two functions satisfying some suitable conditions which will be given later, and $\mathcal{L}_K$ is a nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) = 2\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))K(x-y)dy, \qquad x \in \mathbb{R}^N.$$

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Here $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ is a measurable function which satisfies

$$\begin{cases} \gamma K(x) \in L^1(\mathbb{R}^N) \text{ with } \gamma(x) = \min\{|x|^p, 1\}; \\ \text{there exists } \theta > 0 \text{ such that } K(x) \geq \theta |x|^{-(N+p\theta)} \text{ for any } x \in \mathbb{R}^N \backslash \{0\}; \\ K(x) = K(-x) \text{ for any } x \in \mathbb{R}^N \backslash \{0\}. \end{cases} \tag{1.2}$$

A typical model for K is given by the singular kernel $K(x) = |x|^{-(N+ps)}$. In this case $\mathcal{L}_K u(x) = (-\triangle)^s_p u(x)$ is the fractional p-Laplacian operator which (up to normalization factors) can be defined as

$$(-\Delta)^s_p u(x) = 2 \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \quad \text{for } x \in \mathbb{R}^N. \tag{1.3}$$

In problem (1.1), for $p = 2$ and the function $M \equiv 1$, via variational methods, several existence results were proved in a series of papers of Servadei and Valdinoci [21–28].

Recently, Fiscella and Valdinoci [11] established the existence of a nontrivial solution to the following fractional Laplacian Kirchoff type problem

$$\begin{cases} -M\left(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y) dx dy\right) \widetilde{\mathcal{L}}_K u(x) = \lambda f(x, u) + |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $2^* = \frac{2N}{N-2s}$, $N > 2s$ with $s \in (0, 1)$. M and f are two continuous functions under some suitable assumptions, and the operator $\widetilde{\mathcal{L}}_K$ is defined as

$$\widetilde{\mathcal{L}}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^N} (u(x + y) + u(x - y) - 2u(x)) K(y) dy, \quad x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}^+$ is a measurable function satisfying properties in (1.2) replaced by $p = 2$. In [11], the authors first provided a detailed discussion about the physical meaning underlying the fractional Kirchhoff problems and their applications. They supposed that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and continuous function, and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in \mathbb{R}^+$. Based on the truncated skill and the Mountain Pass Theorem, they obtained the existence of a non-negative solution to problem (1.4) for any $\lambda > \lambda^* > 0$, where λ^* is an appropriate threshold.

Moreover, Sun and Teng [29] obtained the existence and multiplicity of solutions for a Kirchhoff type problem when $K(x) = |x|^{-(N+2s)}$ and $p = 2$, by the Mountain Pass Theorem and the symmetric Mountain Pass Theorem together with truncation techniques.

In the very recent paper [3], Autuori, Fiscella and Pucci established the existence and the asymptotic behavior of non-negative solutions to problem (1.4) under different assumptions on M, the Kirchhoff function M can be zero at zero, that is, the problem is degenerate case.

For the quasi-linear problem, if the Kirchhoff function $M \equiv 1$ and for any $p > 1$, consider the following problem

$$\begin{cases} (-\Delta)^s_p u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.5}$$

Some results have been obtained for problem (1.5). In the works of Franzina–Palatucci [8] and Lindgren–Linqvist [15], the eigenvalue problem associated with $(-\Delta)^s_p$ is studied, and particularly some properties of the first eigenvalue and of the higher order (variational) eigenvalues are obtained. Then, Iannizzotto–Squassina [16] obtained some Weyl-type estimates for the asymptotic behaviour of variational eigenvalues λ_j defined by a suitable cohomological index. From the point of view of regularity theory, some results can be found in [15] even though that work is mostly focused on the case when p is large and the solutions inherit some regularity directly from the functional embeddings themselves. Moreover, Goyal and Sreenadh [13] studied the existence and multiplicity of non-negative solutions to problem (1.5) when the nonlinearity is subcritical growth with concave-convex nonlinearities and sign changing weight. Furthermore, the existence of solutions has been also considered in [4,5,7,12,14–20,30] and references therein.

When $s = 1$, problem (1.1) reduces to a p-Kirchhoff type problem. It has been studied often in the literature, where different methods were proposed to analyze the question of the existence and the multiplicity of solutions and related qualitative properties, see [2,6,10,18] and references therein. In particular, the existence of solutions for p-Kirchhoff problem with a critical nonlinearity has been obtained in [18].

