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The Dirichlet problem for discontinuous perturbations of the mean curvature operator in Minkowski space

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Abstract. Using the critical point theory for convex, lower semicontinuous perturbations of locally Lipschitz functionals, we prove the solvability of the discontinuous Dirichlet problem involving the operator $u \mapsto \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)$.

Keywords: nonsmooth critical point theory, discontinuous Dirichlet problem, mean curvature operator, Palais–Smale condition.

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1 Introduction

Let Ω be an open bounded set in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$ of class C^2 and $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying the growth condition

$$|f(x,s)| \leq C(1+|s|^{q-1}), \quad \text{a.e. } x \in \Omega \ \text{ and all } \ s \in \mathbb{R}, \tag{1.1}$$

with some $q \in (1,\infty)$ and C a positive constant. For a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, we denote

$$\underline{f}(x,s) := \lim_{\delta \searrow 0} \operatorname{ess\,inf}\{f(x,t) : |t-s| < \delta\}$$

and

$$\overline{f}(x,s) := \lim_{\delta \searrow 0} \operatorname{ess\,sup}\{f(x,t) : |t-s| < \delta\}.$$

In this paper we consider the discontinuous Dirichlet problem with mean curvature operator in Minkowski space:

$$\mathcal{M}(u) := \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) \in \left[\underline{f}(x,u), \overline{f}(x,u)\right] \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{1.2}$$

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We assume that

$$\underline{f} \text{ and } \overline{f} \text{ are } N\text{-measurable} \tag{1.3}$$

(recall, a function $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is called *N-measurable* if $h(\cdot, v(\cdot))\colon \Omega \to \mathbb{R}$ is measurable whenever $v\colon \Omega \to \mathbb{R}$ is measurable [3]).

By *a solution* of (1.2) we mean a function $u \in W^{2,p}(\Omega)$ for some $p > N$, such that $\|\nabla u\|_\infty < 1$, which satisfies

$$\mathcal{M}(u)(x) \in \left[\underline{f}(x, u(x)), \overline{f}(x, u(x))\right], \quad \text{a.e. } \ x \in \Omega$$

and vanishes on $\partial\Omega$. At our best knowledge, this type of solutions, but for differential inclusions was firstly considered by A. F. Filippov [7]. Also, for partial differential inclusions we refer the reader to the pioneering works of I. Massabo and C. A. Stuart [12], J. Rauch [14], C. A. Stuart and J. F. Toland [16].

This work is motivated by the recent advances in the study of boundary value problems involving the operator $\mathcal{M}$ (see [2, 6] and the references therein) and by the seminal paper of K.-C. Chang [4] where the classical critical point theory is extended to locally Lipschitz functionals in order to study the problem

$$\Delta u \in \left[\underline{f}(x, u), \overline{f}(x, u)\right] \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0.$$

It is worth to point out that the operators $\mathcal{M}$ and Δ have essentially different structures and the theory developed in [4] appears as not being applicable to problem (1.2). Thus, we shall use a more general critical point theory, namely the one concerning convex, lower semicontinuous perturbations of locally Lipschitz functionals, which was developed by D. Motreanu and P. D. Panagiotopoulos [13] (also, see [10, 11]). It should be noticed that, using this theory, various existence results concerning Filippov type solutions for Dirichlet, periodic and Neumann problems involving the "p-relativistic" operator

$$u \mapsto \left(\frac{|u'|^{p-2}u'}{(1 - |u'|^p)^{1-1/p}} \right)'$$

were obtained in the recent paper [9].

