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Nonoscillatory solutions for super-linear Emden–Fowler type dynamic equations on time scales

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Abstract. In this paper, we consider the following Emden–Fowler type dynamic equations on time scales

$$\left(a(t)|x^{\Delta}(t)|^{\alpha}\operatorname{sgn} x^{\Delta}(t)\right)^{\Delta} + b(t)|x(t)|^{\beta}\operatorname{sgn} x(t) = 0,$$

when $\alpha < \beta$. The classification of the nonoscillatory solutions are investigated and some necessary and sufficient conditions of the existence of oscillatory and nonoscillatory solutions are given by using the Schauder–Tychonoff fixed point theorem. Three possibilities of two classes of double integrals which are not only related to the coefficients of the equation but also linked with the classification of the nonoscillatory solutions and oscillation of solutions are put forward. Moreover, an important property of the intermediate solutions on time scales is indicated. At last, an example is given to illustrate our main results.

Keywords: Emden–Fowler type dynamic equations, intermediate solutions, time scales, nonoscillatory solutions.

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1 Introduction

Emden–Fowler dynamic equations originated in the early 20th century and they were established in the early research of gas dynamics in astrophysics [8]. They also occur in the study of fluid mechanics, relativity, nuclear physics and chemical reaction systems, one can see the survey article by Wong [15] for detailed background of the generalized Emden–Fowler equation. With the development of science and technology, the super-linear Emden–Fowler type dynamic equations on time scales have played an important and extensive role in physics and engineering technology. We refer the reader to [16] and the references cited therein. The basic theorems and applications can be found in Agarwal *et al.* [1]. In the recent years, there have been lots of results for Emden–Fowler type equations in [2, 4, 7, 9].

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In 2007 and 2011, Cecchi *et al.* [3] and Naito [13] studied the asymptotic behavior of nonoscillatory solutions for differential equations of the following forms

$$\big(a(t)\phi(x')\big)' + b(t)\phi(x) = 0,$$

and

$$(p(t)|x'|^{\alpha} \operatorname{sgn} x')' + q(t)|x|^{\beta} \operatorname{sgn} x = 0,$$

respectively. In 2008, Cecchi *et al.* [2] considered the intermediate solutions for Emden–Fowler type equations

$$\big(a(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)\big)' + b(t)|x(t)|^{\beta} \operatorname{sgn} x(t) = 0.$$

In 2010, Kamo and Usami [11] discussed the slowly decaying positive solutions for the quasi-linear ordinary differential equations

$$(p(t)|u'|^{\alpha-1}u')' + q(t)|u|^{\lambda-1}u = 0.$$

In 2011, Jia *et al.* [10] discussed oscillatory solutions for the second order super-linear dynamic equations on time scales

$$x^{\Delta\Delta}(t) + p(t)f\big(x(\sigma(t))\big) = 0.$$

In 2011, Erbe *et al.* [7] considered the asymptotic behavior of solutions for Emden–Fowler equations on time scales

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(t) = 0, \qquad \alpha > 0, \tag{1.1}$$

where $p \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, α is the quotient of odd positive integers, and $\mathbb{T}$ denotes a time scale which is unbounded from above. This article proposed an important property about the solution of (1.1) under the condition $\int_{t_0}^{\infty} t^{\alpha}|p(t)|\,\Delta t < \infty$.

Zhou and Lan gave a classification of nonoscillatory solutions for the second-order neutral delay dynamic equations on time scales

$$[x(t) - c(t)x(t-\tau)]^{\Delta\Delta} + f(t, x(g_1(t)), \dots, x(g_m(t))) = 0, \qquad t \in \mathbb{T},$$

and some existence results of each kind of nonoscillatory solutions were also established in [17].

In 2014, Došlá and Marini [4] studied the nonoscillatory solutions for second order Emden–Fowler type differential equation

$$\big(a(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)\big)' + b(t)|x(t)|^{\beta} \operatorname{sgn} x(t) = 0.$$

This article has an important and far-reaching influence because it solved the open problem on the possible coexistence of three types of nonoscillatory solutions for super-linear Emden–Fowler differential equations. However, to the best of our knowledge, the coexistence of nonoscillatory solutions for dynamic equations on time scales has been scarcely investigated.

Motivated by [4], we consider the second order super-linear dynamic equations on time scales

$$\big(a(t)|x^{\Delta}(t)|^{\alpha} \operatorname{sgn} x^{\Delta}(t)\big)^{\Delta} + b(t)|x(t)|^{\beta} \operatorname{sgn} x(t) = 0, \tag{1.2}$$

where $0 < \alpha < \beta$ are constants and $a(t) > 0$, $b(t) \geq 0$ are rd-continuous functions on $[0,\infty)_{\mathbb{T}}$, and

$$I_a = \int_0^{\infty} \frac{1}{a^{1/\alpha}(s)}\,\Delta s = \infty, \qquad I_b = \int_0^{\infty} b(s)\,\Delta s < \infty.$$

When $a(t) \equiv 1$, that is equation

$$\left(|x^{\Delta}(t)|^{\alpha} \operatorname{sgn} x^{\Delta}(t)\right)^{\Delta} + b(t)|x(t)|^{\beta} \operatorname{sgn} x(t) = 0. \tag{1.3}$$

If $\alpha \neq \beta$, then the prototype of (1.3) is the Emden–Fowler equation

$$x^{\Delta\Delta} + b(t)|x|^{\beta} \operatorname{sgn} x = 0. \tag{1.4}$$

Moreover, the half-linear case of (1.2) is the following form

$$\left(a(t)|x^{\Delta}(t)|^{\alpha} \operatorname{sgn}\ x^{\Delta}(t)\right)^{\Delta} + b(t)|x(t)|^{\alpha} \operatorname{sgn}\ x(t) = 0. \tag{1.5}$$

We will consider only the eventually positive solutions of (1.2) in the following section and denote

$$x^{[1]}(t) = a(t)|x^{\Delta}(t)|^{\alpha} \operatorname{sgn} x^{\Delta}(t)$$

for convenience.