Inspired by the above mentioned works, we will use Ekeland's variational principle to investigate the existence of solutions for problem (1.1). We suppose that the function $M : (0, +\infty) \to (0, +\infty)$ is continuous and satisfies the following conditions:

(M_1) $M \in L^1(0, \sigma)$ with $\sigma > 0$;

(M_2) there exist $0 < \beta \le \frac{1}{p}$ and a positive constant c_1 such that $\widetilde{M}(t) \ge c_1 t^\beta$ for $t > 0$, where $\widetilde{M}(t) = \int_0^t M(\tau)d\tau$;

(M_3) there exists $\alpha > \frac{p*}{p}$ such that $\limsup_{t\to 0^+} t^{-\alpha}\widetilde{M}(t) < \infty$.

Moreover, the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

(F_1) there is a positive constant c_2 such that $|f(x,t)| \le c_2(1 + |t|^{q-1})$, where $p < q < p^*$;

(F_2) there exist positive constant c_3 and $0 < \delta < p\alpha$ such that $F(x,t) \ge c_3|t|^\delta$ as $t \to 0$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

Our result can be stated as follows.

Theorem 1.1. *Let $s \in (0,1)$ be fixed, $N > ps$ and Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary. Let K be a function satisfying condition (1.2), functions M and f satisfy (M_1)–(M_3) and (F_1)–(F_2), then problem (1.1) has a nontrivial solution.*

Remark 1.2. (i) In some works, it is assumed that $M(t) \ge M(0) > 0$ for $t \ge 0$, which is not necessary for our result.
(ii) To the best of our knowledge, it seems that this is the first result about the existence of solutions for the fractional p-Laplacian Kirchhoff type problem.

2 Proof of the main result

Before we prove our main result, let us introduce some notations and the functional space which we will use in the following.

We define $W^{s,p}(\Omega)$, the usual fractional Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dx\, dy\right)^{1/p}. \qquad (2.1)$$

Define the functional space

$$X = \Big\{ u \mid u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega), \\ \text{and } (u(x) - u(y)) \sqrt[p]{K(x-y)} \text{ is in } L^p(Q, dxdy) \Big\},$$

where $Q = \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \backslash \Omega$. The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_Q |u(x) - u(y)|^p K(x-y) dx\, dy \right)^{1/p}. \tag{2.2}$$

Set

$$Z = \{u \in X \ : \ u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\},$$

and the norm

$$\|u\|_Z = \left(\int_Q |u(x) - u(y)|^p K(x-y) dx\, dy \right)^{1/p}. \tag{2.3}$$

By [13, Lemma 2.5], the space $(Z, \|\cdot\|_Z)$ is a reflexive Banach space.

Definition 2.1. We say that u is a weak solution of problem (1.1), if u satisfies

$$\begin{aligned} M\left(\|u\|_Z^p\right) \int_Q |u(x) - u(y)|^{p-2} \left(u(x) - u(y)\right) \left(\phi(x) - \phi(y)\right) K(x-y) dxdy \\ = \int_\Omega f(x,u)\phi dx + \int_\Omega |u|^{p^*-2} u\phi dx \end{aligned} \tag{2.4}$$

for all $\phi \in Z$.

In the sequel we will omit the term *weak* when referring to solutions that satisfy the conditions of Definition 2.1. In fact, every weak solution of (1.1) is in $L^\infty(\Omega)$ by the result of [17, Theorem 3.1].

Looking for a solution of problem (1.1) is equivalent to finding a critical point of the associated Euler–Lagrange functional $J : Z \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p} \widetilde{M}\left(\|u\|_Z^p\right) - \int_\Omega F(x, u(x)) dx - \frac{1}{p^*} \int_\Omega |u(x)|^{p^*} dx, \tag{2.5}$$

for all $u \in Z$. Note that J is a $C^1(Z)$ function for any $u \in Z$, and

$$\begin{aligned} J'(u)\phi = M\left(\|u\|_Z^p\right) \int_Q |u(x) - u(y)|^{p-2} \left(u(x) - u(y)\right) \left(\phi(x) - \phi(y)\right) K(x-y) dxdy \\ - \int_\Omega f(x, u(x))\phi(x) dx - \int_\Omega |u(x)|^{p^*-2} u(x)\phi(x) dx \end{aligned}$$

for any $\phi \in Z$.

Lemma 2.2 ([13]). *Let $K : \mathbb{R}^N \backslash \{0\} \to (0, \infty)$ be a function satisfying* (1.2).