A first existence result for the Dirichlet problem involving the operator $\mathcal{M}$ was obtained by F. Flaherty in [8], where it is shown that problem

$$\mathcal{M}(u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = \varphi,$$

has at least one solution, provided that $\partial\Omega$ has non-negative mean curvature and $\varphi \in C^2(\overline{\Omega})$ with $\|\nabla\varphi\|_\infty < 1$. The result was generalized in [1] by R. Bartnik and L. Simon, proving that problem

$$\mathcal{M}(u) = g(x, u) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0 \tag{1.4}$$

is solvable, provided that the Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is bounded. More general, if g satisfies the L^∞-growth condition:

for each $\rho > 0$ there is some $\alpha_\rho \in L^\infty(\Omega)$ such that

$$|g(x, s)| \le \alpha_\rho(x) \quad \text{for a.e. } x \in \Omega, \ \forall\, s \in \mathbb{R} \text{ with } |s| \le \rho,$$

it is shown in [2, Theorem 2.1] that (1.4) is still solvable. The approach in [2] relies on Szulkin's critical point theory [17]. The aim of the present paper is to obtain a similar result for the discontinuous problem (1.2). Precisely, we show in the main result (Theorem 4.1) that under assumptions (1.1) and (1.3) problem (1.2) always has at least one solution.

The rest of the paper is organized as follows. In Section 2 we recall some notions from nonsmooth analysis which will be needed in the sequel. The variational formulation of problem (1.2) is a key step in our approach and it is given in Section 3. Section 4 is devoted to the proof of the main result.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A functional $\mathcal{G}\colon X \to \mathbb{R}$ is called *locally Lipschitz* if for each $u \in X$, there is a neighborhood $\mathcal{N}_u$ of u and a constant $k > 0$ depending on $\mathcal{N}_u$ such that

$$|\mathcal{G}(w) - \mathcal{G}(z)| \le k\|w - z\|, \qquad \forall\, w, z \in \mathcal{N}_u.$$

For such a function $\mathcal{G}$, the *generalized directional derivative* at $u \in X$ in the direction of $v \in X$ is defined by

$$\mathcal{G}^0(u; v) = \limsup_{w \to u,\ t \searrow 0} \frac{\mathcal{G}(w + tv) - \mathcal{G}(w)}{t}$$

and the *generalized gradient* (in the sense of Clarke [5]) of $\mathcal{G}$ at $u \in X$ is defined as being the subset of X^*

$$\partial\mathcal{G}(u) = \left\{\eta \in X^* :\ \mathcal{G}^0(u; v) \ge \langle \eta, v\rangle,\ \ \forall\, v \in X\right\},$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. For more details concerning the properties of the generalized directional derivative and of the generalized gradient we refer to [5].

If $\mathcal{I}\colon X \to (-\infty, +\infty]$ is a functional having the structure

$$\mathcal{I} = \Phi + \mathcal{G}, \tag{2.1}$$

with $\mathcal{G}\colon X \to \mathbb{R}$ locally Lipschitz and $\Phi\colon X \to (-\infty, +\infty]$ proper, convex and lower semicontinuous, then an element $u \in X$ is said to be *a critical point* of $\mathcal{I}$ provided that

$$\mathcal{G}^0(u; v - u) + \Phi(v) - \Phi(u) \ge 0, \qquad \forall\, v \in X.$$

The number $c = \mathcal{I}(u)$ is called *a critical value* of $\mathcal{I}$ corresponding to the critical point u. According to Kourogenis *et al.* [10], $u \in X$ is a critical point of $\mathcal{I}$ iff

$$0 \in \partial\mathcal{G}(u) + \bar{\partial}\Phi(u),$$

where $\bar{\partial}\Phi(u)$ stands for the subdifferential of Φ at $u \in X$ in the sense of convex analysis [15], i.e.,

$$\bar{\partial}\Phi(u) = \{\eta \in X^* :\ \Phi(v) - \Phi(u) \ge \langle \eta, v - u\rangle,\ \ \forall\, v \in X\}.$$

Also, $\mathcal{I}$ in (2.1) is said *to satisfy the Palais–Smale condition* (in short, *(PS) condition*) if every sequence $(u_n) \subset X$ for which $(\mathcal{I}(u_n))$ is bounded and

$$\mathcal{G}^0(u_n; v - u_n) + \Phi(v) - \Phi(u_n) \ge -\varepsilon_n\|v - u_n\|, \qquad \forall\, v \in X,$$

for a sequence $(\varepsilon_n) \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$, possesses a convergent subsequence.