The main work of this article can be listed as follows. Firstly, we improve the result in [6]. We will show that the case when the solution is a constant as $x^{[1]}(t)$ tends to a constant is impossible. Secondly, we investigate the necessary and sufficient conditions for the existence of oscillatory and nonoscillatory solutions by methods different from [6]. Thirdly, we present an important property about intermediate solutions on time scales which generalize the related contributions to the subject in [4]. The research about the second order super-linear dynamic equations on time scales unifies the cases of differential equations and difference equations.

The paper is organized as follows. In Section 2, we introduce some definitions and a lemma about oscillatory and nonoscillatory solutions and the Schauder–Tychonoff fixed point theorem. In Section 3, we investigate the classification of the nonoscillatory solutions. Then we give some necessary and sufficient conditions for the existence of some oscillatory and nonoscillatory solutions by the Schauder–Tychonoff fixed point theorem. We propose three possibilities of two classes of double integrals which is related to the coefficients of the equation and an important property of the intermediate solutions. Moreover, an example is given to illustrate our main results.

2 Preliminaries

In this section, we collect some definitions and a lemma about dynamic equations on time scales.

Definition 2.1 ([14]). We say that a nontrivial solution x of (1.2) has a generalized zero at t, if $x(t)x(\sigma(t)) \leq 0$. If $x(t) = 0$ we say that solution x has a common zero at t.

Definition 2.2 ([14]). We say that a solution x of equation (1.2) is nonoscillatory on $\mathbb{T}$, if there exists $\tau \in \mathbb{T}$ such that there does not exist any generalized zero at t for $t \in [\tau, \infty)_{\mathbb{T}}$.

A nontrivial solution x of equation (1.2) is called oscillatory on $\mathbb{T}$, if for every $\tau \in \mathbb{T}$ has x a generalized zero on $[\tau, \infty)_{\mathbb{T}}$.

Definition 2.3 ([6]). We say that equation (1.2) is super-linear, if there exists a constant $\gamma > 0$ such that $|v^{-\gamma}||b(s)v^{\beta}|$ is nondecreasing in $|v|$ for each fixed s and

$$\int_M^{\infty} \frac{\Delta v}{v^{\gamma/\alpha}} < \infty \quad \text{for any } M > 0. \tag{2.1}$$

Lemma 2.4 (Schauder–Tychonoff fixed point theorem [12]). *Let X be a locally convex space, $K \subset X$ be nonempty and convex, $S \subset K$, S be compact. Given a continuous map $F\colon K \to S$, then there exists $\tilde{x} \in S$ such that $F(\tilde{x}) = \tilde{x}$.*

3 Main results

In this section, we investigate the classification of the nonoscillatory solutions. Then we give some necessary and sufficient conditions for the existence of some oscillatory and nonoscillatory solutions by Schauder–Tychonoff fixed point theorem. We also present three possibilities of two classes of double integrals which is related to the coefficients of the equation and an important property of the intermediate solutions. We extend some results of [4] to time scales.

Theorem 3.1. *The class $\mathbb{P}$ of all eventually positive solutions of* (1.2) *can be divided into three subclasses:*

$$M^+_{\infty,\ell} = \{x \in \mathbb{P} : x(\infty) = \infty,\ x^{[1]}(\infty) = \ell,\ 0 < \ell < \infty\},$$
$$M^+_{\infty,0} = \{x \in \mathbb{P} : x(\infty) = \infty,\ x^{[1]}(\infty) = 0\},$$
$$M^+_{\ell,0} = \{x \in \mathbb{P} : x(\infty) = \ell,\ x^{[1]}(\infty) = 0,\ 0 < \ell < \infty\}.$$

The superscript symbol "+" means that solutions are eventually positive increasing. We call solutions in $M^+_{\infty,\ell}$, $M^+_{\infty,0}$, $M^+_{\ell,0}$ dominant solutions, intermediate solutions and subdominant solutions.

Proof. Let $x(t)$ be a positive solution of (1.2) for large t. Then there exists a $t_0 > 0$ such that $x(t) > 0$ as $t \geq t_0$. From (1.2) we have

$$(x^{[1]}(t))^\Delta = -b(t)|x(t)|^\beta \text{sgn}\, x(t) \leq 0,$$

so $x^{[1]}(t)$ is non-increasing for $t \geq t_0$, which implies that $x^{[1]}(t)$ is eventually positive or negative.

We conclude that $x^{[1]}(t) \geq 0$, $t \geq t_0$. Otherwise, if $x^{[1]}(t) < 0$ for $t \geq t_0$, then there is a positive number c, such that $x^{[1]}(t) \leq -c$. Integrating the last inequality from 0 to t and letting $t \to \infty$, we have $x(t) \to -\infty$ as $t \to \infty$ since $x^\Delta(t) < 0$, which is contrary to the assumption of the eventually positive solution. So $x^{[1]}(t) \geq 0$, i.e. $x^\Delta(t) \geq 0$.

Thus for any ℓ, $0 < \ell < \infty$, the possible cases of $x^{[1]}(t)$ and $x(t)$ when $t \to \infty$ are as follows:

(i) $x(\infty) = \ell,\ x^{[1]}(\infty) = \ell$;

(ii) $x(\infty) = \infty,\ x^{[1]}(\infty) = \ell$;

(iii) $x(\infty) = \infty,\ x^{[1]}(\infty) = 0$;

(iv) $x(\infty) = \ell,\ x^{[1]}(\infty) = 0$.