(i) If $\{u_n\}$ is a bounded sequence in Z, then there exists $u \in L^m(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \to u$ in $L^m(\mathbb{R}^N)$ as $n \to \infty$ for any $m \in [1, p^*)$;

(ii) There exists a positive constant S depending on N and s, such that for every $u \in Z$, we have

$$\|u\|_{L^{p^*}(\Omega)}^p = \|u\|_{L^{p^*}(\mathbb{R}^N)}^p \le S^{-1} \|u\|_Z^p, \tag{2.6}$$

where $p^* = \frac{Np}{N-ps}$ is the fractional critical exponent.

Lemma 2.3. *There exist $\kappa, \rho > 0$ such that $J(u) \geq \kappa$ for $\|u\|_Z = \rho$.*

Proof. From assumptions (M_2) and (F_1), Hölder's inequality and (2.6), we get

$$\begin{aligned} J(u) &= \frac{1}{p}\widetilde{M}\left(\|u\|_Z^p\right) - \int_\Omega F(x,u(x))dx - \frac{1}{p^*}\int_\Omega |u(x)|^{p^*}dx \\ &\geq \frac{1}{p}c_0\|u\|_Z^{p\beta} - C\int_\Omega |u(x)|dx - C\int_\Omega |u(x)|^q dx - \frac{1}{p^*}\int_\Omega |u(x)|^{p^*}dx \\ &\geq \frac{1}{p}c_0\|u\|_Z^{p\beta} - C|\Omega|^{\frac{p^*-1}{p^*}}\left(\int_\Omega |u(x)|^{p^*}dx\right)^{\frac{1}{p^*}} \\ &\quad - C|\Omega|^{\frac{p^*-q}{p^*}}\left(\int_\Omega |u(x)|^{p^*}dx\right)^{\frac{q}{p^*}} - \frac{1}{p^*}\int_\Omega |u(x)|^{p^*}dx \\ &\geq \frac{1}{p}c_0\|u\|_Z^{p\beta} - C|\Omega|^{\frac{p^*-1}{p^*}}S^{-\frac{1}{p}}\|u(x)\|_Z - C|\Omega|^{\frac{p^*-q}{p^*}}S^{-\frac{q}{p}}\|u(x)\|_Z^q - \frac{1}{p^*}S^{-\frac{p^*}{p}}\|u(x)\|_Z^{p^*}. \end{aligned}$$

Since $0 < p\beta \leq 1 < p < q < p^*$, then there exist $\kappa, \rho > 0$ such that $J(u) \geq \kappa$ for $\|u\|_Z = \rho$. □

Lemma 2.4. *The functional $J(u)$ is bounded from below in $\bar{B}_r(0)$, where $\bar{B}_r(0) = \{u \in Z : \|u\|_Z \leq r\}$. Moreover, $\tilde{c} := \inf_{u\in\bar{B}_r(0)} J(u) < 0$.*

Proof. By the definition of J, we can get that $J(u)$ is bounded from below in $\bar{B}_r(0)$. Now, we show that $\tilde{c} := \inf_{u\in\bar{B}_r(0)} J(u) < 0$. In fact, by conditions (M_3) and (F_2), for $v \in C_0^\infty(\Omega)\backslash\{0\}$ with $\|v\|_Z = 1$ and $t > 0$, we have

$$\begin{aligned} J(tv) &= \frac{1}{p}\widetilde{M}\left(\|tv\|_Z^p\right) - \int_\Omega F(x,tv(x))dx - \frac{1}{p^*}\int_\Omega |tv(x)|^{p^*}dx \\ &\leq C_1 t^{p\alpha}\|v\|_Z^{p\alpha} - C_2 t^\delta\int_\Omega |v|^\delta dx - C_3 t^{p^*}\int_\Omega |v|^{p^*}dx < 0 \end{aligned}$$

for t sufficiently small, where C_i, $i = 1,2,3$ are some positive constants. Then we get $\tilde{c} < 0$. □

Proof of Theorem 1.1. We apply Ekeland's variational principle [1] to functional J on $\bar{B}_r(0)$ endowed with distance $\tau(u,w) = \|u - w\|_Z$, then there is a sequence $\{u_n\} \subset \bar{B}_r(0)$ such that

$$J(u_n) \to \inf_{u\in\bar{B}_r(0)} J(u) = \tilde{c}.$$

We infer that

$$J(u_n) - J(w) \leq \frac{\|u_n - w\|_Z}{n} \quad \text{for all } \ w \neq u_n.$$

Since $J \in C^1(Z,\mathbb{R})$, and $J(0) = 0$, we have $J'(u_n) \to 0$ as $n \to \infty$. Thus

$$J(u_n) \to \tilde{c} \quad \text{and} \quad J'(u_n) \to 0 \quad \text{as } n \to \infty.$$