Theorem 2.1. ([11, Theorem 1]) *If $\mathcal{I}$ is bounded from below and satisfies the (PS) condition then $c = \inf_X \mathcal{I}$ is a critical value of $\mathcal{I}$.*

3 The variational setting

In the sequel we shall give the variational formulation of problem (1.2). With this aim, we introduce the set

$$K_0 = \left\{ v \in W^{1,\infty}(\Omega) : \ \|\nabla v\|_\infty \le 1, \ v = 0 \text{ on } \partial\Omega \right\}.$$

Notice that since $W^{1,\infty}(\Omega)$ is continuously (in fact, compactly) embedded into $C(\overline{\Omega})$, the evaluation at $\partial\Omega$ is understood in the usual sense. According to [2], K_0 is compact in $C(\overline{\Omega})$ and one has

$$\|v\|_\infty \le c(\Omega) \quad \text{for all } v \in K_0, \tag{3.1}$$

with $c(\Omega)$ a positive constant. Also, the functional $\Psi\colon C(\overline{\Omega}) \to (-\infty, +\infty]$ given by

$$\Psi(v) = \begin{cases} \displaystyle\int_\Omega \left[1 - \sqrt{1 - |\nabla v|^2}\right], & \text{for } v \in K_0, \\ +\infty, & \text{for } v \in C(\overline{\Omega}) \setminus K_0 \end{cases} \tag{3.2}$$

is proper, convex and lower semicontinuous [2, Lemma 2.4].

Having in view the growth condition (1.1), we define $\widehat{\mathcal{F}}\colon L^q(\Omega) \to \mathbb{R}$ by

$$\widehat{\mathcal{F}}(v) = \int_\Omega F(x,v), \qquad \forall\, v \in L^q(\Omega),$$

where

$$F(x,s) = \int_0^s f(x,\xi)\, d\xi \qquad (x \in \Omega, \ s \in \mathbb{R})$$

and, on account of the embedding $C(\overline{\Omega}) \subset L^q(\Omega)$, we introduce the functional

$$\mathcal{F} = \widehat{\mathcal{F}}|_{C(\overline{\Omega})}. \tag{3.3}$$

From [4, Theorem 2.1], one has that $\widehat{\mathcal{F}}$ is locally Lipschitz in $L^q(\Omega)$ and

$$\partial\widehat{\mathcal{F}}(v) \subset \left[\underline{f}(\cdot, v(\cdot)), \overline{f}(\cdot, v(\cdot))\right], \tag{3.4}$$

for all $v \in L^q(\Omega)$. Then, by the continuity of the embedding $C(\overline{\Omega}) \subset L^q(\Omega)$ it is clear that $\mathcal{F}$ is locally Lipschitz on $C(\overline{\Omega})$. Also, since $C(\overline{\Omega})$ is dense in $L^q(\Omega)$, the following holds (see [5, p. 47]):

$$\partial\widehat{\mathcal{F}}(v) = \partial\mathcal{F}(v), \qquad \forall\, v \in C(\overline{\Omega}). \tag{3.5}$$

Lemma 3.1. *Let $v \in K_0$. If $\ell \in \partial\mathcal{F}(v)$, then there is some $\zeta_\ell \in L^\infty(\Omega)$ such that $\zeta_\ell(x) \in \left[\underline{f}(x, v(x)), \overline{f}(x, v(x))\right]$ for a.e. $x \in \Omega$ and*

$$\langle \ell, w \rangle = \int_\Omega \zeta_\ell w \tag{3.6}$$

for all $w \in C(\overline{\Omega})$.

Proof. From (3.5) and (3.4) we infer that there is a function $\zeta_\ell \in L^{q'}(\Omega)$ with $1/q + 1/q' = 1$, such that $\zeta_\ell(x) \in \left[\underline{f}(x, v(x)), \overline{f}(x, v(x))\right]$ for a.e. $x \in \Omega$ and (3.6) holds true for all $w \in L^q(\Omega)$. To see that $\zeta_\ell \in L^\infty(\Omega)$, from (1.1) and (3.1), one gets

$$-C_1 \le \underline{f}(x, v(x)) \le \overline{f}(x, v(x)) \le C_1, \quad \text{for a.e. } x \in \Omega,$$

with $C_1 = C(1 + c(\Omega)^{q-1})$. This shows that $|\zeta_\ell(x)| \le C_1$ for a.e. $x \in \Omega$ and the proof is complete. □

The functional framework of Section 2 fits the following choices: $X = C(\overline{\Omega})$, $\Phi = \Psi$ in (3.2), $\mathcal{G} = \mathcal{F}$ in (3.3) and

$$\mathcal{I} := \Psi + \mathcal{F}.$$

Notice that, the compactness of $K_0 \subset C(\overline{\Omega})$ implies that $\mathcal{I}$ satisfies the (PS) condition.