Now we prove the case (i) is impossible. If $\lim_{t\to\infty} x^{[1]}(t) = \ell$, then there exists $t_0 > 0$ such that $x^{[1]}(t) \geq \frac{\ell}{2}$ for $t \geq t_0$, i.e. $x^\Delta(t) \geq \frac{\ell^{1/\alpha}}{(2a)^{1/\alpha}}$. Integrating the last inequality and by the condition $I_a = \infty$, we have $x(\infty) = \infty$. The proof is completed. □

Denote

$$J = \int_0^\infty \frac{1}{a^{1/\alpha}(s)} \left(\int_s^\infty b(r)\Delta r \right)^{1/\alpha} \Delta s, \qquad K = \int_0^\infty b(s) \left(\int_0^s \frac{1}{a^{1/\alpha}(r)} \Delta r \right)^\beta \Delta s.$$
$$J_m = \int_0^\infty \frac{1}{a^{1/\alpha}(s)} \left(\int_s^\infty b(r)\Delta r \right)^{1/m} \Delta s, \qquad K_m = \int_0^\infty b(s) \left(\int_0^s \frac{1}{a^{1/\alpha}(r)} \Delta r \right)^m \Delta s.$$

Now we give some sufficient and necessary conditions for the existence of oscillatory and nonoscillatory solutions.

Theorem 3.2. *The following hold for* (1.2)*:*

i_1) *The class* $\mathbb{M}^+_{\ell,0}$ *is nonempty if and only if* $J < \infty$. *Moreover, for any* ℓ, $0 < \ell < \infty$, *there exists* $x \in \mathbb{M}^+_{\ell,0}$ *such that* $\lim_{t\to\infty} x(t) = \ell$.

i_2) *The class* $\mathbb{M}^+_{\infty,\ell}$ *is nonempty if and only if* $K < \infty$. *Moreover, for any* ℓ, $0 < \ell < \infty$, *there exists* $x \in \mathbb{M}^+_{\infty,\ell}$ *such that* $\lim_{t\to\infty} x^{[1]}(t) = \ell$.

i_3) *Let* $\alpha < \beta$. *Equation* (1.2) *is oscillatory if and only if* $J = \infty$.

i_4) *Let* $\alpha > \beta$. *Equation* (1.2) *is oscillatory if and only if* $K = \infty$.

Proof. i_1) (The "only if" part) Let $x(t)$ be a nonoscillatory solution in $\mathbb{M}^+_{\ell,0}$, $t > t_0 > 0$. i.e. $x(\infty) = \ell$, $x^{[1]}(\infty) = 0$. Integrating equation (1.2) from t to ∞, we get

$$a(t)(x^\Delta(t))^\alpha = \int_t^\infty b(t)x^\beta(t)\,\Delta t,$$

i.e.

$$x^\Delta(t) = \frac{1}{a^{1/\alpha}(t)}\left(\int_t^\infty b(s)x^\beta(s)\,\Delta s\right)^{1/\alpha} \geq \frac{1}{a^{1/\alpha}(t)}\left(\frac{\ell}{2}\right)^{\beta/\alpha}\left(\int_t^\infty b(s)\,\Delta s\right)^{1/\alpha}.$$

Integrating the above inequality from 0 to ∞, we obtain

$$x(\infty) \geq x(0) + \left(\frac{\ell}{2}\right)^{\beta/\alpha}\int_0^\infty \frac{1}{a^{1/\alpha}(s)}\left(\int_s^\infty b(r)\,\Delta r\right)^{1/\alpha}\Delta s.$$

By contrary, if $J = \infty$, then $x(\infty) = \infty$, which is a contradiction with $x(\infty) = \ell$, $0 < \ell < \infty$. So $J < \infty$.

(The "if" part) Suppose $J < \infty$, then there exist $c > 0$, $t_1 > 0$ such that

$$c^{\beta/\alpha}\int_{t_1}^\infty \frac{1}{a^{1/\alpha}(t)}\left(\int_t^\infty b(s)\,\Delta s\right)^{1/\alpha}\Delta t \leq \frac{c}{2}. \tag{3.1}$$

Define $X = \{x \in C_{rd}[t_1,\infty)_{\mathbb{T}} : \frac{c}{2} \leq x(t) \leq c,\ t \geq t_1\}$ and

$$Tx(t) = c - \int_t^\infty \frac{1}{a^{1/\alpha}(s)}\left(\int_s^\infty b(r)x^\beta(r)\,\Delta r\right)^{1/\alpha}\Delta s, \qquad t \geq t_1.$$

Now, we separate the proof into the following two steps.

(i) T maps X into itself. If $x \in X$, then from (3.1) we get

$$0 \leq \int_t^\infty \frac{1}{a^{1/\alpha}(s)}\left(\int_s^\infty b(r)x^\beta(r)\,\Delta r\right)^{1/\alpha}\Delta s \leq \frac{c}{2}.$$

So $\frac{c}{2} \leq Tx \leq c$, which implies that $Tx \in X$.

(ii) T is continuous. Obviously Tx is compact. Let $\{x_n\}$ be a sequence of measurable functions of X converging to $x \in X$ as $n \to \infty$ in the topology of $C_{rd}[t_1,\infty)_{\mathbb{T}}$.

Since $0 \leq x_n(t) \leq c$, we get $\frac{c}{2} \leq Tx_n(t) \leq c$. The Lebesgue dominated convergence theorem shows that $Tx_n(t_1) \to Tx(t_1)$, which implies $Tx_n(t) \to Tx(t)$ uniformly on $[t_1,\infty)$, that is T is continuous.