$\{u_n\} \subset \bar{B}_r(0)$, so $\{u_n\}$ is bounded in Z, then $\{u_n\}$ is a bounded $(PS)_{\tilde{c}}$ sequence for J. Up to a subsequence, still denoted by $\{u_n\}$, such that u_n converges to some function u weakly in Z. From Lemma 2.2, $u_n \to u$ strongly in $L^p(\mathbb{R}^N)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Therefore, the sequence $|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))K(x-y)^{\frac{p-1}{p}}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^{2N})$ and it converges to $|u(x) - u(y)|^{p-2}(u(x) - u(y))K(x-y)^{\frac{p-1}{p}}$ almost everywhere in $\mathbb{R}^{2N}$. Moreover, $(\phi(x) - \phi(y))K(x-y)^{1/p} \in L^p(\mathbb{R}^{2N})$, thus

$$\int_Q |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\phi(x) - \phi(y))K(x-y)dxdy$$

converges to

$$\int_Q |u(x)-u(y)|^{p-2}\,(u(x)-u(y))\,(\phi(x)-\phi(y))\,K(x-y)dxdy$$

as $n \to \infty$.

On the other hand, from Lemma 2.2, we have that $u_n \to u$ in $L^m(\mathbb{R}^N)$ as $n \to \infty$ for any $m \in [1, p^*)$. By the conditions on the nonlinearity f, we get

$$\int_\Omega f(x,u_n(x))\phi(x)dx \to \int_\Omega f(x,u(x))\phi(x)dx, \quad \text{as } n \to \infty.$$

Next we claim that for every $\phi \in Z$, as $n \to \infty$,

$$\int_\Omega |u_n|^{p^*-2}u_n\phi dx \to \int_\Omega |u|^{p^*-2}u\phi dx. \tag{2.7}$$

Indeed, $u_n \to u$ a.e. in Ω as $n \to \infty$, since $u_n \to u$ weakly in Z. By the Egoroff theorem, for every $\delta > 0$, there exists Ω_δ such that $u_n \to u$ uniformly in $\Omega\backslash\Omega_\delta$ and $|\Omega_\delta| < \delta$, where $|\Omega_\delta|$ is the Lebesgue measure of Ω_δ. This together with the Lebesgue dominated convergence theorem implies

$$\lim_{n\to\infty}\int_{\Omega\backslash\Omega_\delta} |u_n|^{p^*-2}u_n\phi dx = \int_{\Omega\backslash\Omega_\delta} |u|^{p^*-2}u\phi dx \quad \text{for every } \phi \in Z. \tag{2.8}$$

Furthermore, for every $\phi \in Z$, and for every $\epsilon > 0$, by the absolute continuity of the integral, we can take δ small enough, such that

$$\int_{\Omega_\delta} \left||u_n|^{p^*-2}u_n - |u|^{p^*-2}u\right| |\phi|dx \le \frac{\epsilon}{2}.$$

For this δ, by (2.8), we obtain

$$\int_{\Omega\backslash\Omega_\delta} \left||u_n|^{p^*-2}u_n - |u|^{p^*-2}u\right| |\phi|dx \le \frac{\epsilon}{2},$$

for n large enough. So (2.7) holds. Thus we get

$$\langle J'(u),\phi\rangle_Z = \lim_{n\to\infty}\langle J'(u_n),\phi\rangle_Z \quad \forall\, \phi \in Z.$$

Then u is a solution of problem (1.1).

Finally, we prove that $u \neq 0$. Since $J(u_n) \to \tilde{c}$ as $n \to \infty$, we find

$$\begin{aligned}\tilde{c} + o(1) = J(u_n) &\ge C_1\|u_n\|_Z^{p\beta} - C_2\|u_n\|_Z - C_3\|u_n\|_Z^q - C_4\|u_n\|_Z^{p^*}\\ &\ge -C_2\|u_n\|_Z - C_3\|u_n\|_Z^q - C_4\|u_n\|_Z^{p^*},\end{aligned}$$

where C_i, $i = 1, \dots, 4$, are some positive constants. The last inequality yields that

$$C_2\|u_n\|_Z + C_3\|u_n\|_Z^q + C_4\|u_n\|_Z^{p^*} \ge -\tilde{c} > 0.$$

Since $u_n \to u$ in Z as $n \to \infty$, we then get $u \neq 0$. □

Acknowledgements

The authors have been supported by Fundamental Research Funds for the Central Universities XDJK2015C042, XDJK2015C043, SWU114040 and SWU114041, and National Natural Science Foundation of China (No. 11501468, No. 11501469).

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