4 Main result

We have the following theorem.

Theorem 4.1. *Assume that* (1.1) *and* (1.3) *hold true. If u is a critical point of $\mathcal{I}$, then u is a solution of problem* (1.2)*. Moreover, $\mathcal{I}$ is bounded from below and attains its infimum at some $u_0 \in K_0$, which solves problem* (1.2)*.*

Proof. Let u be a critical point of $\mathcal{I}$. Then $u \in K_0$ and there exist $h_u \in \overline{\partial}\Psi(u)$ and $\ell_u \in \partial\mathcal{F}(u)$ such that

$$\langle h_u, w\rangle + \langle \ell_u, w\rangle = 0, \qquad \forall\, w \in C(\overline{\Omega}).$$

This and the fact that $h_u \in \overline{\partial}\Psi(u)$ yield

$$\Psi(w) - \Psi(u) + \langle \ell_u, w - u\rangle \geq 0, \qquad \forall\, w \in C(\overline{\Omega}). \tag{4.1}$$

Using Lemma 3.1 we deduce that there is some $\zeta_u = \zeta(\ell_u) \in L^\infty(\Omega)$ such that

$$\zeta_u(x) \in \left[\underline{f}(x, u(x)), \overline{f}(x, u(x))\right], \quad \text{a.e. } x \in \Omega \tag{4.2}$$

and

$$\langle \ell_u, w\rangle = \int_\Omega \zeta_u w, \qquad \forall\, w \in C(\overline{\Omega}). \tag{4.3}$$

By virtue of (4.3), inequality (4.1) becomes

$$\Psi(w) - \Psi(u) + \int_\Omega \zeta_u(w - u) \geq 0, \qquad \forall\, w \in C(\overline{\Omega}). \tag{4.4}$$

On account of Lemma 2.2 in [6], for each function $e \in L^\infty(\Omega)$, the Dirichlet problem

$$\mathcal{M}(v) = e(x) \quad \text{in } \Omega, \qquad v|_{\partial\Omega} = 0$$

has a unique solution $v_e \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. Then, from Lemma 2.3 in [2], one has that v_e is the unique solution in K_0 of the variational inequality

$$\int_\Omega \left[\sqrt{1 - |\nabla v|^2} - \sqrt{1 - |\nabla w|^2} + e(w - v)\right] \geq 0, \qquad \forall\, w \in K_0$$

and hence,

$$\Psi(w) - \Psi(v_e) + \int_\Omega e(w - v_e) \geq 0, \qquad \forall\, w \in C(\overline{\Omega}).$$

From this and (4.4), we infer that $u = v_e$, with $e = \zeta_u$. But, on account of (4.2), this means that u solves problem (1.2).

Next, for arbitrary $u \in K_0$, by (1.1) and (3.1), the primitive F satisfies

$$|F(x, u(x))| \leq C\left(c(\Omega) + c(\Omega)^q/q\right) =: C_2, \quad \text{for a.e. } x \in \Omega.$$

Hence,

$$|\mathcal{F}(u)| \le \int_{\Omega} |F(x,u)| \le C_2 \,\mathrm{vol}(\Omega), \qquad \forall\, u \in K_0.$$

We deduce that the functional $\mathcal{I}$ is bounded from below on $C(\overline{\Omega})$. Then, using that $\mathcal{I}$ verifies the (PS) condition and Theorem 2.1, we have that

$$c = \inf_{C(\overline{\Omega})} \mathcal{I} = \inf_{K_0} \mathcal{I}$$

is a critical value of $\mathcal{I}$ and the proof is complete. □

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