Therefore, applying the Schauder–Tychonoff fixed point theorem, we see that there exists an element $x \in X$ such that $x = Tx$, which shows that $x(t)$ is a positive solution of equation

(1.2) for $t \geq t_1$. Since $x(t)$ is uniformly bounded, then from Theorem 3.1 we get that $x(t)$ is an element in $\mathbb{M}^+_{\ell,0}$.

i_2) (The "only if" part) Let $x(t)$ be a nonoscillatory solution in $M^+_{\infty,\ell}$, $t > t_1 > 0$, i.e. $x(\infty) = \infty$, $x^{[1]}(\infty) = \ell$. According to the definition of limit, we know

$$a(t)(x^\Delta(t))^\alpha \geq \frac{\ell}{2}, \quad \text{i.e.} \quad x^\Delta(t) \geq \frac{(\frac{\ell}{2})^{1/\alpha}}{a^{1/\alpha}(t)}.$$

Integrating from 0 to t, we get

$$x(t) \geq \left(\frac{\ell}{2}\right)^{1/\alpha} \int_0^t \frac{1}{a^{1/\alpha}(s)}\,\Delta s.$$

Substituting the above to equation (1.2) and integrating from 0 to ∞, we have

$$\begin{aligned}\int_0^\infty (x^{[1]}(t))^\Delta\,\Delta t &= -\int_0^\infty b(t)x^\beta(t)\,\Delta t \\ &\leq -\left(\frac{\ell}{2}\right)^{1/\alpha} \int_0^\infty b(t)\left(\int_0^t \frac{1}{a^{1/\alpha}(s)}\,\Delta s\right)^\beta \Delta t,\end{aligned}$$

i.e.

$$x^{[1]}(\infty) \leq x^{[1]}(0) - \left(\frac{\ell}{2}\right)^{1/\alpha} \int_0^\infty b(t)\left(\int_0^t \frac{1}{a^{1/\alpha}(s)}\,\Delta s\right)^\beta \Delta t.$$

By contrary, if $K = \infty$, then $x^{[1]}(\infty) \leq -\infty$, which is a contradiction with the condition $x^{[1]}(\infty) = \ell$, $0 < \ell < \infty$. So $K < \infty$.

(The "if" part) Suppose $K < \infty$ holds. We can choose proper $\ell > 0$ such that

$$\int_{t_1}^\infty b(t)\left(\int_{t_1}^t \frac{(2\ell)^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s\right)^\beta \Delta t \leq \ell.$$

Define the subset X of $C_{rd}[t_1,\infty)_{\mathbb{T}}$ and the mapping $A\colon X \to C_{rd}[t_1,\infty)_{\mathbb{T}}$ by

$$X = \left\{x \in C_{rd}[t_1,\infty)_{\mathbb{T}} : \int_{t_1}^t \frac{\ell^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s \leq x(t) \leq \int_{t_1}^t \frac{(2\ell)^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s,\ t \geq t_1\right\}$$

and

$$Ax(t) = \int_{t_1}^t \frac{(\ell + \int_s^\infty b(r)x^\beta(r)\Delta r)^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s, \qquad t \geq t_1. \tag{3.2}$$

Now, in order to use the Schauder–Tychonoff fixed point theorem in Lemma 2.4 we separate the proof into the following three steps.

(i) A maps X into itself. For any $x \in X$, we have

$$0 \leq \int_s^\infty b(r)x^\beta(r)\,\Delta r \leq \int_s^\infty b(r)\left(\int_{t_1}^t \frac{(2\ell)^{1/\alpha}}{a^{1/\alpha}(u)}\Delta u\right)^\beta \Delta r \leq \ell, \qquad s \geq t_1, \tag{3.3}$$

from (3.2) and (3.3) we obtain

$$\int_{t_1}^t \frac{\ell^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s \leq Ax(t) \leq \int_{t_1}^t \frac{(2\ell)^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s, \qquad t \geq t_1,$$

which implies that $Ax \in X$.

(ii) AX is compact. Since A maps X into itself, we only need to illustrate X is compact. Let

$$y(t) = \frac{x(t)}{\int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\Delta s}.$$

Then

$$\ell^{1/\alpha} \le y(t) \le (2\ell)^{1/\alpha}.$$

For any $x_n \in X$, since

$$y_n = \frac{x_n(t)}{\int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\Delta s}$$

is bounded, from the compactness theorem, we can know that there exists a convergent subsequence y_{n_k}. So for any $x_n \in X$ there exists a convergent subsequence x_{n_k}, which shows that X is compact.

(iii) A is continuous. Let $\{x_n\}$ be a sequence of measurable functions of X converging to $x \in X$ as $n \to \infty$ in the topology of $C_{rd}[t_1, \infty)_{\mathbb{T}}$.

From $0 \le x_n(t) \le \int_{t_1}^{t} \frac{(2\ell)^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s$, we obtain

$$\begin{aligned} 0 &\le \int_{t_1}^{\infty} b(t) x_n^{\beta}(t)\,\Delta t \le \int_{t_1}^{\infty} b(t) \left(\int_{t_1}^{t} \frac{(2\ell)^{1/\alpha}}{a^{1/\alpha}(s)}\,\Delta s \right)^{\beta} \Delta t \\ &\le (2\ell)^{1/\alpha} \int_{0}^{\infty} b(t) \left(\int_{0}^{t} \frac{1}{a^{1/\alpha}(s)}\,\Delta s \right)^{\beta} \Delta t = (2\ell)^{1/\alpha} K < \infty. \end{aligned}$$

The Lebesgue dominated convergence theorem shows that

$$\int_{t_1}^{\infty} b(s) x_n^{\beta}(s)\,\Delta s \to \int_{t_1}^{\infty} b(s) x^{\beta}(s)\,\Delta s \quad \text{as } n \to \infty,$$

i.e. $Ax_n(\infty) \to Ax(\infty)$, which implies $Ax_n(t) \to Ax(t)$ uniformly on $[t_1, \infty)$, that is A is continuous.

Therefore, applying the Schauder–Tychonoff fixed point theorem, we see that there exists an element $x \in X$ such that $x = Ax$, which shows that $x(t)$ is a positive solution of equation (1.2) for $t \ge t_1$. So $x(t)$ is an element in $\mathbb{M}^{+}_{\infty,\ell}$.

i_3) The "only if" part follows from i_1).

To prove the "if" part. Assume for contradiction that (1.2) has a nonoscillatory solution $x(t)$. We may assume without loss of generality that $x(t) > 0$ for $t \ge t_0 > 0$. Integrating (1.2) from t to ∞ and noting that $\lim_{t\to\infty} a(t)(x^{\Delta}(t))^{\alpha} \ge 0$, we have

$$a(t)\big(x^{\Delta}(t)\big)^{\alpha} \ge \int_{t}^{\infty} b(s) x^{\beta}(s)\,\Delta s, \qquad t \ge t_0,$$

which implies

$$x^{\Delta}(t) \ge \frac{1}{a^{1/\alpha}(t)} \left(\int_{t}^{\infty} b(s) x^{\beta}(s)\,\Delta s \right)^{1/\alpha}, \qquad t \ge t_0.$$

Dividing the above by $x^{\gamma/\alpha}(t)$, we obtain

$$\frac{x^{\Delta}(t)}{x^{\gamma/\alpha}(t)} \ge \frac{1}{a^{1/\alpha}(t)} \left(\int_{t}^{\infty} \frac{b(s) x^{\beta}(s)}{x^{\gamma}(t)}\,\Delta s \right)^{1/\alpha} \ge \frac{1}{a^{1/\alpha}(t)} \left(\int_{t}^{\infty} \frac{b(s) x^{\beta}(s)}{x^{\gamma}(s)}\,\Delta s \right)^{1/\alpha}.$$

Since $x(t)$ is an eventually positive solution, there exists a positive constant c_0 such that $x(t) \geq c_0$ for $t \geq t_0$. We have

$$\frac{b(t)x^{\beta}(t)}{x^{\gamma}(t)} \geq \frac{b(t)c_0^{\beta}}{c_0^{\gamma}}, \qquad t \geq t_0,$$

hence

$$\frac{x^{\Delta}(t)}{x^{\gamma/\alpha}(t)} \geq \left(\frac{1}{c_0^{\gamma}}\frac{1}{a(t)}\int_t^{\infty} b(s)c_0^{\beta}\,\Delta s\right)^{1/\alpha} \geq \frac{1}{c_0^{\gamma/\alpha}}\left(\frac{1}{a(t)}\int_t^{\infty} b(s)c_0^{\beta}\,\Delta s\right)^{1/\alpha}, \qquad t \geq t_0.$$

Integrating above over $[t_0, t]$ and by condition (2.1), we have

$$\frac{1}{c_0^{\gamma/\alpha}}\int_{t_0}^{t}\left(\frac{1}{a(t)}\int_t^{\infty} b(s)c_0^{\beta}\,\Delta s\right)^{1/\alpha} \leq \int_{x(t_0)}^{x(t)}\frac{\Delta v}{v^{\gamma/\alpha}} < \infty,$$

which implies

$$\int_{t_0}^{\infty}\left(\frac{1}{a(t)}\int_t^{\infty} b(s)c_0^{\beta}\,\Delta s\right)^{1/\alpha}\Delta t < \infty.$$

But this contradicts $J = \infty$.

The proof of i_4) is similar to that of i_3), so it is omitted here. The proof is completed. □

Motivated by [5], now we give the following proof.

Lemma 3.3. *If $0 < m \leq 1$, then $J_m = \infty \Rightarrow K_m = \infty$.*

Proof. Let $p = 1/m$. Obviously $p \geq 1$. The integrals J_m, K_m can be written as

$$J_m = \int_0^{\infty}\frac{1}{a^{1/\alpha}(t)}\left(\int_t^{\infty} b(s)\,\Delta s\right)^{p}\Delta t, \qquad K_m = \int_0^{\infty} b(t)\left(\int_0^{t}\frac{1}{a^{1/\alpha}(s)}\,\Delta s\right)^{1/p}\Delta t.$$

Put $\frac{1}{a^{1/\alpha}(t,s)} = 0$ for $s < t$ and $\frac{1}{a^{1/\alpha}(t,s)} = \frac{1}{a^{1/\alpha}(t)}$ for $s \geq t$. Then we obtain

$$\begin{aligned}
\sqrt[p]{J_m} &= \sqrt[p]{\int_0^{\infty}\frac{1}{a^{1/\alpha}(t)}\left(\int_t^{\infty} b(s)\Delta s\right)^{p}\Delta t} = \sqrt[p]{\int_0^{\infty}\left[\int_t^{\infty}\left(\frac{1}{a^{1/\alpha}(t)}\right)^{1/p} b(s)\,\Delta s\right]^{p}\Delta t}\\
&= \sqrt[p]{\int_0^{\infty}\left[\int_0^{\infty}\left(\frac{1}{a^{1/\alpha}(t,s)}\right)^{1/p} b(s)\,\Delta s\right]^{p}\Delta t} \leq \int_0^{\infty}\sqrt[p]{\int_0^{\infty}\frac{1}{a^{1/\alpha}(t,s)}b(s)^{p}\,\Delta t}\,\Delta s\\
&= \int_0^{\infty} b(s)\,\Delta s\sqrt[p]{\int_0^{\infty}\frac{1}{a^{1/\alpha}(t,s)}\,\Delta t} = \int_0^{\infty} b(s)\,\Delta s\sqrt[p]{\int_0^{s}\frac{1}{a^{1/\alpha}(t)}\,\Delta t}\\
&= \int_0^{\infty} b(s)\,\Delta s\sqrt[p]{\int_0^{s}\frac{1}{a^{1/\alpha}(t,s)}\,\Delta s + \int_s^{\infty}\frac{1}{a^{1/\alpha}(t,s)}\,\Delta s}\\
&= \int_0^{\infty} b(s)\,\Delta s\sqrt[p]{\int_0^{s}\frac{1}{a^{1/\alpha}(t,s)}\,\Delta s} \leq \int_0^{\infty} b(s)\,\Delta s\sqrt[p]{\int_0^{\infty}\frac{1}{a^{1/\alpha}(t)}\,\Delta t} = K_m.
\end{aligned}$$

The proof is completed. □

Lemma 3.4. *If $\alpha < \beta$, then $J = \infty \Rightarrow K = \infty$.*

Proof. Consider two cases: (i) $\alpha \le 1$; (ii) $\alpha > 1$.

Case (i): Let $t_0 \ge 0$ be such that $\int_0^t \frac{1}{a^{1/\alpha}(s)}\,\Delta s > 1$ for $t \ge t_0$. Since $J = \infty$, by Lemma 3.3 $K_\alpha = \infty$. Since $\alpha < \beta$, we have $K > K_\alpha = \infty$.

Case (ii): We have

$$J = \int_0^\infty \frac{1}{a^{1/\alpha}(t)} \left(\int_t^\infty b(s)\,\Delta s \right)^{1/\alpha} \Delta t \quad \text{for } t \ge t_0.$$

Integrating by parts we have

$$\begin{aligned} J &= \int_0^\infty \left(\int_0^t \frac{1}{a1/\alpha(s)}\,\Delta s \right)^{\Delta} \left(\int_t^\infty b(s)\,\Delta s \right)^{1/\alpha} \Delta t \\ &= \frac{1}{\alpha} \int_0^\infty b(t) \left(\int_0^{\sigma(t)} \frac{1}{a^{1/\alpha}(s)}\,\Delta s \right) \left(\int_t^\infty b(s)\,\Delta s \right)^{\frac{1-\alpha}{\alpha}} \Delta t. \end{aligned}$$

Since $1 < \alpha < \beta$, by the Hölder inequality, we obtain

$$J \le \frac{1}{\alpha} \left(\int_0^\infty \left(\int_0^{\sigma(t)} \frac{1}{a^{1/\alpha}(s)}\,\Delta s \right)^{\beta} b(t)\,\Delta t \right)^{\frac{1}{\beta}} \times \left(\int_0^\infty \frac{b(t)^{(1-1/\beta)^2}}{\left(\int_t^\infty b(s)\,\Delta s\right)^{\frac{(\alpha-1)\beta}{(\beta-1)\alpha}}}\,\Delta t \right)^{\frac{\beta-1}{\beta}}.$$

Since $\int_0^\infty b(s)\,\Delta s < \infty$, $(1-1/\beta)^2 < 1$ and $\frac{(\alpha-1)\beta}{(\beta-1)\alpha} < 1$, we get

$$\left(\int_0^\infty \frac{b(t)^{(1-1/\beta)^2}}{\left(\int_t^\infty b(s)\,\Delta s\right)^{\frac{(\alpha-1)\beta}{(\beta-1)\alpha}}}\,\Delta t \right)^{\frac{\beta-1}{\beta}} \le \left(\int_0^\infty \frac{b(t)}{\left(\int_t^\infty b(s)\,\Delta s\right)^{\frac{(\alpha-1)\beta}{(\beta-1)\alpha}}}\,\Delta t \right)^{\frac{\beta-1}{\beta}} < M,$$

where M is a finite positive constant. So we can choose proper M such that $J \le \frac{M}{\alpha} K^{1/\beta}$. Since $J = \infty$, this inequality yields the assertion. The proof is completed. □

Theorem 3.5. *The possible cases of mutual behavior of integrals J, K, when $\alpha < \beta$ are as follows:*

- C_1) $J = \infty$, $K = \infty$;
- C_2) $J < \infty$, $K = \infty$ *when* $\alpha < \beta$;
- C_3) $J < \infty$, $K < \infty$.

Proof. The theorem can be easily proved by applying Lemmas 3.3, and 3.4. □

Lemma 3.6. *Let $\mu > 1, \lambda\mu > 1$ and f, g be nonnegative rd-continuous functions on $[t_2, \infty)$. Then*

$$\begin{aligned} &\left(\int_{t_2}^t g(s) \left(\int_s^t f(r)\,\Delta r \right)^{\lambda} \Delta s \right)^{\mu} \\ &\quad \le \lambda^{\mu} \left(\frac{\mu-1}{\lambda\mu-1} \right)^{\mu-1} \left(\int_{t_2}^t f(r) \left(\int_{t_2}^r g(s)\,\Delta s \right)^{\mu} \Delta r \right) \left(\int_{t_2}^t f(r)\,\Delta r \right)^{\lambda\mu-1}. \end{aligned} \tag{3.4}$$

Proof. Consider two cases: (i) $f > 0$; (ii) f has zeros.

Case (i): Integrating by parts, we have

$$\int_{t_2}^t f(s) \left(\int_s^t f(r)\,\Delta r \right)^{\lambda-1} \left(\int_{t_2}^s g(u)\,\Delta u \right) \Delta s = \frac{1}{\lambda} \int_{t_2}^t g(s) \left(\int_{\sigma(s)}^t f(r)\,\Delta r \right)^{\lambda} \Delta s.$$

According to the above equality and Hölder's inequality, we obtain

$$\begin{aligned}&\int_{t_2}^{t} g(s)\left(\int_{\sigma(s)}^{t} f(r)\,\Delta r\right)^{\lambda}\Delta s\\&\quad=\lambda\int_{t_2}^{t} f^{1/p}(s)f^{1/q}(s)\left(\int_{s}^{t} f(r)\Delta r\right)^{\lambda-1}\left(\int_{t_2}^{s} g(u)\,\Delta u\right)\Delta s\\&\quad\le\lambda\left(\int_{t_2}^{t} f(s)\left(\int_{t_2}^{s} g(u)\,\Delta u\right)^{p}\Delta s\right)^{1/p}\left(\int_{t_2}^{t} f(s)\left(\int_{s}^{t} f(r)\,\Delta r\right)^{(\lambda-1)q}\Delta s\right)^{1/q},\end{aligned}$$

where $p>1, 1/p+1/q=1$. Letting $p=\mu$, $q=\frac{\mu}{\mu-1}$, we get

$$\begin{aligned}&\int_{t_2}^{t} g(s)\left(\int_{\sigma(s)}^{t} f(r)\,\Delta r\right)^{\lambda}\Delta s\\&\quad\le\lambda\left(\int_{t_2}^{t} f(s)\left(\int_{t_2}^{s} g(u)\,\Delta u\right)^{\mu}\Delta s\right)^{1/\mu}\left(\int_{t_2}^{t} f(s)\left(\int_{s}^{t} f(r)\,\Delta r\right)^{\gamma}\Delta s\right)^{(\mu-1)/\mu},\end{aligned}\tag{3.5}$$

where $\gamma=(\lambda-1)\mu/(\mu-1)>-1$. Moreover, we have

$$\int_{t_2}^{t} f(s)\left(\int_{s}^{t} f(\tau)\,\Delta\tau\right)^{\gamma}\Delta s=\frac{1}{\gamma+1}\left(\int_{t_2}^{t} f(s)\,\Delta s\right)^{\gamma+1}.$$

Hence, from (3.5) we obtain

$$\begin{aligned}&\int_{t_2}^{t} g(s)\left(\int_{\sigma(s)}^{t} f(r)\,\Delta r\right)^{\lambda}\Delta s\\&\quad\le\lambda\left(\frac{\mu-1}{\lambda\mu-1}\right)^{(\mu-1)/\mu}\left(\int_{t_2}^{t} f(s)\left(\int_{t_2}^{s} g(u)\,\Delta u\right)^{\mu}\Delta s\right)^{1/\mu}\left(\int_{t_2}^{t} f(r)\,\Delta r\right)^{(\lambda\mu-1)/\mu}.\end{aligned}$$

Case (ii): If f has zeros for $t\ge t_2$, for any $s\in[t_2,t)_{\mathbb{T}}$, let $I(s)=\mathrm{cl}\{r\in(s,t):f(r)>0\}$. Since $\int_s^t f(r)\,\Delta r=\int_{I(s)} f(r)\,\Delta r$, the conclusion is true as before. The proof is completed. □

Theorem 3.7. *Let* $1<\alpha<\beta$, *and* $\int_0^\infty s^\beta b(s)\,\Delta s<\infty$. *Then any intermediate solution* x *of* (1.3) *satisfies* $\liminf_{t\to\infty}\frac{tx^\Delta(t)}{x(t)}>0$.

Proof. Integrating equation (1.3), we obtain $x^\Delta(t)=\left(\int_s^\infty b(r)x^\beta(r)\,\Delta r\right)^{1/\alpha}$. We can choose t_3 large enough so that $x(t)>0$, $x^\Delta(t)>0$ and $\int_{t_1}^\infty r^\beta b(r)\,\Delta r<1$ for $t\ge t_1$.

Choose t_4 large enough such that

$$k\left(\int_{t_2}^{\infty} r^\beta b(r)\,\Delta r\right)^{(\beta-\alpha)/\alpha}<1,\tag{3.6}$$

where $k=\frac{1}{\alpha}\left(\frac{\alpha(\beta-1)}{\beta-\alpha}\right)^{(\beta-1)/\beta}$. Let $\eta=\max\{t_3,t_4\}$. Integrating (1.3) twice, we obtain

$$x(t)-x(\eta)=\int_{\eta}^{t}\left(\int_{\sigma(s)}^{t} b(r)x^\beta(r)\,\Delta r+\int_{t}^{\infty} b(r)x^\beta(r)\,\Delta r\right)^{1/\alpha}\Delta s$$

for $t\ge\eta$.

From the inequality
$$(X+Y)^{1/\alpha} \le X^{1/\alpha} + Y^{1/\alpha},$$
where X, Y are positive numbers, we obtain
$$x(t) - x(\eta) \le \int_\eta^t \left(\int_{\sigma(s)}^t b(r)x^\beta(r)\,\Delta r \right)^{1/\alpha} \Delta s + t\left(\int_t^\infty b(r)x^\beta(r)\,\Delta r \right)^{1/\alpha} \Delta s.$$
Let $f(r) = b(r)x^\beta(r)$, $g(s) \equiv 1$, $\lambda = 1/\alpha$ and $\mu = \beta$ and by Lemma 3.6 we have
$$x(t) - x(\eta) \le k\left(\int_\eta^t r^\beta b(r)x^\beta(r)\,\Delta r \right)^{1/\beta} \left(\int_\eta^t b(r)x^\beta(r)\,\Delta r \right)^{(\beta-\alpha)/\alpha} + t\left(\int_t^\infty b(r)x^\beta(r)\,\Delta r \right)^{1/\alpha},$$
from (3.6),
$$\begin{aligned} x(t) - x(\eta) &\le \left(\int_\eta^t r^\beta b(r)x^\beta(r)\,\Delta r \right)^{1/\beta} + t\left(\int_t^\infty b(r)x^\beta(r)\,\Delta r \right)^{1/\alpha} \\ &= \left(\int_\eta^t r^\beta b(r)x^\beta(r)\,\Delta r \right)^{1/\beta} + tx^\Delta(t). \end{aligned}$$
Since
$$\left(\int_\eta^t r^\beta b(r)x^\beta(r)\,\Delta r \right)^{1/\beta} \le x(t)\left(\int_\eta^\infty r^\beta b(r)\,\Delta r \right)^{1/\beta},$$
we obtain
$$1 - \frac{x(\eta)}{x(t)} \le \left(\int_\eta^\infty r^\beta b(r)\,\Delta r \right)^{1/\beta} + \frac{tx^\Delta(t)}{x(t)},$$
i.e.
$$\frac{tx^\Delta(t)}{x(t)} \ge 1 - \frac{x(\eta)}{x(t)} - \left(\int_\eta^\infty r^\beta b(r)\,\Delta r \right)^{1/\beta}.$$
We obtain the assertion from (3.6). The proof is completed. □

4 Examples

In this section, we will present an example to illustrate our main results.

Example 4.1. Let $\mathbb{T} = \mathbb{R}$. Consider the following Emden–Fowler equation
$$x'' + \frac{1}{(t+2)^3}|x|^2 \operatorname{sgn} x = 0, \qquad t \ge 0. \tag{4.1}$$
We have $\alpha = 1$, $\beta = 2$, $a(t) = 1$ and $b(t) = \frac{1}{(t+2)^3}$ Thus
$$\begin{aligned} J &= \int_0^\infty \frac{1}{a^{1/\alpha}(t)} \left(\int_t^\infty b(s)\,ds \right)^{1/\alpha} dt \\ &= \int_0^\infty \int_t^\infty \frac{1}{(s+2)^3}\,ds\,dt \\ &= 0 < \infty. \end{aligned} \tag{4.2}$$
From Theorem 3.2 we can get that (4.1) above has subdominant solutions for $t \ge 0$.

5 Conclusion

At the end of this paper, let us suggest the further possible research in the theory of dynamic equations, concretely for Emden–Fowler dynamic equations. First, the coexistence of three classes of nonoscillatory solutions can be studied. Second, the sufficient and necessary conditions for the existence of intermediate solutions may be established. Third, the cases of sub-linear and half-linear about the corresponding conclusions can also be considered.

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References

[1] R. P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.* **141**(2002), 1–26. MR1908825; url

[2] M. Cecchi, Z. Došlá, M. Marini, Intermediate solutions for Emden–Fowler type equations: continuous versus discrete, *Adv. Dyn. Syst. Appl.* **3**(2008), 161–176. MR2547667

[3] M. Cecchi, Z. Došlá, M. Marini, On intermediate solutions and the Wronskian for half-linear differential equations, *J. Math. Anal. Appl.* **336**(2007), 905–918. MR2352988; url

[4] Z. Došlá, M. Marini, On super-linear Emden–Fowler type differential equations, *J. Math. Anal. Appl.* **416**(2014), 497–510. MR3188719; url

[5] Z. Došlá, I. Vrkoč, On an extension of the Fubini theorem and its applications in ODEs, *Nonlinear Anal.* **57**(2004), 531–548. MR2062993; url

[6] Á. Elbert, T. Kusano, Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations, *Acta Math. Hungar.* **56**(1990), 325–336. MR1111319; url

[7] L. Erbe, B. Jia, A. Peterson, On the asymptotic behavior of solutions of Emden–Fowler equations on time scales, *Ann. Mat. Pura Appl. (4)* **191**(2012), 205–217. MR2909796; url

[8] R. H. Fowler, Emden's equation: The solution of Emden's and similar differential equations, *Mon. Not. R. Astron. Soc.* **91**(1930), 63–91. url

[9] Z. Han, S. Sun, B. Shi, Oscillation criteria for a class of second order Emden–Fowler delay dynamic equations on time scales, *J. Math. Anal. Appl.* **334**(2007), 847–858. MR2338632; url

[10] B. Jia, L. Erbe, A. Peterson, Kiguradze-type oscillation theorems for second order superlinear dynamic equations on time scales, *Canad. Math. Bull.* **54**(2011), 580–592. MR2894509; url

[11] K. Kamo, H. Usami, Characterization of slowly decaying positive solutions of second-order quasilinear ordinary differential equations with sub-homogeneity, *Bull. Lond. Math. Soc.* **42**(2010), 420–428. MR2651937; url

[12] Y. Liu, J. Zhang, J. Yan, Existence of oscillatory solutions of second order delay differential equations, *J. Comput. Appl. Math.* **277**(2015), 17–22. MR3272160; url

[13] M. Naito, On the asymptotic behavior of nonoscillatory solutions of second order quasilinear ordinary differential equations, *J. Math. Anal. Appl.* **381**(2011), 315–327. MR2796212; url

[14] J. Vítovec, Critical oscillation constant for Euler-type dynamic equations on time scales, *Appl. Math. Comput.* **243**(2014), 838–848. MR3244531; url

[15] J. S. W. Wong, On the generalized Emden–Fowler equation, *SIAM Rev.* **17**(1975), 339–360. MR0367368

[16] J. Yu, Z. Guo, Boundary value problems of discrete generalized Emden–Fowler equation, *Sci. China Ser. A* **49**(2006), 1303–1314. MR2287260; url

[17] Y. Zhou, Y. Lan, Classification and existence of nonoscillatory solutions of the second-order neutral delay dynamic equations on time scales, *Journal of Mathematical Sciences* **198**(2014), 279–295. url